14. Newtonian Cosmology

Textbook: §27.1

Olbers paradox

Why is the sky dark?

With reference to stars, the main answer is "the finite age of the universe" (plus the fact that the speed of light is finite).

Cosmological principle

The universe is isotropic and homogeneous (on large scales).

Expansion of the universe, Hubble constant

Given the cosmological principle, expansion can only be by way of a Hubble-like law, in which the current positions

$$\boldsymbol{x}(t) = R(t)\boldsymbol{\varpi} \; , \tag{14.1}$$

with $\boldsymbol{\varpi}$ co-moving positions that do not depend on time (on large scales) and $R(t)$ a time-dependent scale factor. For the Hubble constant,

$$\boldsymbol{v}(t) = \dot{\boldsymbol{x}}(t) = H(t)\boldsymbol{x}(t) \qquad \Rightarrow \qquad H(t) = \frac{\dot{R}(t)}{R(t)} \; . \tag{14.2}$$

Note that $H_0 = H(t_0)$ is the present value of the Hubble constant ($t_0 =$ "now"), and by definition $R(t_0) = 1$.

Friedmann equation

$$\left(H^2 - \frac{8}{3}\pi G\rho\right) R^2 = -kc^2 \qquad \text{with} \qquad \begin{cases} k > 0 \;\Rightarrow\; \text{closed} \\ k = 0 \;\Rightarrow\; \text{flat} \\ k < 0 \;\Rightarrow\; \text{open} \end{cases} \tag{14.3}$$

Critical density

The density required to yield a flat universe:

$$\rho_{\mathrm{c}} = \frac{3H^2}{8\pi G} \tag{14.4}$$

Associated is a *density parameter*, the density relative to the critical density:

$$\Omega(t) \equiv \frac{\rho(t)}{\rho_{\mathrm{c}}(t)} \tag{14.5}$$

Redshifts, wavelengths, and temperatures

$$1 + z = \frac{\lambda_{\mathrm{obs}}}{\lambda_{\mathrm{em}}} \qquad \Rightarrow \qquad \lambda \propto R \tag{14.6}$$

$$p \propto \frac{1}{\lambda} \propto \frac{1}{R} \quad \Rightarrow \quad \begin{cases} \epsilon \propto \dfrac{1}{R} & \Rightarrow \quad T \propto \dfrac{1}{R} \propto 1+z & \text{(relativistic)} \\ \epsilon \propto \dfrac{1}{R^2} & \Rightarrow \quad T \propto \dfrac{1}{R^2} \propto (1+z)^2 & \text{(non-relativistic)} \end{cases} \tag{14.7}$$

Matter-dominated, flat Universe

$$R = (6\pi G \rho_c)^{1/3}\, t^{2/3} = \left(\frac{3t}{2t_H}\right)^{2/3} \qquad \Rightarrow \qquad \frac{t}{t_H} = \frac{2}{3}\left(\frac{1}{1+z}\right)^{3/2} \tag{14.8}$$

For next time:

- If we found some stars or other objects that had ages $t_{\rm star} > t_H$, what would this imply?
- Read ahead on relativity (§27.3, up to Eq. 27.65)

15. Relativistic Cosmology

Textbook: §27.3

Robertson-Walker metric

The most general metric possible for describing an isotropic and homogeneous Universe,

$$(\mathrm{d}s)^2 = (c\mathrm{d}t)^2 - R^2(t)\left[\left(\frac{\mathrm{d}\varpi}{\sqrt{1-k\varpi^2}}\right)^2 + (\varpi\mathrm{d}\theta)^2 + (\varpi\sin\theta\,\mathrm{d}\phi)^2\right] \tag{15.1}$$

where k is the current curvature of the Universe (as a function of time: $K(t) = k/R^2(t)$).

Friedmann equation

Solving Einstein's field equations for an isotropic, homogeneous universe leads to exactly the same Friedmann equation as we found for the Newtonian case, so all the results derived there remain valid — only the interpretation of k has changed. The density ρ combines rest mass and the mass-equivalent of energy.

