1. Hydrostatic equilibrium and virial theorem

Textbook: §10.1, 2.4

Assumed known: §2.1–2.3

Equation of motion in spherical symmetry

$$\rho \frac{\mathrm{d}^2 r}{\mathrm{d}t^2} = -\frac{GM_r \rho}{r^2} - \frac{\mathrm{d}P}{\mathrm{d}r} \tag{1.1}$$

Hydrostatic equilibrium

$$\frac{\mathrm{d}P}{\mathrm{d}r} = -\frac{GM_r \rho}{r^2} \tag{1.2}$$

Mass conservation

$$\frac{\mathrm{d}M_r}{\mathrm{d}r} = 4\pi r^2 \rho \tag{1.3}$$

Virial Theorem

$$E_{\mathrm{pot}} = -2E_{\mathrm{kin}} \qquad \text{or} \qquad E_{\mathrm{tot}} = \frac{1}{2}E_{\mathrm{pot}} \tag{1.4}$$

Derivation for gaseous spheres: multiply equation of hydrostatic equilibrium by r on both sides, integrate over sphere, and relate pressure to kinetic energy (easiest to verify for ideal gas).

For next time

- Read derivation of virial theorem for set of particles (§2.4).
- Remind yourself of interstellar dust and gas, and extinction (§12.1).
- Remind yourself about thermodynamics, in particular adiabatic processes (bottom of p. 317 to p. 321).

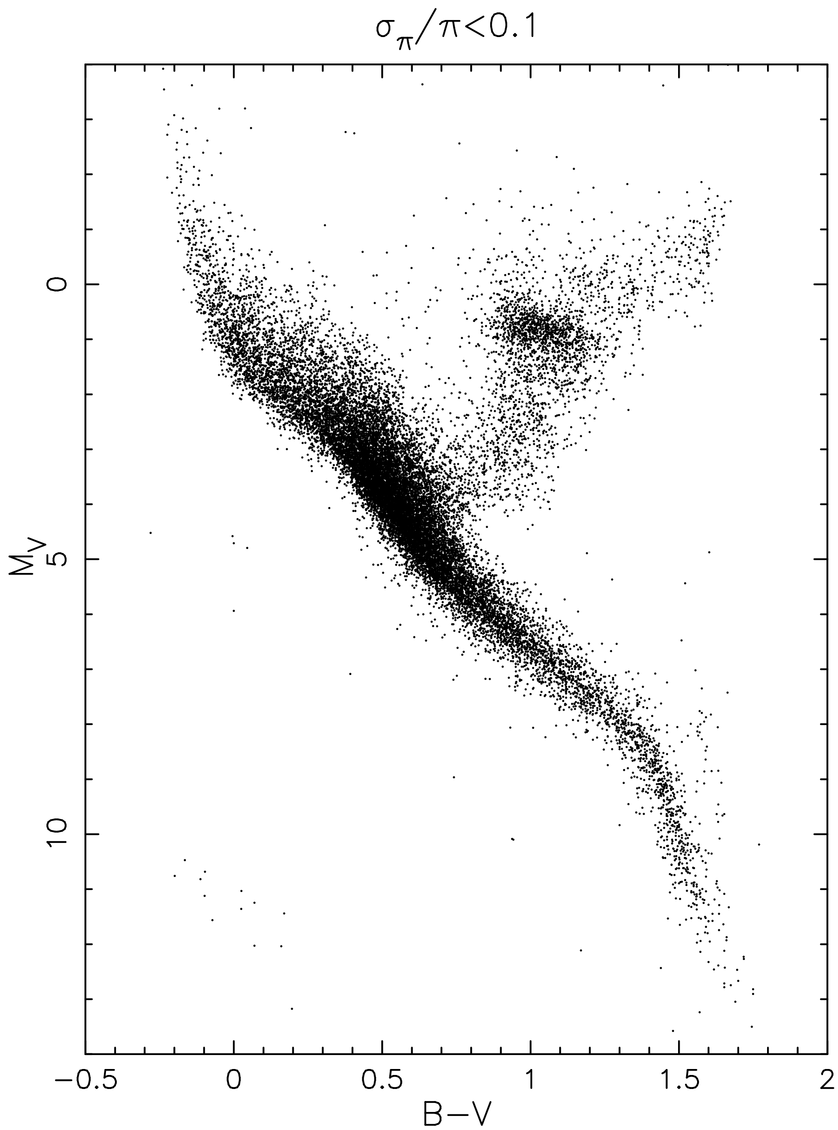


Fig. 1.1. The HRD of nearby stars, with colours and distances measured by the Hipparcos satellite. Taken from Verbunt (2000, first-year lecture notes, Utrecht University).

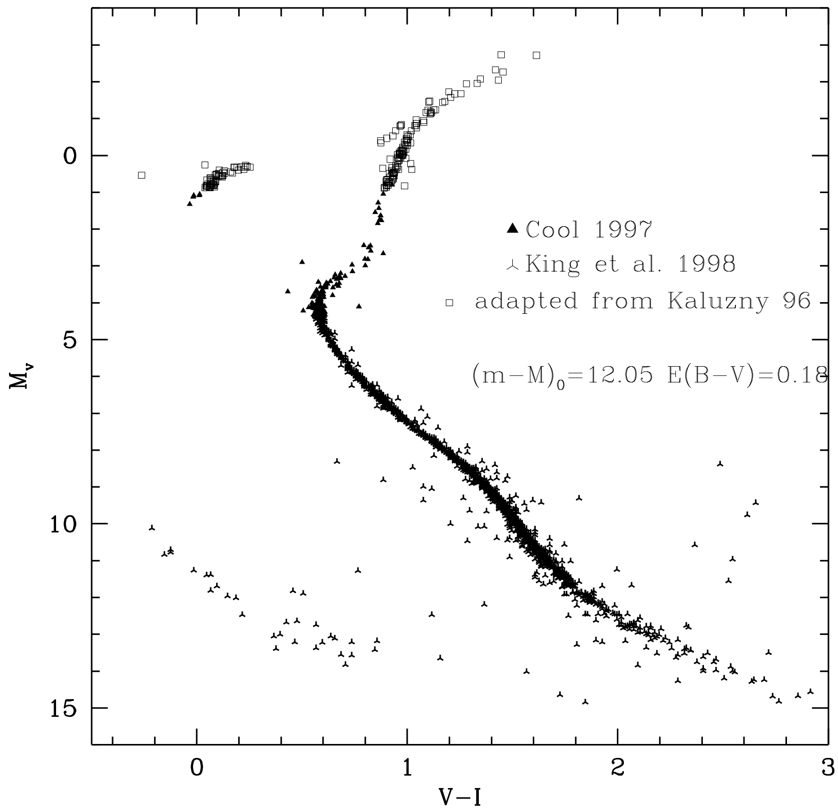


Fig. 1.2. Observed HRD of the stars in NGC 6397. Taken from D'Antona (1999, in "The Galactic Halo: from Globular Clusters to Field Stars", 35th Liege Int. Astroph. Colloquium).

