Syntax and Semantics of FuzzyDL

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1. Comments Any line beginning with # or % is considered a comment.

2. Fuzzy operators. $\ominus, \oplus, \ominus$ and $\Rightarrow$ denote a t-norm, t-conorm, negation function and implication function respectively; $\alpha, \beta \in [0, 1]$.

Łukasiewicz negation	$\ominus_{Ł}\alpha$	$1-\alpha$
Gödel t-norm	$\alpha \otimes_G \beta$	$\min\{\alpha, \beta\}$
Łukasiewicz t-norm	$\alpha \otimes_{Ł} \beta$	$\max\{\alpha+\beta-1, 0\}$
Gödel t-conorm	$\alpha \oplus_G \beta$	$\max\{\alpha, \beta\}$
Łukasiewicz t-conorm	$\alpha \oplus_{Ł} \beta$	$\min\{\alpha+\beta, 1\}$
Gödel implication	$\alpha \Rightarrow_G \beta$	$\begin{cases} 1, & \text{if } \alpha \leq \beta \\ \beta, & \text{if } \alpha > \beta \end{cases}$
Łukasiewicz implication	$\alpha \Rightarrow_{Ł} \beta$	$\min\{1, 1-\alpha+\beta\}$
Kleene-Dienes implication	$\alpha \Rightarrow_{KD} \beta$	$\max\{1-\alpha, \beta\}$
Zadeh'set inclusion	$\alpha \Rightarrow_Z \beta$	1 iff $\alpha \leq \beta$, 0 otherwise

The reasoner can accept three different semantics, which are used to interpret $\ominus, \oplus, \ominus$ and $\Rightarrow$.

- Zadeh semantics: Łukasiewicz negation, Gödel t-norm, Gödel t-conorm and Kleene-Dienes implication (except in GCIs and concept implication, where we use Zadeh implication). This semantics is included for compatibility with earlier papers about fuzzy description logics.
- Łukasiewicz semantics: Łukasiewicz negation, Łukasiewicz t-norm, Łukasiewicz t-conorm and Łukasiewicz implication.
- Classical semantics: classical (crisp) conjunction, disjunction, negation and implication.

Syntax to define the semantics of the knowledge base:

(define-fuzzy-logic [lukasiewicz — zadeh — classical])

3. Truth constants. Truth constants can be defined as follows (and later on, they can be used as the lower bound of a fuzzy axiom): (define-truth-constant constant n), where n is a rational number in $[0, 1]$.

4. Concept modifiers. Modifiers change the membership function of a fuzzy concept.

(define-modifier CM linear-modifier(c))	linear hedge with $c > 0$ (Figure 1 (f))
(define-modifier CM triangular-modifier(a,b,c))	triangular function (Figure 1 (d))

5. Concrete Fuzzy Concepts. Concrete Fuzzy Concepts (CFCs) define a name for a fuzzy set with an explicit fuzzy membership function (we assume $a \leq b \leq c \leq d$).

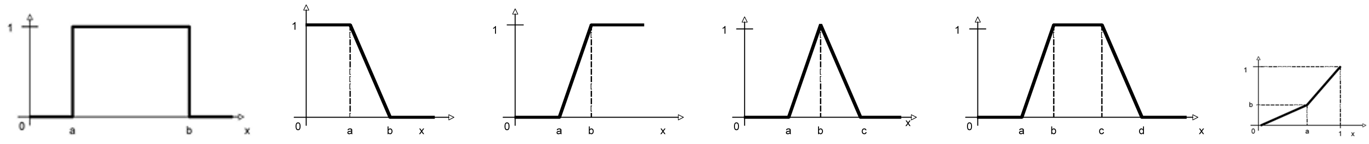

Figure 1: (a) Crisp value; (b) L-function; (c) R-function; (d) (b) Triangular function; (e) Trapezoidal function; (f) Linear hedge

(define-fuzzy-concept CFC crisp(k1,k2,a,b))	crisp interval (Figure 1 (a))
(define-fuzzy-concept CFC left-shoulder(k1,k2,a,b))	left-shoulder function (Figure 1 (b))
(define-fuzzy-concept CFC right-shoulder(k1,k2,a,b))	right-shoulder function (Figure 1 (c))
(define-fuzzy-number CFC triangular(k1,k2,a,b,c))	triangular function (Figure 1 (d))
(define-fuzzy-concept CFC trapezoidal(k1,k2,a,b,c,d))	trapezoidal function (Figure 1 (e))
(define-fuzzy-concept CFC linear(k1,k2,a,b))	linear function (Figure 1 (f))
(define-fuzzy-concept CFC modified(mod,F))	modified datatype

