PHY 407 Lab 8

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1 Question 1

Here, write $\vec{u} = (u, \eta)$

$$\vec{F}(\vec{u}) = \left(g\eta + \frac{u^2}{2}, u(\eta + H)\right) \tag{1}$$

2 Question 2

$$\begin{aligned} u_j^{n+1} &= u_j^n - \frac{\Delta t}{2\Delta x}\left(g(\eta_{j+1}^n - \eta_{j-1}^n) + \frac{1}{2}((u_{j+1}^n)^2 - (u_{j-1}^n)^2)\right) \\ \eta_j^{n+1} &= \eta_j^n - \frac{\Delta t}{2\Delta x}\left(u_{j+1}^n(\eta_{j+1}^n + H_{j+1}) - u_{j-1}^n(\eta_{j-1}^n + H_{j-1})\right) \end{aligned}$$

3 Question 3

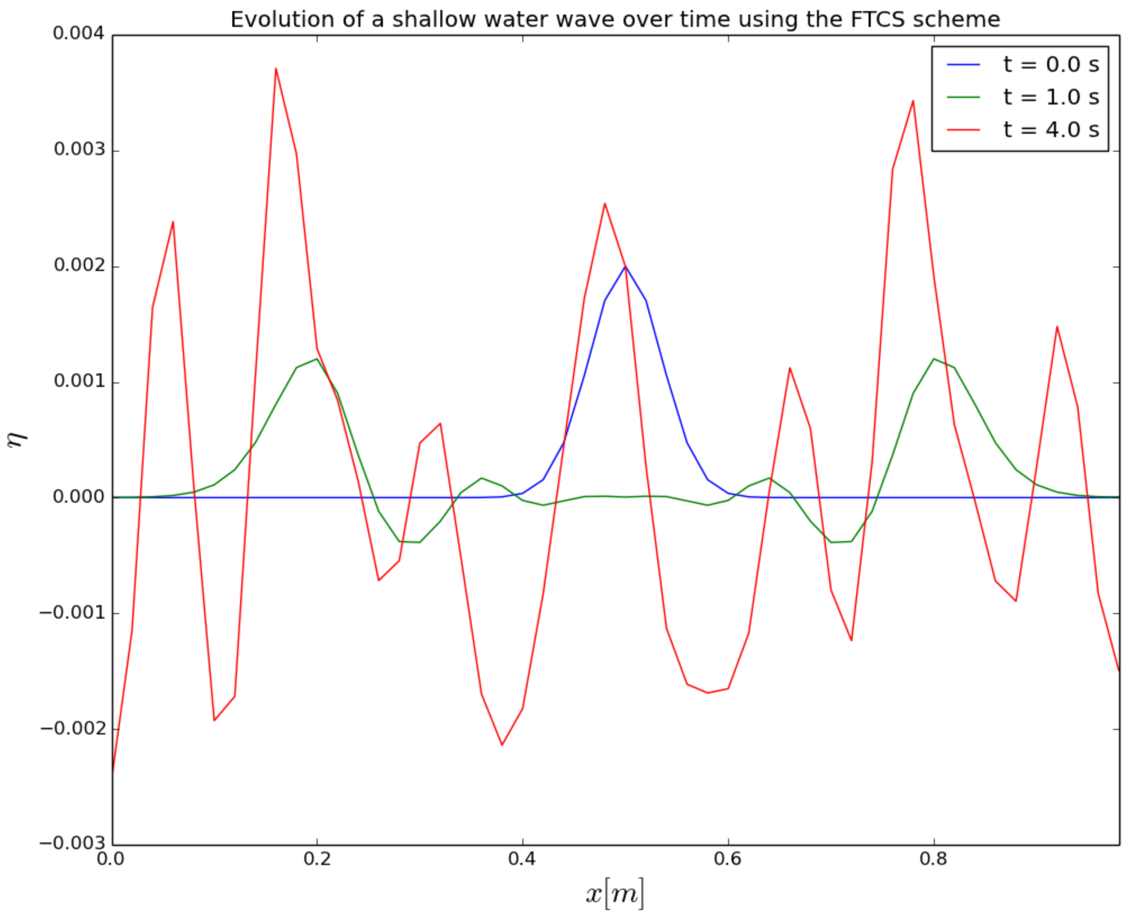

Figure 1

4 Question 4

Start by linearizing Equation 1:

$$\vec{F}(u, \eta) \approx (g\eta, uH)\,. \tag{2}$$

Write u and η as Fourier series:

$$\begin{aligned} u(x,t) &= \sum_{k=1}^{\infty} c_k(t) e^{ikx}, \\ \eta(x,t) &= \sum_{k=1}^{\infty} d_k(t) e^{ikx}. \end{aligned}$$