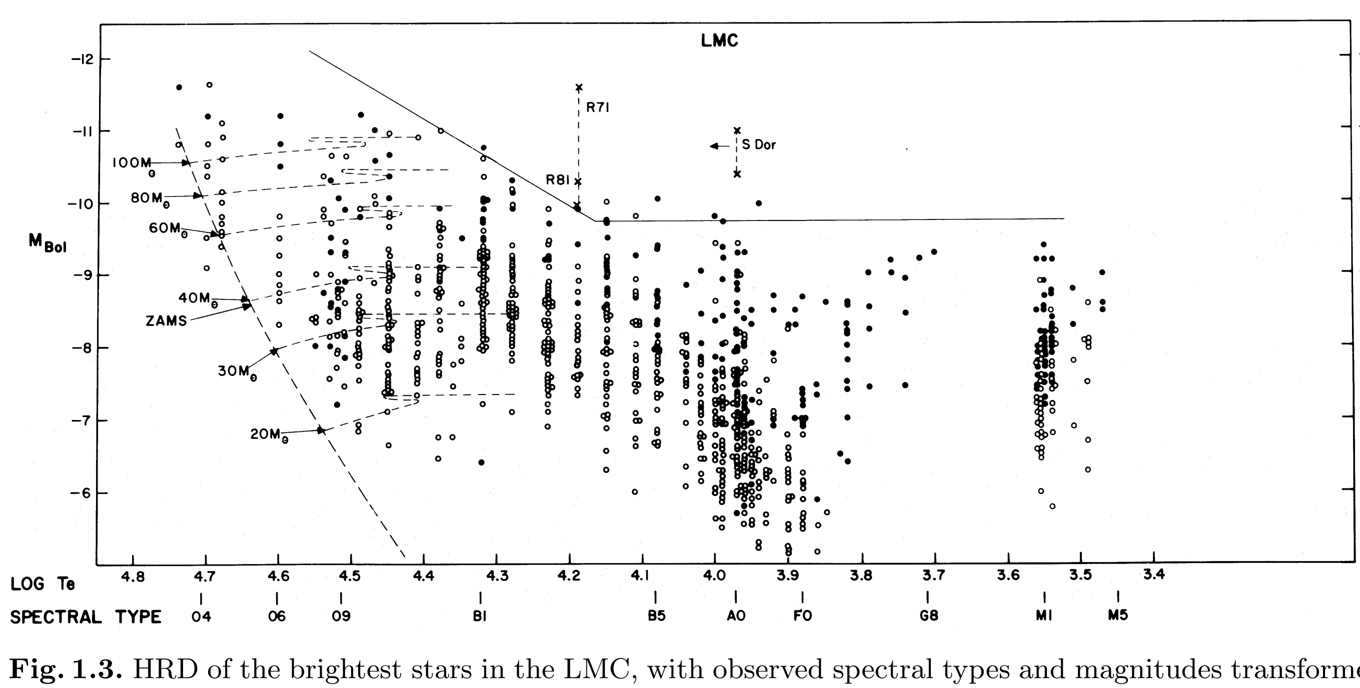


Fig. 1.3. HRD of the brightest stars in the LMC, with observed spectral types and magnitudes transformed to temperatures and luminosities. Overdrawn is the empirical upper limit to the luminosity, as well as a theoretical main sequence. Taken from Humphreys & Davidson (1979, ApJ 232, 409).

2. Star formation

Textbook: §12.2

Jeans Mass and Radius

$$M_{\rm J} = \left(\frac{3}{4\pi}\right)^{1/2}\left(\frac{5k}{G\mu m_{\rm H}}\right)^{3/2}\frac{T^{3/2}}{\rho^{1/2}} = 29\,M_\odot\,\mu^{-3/2}\left(\frac{T}{10\,{\rm K}}\right)^{3/2}\left(\frac{n}{10^4\,{\rm cm}^{-3}}\right)^{-1/2}, \tag{2.1}$$

$$R_{\rm J} = \left(\frac{3}{4\pi}\frac{5k}{G\mu m_{\rm H}}\right)^{1/2}\frac{T^{1/2}}{\rho^{1/2}} = 0.30\,{\rm pc}\,\mu^{-1/2}\left(\frac{T}{10\,{\rm K}}\right)^{1/2}\left(\frac{n}{10^4\,{\rm cm}^{-3}}\right)^{-1/2}. \tag{2.2}$$

Dynamical or free-fall timescale

$$t_{\rm ff} \simeq \sqrt{\frac{R^3}{GM}} \qquad \left(\text{exact: } \sqrt{\frac{3\pi}{32G\rho}};\ \text{often also: } \sim\sqrt{\frac{1}{G\rho}}\right). \tag{2.3}$$