6. Fuzzy Numbers. Firstly, if fuzzy numbers are used, one has to define the range $[k_1, k_2] \subseteq \mathbb{R}$ as follows:

```
(define-fuzzy-number-range k1 k2)
```

Let `fi` be a fuzzy number (a_i, b_i, c_i) $(a \leq b \leq c)$, and $n \in \mathbb{R}$. Valid fuzzy number expressions (see Figure 1 (d)) are:

name	fuzzy number definition	*name*
(a, b, c)	fuzzy number	(a, b, c)
n	real number	(n, n, n)
(f+ f1 f2 ... fn)	addition	$(\sum_{i=1}^{n} a_i, \sum_{i=1}^{n} b_i, \sum_{i=1}^{n} c_i)$
(f- f1 f2)	substraction	$(a_1 - c_2, b_1 - b_2, c_1 - a_2)$
(f* f1 f2 ... fn)	product	$(\prod_{i=1}^{n} a_i, \prod_{i=1}^{n} b_i, \prod_{i=1}^{n} c_i)$
(f/ f1 f2)	division	$(a_1/c_2, b_1/b_2, c_1/a_2)$

Fuzzy numbers can be named as:

```
(define-fuzzy-number name fuzzyNumberExpression)
```

7. Features. Features are functional datatype attributes.

(functional F)	Firstly, the feature is defined. Then we set the range
(range F *integer* k1 k2)	The range are integer numbers in $[k_1, k_2]$
(range F *real* k1 k2)	The range are rational number in $[k_1, k_2]$
(range F *string*)	The range are strings
(range F *boolean*)	The range are booleans

8. Datatype restrictions.

(>= F var)	at least datatype restriction	$\sup_{b\in\Delta_{\mathbf{D}}}[F^{\mathcal{I}}(x,b)\otimes(b\geq var)]$
(>= F $f(F_1,\ldots,F_n)$)	at least datatype restriction	$\sup_{b\in\Delta_{\mathbf{D}}}[F^{\mathcal{I}}(x,b)\otimes(b\geq f(F_1,\ldots,F_n)^{\mathcal{I}})]$
(>= F FN)	at least datatype restriction	$\sup_{b,b'\in\Delta_{\mathbf{D}}}[F^{\mathcal{I}}(x,b)\otimes(b\geq b')\otimes FN^{\mathcal{I}}(b')]$
(<= F var)	at most datatype restriction	$\sup_{b\in\Delta_{\mathbf{D}}}[F^{\mathcal{I}}(x,b)\otimes(b\leq var)]$
(<= F $f(F_1,\ldots,F_n)$)	at most datatype restriction	$\sup_{b\in\Delta_{\mathbf{D}}}[F^{\mathcal{I}}(x,b)\otimes(b\leq f(F_1,\ldots,F_n)^{\mathcal{I}})]$
(<= F FN)	at most datatype restriction	$\sup_{b,b'\in\Delta_{\mathbf{D}}}[F^{\mathcal{I}}(x,b)\otimes(b\leq b')\otimes FN^{\mathcal{I}}(b')]$
(= F var)	exact datatype restriction	$\sup_{b\in\Delta_{\mathbf{D}}}[F^{\mathcal{I}}(x,b)\otimes(b=var)]$
(= F $f(F_1,\ldots,F_n)$)	exact datatype restriction	$\sup_{b\in\Delta_{\mathbf{D}}}[F^{\mathcal{I}}(x,b)\otimes(b=f(F_1,\ldots,F_n)^{\mathcal{I}})]$
(= F FN)	exact datatype restriction	$\sup_{b\in\Delta_{\mathbf{D}}}[F^{\mathcal{I}}(x,b)\otimes FN^{\mathcal{I}}(b)]$

In datatype restrictions, the variable *var* may be replaced with a value, i.e., an integer, a real, a string, or a boolean constant (true, false) depending on the range of the feature F, although in the case of booleans, <= and >= are not allowed.

