USC, CSCI570

Exam III - Review Session Solution

Hanqing Zeng

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Problem 1

- $Set-Intersection \in NP$:

  Given a certificate H, we can check if H intersects every S_i in polynomial time.

  Brute force approach: for all elements e in S_i, and for all S_i in $\{S_1, S_2, ..., S_n\}$, go through H to check if it covers e. $\Rightarrow$ polynomial in terms of b, n and number of elements in each set.

- $Vertex-Cover \le_P Set-Intersection$:

  Given an undirected graph $G = (V, E)$, for every edge $e_i = (v_j, v_k)$, we construct a set $S_i = \{v_j, v_k\}$. A graph of m edges is mapped to a family of m sets, where each set contains exactly two elements.

  Vertex-Cover of size at most $b \Longleftrightarrow H$ intersects with every S_i where $|H| \le b$.

- $Proof$:

  "$\Longrightarrow$": Let the vertex-cover be V_{vc}. By definition of vertex-cover, we get $\forall e_i = (v_j, v_k)$, $v_j \in V_{vc}$ or $v_k \in V_{vc}$. Let $H = V_{vc}$. By the mapping, we get $\forall S_i = \{v_j, v_k\}$, $v_j \in H$ or $v_k \in H$. H is a valid solution.

  "$\Longleftarrow$": H intersects with every set, so $\forall S_i = \{v_j, v_k\}$, $v_j \in H$ or $v_k \in H$. Let a $V_{vc} = H$. By the mapping, we get $\forall e_i = (v_j, v_k)$, $v_j \in V_{vc}$ or $v_k \in V_{vc}$. V_{vc} is a valid vertex-cover.

Problem 2

SPAN-2

- $SPAN-2 \in NP$:

  Go through every node in the spanning tree and check its degree $\Rightarrow$ $O(n)$.

- $Hamiltonian-Path \leq_P SPAN-2$:

  Given a graph G, a Hamiltonian path in G can be found $\Longleftrightarrow$ a $SPAN-2$ spanning tree can be found in G.

- $Proof$:

  "$\Longrightarrow$": Set the starting node of the Hamiltonian path as the root of the spanning tree. The tree is then just a simple path. The root and leaf of the tree have degree of 1, and all the intermediate nodes have degree of 2. The spanning tree is a $SPAN-2$ tree.

  "$\Longleftarrow$": For any intermediate node in $SPAN-2$ tree, it can only have one parent and exactly one child. For the root node, it can have one or two children. By setting one leaf node as the starting point of the Hamiltonian path, we can construct the Hamiltonian path by traversing the $SPAN-2$ tree.

SPAN-3

- $SPAN-3 \in NP$:

  Same as $SPAN-2$.

- $SPAN-2 \leq_P SPAN-3$:

  Given a graph $G_2 = (V_2, E_2)$, construct a new graph $G_3 = (V_3, E_3)$, by the following method. $\forall v_i \in V_2$, add a new vertex v_i', and a new edge $e_i' = (v_i, v_i')$.

  $SPAN-2$ tree can be found in $G \Longleftrightarrow SPAN-3$ tree can be found in G'.

- $Proof$:

  Define V' as the set of vertices consisting of v_i'. So $V_3 = V_2 \cup V'$. Let the function $DEG(V, T)$ denote the maximum degree of the vertices in V, for a spanning tree T.

  "$\Longrightarrow$": Given a $SPAN-2$ in G, we can construct $SPAN-3$ by adding the v_i' and e_i'. The degree of each v_i in $SPAN-2$ now is incremented by 1 (incurred by e_i'). So $DEG(V_3, T_{SPAN-3}) = DEG(V_2, T_{SPAN-2}) + 1 = 3$. The degree of v_i' is 1 (leaf node). The new tree is a $SPAN-3$ tree.

  "$\Longleftarrow$": Since the degree of vertices in V' is 1, and a spanning tree in G_3 must contain all the vertices in V', so the vertices in V' are leaf nodes in the spanning tree of G_3. By removing V' in a $SPAN-3$ tree of G_3 (call it T_{SPAN-3}), we get a spanning tree T_{G_2} of G_2. Thus, $DEG(V_2, T_{G_2}) = DEG(V_3, T_{SPAN-3}) - 1 = 2$. So T_{G_2} is a $SPAN-2$ tree of G_2.