Pulsation time scale

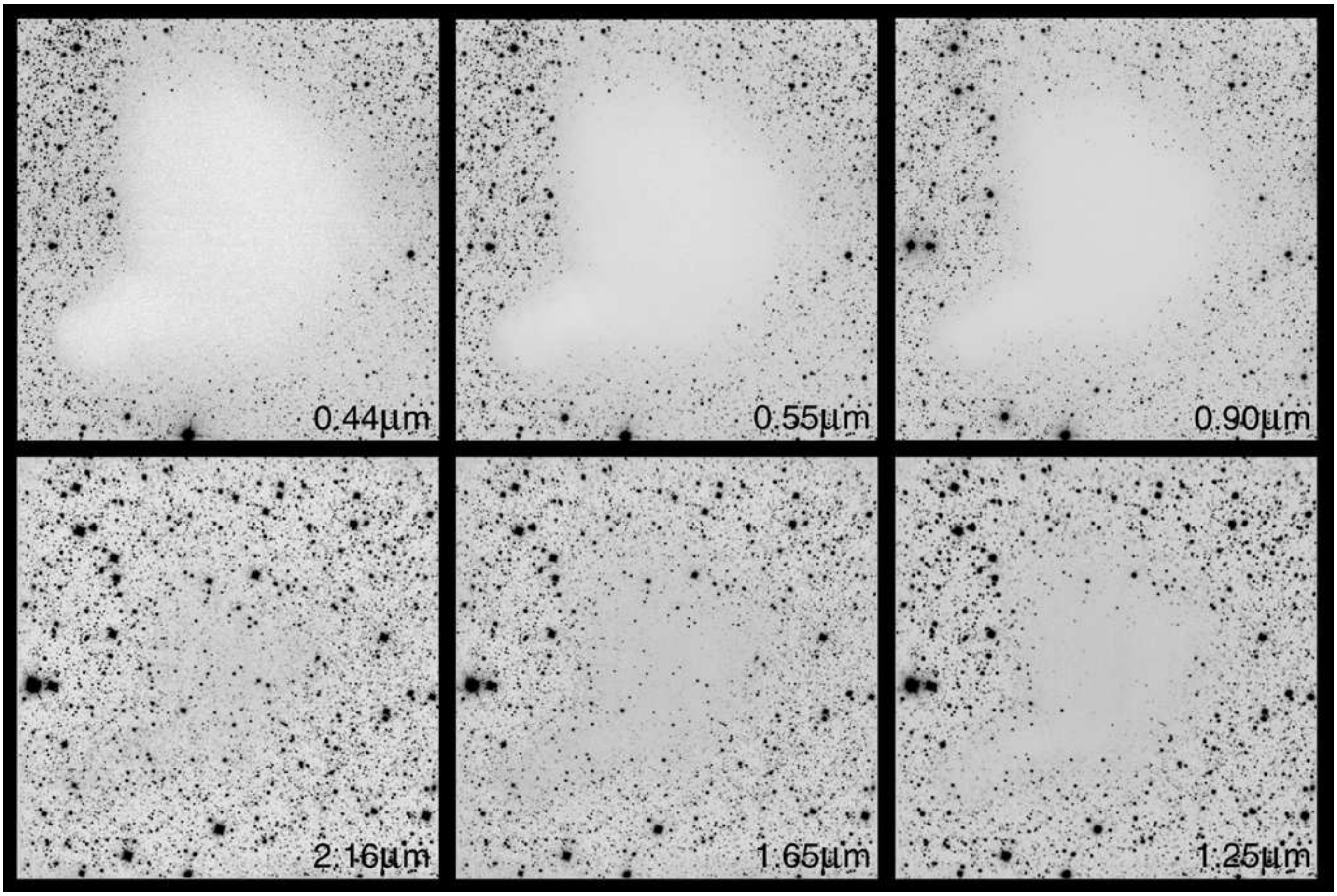
$$t_{\rm puls} \simeq \frac{R}{c_{\rm s}} \simeq \sqrt{\frac{1}{\gamma G\rho}} \qquad \left(\text{usually simply: } \sim\sqrt{\frac{1}{G\rho}}\right). \tag{2.4}$$

Contraction or Kelvin-Helmholtz timescale

$$t_{\rm KH} = \frac{-E_{\rm pot}}{L} \simeq \frac{GM^2}{RL} \tag{2.5}$$

For next time

- Think about initial mass-radius relation;
- Remind yourself about pressure integral (§10.2, in particular eq. 10.9), as well as mean molecular weight; and
- Remind yourself of relativistic energy and momentum (§4.4)



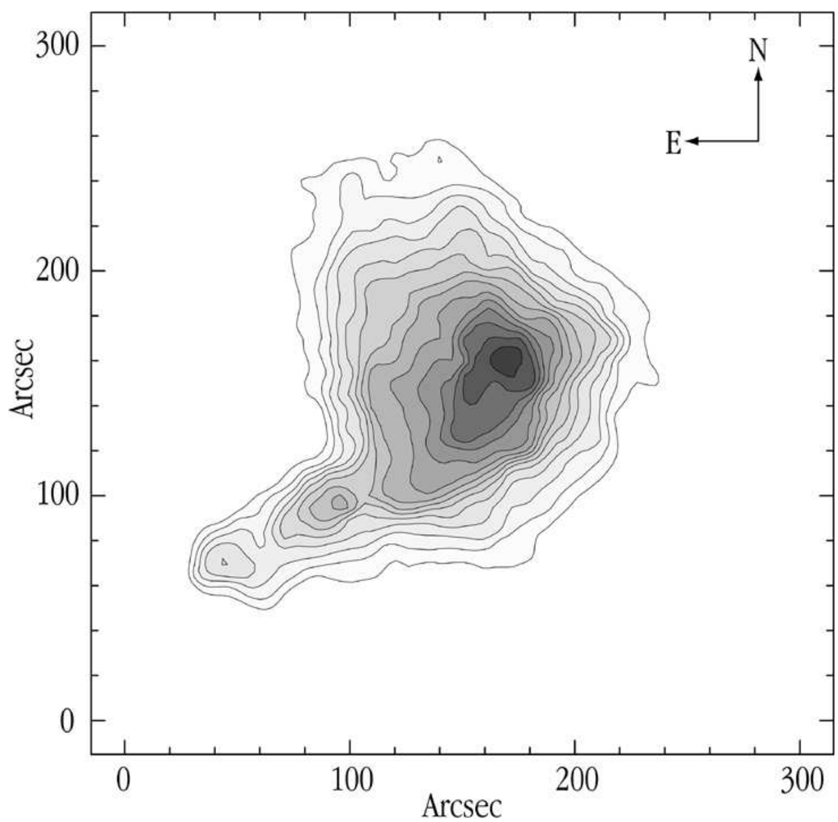


Fig. 2.1. (top) Images towards the dark globule Barnard 68 in B, V, I, J, H, and K (clockwise from upper left to lower left). The images are 4.′9 on the side; North is up, East to the left.

Fig. 2.2. (left) Inferred extinction through Barnard 68. Contours of V-band optical depth are shown, starting at 4 and increasing in steps of 2. Both pictures are taken from ESO press release 29/99.