We can now approximate our timestepping as follows:

$$
\begin{aligned}
u(x, t+\Delta t) &= \sum_{k=1}^{\infty} c_k(t+\Delta t) e^{ikx} \\
&= \sum_{k=1}^{\infty} c_k(t) e^{ikx} - \frac{\Delta t\, g}{2\Delta x}\left(\sum_{k=1}^{\infty} d_k(t) e^{ik(x+\Delta x)} - \sum_{k=1}^{\infty} d_k(t) e^{ik(x-\Delta x)}\right) \\
\eta(x, t+\Delta t) &= \sum_{k=1}^{\infty} d_k(t+\Delta t) e^{ikx} \\
&= \sum_{k=1}^{\infty} d_k(t) e^{ikx} - \frac{\Delta t\, H}{2\Delta x}\left(\sum_{k=1}^{\infty} c_k(t) e^{ik(x+\Delta x)} - \sum_{k=1}^{\infty} c_k(t) e^{ik(x-\Delta x)}\right) \\
\Longrightarrow c_k(t+\Delta t) &= c_k(t) - d_k(t)\frac{\Delta t\, g}{2\Delta x}\left(e^{ik\Delta x} - e^{-ik\Delta x}\right), \\
d_k(t+\Delta t) &= d_k(t) - c_k(t)\frac{\Delta t\, H}{2\Delta x}\left(e^{ik\Delta x} - e^{-ik\Delta x}\right)
\end{aligned}
$$

$$
\vec{c} = (t+\Delta t) = \begin{pmatrix} c_k(t+\Delta t) \\ d_k(t+\Delta t) \end{pmatrix} = \begin{pmatrix} 1 & -\frac{\Delta t\, g}{\Delta x}\sin(k\Delta x) \\ -\frac{\Delta t\, H}{\Delta x}\sin(k\Delta x) & 1 \end{pmatrix} \begin{pmatrix} c_k(t) \\ d_k(t) \end{pmatrix}
$$

If we call the 2×2 matrix above $\mathbf{A}$, we find the eigenvalues as follows:

$$
(det)(\mathbf{A} - \lambda \mathbf{I}) = \begin{vmatrix} 1-\lambda & -\frac{\Delta t\, g}{\Delta x}\sin(k\Delta x) \\ -\frac{\Delta t\, H}{\Delta x}\sin(k\Delta x) & 1-\lambda \end{vmatrix} = 0
$$

We find that the eigenvalues are:

$$
\begin{aligned}
\lambda_1 &= 1 + \sqrt{gH}\left(\frac{\Delta t}{\Delta x}\sin(k\Delta x)\right) \\
\lambda_2 &= 1 - \sqrt{gH}\left(\frac{\Delta t}{\Delta x}\sin(k\Delta x)\right)
\end{aligned}
$$

For stability, we need $|\lambda| < 1$. We know that $-1 \leq \sin(x) \leq 1$, so in the the case of λ_1, we need $\sin(k\Delta x) < 0$. This means we need $\frac{(2n-1)\pi}{k} < \Delta x < \frac{2n\pi}{k}$ (n an integer). In the case of λ_2, we need $\sin(k\Delta x) > 0$. This means we need $\frac{2n\pi}{k} < \Delta x < \frac{(2n+1)\pi}{k}$ (n an integer).

Obviously Δx cannot simulataneously satisfy both of these conditions, so we expect the FTCS scheme to be unstable for this system (even under a linear approximation).