For non-relativistic particles, the kinetic energy of a particle is negligible compared to its rest mass, and thus,

$$\text{matter:} \qquad \rho_\mathrm{m} = \rho_{\mathrm{m},0}\,R^{-3}\,, \qquad\qquad \text{where } \rho_{\mathrm{m},0} \text{ is constant.} \tag{15.2}$$

For photons and extreme-relativistic particles, not only is the number density diluted by $1/R^3$, the wavelength is stretched proportionally to R (as one can see also from $\rho_\gamma = u/c^2$, where the radiation energy density is $u = aT^4$, combined with $T \propto 1/R$). Thus,

$$\text{radiation:} \qquad \rho_\gamma = \rho_{\gamma,0}\,R^{-4}\,, \qquad\qquad \text{where } \rho_{\gamma,0} \text{ is constant} \tag{15.3}$$

Note that the above relations assume that matter and radiation are decoupled, and can have different temperatures. At present, $\rho_\gamma \ll \rho_\mathrm{m}$, but with ρ_m dominated by *dark matter*.

Present observations of distant supernovae and of the cosmic microwave background indicate an additional, more exotic contribution than the previously-assumed photons, baryonic matter, neutrinos, and dark matter. This exotic *dark energy* appears to be causing the expansion of the universe to accelerate. The simplest way to obtain such an effect in the Friedmann equation is to insert a *cosmological constant* Λ, as follows,

$$\left[H^2 - \frac{8}{3}\pi G\rho - \frac{1}{3}\Lambda c^2\right]R^2 = -kc^2 \qquad \left(\text{with} \quad H = \frac{\dot{R}}{R}\,, \quad t_\mathrm{H} \equiv \frac{1}{H_0} \quad \text{as before}\right), \tag{15.4}$$

where for simplicity we assume that Λ is a constant, independent of t (unlike R, ρ, and H). Since $\rho_\mathrm{m} \propto 1/R^3$, the cosmological constant Λ must be negligible at early times. And since $\rho_\gamma \propto 1/R^4$, radiation dominates over matter at early enough times (of course, at high enough temperatures particles can also become extremely relativistic).

It is often useful to write the Friedmann equation in terms of current values of $\Omega_{\rm M}$, Ω_Λ, etc., as follows:

$$\left(\frac{\dot{R}}{R}\right)^2 = H^2 = H_0^2\left(\frac{\Omega_{\rm rad,0}}{R^4} + \frac{\Omega_{\rm M,0}}{R^3} + \frac{\Omega_{k,0}}{R^2} + \Omega_{\Lambda,0}\right) \quad \text{with} \quad \begin{cases} \Omega_{\rm rad,0} & = \dfrac{8\pi G a T_0^4}{3c^2 H_0^2}\,, \\ \Omega_{\rm M,0} & = \dfrac{8\pi G \rho_{\rm M,0}}{3H_0^2}\,, \\ \Omega_{k,0} & = \dfrac{kc^2}{H_0^2}\,, \\ \Omega_\Lambda & = \dfrac{\Lambda c^2}{3H_0^2}\,. \end{cases} \tag{15.5}$$

Note that $\Omega_{\rm rad,0} + \Omega_{\rm M,0} + \Omega_{k,0} + \Omega_{\Lambda,0} = 1$.

Solutions for special cases

The Friedmann equation can often be solved for flat Universes ($k = 0$). Examples are (below, $t_H \equiv 1/H$ is the Hubble time):

- $k = 0$, $\Omega_{\rm M} = 1$ (flat, matter-dominated):

$$R = (6\pi G \rho_{\rm c})^{1/3}\, t^{2/3} = \left(\frac{3t}{2t_H}\right)^{2/3} \qquad \Rightarrow \qquad \frac{t}{t_H} = \frac{2}{3}\left(\frac{1}{1+z}\right)^{3/2} \tag{15.6}$$

- $k = 0$, $\Omega_\Lambda = 1$: (flat, cosmological-constant dominated):

$$R = \exp\left((t - t_0)\sqrt{\frac{1}{3}\Lambda c^2}\right) = \exp\left(\frac{t - t_0}{t_H}\right) \tag{15.7}$$

- $k = 0$, $\Omega_{\rm M} + \Omega_\Lambda = 1$ (flat universe with both matter and cosmological constant):

$$R = \left(\frac{\Omega_{\rm M,0}}{\Omega_{\Lambda,0}}\right)^{1/3} \sinh^{2/3}\left(\frac{3t}{2t_{H,0}}\Omega_{\Lambda,0}^{1/2}\right), \tag{15.8}$$

  where now we have to write this in terms of the *current* values of H, $\Omega_{\rm M}$ and Ω_Λ, since these values change with time. (*In case you want to derive this yourself: start with Eq. 15.5, subsitute* $x^2 = R^3\Omega_{\Lambda,0}/\Omega_{\rm M,0}$ *and use that* $\mathrm{d}\,\mathrm{arcsinh}\,x/\mathrm{d}\,x = (1 + x^2)^{-1/2}$.)