3. Equation of state

Textbook: §10.2, 16.3

General expressions for number density, pressure, and energy

Given a momentum distribution $n(p)\mathrm{d}p$, then the particle density n, kinetic energy density U, and pressure P are given by

$$n = \int_0^\infty n(p)\mathrm{d}p, \tag{3.1}$$

$$U = \int_0^\infty n(p)\epsilon_p\,\mathrm{d}p, \tag{3.2}$$

$$P = \frac{1}{3}\int_0^\infty n(p)v_p p\,\mathrm{d}p. \tag{3.3}$$

For non-relativistic particles, $v_p = p/m$ and $\epsilon_p = p^2/2m$, while for (extremely) relativistic particles, $v_p \simeq c$, and $\epsilon_p = pc$. Hence,

$$P_{\mathrm{NR}} = \frac{1}{3}\int_0^\infty 2\epsilon_p n(p)\mathrm{d}p \qquad \Rightarrow \qquad P = \frac{2}{3}U, \tag{3.4}$$

$$P_{\mathrm{ER}} = \frac{1}{3}\int_0^\infty \epsilon_p n(p)\mathrm{d}p \qquad \Rightarrow \qquad P = \frac{1}{3}U. \tag{3.5}$$

General momentum distribution

$$n(p)\mathrm{d}p = n(\epsilon)\frac{g}{h^3}4\pi p^2\mathrm{d}p \qquad \text{(where g is the statistical weight).} \tag{3.6}$$

Here, $n(\epsilon)$ depends on the nature of the particles:

$$n(\epsilon) = \begin{cases} \dfrac{1}{\mathrm{e}^{(\epsilon-\mu)/kT}+0} & \text{classical; Maxwell-Boltzmann statistics,} \\ \dfrac{1}{\mathrm{e}^{(\epsilon-\mu)/kT}+1} & \text{fermions; Fermi-Dirac statistics,} \\ \dfrac{1}{\mathrm{e}^{(\epsilon-\mu)/kT}-1} & \text{bosons; Bose-Einstein statistics.} \end{cases} \tag{3.7}$$

Here, μ is the chemical potential; one can view the latter as a normalisation term that ensures $\int_0^\infty n(p)\mathrm{d}p = n$.

Classical: Maxwellian

After solving for μ, one recovers the Maxwellian momentum distribution:

$$n(p)\mathrm{d}p = n\frac{4\pi p^2\mathrm{d}p}{(2\pi mkT)^{3/2}}\mathrm{e}^{-p^2/2mkT} \tag{3.8}$$

Bosons: application to photons

For photons, the normalisation is not by total number of particles, but by energy; one finds $\mu = 0$. The statistical weight is $g = 2$ (two senses of polarisation). With $\epsilon = h\nu$ and $p = h\nu/c$, one finds for

$n(\nu)\mathrm{d}\nu$ and $U(\nu)\mathrm{d}\nu = h\nu n(\nu)\mathrm{d}\nu$,

$$n(\nu)\mathrm{d}\nu = \frac{4\pi\nu^2\mathrm{d}\nu}{c^3}\frac{2}{\mathrm{e}^{h\nu/kT}-1}; \tag{3.9}$$

$$U(\nu)\mathrm{d}\nu = \frac{8\pi h\nu^3}{c^3}\frac{\mathrm{d}\nu}{\mathrm{e}^{h\nu/kT}-1}. \tag{3.10}$$

Fermions: application to electrons

$$n(p)\mathrm{d}p = \frac{g}{h^3}\frac{4\pi p^2\mathrm{d}p}{\mathrm{e}^{(\epsilon-\mu)/kT}+1}. \tag{3.11}$$

Complete degeneracy

$$n(\epsilon) = \left\{\begin{array}{l} 1 \text{ for } \epsilon < \epsilon_\mathrm{F} \\ 0 \text{ for } \epsilon > \epsilon_\mathrm{F} \end{array}\right\} \quad \Leftrightarrow \quad n(p) = \left\{\begin{array}{ll} \frac{g}{h^3}4\pi p^2\mathrm{d}p & \text{for } p < p_\mathrm{F} \\ 0 & \text{for } p > p_\mathrm{F} \end{array}\right. . \tag{3.12}$$

Expressing p_F as a function of the number density n,

$$p_\mathrm{F} = h\left(\frac{3n}{4\pi g}\right)^{1/3}. \tag{3.13}$$

NRCD: non-relativistic complete degeneracy

For non-relativistic particles, one has $\epsilon_p = p^2/2m$, and thus $P = \frac{2}{3}U$. Hence,

$$P = \frac{2}{3}\int_0^{p_\mathrm{F}} n(p)\epsilon_p\,\mathrm{d}p = \frac{1}{20}\left(\frac{3}{\pi}\right)^{2/3}\frac{h^2}{m}n^{5/3}. \tag{3.14}$$

For electrons: $P_\mathrm{e} = K_1(\rho/\mu_\mathrm{e}m_\mathrm{H})^{5/3}$ with $K_1/m_\mathrm{H}^{5/3} = 9.91\times10^{12}$ (cgs). (3.15)

ERCD: extremely relativistic complete degeneracy

For relativistic particles, $\epsilon_p = pc$, and thus $P = \frac{1}{3}U$ (Eq. 3.5). Hence,

$$P = \frac{1}{3}\int_0^{p_\mathrm{F}} n(p)\epsilon_p\,\mathrm{d}p = \frac{1}{8}\left(\frac{3}{\pi}\right)^{1/3}hc\,n^{4/3}. \tag{3.16}$$

For electrons: $P_\mathrm{e} = K_2(\rho/\mu_\mathrm{e}m_\mathrm{H})^{4/3}$ with $K_2/m_\mathrm{H}^{4/3} = 1.231\times10^{15}$ (cgs). (3.17)

For next time

– Is degeneracy importantant for daily materials?