Furthermore, f is defined as follows:

$$\begin{array}{rl} f(F_1,\ldots,F_n)\rightarrow & F \\ & real \\ & (nF)\mid(n*F) \\ & (F_1-F_2) \\ & (F_1+F_2+\ldots+F_n) \end{array}$$

9. Constraints. Constraints are of the form (constraints ⟨ constraint-i ⟩ +), where ⟨ constraint-i ⟩ is one of the following (with $OP = \geq \mid \leq \mid =.$):

(a1 * var1 + … + ak * vark OP number)	linear inequation	$a_1 var_1+\ldots+a_k*var_k\ OP\ number$
(binary var)	binary variable	$var\in\{0,1\}$
(free var)	binary variable	$var\in(-\infty,\infty)$

10. Show statements.

(show-concrete-fillers F1 … Fn)	show value of the fillers of $F_1\ldots F_n$
(show-concrete-fillers-for a F1 … Fn)	show value of the fillers of $F_1\ldots F_n$ for a
(show-concrete-instance-for a F C1 … Cn)	show degrees of being the F filler of a an instance of C_i
(show-abstract-fillers R1 … Rn)	show fillers of $R_1\ldots R_n$ and membership to any concept
(show-abstract-fillers-for a R1 … Rn)	show fillers of $R_1\ldots R_n$ for a and membership to any concept
(show-concepts a1 … an)	show membership of $a_1\ldots a_n$ to any concept
(show-instances C1 … Cn)	show value of the instances of the concepts $C_1\ldots C_n$
(show-variables x1 … xn)	show value of the variables $x_1\ldots x_n$
(show-language)	show language of the KB, from $\mathcal{ALC}$ to $\mathcal{SHIF}(D)$

where C_i is the name of a defined concrete fuzzy concept. We assume that an abstract role R appears in at most one statement of the forms show-abstract-fillers? or show-abstract-fillers-for?.

11. Crisp declarations.

(crisp-concept C1 … Cn)	concepts $C_1\ldots C_n$ are crisp
(crisp-role R1 … Rn)	roles $R_1\ldots R_n$ are crisp