16. Big Bang Nucleosynthesis

Textbook: §28.1, up to p. 1292.

Ratio of photons to baryons

The number density of photons is given by,

$$n_\gamma = \int_0^\infty \frac{4\pi\nu^2 \mathrm{d}\nu}{c^3} \frac{2}{\mathrm{e}^{h\nu/kT}-1} = 8\pi \left(\frac{kT}{hc}\right)^3 \int_0^\infty \frac{x^2 \mathrm{d}x}{\mathrm{e}^x - 1} = 4.2\times 10^8\,\mathrm{m}^{-3} \left(\frac{T}{2.736\,\mathrm{K}}\right)^3, \tag{16.1}$$

where we used $\int_0^\infty [x^2/(\mathrm{e}^x-1)]\,\mathrm{d}x = 2\zeta(3) = 2.404$, from a table of integrals.

The present baryon number density is

$$n_{\mathrm{B},0} = \frac{\Omega_{\mathrm{B},0}\,\rho_{\mathrm{c},0}}{m_\mathrm{H}} = 6.3\ \mathrm{m}^{-3}\,\Omega_{\mathrm{B},0} \left(\frac{H_0}{75\,\mathrm{km\,s^{-1}\,Mpc^{-1}}}\right)^2. \tag{16.2}$$

Thus, the present ratio of baryons to photons (and past, back to e^+e^- annihilation!) is

$$\eta \equiv \frac{n_{\mathrm{B},0}}{n_{\gamma,0}} \simeq 1.5\times 10^{-8}\,\Omega_{\mathrm{B},0} \left(\frac{H_0}{75\,\mathrm{km\,s^{-1}\,Mpc^{-1}}}\right)^2 . \tag{16.3}$$

Radiation-dominated period

For radiation, the equivalent density is $\rho_\mathrm{rad} = aT^4/c^2$. Since $\rho_\mathrm{rad} \propto R^{-4}$, it dominates at early times; then,

$$\frac{\dot{R}}{R} = \left(\frac{8\pi G}{3}\frac{aT^4}{c^2}\right)^{1/2} . \tag{16.4}$$

For radiation, $T \propto R^{-1}$ (see Eq. 14.7: relativistic case), so $\dot{T}/T = -\dot{R}/R$; integrating, we get

$$T = \left(\frac{3c^2}{32\pi Ga}\right)^{1/4} t^{-1/2} . \tag{16.5}$$

The above is not quite right, as one should include the contribution from neutrinos, as well as, at early times, the sea of e^+e^- pairs. For each neutrino family the energy density is $u_{\nu,\bar\nu} = \frac{7}{8}aT^4$; relativistic electrons or positron (which have two spin states) have $u_{e+} = u_{e-} = \frac{7}{8}aT^4$. These can be included by taking $a' = a(1+\frac{7}{8}[N_\nu+2])$ instead of a in the radiation energy density above, where $N_\nu = 3$ is the number of neutrino families (one can include in N_ν possible other weakly interacting particles that are relativistic at early times). A further complication is the energy dumped in the radiation field as the e^+e^- pairs annihilate. Neutrinos do not share in this added energy, so that one must then use $a' = a(1+\frac{7}{8}N_\nu(T_\nu/T)^4)$, where $T_\nu/T \approx (4/11)^{1/3}$.

Neutron to proton ratio

At high T, the reactions changing proton into neutrons and vice versa proceed quickly, and hence the number densities follow their equilibrium ratio:

$$\frac{n_\mathrm{n}}{n_\mathrm{p}} = \mathrm{e}^{-Q/kT} \qquad \text{with} \qquad Q = (m_\mathrm{n} - m_\mathrm{p})c^2 = 1.293\ \mathrm{MeV}. \tag{16.6}$$

This ratio will 'freeze out' at $T \simeq 10^{10}\,\mathrm{K}$, when the reaction speeds become too slow.