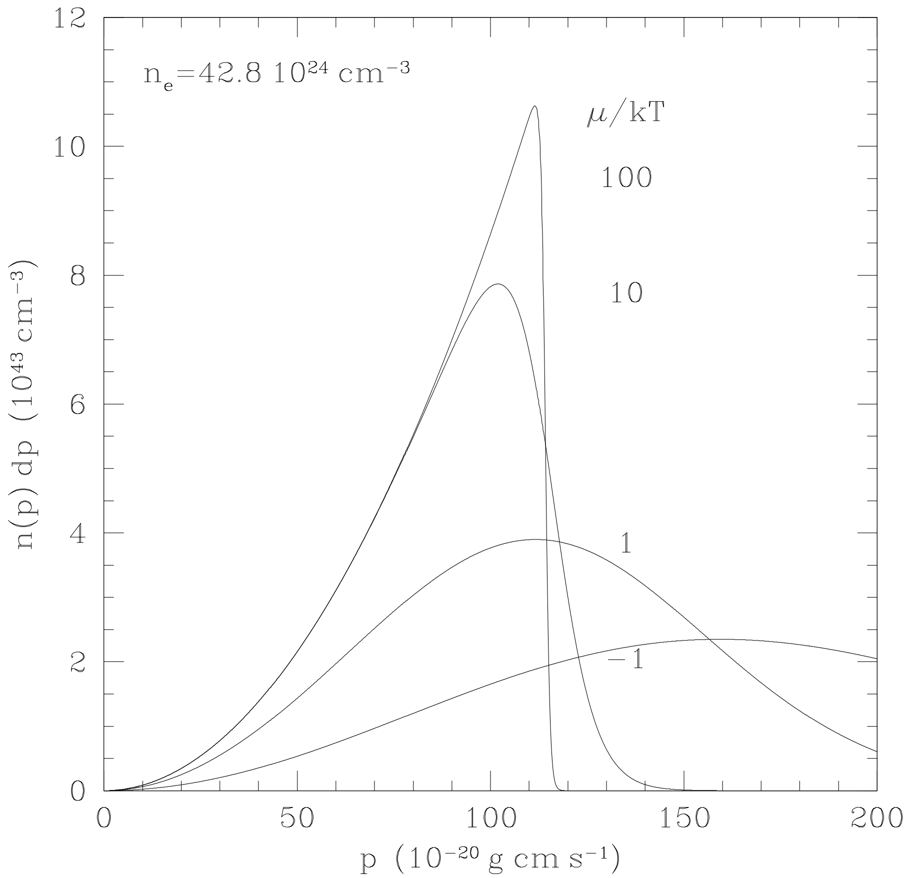


Fig. 3.1. The distribution of the number of particles $n(p)$ as a function of momentum p for a number of values of μ/kT.

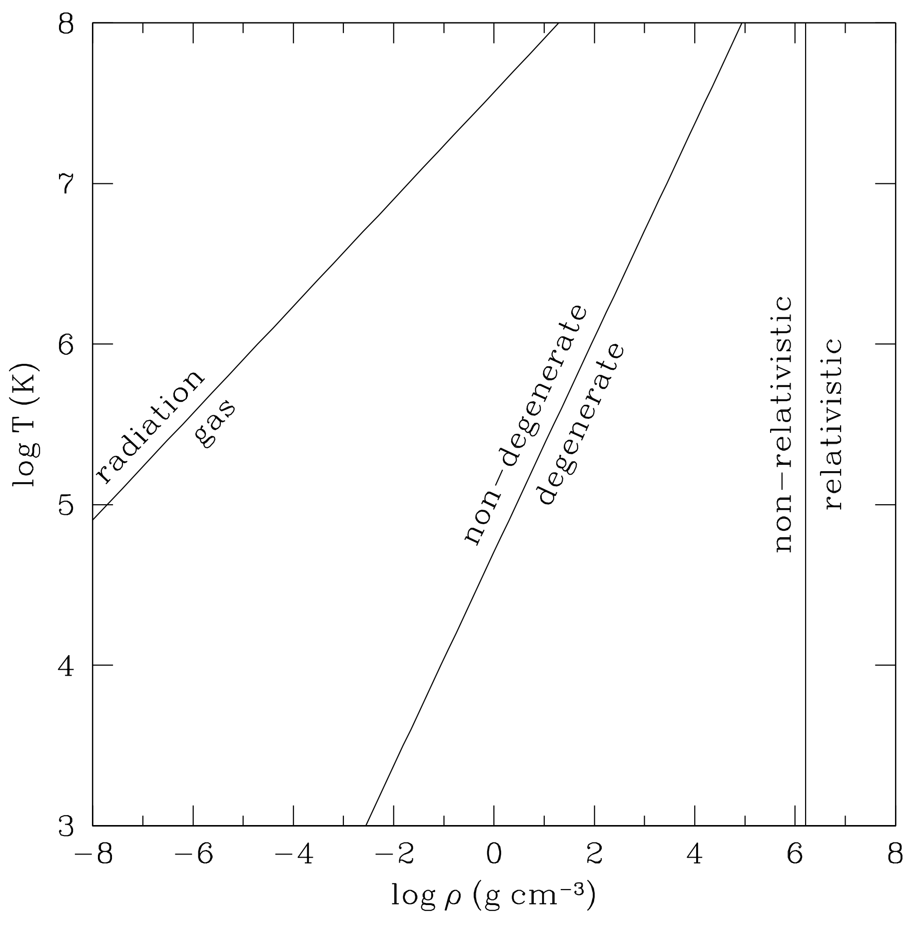


Fig. 3.2. The T, ρ diagram for $X = 0.7$ and $Z = 0.02$, with the areas indicated where matter behaves as an ideal gas ($P \propto nT$), non-relativistic degenerate gas ($P \propto n_e^{5/3}$), relativistic degenerate gas ($P \propto n_e^{4/3}$), or radiation-dominated gas. Note that these are not "sharp" boundaries. Also, at high enough T relativistic effects will become significant at all densities, not just for degenerate matter.

