APM 346 Problem Set 1

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1 Question 1

Classify and solve:

$$xu_x - yu_y = 0 \iff u_x - \frac{y}{x}u_y = 0$$

The above is true as long as $x \neq 0$, which we assume.

Classification: $\boxed{\text{first order, linear, homogeneous PDE}}$

We can rewrite it as: $(1, -\frac{y}{x}) \cdot \nabla u = 0$, and we can find a general solution for it with the methods of characteristics. This modified form of the equation makes it obvious that u is constant on curves γ with the property $\frac{d}{dt}(x(t), y(t)) = (1, -\frac{y(t)}{x(t)})$. This set of two ODEs is easy to solve, and we find $x(t) = t$, $y(t) = \frac{C}{t} = \frac{C}{x(t)}$, where C is a constant of integration (see below for solution for $y(t)$).

$$\begin{aligned}
\frac{dy(t)}{dt} &= -\frac{y(t)}{x(t)} \\
\frac{dy(t)}{dt} &= -\frac{y(t)}{t} \\
\frac{dy(t)}{y(t)} &= -\frac{dt}{t} \\
y(t) &= \frac{C}{t}
\end{aligned}$$

So $u(x, y)$ is constant on $\gamma = (x, \frac{C}{x})$ by $\frac{d}{dx}u(\gamma(x)) = \frac{d}{dx}u(x, y(x)) = (1, -\frac{y}{x}) \cdot \nabla u = 0$. And if $u(x, \frac{C}{x}$ is constant everywhere on the curve, it is constant for $x = 1$

$$u(x, \frac{C}{x}) = const = u(1, C)$$

So u depends only on C, which we can express as $C = xy$, $\implies$ $\boxed{u(x,y) = f(xy)}$. In general, u is some function that depends only on the product of x and y.

2 Question 2

Classify and find general solution for:

$$au_x + bu_y + cu + d = 0$$

For a, b, c, d all constant.

Classification: $\boxed{\text{first order, linear, inhomogenenous PDE}}$

As usual, use the method of characteristics. It is useful to rearrange the PDE as follows:

$$u_x + \frac{b}{a}u_y = -\frac{c}{a}u - \frac{d}{a}$$

The lefthand side can be written $(1, \frac{b}{a}) \cdot \nabla u$, and so we want to find the curve $(x, y(x))$ that satisfies $\frac{d}{dx}(x, y(x)) = (1, \frac{b}{a})$. Obviously, we want $y(x) = \frac{b}{a}x + c_1$. With this choice for $y(x)$, our PDE becomes the following ODE:

$$\frac{d}{dx}u(x, y(x)) + \frac{c}{a}u(x, y(x)) = -\frac{d}{a} \tag{1}$$

which we can solve using the method of integrating factors. For simplicity, we define $U(x) = u(x, y(x))$, and try to find $\mu(x)$ such that $\frac{d}{dx}(\mu(x)U(x)) = \mu(x)\frac{d}{dx}U(x) + \frac{c}{a}\mu(x)U(x)$. This looks like the product rule - what we really want is $\mu(x)$ such that $\frac{d}{dx}\mu(x) = \frac{c}{a}\mu(x)$, which implies $\mu(x) = e^{cx/a}$. Multiplying Equation 1 by this μ, we find:

$$\begin{aligned}
\mu\frac{d}{dx}U(x) + \mu\frac{c}{a}U(x) = \frac{d}{dx}(\mu U(x)) &= -\frac{d}{a}\mu \\
d(e^{cx/a}U(x)) &= -\frac{d}{a}e^{cx/a}dx \\
e^{cx/a}U(x) &= -\frac{d}{c}e^{cx/a} + c_2 \\
U(x) = u(x, y(x)) &= -\frac{d}{c} + c_2e^{cx/a} \qquad (2)
\end{aligned}$$

Now take Equation 2 at $x = 0$, since we know the behaviour of $y(x)$ at $x = 0$ ($y(0) = c_1$).

$$
\begin{aligned}
u(0, y(0)) &= -\frac{d}{c} + c_2 \\
\text{However we can rewrite}\, u(0, y(0) &= u(0, c_1) = f(c_1) \\
f(c_1) &= -\frac{d}{c} + c_2 \\
\text{We also know}\, c_1 &= y - \frac{b}{a}x \\
\implies f(c_1) &= f(y - \frac{b}{a}x) = -\frac{d}{c} + c_2 \\
\implies c_2 &= f(y - \frac{b}{a}x) + \frac{d}{c} \qquad (3)
\end{aligned}
$$