12. Concept expressions.

top	top concept	1
bottom*	bottom concept	0
A	atomic concept	$A^{\mathcal{I}}(x)$
(and C1 C2)	concept conjunction	${C_1}^{\mathcal{I}}(x) \otimes {C_2}^{\mathcal{I}}(x)$
(g-and C1 C2)	Gödel conjunction	${C_1}^{\mathcal{I}}(x) \otimes_G {C_2}^{\mathcal{I}}(x)$
(l-and C1 C2)	Łukasiewicz conjunction	${C_1}^{\mathcal{I}}(x) \otimes_{Ł} {C_2}^{\mathcal{I}}(x)$
(or C1 C2)	concept disjunction	${C_1}^{\mathcal{I}}(x) \oplus {C_2}^{\mathcal{I}}(x)$
(g-or C1 C2)	Gödel disjunction	${C_1}^{\mathcal{I}}(x) \oplus_G {C_2}^{\mathcal{I}}(x)$
(l-or C1 C2)	Łukasiewicz disjunction	${C_1}^{\mathcal{I}}(x) \oplus_{Ł} {C_2}^{\mathcal{I}}(x)$
(not C1)	concept negation	$\ominus_{Ł} {C_1}^{\mathcal{I}}(x)$
(implies C1 C2)	concept implication	${C_1}^{\mathcal{I}}(x) \Rightarrow {C_2}^{\mathcal{I}}(x)$
(g-implies C1 C2)	Gödel implication	${C_1}^{\mathcal{I}}(x) \Rightarrow_G {C_2}^{\mathcal{I}}(x)$
(l-implies C1 C2)	Łukasiewicz implication	${C_1}^{\mathcal{I}}(x) \Rightarrow_{Ł} {C_2}^{\mathcal{I}}(x)$
(kd-implies C1 C2)	Kleene-Dienes implication	${C_1}^{\mathcal{I}}(x) \Rightarrow_{KD} {C_2}^{\mathcal{I}}(x)$
(all R C1)	universal role restriction	$\inf_{y \in \Delta^{\mathcal{I}}} R^{\mathcal{I}}(x,y) \Rightarrow {C_1}^{\mathcal{I}}(y)$
(some R C1)	existential role restriction	$\sup_{y \in \Delta^{\mathcal{I}}} R^{\mathcal{I}}(x,y) \otimes {C_1}^{\mathcal{I}}(y)$
(ua s C1)	upper approximation	$\sup_{y \in \Delta^{\mathcal{I}}} s^{\mathcal{I}}(x,y) \otimes C^{\mathcal{I}}(y)$
(la s C1)	lower approximation	$\inf_{y \in \Delta^{\mathcal{I}}} s^{\mathcal{I}}(x,y) \Rightarrow C^{\mathcal{I}}(y)$
(tua s C1)	tight upper approximation	$\inf_{z \in X} \{s_i^{\mathcal{I}}(x,z) \Rightarrow \sup_{y \in \Delta^{\mathcal{I}}} \{s_i^{\mathcal{I}}(y,z) \otimes C^{\mathcal{I}}(y)\}\}$
(lua s C1)	loose upper approximation	$\sup_{z \in X} \{s_i^{\mathcal{I}}(x,z) \otimes \sup_{y \in \Delta^{\mathcal{I}}} \{s_i^{\mathcal{I}}(y,z) \otimes C^{\mathcal{I}}(y)\}\}$
(tla s C1)	tight lower approximation	$\inf_{z \in X} \{s_i^{\mathcal{I}}(x,z) \Rightarrow \inf_{y \in \Delta^{\mathcal{I}}} \{s_i^{\mathcal{I}}(y,z) \Rightarrow C^{\mathcal{I}}(y)\}\}$
(lla s C1)	loose lower approximation	$\sup_{z \in X} \{s_i^{\mathcal{I}}(x,z) \otimes \inf_{y \in \Delta^{\mathcal{I}}} \{s_i^{\mathcal{I}}(y,z) \Rightarrow C^{\mathcal{I}}(y)\}\}$
(self S)	local reflexivity concept	$S^{\mathcal{I}}(x)(x,x)$
(CM C1)	modifier applied to concept	$f_{\mathtt{m}}({C_1}^{\mathcal{I}}(x))$
(CFC)	concrete fuzzy concept	$CFC^{\mathcal{I}}(x)$
(FN)	fuzzy number	$FN^{\mathcal{I}}(x)$
([>= var] C1)	threshold concept	$\begin{cases} {C_1}^{\mathcal{I}}(x), & \text{if } {C_1}^{\mathcal{I}}(x) \geq w \\ 0, & \text{otherwise} \end{cases}$
([<= var] C1)	threshold concept	$\begin{cases} {C_1}^{\mathcal{I}}(x), & \text{if } {C_1}^{\mathcal{I}}(x) \leq w \\ 0, & \text{otherwise} \end{cases}$
(n C1)	weighted concept	$n{C_1}^{\mathcal{I}}(x)$
(w-sum (n1 C1) ... (nk Ck))	weighted sum	$n_1 {C_1}^{\mathcal{I}}(x) + \ldots + n_k {C_k}^{\mathcal{I}}(x)$
(w-max (v1 C1) ... (vk Ck))	weighted maximum	$\max_{i=1}^{k} \min\{v_i, x_i\}$
(w-min (v1 C1) ... (vk Ck))	weighted minimum	$\min_{i=1}^{k} \max\{1 - v_i, x_i\}$
(w-sum-zero (n1 C1) ... (nk Ck))	weighted sum-zero	$\begin{cases} 0 & \text{if } {C_i}^{\mathcal{I}}(x) = 0 \text{ for some } i \in \{1, \ldots, n\} \\ \text{w-sum} & \text{otherwise} \end{cases}$
(owa (w1 ... wn) (C1 ... Cn))	OWA aggregation operator	$\sum_{i=1}^{n} w_i y_i$
(q-owa q C1 ... Cn)	quantifier-guided OWA	Same as OWA taking $w_i = q(i/n) - q((i-1)/n)$
(choquet (w1 ... wn) (C1 ... Cn))	Choquet integral	$y_1 \cdot w_1 + \sum_{i=2}^{n} (y_i - y_{i-1}) w_i$
(sugeno (v1 ... vn) (C1 ... Cn))	Sugeno integral	$\max_{i=1}^{n} \{\min\{y_i, mu_i\}\}$
(q-sugeno (v1 ... vn) (C1 ... Cn))	Quasi-Sugeno integral	$\max_{i=1}^{n} \{y_i \otimes_{Ł} mu_i\}$
(DR)	datatype restriction	$DR^{\mathcal{I}}(x)$

where:

- a is an individual
- $n_1, \ldots, n_k \in [0,1]$ with $\sum_{i=1}^{k} n_i \leq 1$,
- $w_1, \ldots, w_k \in [0,1]$ with $\sum_{i=1}^{k} w_i = 1$,
- $v_1, \ldots, v_k \in [0,1]$ with $\max_{i=1}^{k} k_i = 1$,
- q is a quantifier (defined as a right-shoulder or a linear function),

- w is a variable or a real number in $[0,1]$,
- y_i is the i-largest of the ${C_i}^{\mathcal{I}}(x)$,
- mu_i is defined as follows: $mu_1 = ow_1$, $mu_i = ow_i \oplus mu_{i-1}$ for $i \in \{2,\dots,n\}$,
- ow_i is the weight v_i of the i-largest of the ${C_i}^{\mathcal{I}}(x)$.
- Fuzzy numbers can only appear in existential, universal and datatype restrictions.
- In threshold concepts var may be replaced with w.
- Fuzzy relations s should be previously defined as fuzzy similarity relation or a fuzzy equivalence relation as `(define-fuzzy-similarity s)` or `(define-fuzzy-equivalence s)`, respectively.

Important note: The reasoner restricts the calculus to witnessed models.

13. Axioms.

(instance a C1 [d])	concept assertion	${C_1}^{\mathcal{I}}(a^{\mathcal{I}}) \geq d$
(related a b R [d])	role assertion	$R^{\mathcal{I}}(a^{\mathcal{I}}, b^{\mathcal{I}}) \geq d$
(implies C1 C2 [d])	GCI	$\inf_{x \in \Delta^{\mathcal{I}}} {C_1}^{\mathcal{I}}(x) \Rightarrow {C_2}^{\mathcal{I}}(x) \geq d$
(g-implies C1 C2 [d])	Gödel GCI	$\inf_{x \in \Delta^{\mathcal{I}}} {C_1}^{\mathcal{I}}(x) \Rightarrow_G {C_2}^{\mathcal{I}}(x) \geq d$
(kd-implies C1 C2 [d])	Kleene-Dienes GCI	$\inf_{x \in \Delta^{\mathcal{I}}} {C_1}^{\mathcal{I}}(x) \Rightarrow_{KD} {C_2}^{\mathcal{I}}(x) \geq d$
(l-implies C1 C2 [d])	Łukasiewicz GCI	$\inf_{x \in \Delta^{\mathcal{I}}} {C_1}^{\mathcal{I}}(x) \Rightarrow_{Ł} {C_2}^{\mathcal{I}}(x) \geq d$
(z-implies C1 C2 [d])	Zadeh's set inclusion GCI	$\inf_{x \in \Delta^{\mathcal{I}}} {C_1}^{\mathcal{I}}(x) \Rightarrow_Z {C_2}^{\mathcal{I}}(x) \geq d$
(define-concept A C)	concept definition	$\forall_{x \in \Delta^{\mathcal{I}}} A^{\mathcal{I}}(x) = C^{\mathcal{I}}(x)$
(define-primitive-concept A C)	concept subsumption	$\inf_{x \in \Delta^{\mathcal{I}}} A^{\mathcal{I}}(x) \leq C^{\mathcal{I}}(x)$
(equivalent-concepts C1 C2)	concept definition	$\forall_{x \in \Delta^{\mathcal{I}}} {C_1}^{\mathcal{I}}(x) = {C_2}^{\mathcal{I}}(x)$
(disjoint C1 ... Ck)	concept disjointness	$\forall_{i,j \in \{1,\dots,k\}, i<j}$ (implies (g-and Ci Cj) *bottom*)
(disjoint-union C1 ... Ck)	disjoint union	(disjoint C2 ... Ck) and C1 = (or C2 ... Ck)
(range R C1)	range restriction	(implies *top* (all RN C))
(domain R C1)	fomain restriction	(implies (some RN *top*) C)
(functional R)	functional role	$R^{\mathcal{I}}(a,b) = R^{\mathcal{I}}(a,c) \to b = c$
(inverse-functional R)	inverse functional role	$R^{\mathcal{I}}(b,a) = R^{\mathcal{I}}(c,a) \to b = c$
(reflexive R)	reflexive role	$\forall a \in \Delta^{\mathcal{I}}, R^{\mathcal{I}}(a,a) = 1.$
(symmetric R)	symmetric role	$\forall a, b \in \Delta^{\mathcal{I}}, R^{\mathcal{I}}(a,b) = R^{\mathcal{I}}(b,a).$
(transitive R)	transitive role	$\forall_{a,b \in \Delta^{\mathcal{I}}} R^{\mathcal{I}}(a,b) \geq \sup_{c \in \Delta^{\mathcal{I}}} R^{\mathcal{I}}(a,c) \otimes R^{\mathcal{I}}(c,b).$
(implies-role R1 R2 [d])	RIA	$\inf_{x,y \in \Delta^{\mathcal{I}}} {R_1}^{\mathcal{I}}(x,y) \Rightarrow_{Ł} {R_2}^{\mathcal{I}}(x,y) \geq d$
(inverse R1 R2)	inverse role	${R_1}^{\mathcal{I}} \equiv ({R_2}^{\mathcal{I}})^{-}$