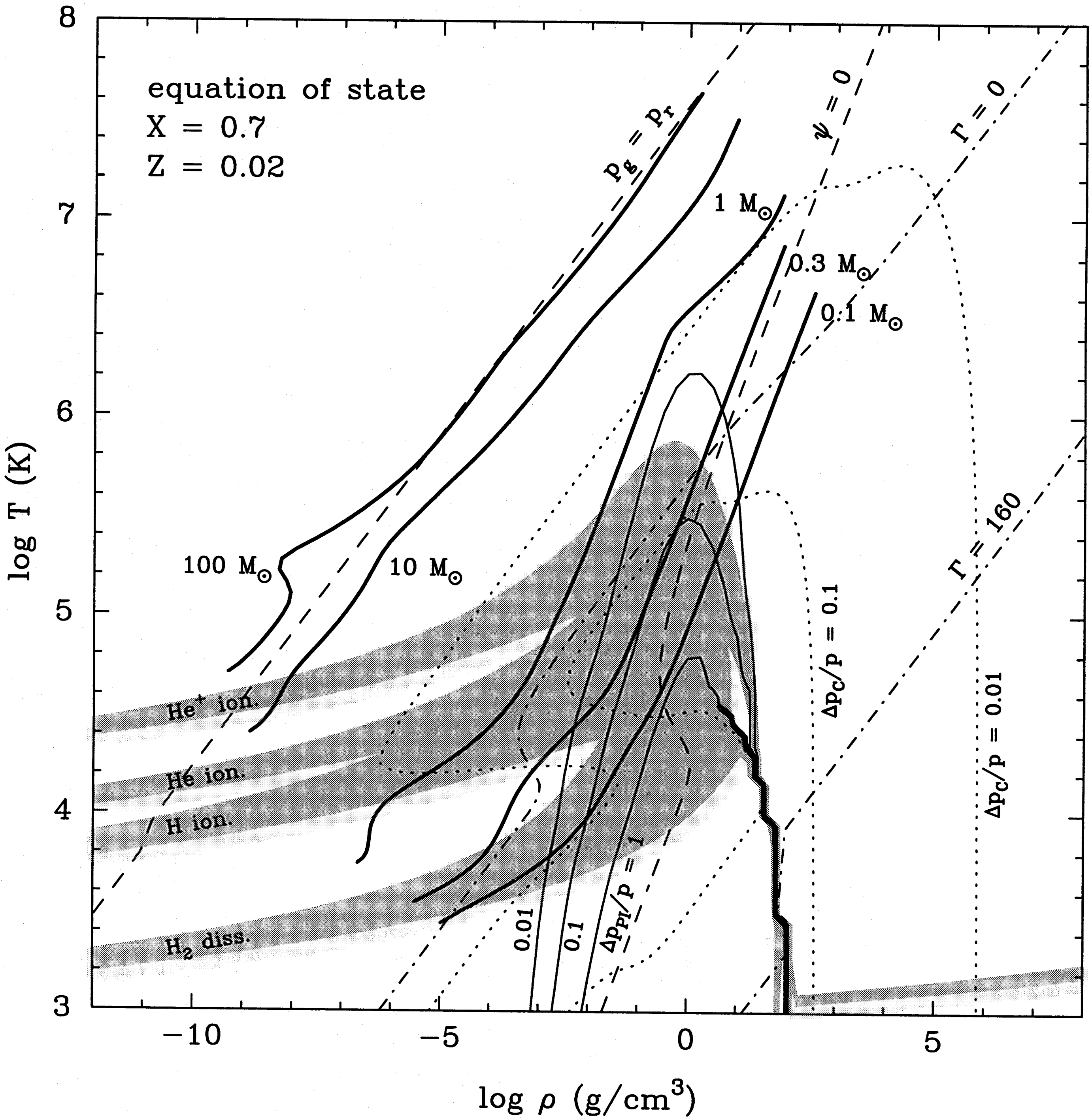


Fig. 3.3. T, ρ diagram for $X = 0.7$ and $Z = 0.02$ from Pols et al. (1995, MNRAS 274, 964). Dashed lines indicates where radiation pressure equals the gas pressure ($P_g = P_r$), and where degeneracy becomes important ($\psi = 0$); note that the latter is defined differently. The shaded regions indicate regions where various ions become ionised. None of the other lines were discussed in the text. Dash-dotted lines indicate constant plasma-interaction parameter Γ; dotted lines constant contribution from Coulomb interactions; thin solid lines constant contribution from pressure ionisation. The thick solid lines indicate the run of temperature as a function of temperature as found in zero-age main sequence (ZAMS) stellar models for several masses.

4. Simple stellar models

Textbook: – p. 334–340, applications in §16.4

Polytropic models

$$\left.\begin{array}{l} M_r = -\dfrac{r^2}{\rho G}\dfrac{\mathrm{d}P}{\mathrm{d}r} \Rightarrow \dfrac{\mathrm{d}M_r}{\mathrm{d}r} = -\dfrac{1}{G}\dfrac{\mathrm{d}}{\mathrm{d}r}\left(\dfrac{r^2}{\rho}\dfrac{\mathrm{d}P}{\mathrm{d}r}\right) \\ \dfrac{\mathrm{d}M_r}{\mathrm{d}r} = 4\pi r^2 \rho \\ P = K\rho^\gamma \end{array}\right\} \Rightarrow \quad \frac{1}{\rho r^2}\frac{\mathrm{d}}{\mathrm{d}r}\left(r^2\rho^{\gamma-2}\frac{d\rho}{\mathrm{d}r}\right) = -\frac{4\pi G}{K\gamma}. \tag{4.1}$$

Making the equation dimensionless, we derive the *Lane-Emden equation* of index n,

$$\left.\begin{array}{l} \rho = \rho_\mathrm{c}\theta^n \quad \text{with} \quad n = \dfrac{1}{\gamma - 1} \quad \left(\text{i.e.,}\gamma = 1 + \dfrac{1}{n}\right) \\ r = \alpha\xi \quad \text{with} \quad \alpha = \left(\dfrac{n+1}{4\pi G}K\rho_\mathrm{c}^{(1/n)-1}\right)^{1/2} \end{array}\right\} \Rightarrow \quad \frac{1}{\xi^2}\frac{\mathrm{d}}{\mathrm{d}\xi}\left(\xi^2\frac{\mathrm{d}\theta}{\mathrm{d}\xi}\right) = -\theta^n. \tag{4.2}$$

The boundary conditions are $\theta_\mathrm{c} = 1$ and $(\mathrm{d}\theta/\mathrm{d}\xi)_\mathrm{c} = 0$.

Solutions of the Lane-Emden equations

In general, the Lane-Emden equation does not have an analytic solution, but needs to be solved numerically. The exceptions are $n = 0$, 1, and 5, for which,

$$\begin{array}{llll} n = 0 \quad (\gamma = \infty): & \theta = 1 - \dfrac{\xi^2}{6} & \Rightarrow & \rho = \rho_\mathrm{c}, \\ n = 1 \quad (\gamma = 2): & \theta = \dfrac{\sin\xi}{\xi} & \Rightarrow & \rho = \rho_\mathrm{c}\dfrac{\sin\alpha r}{\alpha r}, \\ n = 5 \quad (\gamma = 1.20): & \theta = \left(1 + \dfrac{\xi^2}{3}\right)^{-1/2} & \Rightarrow & \rho = \rho_\mathrm{c}\left(1 + \dfrac{(\alpha r)^2}{3}\right)^{-5/2}. \end{array} \tag{4.3}$$

The stellar radius

Since one has $r = \alpha\xi$, the stellar radius is given by

$$R = \alpha\xi_1 = \left[\frac{(n+1)K}{4\pi G}\right]^{1/2} \rho_c^{(1-n)/2n}\xi_1, \tag{4.4}$$

where ξ_1 is the value of ξ for which $\theta(\xi)$ reaches its first zero. In Table 4.1, values of ξ_1 are listed for various n.

