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UNDERSTANDING THE PRIMORDIAL POWER SPECTRUM

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ABSTRACT

The matter power spectrum is a hugely misunderstood aspect of cosmology. Many graduate students have fallen on their swords during their quals because they said the Harrison-Zel'dovich power spectrum is $P(k) \propto k^n$, rather than k^{n-1}. Jason and I have been discussing this topic for a few weeks, and we finally think we understand it. Therefore, I have written this up for posterity. This goes over the primordial power spectrum, the evolution of cold dark matter perturbations in the early universe, and to a minor degree the baryon acoustic oscillations, which are a related concept. The basic gist is that the Harrison-Zel'dovich-Peebles flat power spectrum describes the amplitude of density perturbation modes as those modes become contained by the universe's horizon. From that point on, their evolution is determined by the source of density in the universe at that time—in the radiation-dominated era, they go as k^{-3}, and in the matter-dominated era, they depend linearly on k. These effects can be combined to determine a primordial power spectrum at a given time t which is evolved by a transfer function to yield the observed matter power spectrum. This primordial power spectrum has the form $P(k) \propto k^n$, which is what people often mistake for the flat Harrison-Zel'dovich power spectrum when $n = 1$. In reality, the Harrison-Zel'dovich power spectrum is $P(k) \propto k^{n-1}$, is actually flat for $n = 1$, and describes the power of each mode as it crosses the horizon.

Keywords: cosmology, power spectrum, perturbations

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I. DENSITY PERTURBATIONS

We begin, following Duran et al. (2012), by defining a density perturbation as

$$\Delta(k,t) = \frac{1}{2\pi^2} k^3 P(k) \left(\frac{D_+(t)}{D_+(t_0)} \right)^2 \tag{1}$$

where $D_+(t)$ represents the growing solution to a perturbation in the density field. In the radiation-dominated era, D_+ is constant, and in the matter-dominated era,

$$D_+(t) = D_+(t_{\text{in}}) a(t_{\text{in}}). \tag{2}$$

Here, t_0 is the time at present and t_{in} is the time at which a mode is contained within the event horizon. $a(t)$ is the scale factor. We can define that mode as

$$k_{\text{in}} = \frac{2\pi a(t_{\text{in}})}{r_H(t_{\text{in}})} = \frac{2\pi a(t_{\text{in}})}{c\, t_{\text{in}}}. \tag{3}$$

r_H is of course the radius of the event horizon at time t_{in}.

The Harrison-Zel'dovich-Peebles power spectrum tells us that the amplitude of a perturbation at the time it is contained by the horizon is constant—this is what we mean when we say the HZP power spectrum is flat, with equal power in all modes (Lazarides 1999; Duran et al. 2012). This means that

$$\Delta(k_{\text{in}}, t_{\text{in}}) = \text{const.} = \Delta(k_{\text{eq}}, t_{\text{eq}}), \tag{4}$$

where we have introduced a new point in time, t_{eq}, corresponding to a mode k_{eq}. This is the point at which $\Omega_m = \Omega_r$. Before this time, the universe is radiation-dominated, meaning photons contribute more to the universe's density than matter (not that there are more photons than matter particles–there very well may be between this point and recombination, but radiation contributes more to the universe's density!). After this time, the universe is matter-dominated. What this means is that for $t_{\text{in}} < t_{\text{eq}}$ and $k_{\text{in}} > k_{\text{eq}}$ (as large k corresponds to small spatial modes), perturbations are "frozen-in" once they are contained within the horizon. Past this point, the growing solutions grow linearly with scale factor.

II. PERTURBATION EVOLUTION

Once a perturbation has crossed the horizon (become contained within the observable universe—another way to understand this is that prior to this point opposite ends of a perturbation wavelength were not in causal contact with each other; past this point, they are), its evolution can be described, using Equation 4, by

$$\Delta(k_{\text{eq}}, t_0) = \left(\frac{D_+(t_0)}{D_+(t_{\text{eq}})} \right)^2 \Delta(k_{\text{eq}}, t_{\text{eq}}) = \left(\frac{D_+(t_0)}{D_+(t_{\text{eq}})} \right)^2 \Delta(k_{\text{in}}, t_{\text{in}}) = \left(\frac{D_+(t_{\text{in}})}{D_+(t_{\text{eq}})} \right)^2 \Delta(k_{\text{in}}, t_0), \tag{5}$$

which follows from applying Equation 4 to Equation 1. If we want to compare the power spectrum $P(k)$ at t_{in} and t_{eq}, we can therefore combine Equation 5 and Equation 1, and find