5 Question 5

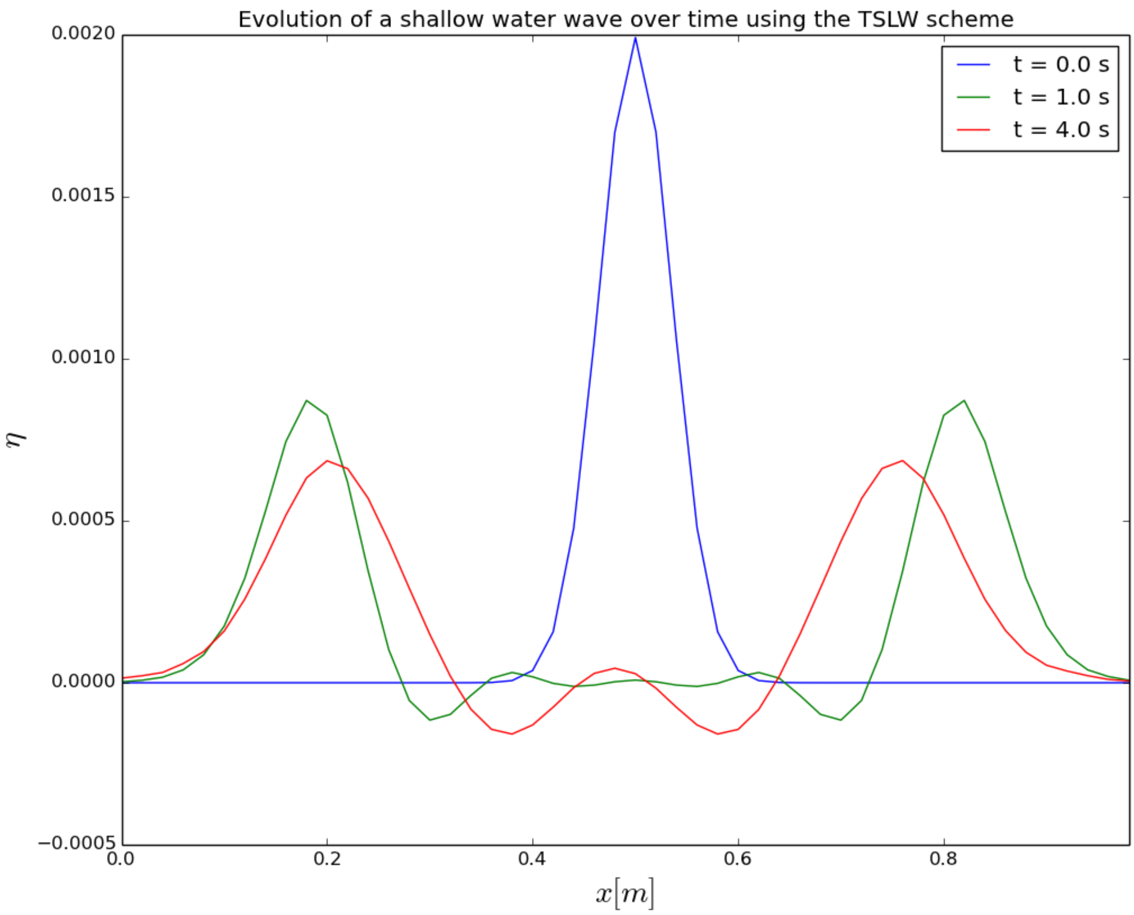

Figure 2

6 Question 6

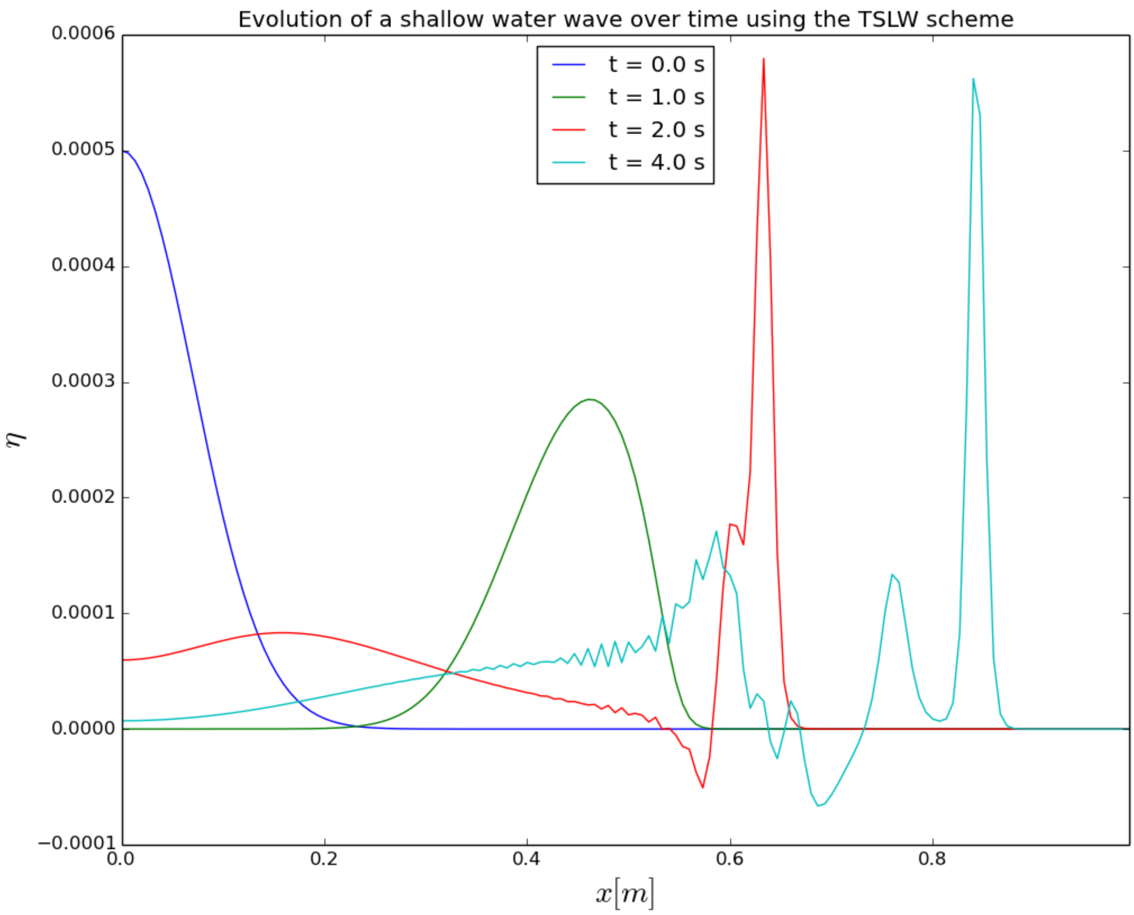

Figure 3