The total mass

Integration of $\rho(r)$ gives the total mass of the star,

$$M = 4\pi\alpha^3\rho_\mathrm{c}\int_0^{\xi_1}\xi^2\theta^n\mathrm{d}\xi = 4\pi\alpha^3\rho_\mathrm{c}\int_0^{\xi_1}\mathrm{d}\left(-\xi^2\frac{d\theta}{d\xi}\right) = 4\pi\left[\frac{(n+1)K}{4\pi G}\right]^{3/2}\rho_\mathrm{c}^{(3-n)/2n}\left(-\xi^2\frac{\mathrm{d}\theta}{\mathrm{d}\xi}\right)_{\xi_1} \tag{4.5}$$

Table 4.1. Constants for the Lane-Emden functions

n	γ	ξ_1	$-\xi^2 \frac{\mathrm{d}\theta_n}{\mathrm{d}\xi}\Big\vert_{\xi_1}$	$\frac{\rho_c}{\overline{\rho}}$	$K\frac{R^{(n-3)/n}}{GM^{(n-1)/n}}$	$\frac{P_c}{GM^2/R^4}$
0.0	∞	2.4494	4.8988	1.0000	...	0.119366
0.5	3	3.7528	3.7871	1.8361	2.270	0.26227
1.0	2	3.14159	3.14159	3.28987	0.63662	0.392699
1.5	5/3	3.65375	2.71406	5.99071	0.42422	0.770140
2.0	3/2	4.35287	2.41105	11.40254	0.36475	1.63818
2.5	7/5	5.35528	2.18720	23.40646	0.35150	3.90906
3.0	4/3	6.89685	2.01824	54.1825	0.36394	11.05066
3.5	9/7	9.53581	1.89056	152.884	0.40104	40.9098
4.0	5/4	14.97155	1.79723	622.408	0.47720	247.558
4.5	11/9	31.83646	1.73780	6189.47	0.65798	4922.125
5.0	6/5	∞	1.73205	∞	∞	∞

Taken from Chandrasekar, 1967, Introduction to the study of stellar structure (Dover: New York), p. 96

where we used the Lane-Emden equation to substitute for θ^n. Values of $(-\xi^2 \mathrm{d}\theta/\mathrm{d}\xi)_{\xi_1}$ are again listed in Table 4.1. By combining the relations for the radius and the mass, one also derives a relation between the radius, mass, and K, which, for given K, gives the mass-radius relation. The appropriate numbers are listed in the table.

The central density and pressure

We can express the central density ρ_c in terms of the mean density $\overline{\rho} = M/\frac{4}{3}\pi R^3$ using the relations for the mass and radius. Solving K from the expressions for the mass and radius, one can also find the ratio of the central pressure to GM^2/R^4. Values of $\rho_c/\overline{\rho}$ and $P_c/(GM^2/R^4)$ are listed in Table 4.1.

The potential energy

Given the polytropic relation, one can also calculate the total potential energy. We just list the result here:

$$E_{\mathrm{pot}} = -\frac{3}{5-n}\frac{GM^2}{R}. \tag{4.6}$$

For next time

– Remind yourself of mean-free path and of basic radiation processes (example 9.2.1, pp 239–247).

