



Faculty of Science and Engineering

Department of Mathematics & Statistics

## END OF SEMESTER ASSESSMENT PAPER

MODULE TITLE: Numerical Partial Differential Equations

MODULE CODE: MS4008

SEMESTER: Autumn 2024

LECTURER/EXAMINER: Prof. N. Kopteva

EXTERNAL EXAMINER: Dr Thierry Platini

DURATION OF EXAMINATION:  $2\frac{1}{2}$  hours

PERCENTAGE OF TOTAL MARKS: 75% (+25 % for continuous assessment)

### INSTRUCTIONS TO CANDIDATES:

- Answer questions 1, 2, and 3. To obtain maximum marks you must show all your work clearly and in detail.
- Standard mathematical tables are provided by the invigilators. Under no circumstances should you use your own tables or be in possession of any written material other than that provided by the invigilators.
- Non-programmable, non-graphical calculators that have been approved by the lecturer are permitted.

**1. Answer parts (a) and (b), and one of parts (c) and (d).****30%**(a) The norm in the space  $H^1(0, 1)$  is given by

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$$\|v\| = \sqrt{\int_0^1 (v'^2(x) + v^2(x)) dx}.$$

Consider the problem:

$$-u'' + 5x^3 u = f \quad \text{for } x \in (0, 1), \quad u'(0) = u(1) = 0.$$

i. Obtain a weak formulation of this problem in the form of

• *Variational problem (VAR):*Find  $u \in V$  such that  $a(u, v) = L(v) \quad \forall v \in V$ .Here a bilinear form  $a(\cdot, \cdot)$ , a linear form  $L(\cdot)$ , and a subspace  $V$  of  $H^1(0, 1)$  should be specified.ii. Show that the bilinear form  $a(\cdot, \cdot)$  is symmetric.iii. Then show that, for some positive constants  $\alpha$  and  $\gamma$ , one has

$$\alpha \|v\|^2 \leq a(v, v) \leq \gamma \|v\|^2 \quad \text{for all } v \in V.$$

Specify the constants  $\alpha$  and  $\gamma$ .

iv. Consider a discretization in the form of

• *Discrete Variational problem (VAR<sup>h</sup>):*Let  $V^h$  be a finite dimensional subspace of  $V$ .Find  $u_h \in V^h$  such that  $a(u_h, v_h) = L(v_h) \quad \forall v_h \in V^h$ .Prove that for the solution  $u$  of problem (VAR) and the solution  $u_h$  of problem (VAR<sup>h</sup>), we have

$$\|u - u_h\|^2 \leq (\gamma/\alpha) \|u - v_h\|^2 \quad \forall v_h \in V^h.$$

(b) Obtain a weak formulation of the following problem:

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$$-u'' + 2u' = f \quad \text{for } x \in (0, 1), \quad -u'(0) + 5u(0) = 7, \quad u(1) = 3.$$

Note: you are expected to specify the space in which  $u$  is found, from which space the arbitrary test functions  $v$  are taken, and which boundary conditions  $u$  and  $v$  are required to satisfy, if any.

(c) Consider the problem:

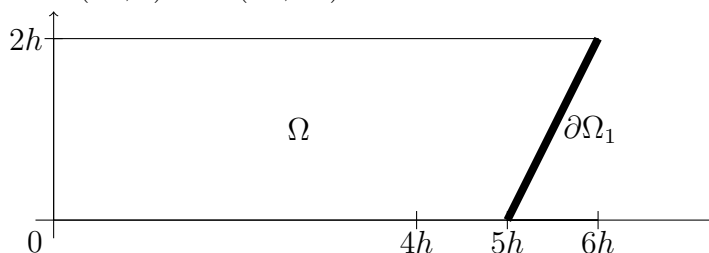
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$$-\left(\frac{\partial^2 u}{\partial x^2} + \frac{\partial^2 u}{\partial y^2}\right) = f \quad \text{for } (x, y) \in \Omega,$$

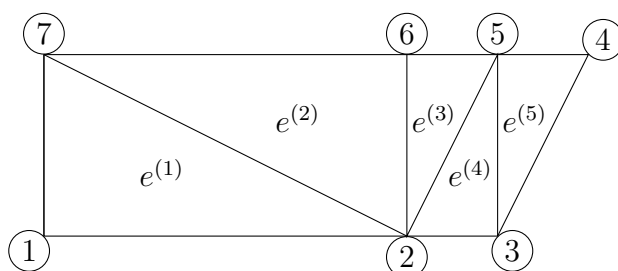
$$u(x, y) = 0 \quad \text{for } (x, y) \in \partial\Omega_1, \quad \frac{\partial u(x, y)}{\partial \mathbf{n}} = 0 \quad \text{for } (x, y) \in \partial\Omega_2,$$

where  $f$  is constant,  $\Omega$  is a two-dimensional domain, the boundary of which  $\partial\Omega$  is the union of disjoint sets  $\partial\Omega_1$  and  $\partial\Omega_2$ .