So substitute Equation 3 into Equation 2:

$$u(x, y(x)) = -\frac{d}{c} + (f(y - \frac{b}{a}x) + \frac{d}{c})e^{-cx/a}$$

The general solution to the PDE is: $\boxed{u(x, y(x)) = -\frac{d}{c} + (f(y - \frac{b}{a}x) + \frac{d}{c})e^{-cx/a}}$

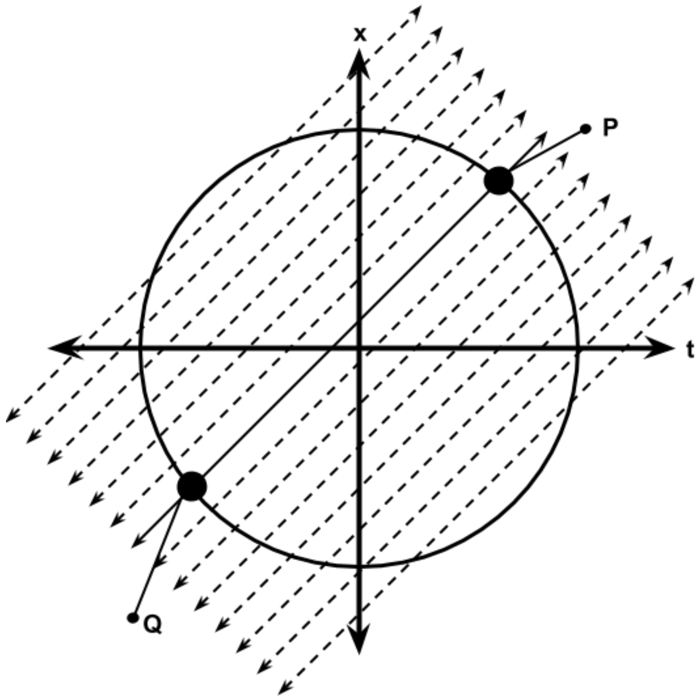
3 Question 3

Classify and determine well-posedness of BVP:

$$\begin{cases} u_t + u_x = 0 : x^2 + t^2 < 1 \\ u(t, x) = x \,\text{for}\, x^2 + t^2 = 1 \end{cases}$$

Classification: $\boxed{\text{first order, linear, homogeneous PDE, subject to Dirichlet boundary condition}}$

As we saw in class, this class of PDE has a general solution $u(t, x) = f(at - bx)$, where for this problem, $a = 1,\ b = 1$. So the boundary condition gives us $u(t, x) = f(t - x) = x$ on $x^2 + t^2 = 1$. Let's think about what we know about u in the (t, x) plane. The figure below shows the characteristic curves (dashed lines), and location of the boundary condition of u (circle).



We choose $P = (t_P, x_P)$ and $Q = (t_Q, x_Q)$ to be points that are both on the boundary and on the same characteristic curve. We know from our PDE that $u(t_P, x_P) = u(t_Q, x_Q) = const$. But our boundary conditions tell us that $u(t_P, x_P) = x_P$ and $u(t_Q, x_Q) = x_Q$. It is obvious from the picture that $x_P \neq x_Q$, and in fact, there are no choices for P, Q such that they both lie on the same characteristic curve and satisfy the boundary condition. $\boxed{\text{The BVP is not well posed}}$ because there is no way to satisfy the boundary condition.

4 Question 4

Classify and determine well-posedness of BVP:

$$\begin{cases} u_t + u_x = 0 \\ u(t = x, x) = 1 \end{cases}$$

Classification: $\boxed{\text{first order, linear, homogeneous PDE, subject to Dirichlet boundary condition}}$

This PDE is the same as Question 3, so has the same general solution. The boundary condition is now: $u(t = x, x) = f(x - x) = f(0) = 1$. As with Question 3, we consider the characteristic curves and the BVP. In this case, the general solution gives us that the characteristic curves look like $x = t + C$. Our boundary condition constrains what happens on the curve $x = t$, which is just the characteristic curve where $C = 0$. The boundary condition tells us that $u(t, x)$ is constant on the curve, but our PDE already tells us this. Crucially, the boundary value reveals nothing about the behaviour of u perpendicular to the characteristic curves, so $\boxed{\text{the BVP is not well posed}}$.