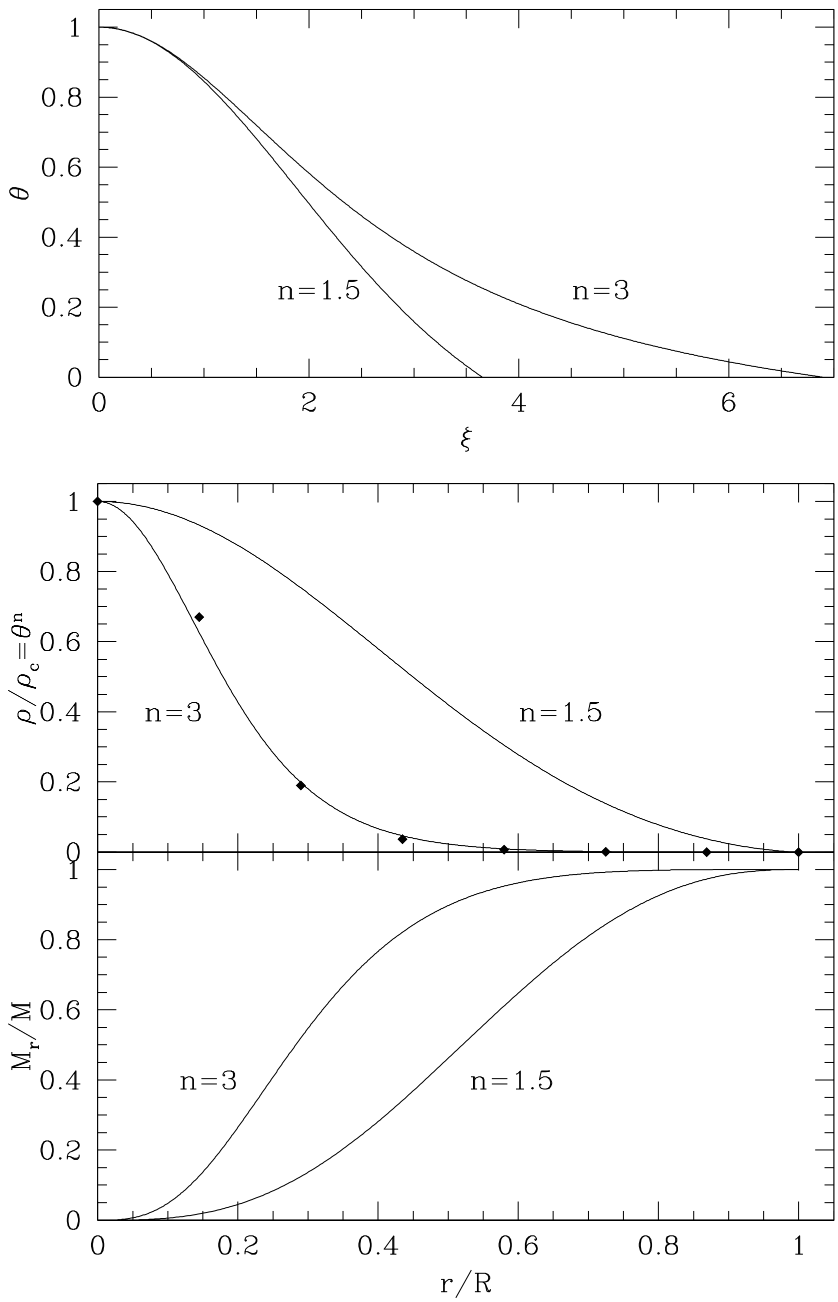


Fig. 4.1. *(top)* Run of $\theta(\xi)$ as a function of ξ for $n = 1.5$ and $n = 3$ (i.e., $\gamma = \frac{5}{3}$ and $\gamma = \frac{4}{3}$). Note that $\xi \propto r$ and $\theta^n \propto \rho$. For non-degenerate stars, $T \propto \theta$. *(middle)* Corresponding run of $\rho(r)/\rho_c$ as a function of r/R. The black dots indicate the values appropriate for the Sun; see Table 4.2. *(bottom)* Run of M_r/M as a function of r/R. Note how much more centrally condensed the $n = 3$ polytrope is compared to the $n = 1.5$ one.

Table 4.2. The run of density of a polytropic model with $n = 3$ and $\gamma = \frac{4}{3}$

ξ	0	1	2	3	4	5	6	6.9011
θ	1	0.855	0.583	0.359	0.209	0.111	0.044	0
r/R_*	0	0.145	0.290	0.435	0.580	0.725	0.869	1
ρ/ρ_c	1	0.625	0.198	0.0463	0.00913	0.00137	0.0000858	0
$(\rho/\rho_c)_\odot$	1	0.67	0.19	0.037	0.0065	0.0011	0.00015	0

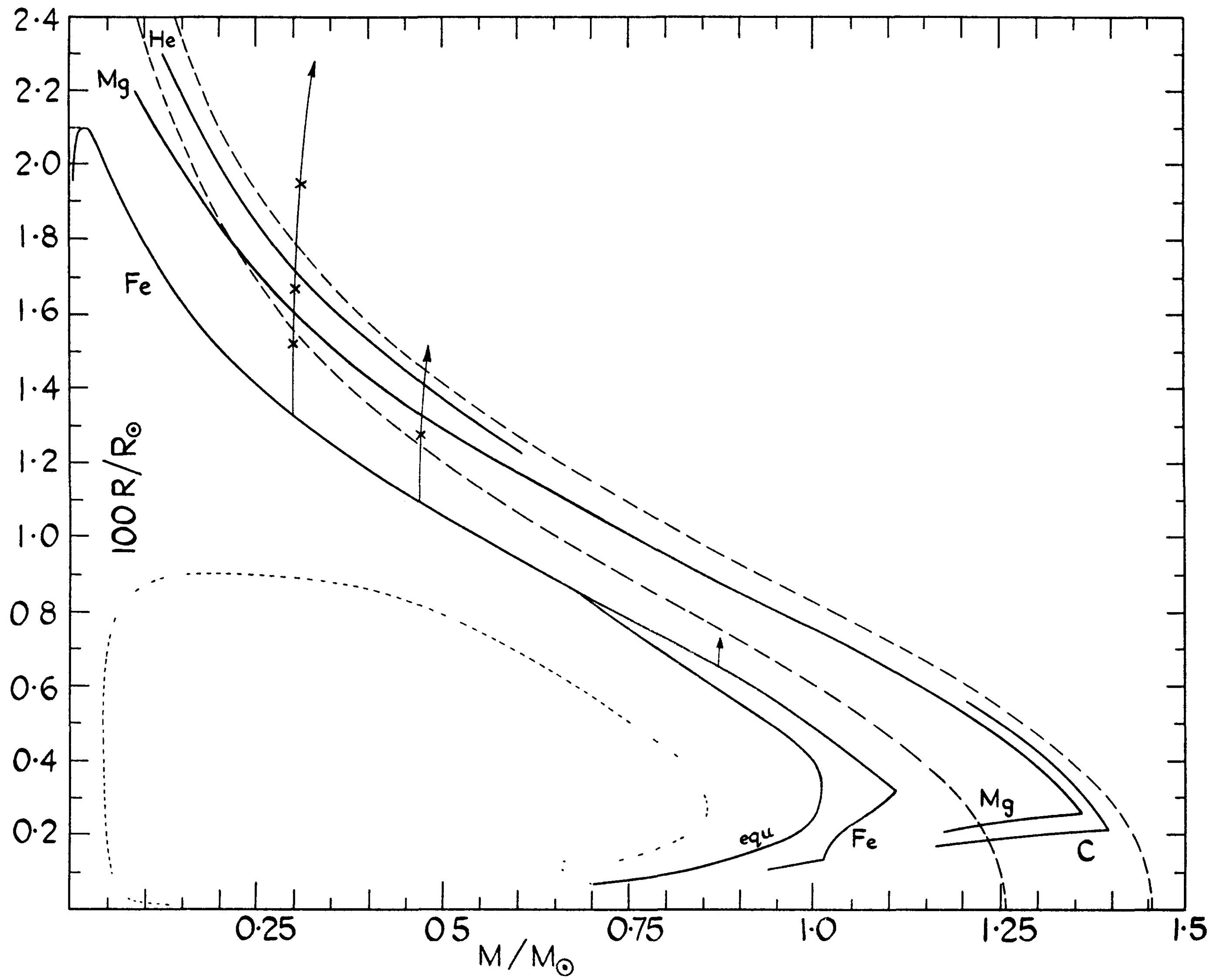


Fig. 4.2. Mass-radius relation for white dwarfs of various compositions. The dashed curves indicate Chandrasekhar models for $\mu_e = 2$ (upper) and 2.15 (lower), in which simple estimates like those discussed in class are used, except that the mildly relativistic regime is treated correctly. The models deviate from these idealized curves because the elements are not become completely ionized, and at very high densities, inverse beta decay becomes important (the curve labelled 'equ' takes into account the resulting changes in elemental abundances). For both reasons, there are variations in μ_e. The arrows indicate the effects of adding a hydrogen atmosphere. The dotted curve is a mass-radius relation for neutron stars. Taken from Hamada & Salpeter (1961, ApJ 134, 683).