Deuterium formation

Deuterium is formed and broken apart by the reaction

$$p + n \leftrightarrow {}^2\mathrm{H} + \gamma \; . \tag{16.7}$$

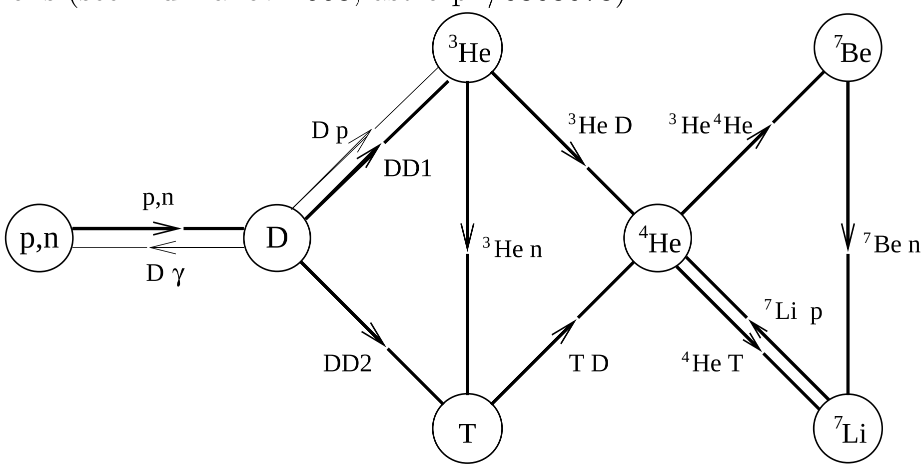
In equilibrium, the abundance can be described by the Saha equation,

$$\frac{n_\mathrm{p} n_\mathrm{n}}{n_\mathrm{D}} = \frac{U_\mathrm{p} U_\mathrm{n}}{U_\mathrm{D}} \frac{(2\pi kT)^{3/2}}{h^3} \left(\frac{m_\mathrm{p} m_\mathrm{n}}{m_\mathrm{D}}\right)^{3/2} \mathrm{e}^{-B/kT} \; , \quad \text{with} \quad B = (m_\mathrm{p} + m_\mathrm{n} - m_\mathrm{D})c^2 = 2.225 \text{ MeV.} \tag{16.8}$$

The statistical weights are $U_\mathrm{p} = U_\mathrm{n} = 2$, and $U_\mathrm{D} = 3$.

Formation of light elements

Once Deuterium becomes present in significant abundances, it is rapidly burned to Helium; thus most remaining neutrons are funnelled though D to end up in ^{4}He (see Figs. 16.1 and 16.2), via the following reactions (see Mukhanov 2003, astro-ph/0303073):



[Note that D = ^{2}H (deuterium) and T = ^{3}H (tritium).] A little ^{7}Li is also formed. Since ^{7}Be is unstable to electron capture, any ^{7}Be created will eventually (much later!) be converted into ^{7}Li, either by a free electron or, after recombination, by the usual K-shell electron capture (when the universe has cooled off enough that electrons combine with ions to yield neutral atoms, ^{7}Be will have an electron-capture half-life of 53.28 days). Any leftover neutrons will of course decay into protons, while leftover tritium will decay into ^{3}He.

Heavier elements do not form because 3-body reactions (such as the triple-α reaction) are too slow at these densities and timescales.

Dependencies

If n_B is *very* small (small η), then all reactions are too slow, and almost nothing forms. If n_B is very big (large η), one still has to wait until the temperature is low enough that D can form, so the final ^{4}He abundance is not very different; however, the D + D and $D + p$ reactions are faster at high density, so less D is left at the end (see Fig. 16.2).