5 Question 5

Use the method of characteristics to solve:

$$\begin{cases} u_x + yu_y + zu_z = 0 \\ u(0, y, z) = y^2 + z^2 \end{cases}$$

As usual, we can rewrite our PDE in terms of ∇u

$$u_x + yu_y + zu_z = 0 \iff (1, y, z) \cdot \nabla u = 0$$

This makes it obvious that u is constant on curves γ with the property:

$$\frac{d}{dt}\gamma = \frac{d}{dt}(x(t), y(t), z(t)) = (1, y(t), z(t))$$

This set of three ODEs is easy to solve, providing

$$x(t) = t \tag{4}$$

$$y(t) = Ce^t = Ce^{x(t)} \tag{5}$$

$$z(t) = De^t = De^{x(t)} \tag{6}$$

where C, D are constants of integration. So $u(x, Ce^x, De^x) = const$, and if this is true, then $u(x, Ce^x, De^x) = u(0, C, D)$. So changes in u depend only on C and D. Thus we can write:

$$u(x, y, z) = f(C, D) = f(ye^{-x}, ze^{-x})$$

by rearranging Equations 5 and 6 for C and D respectively.
We can determine f with the boundary condition:

$$u(0, y, z) = f(y, z) = y^2 + z^2 \implies f(ye^{-x}, ze^{-x}) = e^{-2x}(y^2 + z^2)$$

Final answer: $\boxed{u(x, y, z) = e^{-2x}(y^2 + z^2)}$

6 Question 6

Classify the following PDE and determine whether it is elliptic, parabolic or hyperbolic. Change variables and reduce the PDE so it's principal part takes the canonical form. Then find constant α, β such that $u(\xi, \eta) = e^{\alpha\xi+\beta\eta}v(\xi, \eta)$ eliminates the first derivative terms.

$$u_{xx} + 4u_{xy} + 3u_{yy} + 3u_x - u_y + u = 0 \tag{7}$$

Classification: $\boxed{\text{second order, linear, constant coefficient homogeneous partial differential equation}}$
Calculate discrimnant (d) to further determine type:

$$\begin{aligned} d &= AC - B^2, \text{ where for the above PDE, } A = 1, B = 2, C = 3 \\ d &= (1)(3) - 4 < 0 \end{aligned}$$

A discriminant less than zero means the $\boxed{\text{PDE is hyperbolic}}$.

This means we want a change of variables such that principal part of the PDE takes the form: $u_{xx} - u_{yy}$. We change from (x, y) to (ξ, η).

$$\begin{aligned} \xi &= ax + by \\ \eta &= cx + dy \\ \partial_x &= a\partial_\xi + c\partial_\eta \qquad (8) \\ \partial_y &= b\partial_\xi + d\partial_\eta \qquad (9) \end{aligned}$$

Substitute Equations 8 and 9 into Equation 7 to make the change of variables. Thus our PDE becomes:

$$\begin{aligned} (a\partial_\xi + c\partial_\eta)^2 u &+ 4(a\partial_\xi + c\partial_\eta)(b\partial_\xi + d\partial_\eta)u + 3(b\partial_\xi + d\partial_\eta)^2 u \\ &+ 3(a\partial_\xi + c\partial_\eta)u - (b\partial_\xi + d\partial_\eta)u + u = 0 \end{aligned}$$

Multiply through by C^2 to get the slightly nicer form and expand the multiplication terms:

$$
\begin{aligned}
(a^2\partial_\xi^2 + 2ac\partial_{\xi\eta} + c^2\partial_\eta^2)u &+ (4ab\partial_\xi^2 + 4ad\partial_{\xi\eta} + 4bc\partial_{\xi\eta} + 4cd\partial_\eta^2)u + (3b^2\partial_\xi^2 + 6bd\partial_{\xi\eta} + 3d^2\partial_\eta^2)u \\
&+ 3(a\partial_\xi + c\partial_\eta)u - (b\partial_\xi + d\partial_\eta)u + u = 0 \\
(a^2 + 4ab + 3b^2)\partial_\xi^2 u &+ (2ac + 4ad + 4bc + 6bd)\partial_{\xi\eta}u + (c^2 + 4cd + 3d^2)\partial_\eta^2 u \\
&+ (3a - b)\partial_\xi u + (3c - d)\partial_\eta u + u = 0
\end{aligned}
$$

The principal part (pr, the part we want to reduce to a hyperbolic form) is the first line of the PDE above, and for now this is the only part we will concern ourselves with for the continuing algebra.