This problem is posed in the trapezoidal domain  $\Omega$  (see below) with vertices  $(0, 0)$ ,  $(5h, 0)$ ,  $(6h, 2h)$ , and  $(0, 2h)$ , while  $\partial\Omega_1$  is the part of the boundary joining the vertices  $(5h, 0)$  and  $(6h, 2h)$ :



The above problem is discretized using *linear finite elements* with the following triangulation:

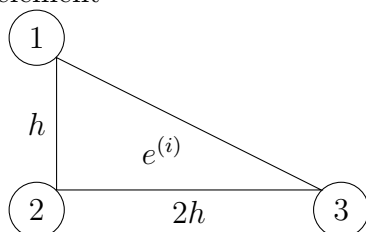


Each  $e^{(i)}$ , for  $i = 1, \dots, 5$ , is a right-angled triangle similar to the one shown below.

For this discretization:

- Find the global stiffness matrix  $K_{(f)}$  and the global load vector  $F_{(f)}$  in which the boundary conditions are still to be addressed.
- Find the global stiffness matrix  $K$  and the global load vector  $F$  which take the boundary conditions into consideration.
- Then write the numerical method as a linear system  $KU = F$ . For each entry of the unknown vector  $U$ , specify with which mesh node it is associated.

Note that for the linear element



the local stiffness matrix  $K^{(i)}$  and the local load vector  $F^{(i)}$  are given by

$$K^{(i)} = \frac{1}{4} \begin{bmatrix} 4 & -4 & 0 \\ -4 & 5 & -1 \\ 0 & -1 & 1 \end{bmatrix}, \quad F^{(i)} = \frac{f}{3} \begin{bmatrix} h^2 \\ h^2 \\ h^2 \end{bmatrix}.$$

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(d) Recall the problem from part (b):

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$$-u'' + 2u' = f \quad \text{for } x \in (0, 1), \quad -u'(0) + 5u(0) = 7, \quad u(1) = 3.$$

- i. Suppose this problem is discretized using piecewise **linear** finite elements with the local shape functions  $\phi_1^{(i)}$  and  $\phi_2^{(i)}$ , defined on each mesh element  $e^{(i)} = (x_i, x_{i+1})$ , with  $h_i = x_{i+1} - x_i$ , by

$$\phi_k^{(i)} = \varphi_k \left( \frac{x - x_i}{h_i} \right), \quad k = 1, 2, \quad \varphi_1(t) = 1 - t, \quad \varphi_2(t) = t.$$

Find the local stiffness matrix  $K^{(i)}$  and the local load vector  $F^{(i)}$ , assuming that  $f$  is constant.

- ii. On the mesh  $0 = x_1 < x_2 < x_3 < x_4 < x_5 = 1$ , with  $h_i = x_{i+1} - x_i$ , write the above numerical method as a linear system  $KU = F$ , where  $K$  is the global stiffness matrix,  $F$  is the global load vector, and  $U$  is the computed-solution vector.

(Note: do not forget to address the boundary conditions.)

**2. Answer parts (a), (b), (d), and one of parts (c) and (e).**

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Let  $a$  be a positive constant. In the square domain  $\Omega = (0, 1) \times (0, 1)$  with the boundary  $\partial\Omega$  consider the problem:

$$\begin{aligned} Lu = -a \left( \frac{\partial^2 u}{\partial x^2} + \frac{\partial^2 u}{\partial y^2} \right) - (x + 3) \frac{\partial u}{\partial y} &= f(x, y) & \text{for } (x, y) \in \Omega, \\ u(x, y) &= 0 & \text{for } (x, y) \in \partial\Omega. \end{aligned}$$