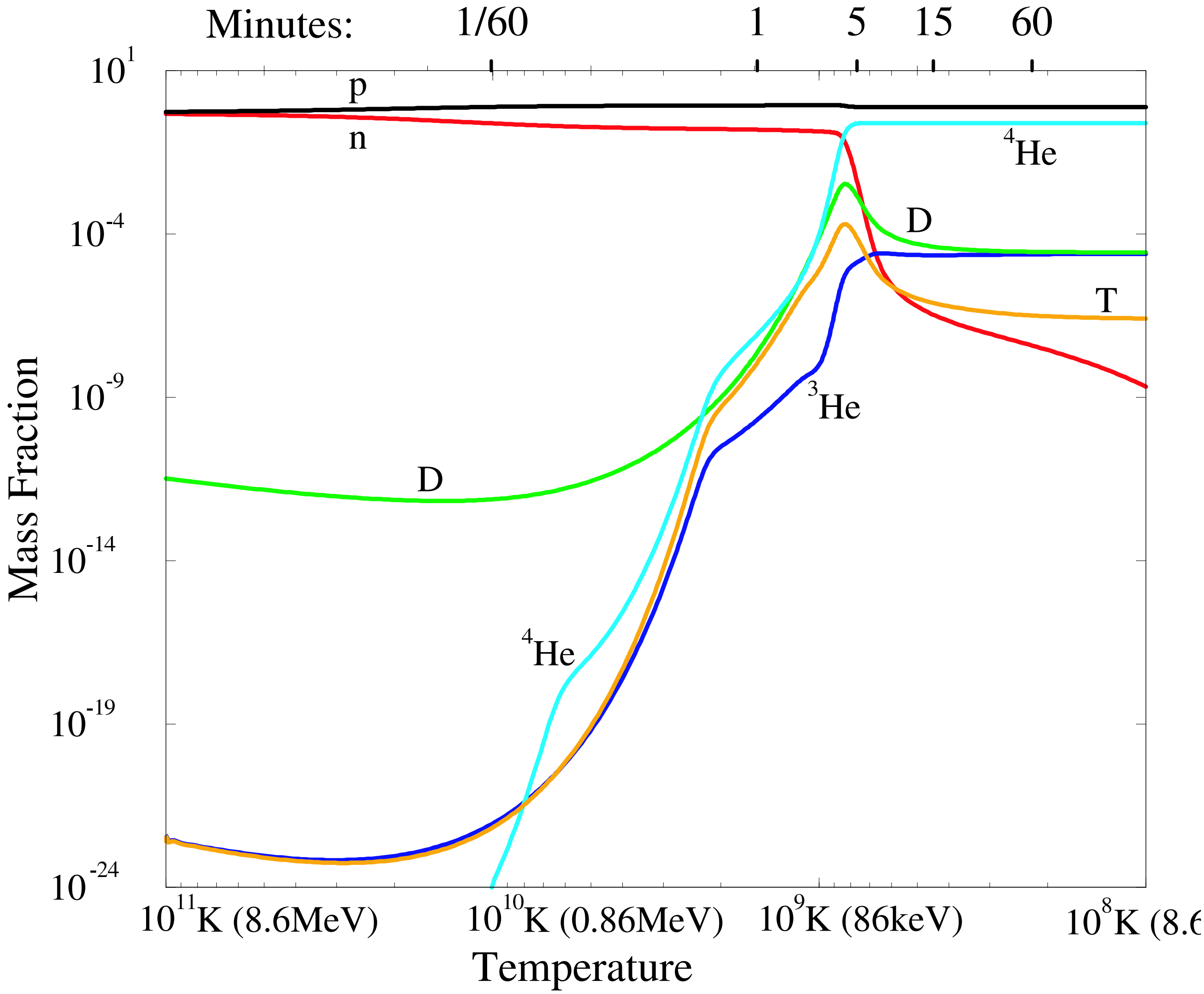


Fig. 16.1. Big bang nucleosynthesis: light element abundances (mass fractions) as a function of time (upper scale) or temperature (lower scale) in the early uinverse, for a case with present $\Omega_{\rm B} = 0.05$ and $H_0 = 75\ {\rm km\,s^{-1}\,Mpc^{-1}}$, i.e., $\eta = 7.5 \times 10^{-10}$, just slightly higher than the "best" value of $\eta \simeq 6 \times 10^{-10}$ (note that D is deuterium, T is tritium). Note the sharp peak in the D abundance at $t \approx 4$ minutes ($T \approx 9 \times 10^{8}$ K), corresponding to the time when the ^{4}He abundance rises to its final value. From Mukhanov (2003), astro-ph/0303073.