$$
pr = (a^2 + 4ab + 3b^2)\partial_\xi^2 u \quad + \quad (2ac + 4ad + 4bc + 6bd)\partial_{\xi\eta}u + (c^2 + 4cd + 3d^2)\partial_\eta^2 u
$$

To put this in hyperbolic form, use the fact that $\partial_\xi^2 - m\partial_\eta^2$ (m a constant), is still hyperbolic. Thus if $m = g/h$ ($h \neq 0$), $h\partial_\xi^2 - g\partial_\eta^2$ is also hyperbolic. So the only conditions we have on our constants are as follows:

$$
\begin{aligned}
0 &> a^2 + 4ab + 3b^2 \qquad (10) \\
0 &= 2ac + 4ad + 4bc + 6bd \qquad (11) \\
0 &< c^2 + 4cd + 3d^2 \qquad (12)
\end{aligned}
$$

Start with Equation 12, and set the right hand side equal to $f(c, d)$. Right away we know that c or d, must be less than zero. Since d is an arbitrary constant, set $d = -D$. The easiest way to ensure $f(c, d) < 0$ is to find its minima by setting its partial derivatives with respect to c and c equal to zero.

$$
\begin{aligned}
\frac{\partial f}{\partial c} &= 2c - 4D = 0 \\
\frac{\partial f}{\partial d} &= -4c + 6D = 0 \\
\implies -4c + 6D &= 2c - 4D \\
c &= \frac{5}{3}D \qquad (13)
\end{aligned}
$$

A quick check shows that if c is given by Equation 13, then $f(c, D) = (\frac{25}{9} - \frac{60}{9} + \frac{27}{9})D^2 = -\frac{8}{9}D^2 < 0$ for D real.

So choose $D = 3$, $\implies c = 5$, and so our original $d = -3$. Now find a and b such that Equation 11 is satisfied.

$$0 = 10a - 12a + 20b - 18b \implies a = b$$

Choose $a = 1, \implies b = 1$. Now ensure that the inequality in Equation 10 is satisfied: $a^2 + 4ab + 3b^2$ $= 1 + 4 + 3 > 0$. So the final set of constants to make the linear change of coordinates: $a = 1$, $b = 1$, $c = 5$, $d = -3$. We can now rewrite Equation 7 as:

$$8u_{\xi\xi} - 8u_{\eta\eta} + 2u_{\xi} + 18u_{\eta} + u = 0 \tag{14}$$

So with the principal part in a canonical form, Equation 7 is: $\boxed{8u_{\xi\xi} - 8u_{\eta\eta} + 2u_{\xi} + 18u_{\eta} + u = 0}$

Now, attempt to find α, β for $u(\xi, \eta) = e^{\alpha\xi - \beta\eta}v(\xi, \eta)$ such that the first-derivative terms in Equation 14 are eliminated. Start by setting $\tau = \tau(\xi, \eta) = \alpha\xi - \beta\eta$ and taking appropriate derivatives of u.

$$\begin{aligned}
u_{\xi} &= \alpha e^{\tau} v + e^{\tau} v_{\xi} \\
u_{\eta} &= \beta e^{\tau} v + e^{\tau} v_{\eta} \\
u_{\xi\xi} &= \alpha^2 e^{\tau} v + 2\alpha e^{\tau} v_{\xi} + e^{\tau} v_{\xi\xi} \\
u_{\eta\eta} &= \beta^2 e^{\tau} v + 2\beta e^{\tau} v_{\eta} + e^{\tau} v_{\eta\eta}
\end{aligned}$$

Substitute the above expressions for derivatives of u into Equation 14 and divide through by e^{τ}

$$\begin{aligned}
8(\alpha^2 v + 2\alpha v_{\xi} + v_{\xi\xi}) - 8(\beta^2 v + 2\beta v_{\eta} + v_{\eta\eta}) + 2(\alpha v + v_{\xi}) + 18(\beta v + v_{\eta}) + v &= 0 \\
8v_{\xi\xi} - 8v_{\eta\eta} + (16\alpha + 2)v_{\xi} + (-16\beta + 18)v_{\eta} + (8\alpha^2 - 8\beta^2 + 2\alpha + 18\beta + 1)v &= 0 \\
v_{\xi\xi} - v_{\eta\eta} + 11v &= 0
\end{aligned}$$

Where the last line follows if one chooses $\boxed{\alpha = -1/8,\ \beta = 9/8}$ to eliminate the first derivative terms.