where d is the degree and can be: *(i)* a variable, *(ii)* an already defined truth constant, *(iii)* a rational number in $[0,1]$, *(iv)* a linear expression.
Notes: Transitive roles cannot be functional. In Zadeh logic, $\Rightarrow$ is Zadeh's set inclusion.

14. Queries.

(sat?)	Is $\mathcal{K}$ consistent?
(max-instance? a C)	$\sup\{n \mid \mathcal{K} \models$ (instance a C n)$\}$
(min-instance? a C)	$\inf\{n \mid \mathcal{K} \models$ (instance a C n)$\}$
(all-instances? C)	(min-instance? a C) for every individual of $\mathcal{K}$
(max-related? a b R)	$\sup\{n \mid \mathcal{K} \models$ (related a b R n)$\}$
(min-related? a b R)	$\inf\{n \mid \mathcal{K} \models$ (related a b R n)$\}$
(max-subs? C D)	$\sup\{n \mid \mathcal{K} \models$ (implies D C n)$\}$
(min-subs? C D)	$\inf\{n \mid \mathcal{K} \models$ (implies D C n)$\}$
(max-g-subs? C D)	$\sup\{n \mid \mathcal{K} \models$ (g-implies D C n)$\}$
(min-g-subs? C D)	$\inf\{n \mid \mathcal{K} \models$ (g-implies D C n)$\}$
(max-l-subs? C D)	$\sup\{n \mid \mathcal{K} \models$ (l-implies D C n)$\}$
(min-l-subs? C D)	$\inf\{n \mid \mathcal{K} \models$ (l-implies D C n)$\}$
(max-kd-subs? C D)	$\sup\{n \mid \mathcal{K} \models$ (kd-implies D C n)$\}$
(min-kd-subs? C D)	$\inf\{n \mid \mathcal{K} \models$ (kd-implies D C n)$\}$
(max-sat? C [a])	$\sup_{\mathcal{I}} \sup_{a \in \Delta^{\mathcal{I}}} C^{\mathcal{I}}(a)$
(min-sat? C [a])	$\inf_{\mathcal{I}} \sup_{a \in \Delta^{\mathcal{I}}} C^{\mathcal{I}}(a)$
(max-var? var)	$\sup\{$var $\mid \mathcal{K}$ is consistent$\}$
(min-var? var)	$\inf\{$var $\mid \mathcal{K}$ is consistent$\}$
(defuzzify-lom? C_m a F)	Defuzzify the value of F using largest of the maxima
(defuzzify-mom? C_m a F)	Defuzzify the value of F using middle of the maxima
(defuzzify-som? C_m a F)	Defuzzify the value of F using smallest of the maxima
(bnp? f)	Computes the Best Non-Fuzzy Performance (BNP) of fuzzy number f

where concept C_m represents several Mamdani/Rules IF-THEN fuzzy rules expressing how to obtain the value of concrete feature F.