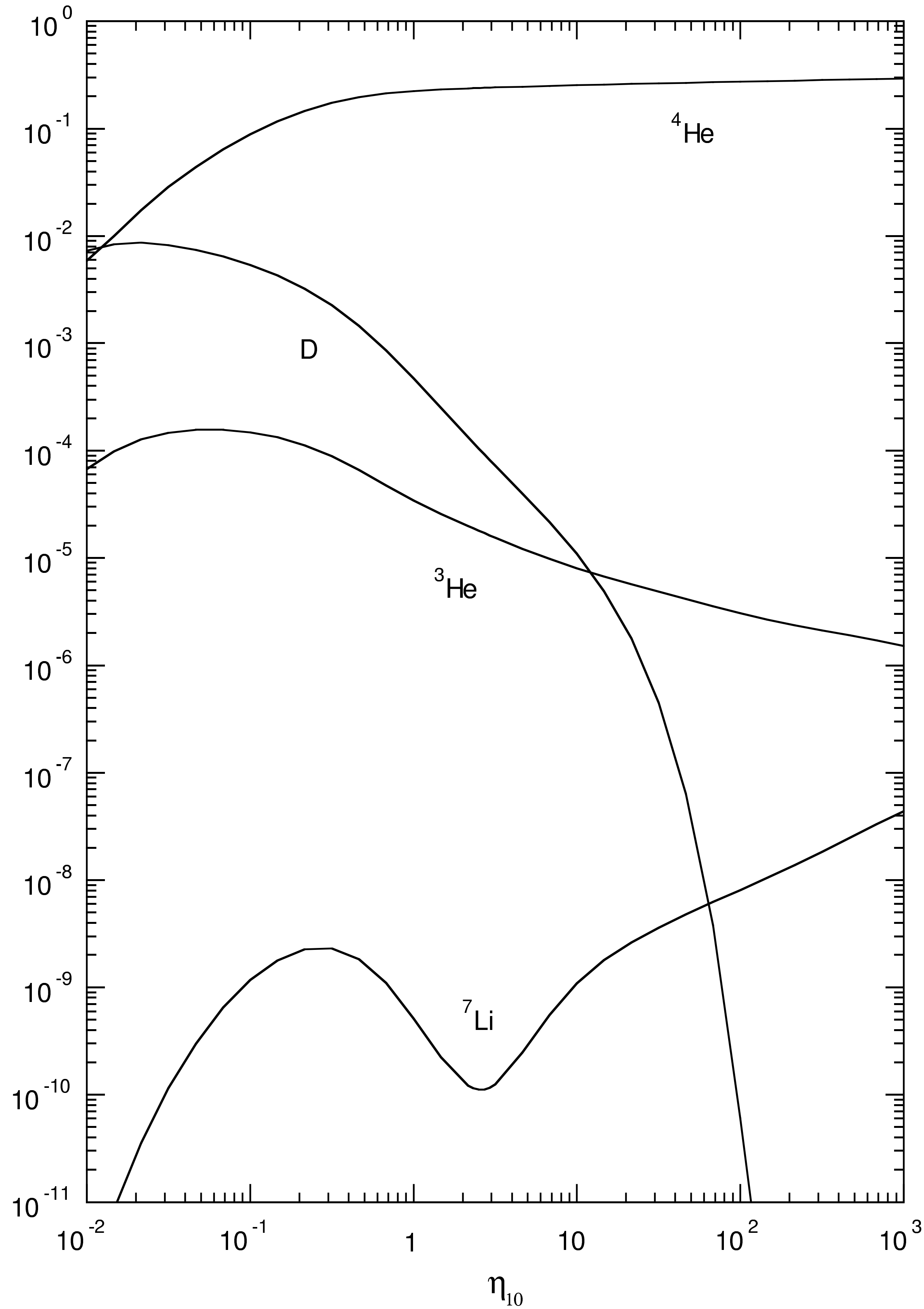


Fig. 16.2. Expected final abundances (mass fractions) for the light elements from big bang nucleosynthesis, as a function of baryon-to-photon ratio η (in units of 10^{-10}: note that $\eta_{10} \equiv \eta/10^{-10}$). From Mukhanov (2003), astro-ph/0303073.