This problem is discretized on the uniform mesh  $\{(x_i, y_j)\}_{i,j=1,\dots,N+1}$ , where  $x_i = (i - 1)h$ ,  $y_j = (j - 1)h$ ,  $h = 1/N$ , by the finite difference method:

$$\begin{aligned} L^h U_{i,j} &= -a \left( \frac{U_{i-1,j} + U_{i+1,j} + U_{i,j+1} + U_{i,j-1} - 4U_{i,j}}{h^2} \right) - (x_i + 3) \left( \frac{U_{i,j+1} - U_{i,j}}{h} \right) \\ &= f(x_i, y_j) \end{aligned}$$

for  $i, j = 2, \dots, N$ , with the boundary conditions:

$$U_{ij} = 0 \quad \text{for } (x_i, y_j) \in \partial\Omega.$$

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- (a) Specify the local truncation error  $r_{ij}$  of this method associated with the mesh node  $(x_i, y_j)$ . Obtain a truncation error estimate of the form  $r_{ij} = O(h^p)$  for some  $p > 0$ . 5%

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- (b) Show that the finite difference operator  $L^h$ , possibly under a certain condition that involves  $h$  and  $a$ , satisfies the discrete maximum principle of the form: 6%

$$\left. \begin{array}{l} L^h V_{ij} \leq 0 \quad \forall i, j = 2, \dots, N \\ V_{ij} \leq 0 \quad \forall (x_i, y_j) \in \partial\Omega \end{array} \right\} \Rightarrow V_{ij} \leq 0 \quad \forall i, j = 1, \dots, N+1.$$

Specify the discrete maximum principle condition on  $h$ , if one was used in your proof.

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- (c) Using the result of part (b), show that the finite difference operator  $L^h$  satisfies the discrete comparison principle of the form: 6%

$$\left. \begin{array}{l} |L^h W_{ij}| \leq L^h V_{ij} \quad \forall i, j = 2, \dots, N \\ |W_{ij}| \leq V_{ij} \quad \forall (x_i, y_j) \in \partial\Omega \end{array} \right\} \Rightarrow |W_{ij}| \leq V_{ij} \quad \forall i, j = 1, \dots, N+1.$$

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- (d) Using the discrete comparison principle described in part (c), show that 8%

$$W_{ij} = 0 \quad \forall (x_i, y_j) \in \partial\Omega \Rightarrow \max_{i,j=1,\dots,N+1} |W_{ij}| \leq C_0 \max_{i,j=2,\dots,N} |L^h W_{ij}|$$

for some positive constant  $C_0$ . Specify this constant.

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- (e) Using the result of part (a) and the property described in part (d), estimate the error of the finite difference method 6%

$$\max_{i,j=1,\dots,N+1} |U_{ij} - u(x_i, y_j)|,$$

where  $U_{ij}$  is the computed solution and  $u(x_i, y_j)$  is the exact solution at the mesh node  $(x_i, y_j)$ .

**3. Answer parts (a) and (d), and one of parts (b) and (c).****20%**

Consider the problem

$$\begin{aligned}
 u_t &= u_{xx} && \text{for } x \in (0, 1), \, t > 0, \\
 u(0, t) &= u(1, t) = 0 && \text{for } t > 0, \\
 u(x, 0) &= g(x) && \text{for } x \in [0, 1].
 \end{aligned}$$

This problem is discretized on the uniform mesh

$$\{(x_j, t_m), j = 1, \dots, N+1, m = 1, 2, \dots\},$$

where  $x_j = (j-1)h$ , with  $h = 1/N$ , and  $t_m = (m-1)k$ . Let  $U_j^m$  be the computed solution associated with the point  $(x_j, t_m)$ .

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- (a) Using Von Neumann's method, find out whether the following backwards differentiation method is unconditionally stable, unconditionally unstable, or conditionally stable. If it is conditionally stable, find the stability condition.

9%

$$\frac{1}{k} \left( \frac{3}{2} U_j^{m+1} - 2 U_j^m + \frac{1}{2} U_j^{m-1} \right) = \frac{U_{j-1}^{m+1} - 2 U_j^{m+1} + U_{j+1}^{m+1}}{h^2}.$$

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- (b) Using Von Neumann's method, find out whether the following method is unconditionally stable, unconditionally unstable, or conditionally stable. If it is conditionally stable, find the stability condition.

9%

$$\frac{U_j^{m+1} - U_j^m}{k} = 0.4 \frac{U_{j-1}^{m+1} - 2 U_j^{m+1} + U_{j+1}^{m+1}}{h^2} + 0.6 \frac{U_{j-1}^m - 2 U_j^m + U_{j+1}^m}{h^2}.$$

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- (c) Estimate the local truncation errors of the methods in parts (a) and (b).

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- (d) For each of the methods in parts (a) and (b), specify whether it is implicit or explicit. Explain your answer.

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