$$P(k) = 2\pi^2 \Delta(k,t)\, k^{-3} \left(\frac{D_+(t_0)}{D_+(t)}\right)^2 \tag{6}$$

$$\begin{aligned} P(k_{\text{eq}}) &\propto \Delta(k_{\text{eq}}, t_0)\, k_{\text{eq}}^{-3} \left(\frac{D_+(t_0)}{D_+(t_{\text{eq}})}\right)^2 \\ &\propto k_{\text{eq}}^{-3} \left(\frac{D_+(t_0)}{D_+(t_{\text{eq}})}\right)^2 \left(\frac{D_+(t_{\text{in}})}{D_+(t_{\text{eq}})}\right)^2 \Delta(k_{\text{in}}, t_0) \end{aligned} \tag{7}$$

$$\Delta(k_{\text{in}}, t_0) \propto k_{\text{in}}^3 \left(\frac{D_+(t_{\text{eq}})}{D_+(t_0)}\right)^2 P(k_{\text{in}})$$

$$P(k_{\text{eq}}) \propto \left(\frac{k_{\text{eq}}}{k_{\text{in}}}\right)^{-3} \left(\frac{D_+(t_{\text{in}})}{D_+(t_{\text{eq}})}\right)^2 \left(\frac{D_+(t_0)}{D_+(t_{\text{eq}})}\right)^2 \left(\frac{D_+(t_{\text{eq}})}{D_+(t_0)}\right)^2 P(k_{\text{in}})$$

$$P(k_{\text{eq}}) \propto \left(\frac{k_{\text{eq}}}{k_{\text{in}}}\right)^{-3} \left(\frac{D_+(t_{\text{in}})}{D_+(t_{\text{eq}})}\right)^2 P(k_{\text{in}}) \tag{8}$$

At this point, we need to start to make statements about $D_+(t)$. What Equation 8 looks like depends on what regime we're in. Note that we have still not said anything general about $P(k)$ beyond noting that $\Delta(k,t)$ is constant for all k at the horizon.

II.A. *Radiation-dominated Era*

If radiation dominates the universe's density, then D_+ is constant. I believe this is a consequence of the solutions to the fluid dynamics equations of the early universe when subjected to perturbation theory; in this regime, Lazarides (1999) says perturbations grow as

$$\delta(t) = B_1 + B_2 \ln(t). \tag{9}$$

In this case, ignoring the logarithmic second term, $D_+(t_{\text{in}}) = D_+(t_{\text{eq}})$. This means that Equation 8 becomes

$$P(k_{\text{eq}}) \propto \left(\frac{k_{\text{eq}}}{k_{\text{in}}}\right)^{-3} P(k_{\text{in}}). \tag{10}$$

In other words, this suggests that in the radiation-dominated era, modes that are contained within the horizon evolve such that $P(k) \propto k^{-3}$.

II.B. *Matter-dominated Era*

If matter dominates the universe's density, then D_+ depends on the scale factor a, as shown in Equation 2. To determine how this depends on t, we look to the FLRW equations:

$$H^2 \propto \frac{\Omega_r}{a^4} + \frac{\Omega_m}{a^3} + \frac{\Omega_k}{a^2} + \Omega_\lambda \tag{11}$$

$$\left(\frac{\dot{a}}{a}\right)^2 \propto \frac{\Omega_r}{a^4} + \frac{\Omega_m}{a^3} + \frac{\Omega_k}{a^2} + \Omega_\lambda \tag{12}$$

If we approximate the matter-dominated era as having zero density in anything but matter, then

$$\dot{a} \propto a^{-1/2}\,\Omega_m \tag{13}$$

$$\int a^{1/2}\mathrm{d}a \propto \Omega_m \int \mathrm{d}t$$

$$a \propto t^{2/3}. \tag{14}$$

Therefore, using Equation 2, we find

$$\frac{D_+(t_{\text{in}})}{D_+(t_{\text{eq}})} = \left(\frac{t_{\text{in}}}{t_{\text{eq}}}\right)^{2/3}. \tag{15}$$

If we plug Equation 14 into Equation 3, we find that

$$k_{\text{in}} \propto t_{\text{in}}^{-1/3} \tag{16}$$

and therefore

$$\frac{D_+(t_{\text{in}})}{D_+(t_{\text{eq}})} = \left(\frac{k_{\text{eq}}}{k_{\text{in}}}\right)^2 \tag{17}$$

If we plug Equation 17 back into Equation 8, we find

$$P(k_{\text{eq}}) \propto \left(\frac{k_{\text{eq}}}{k_{\text{in}}}\right)^{-3} \left(\frac{k_{\text{eq}}}{k_{\text{in}}}\right)^{4} P(k_{\text{in}})$$

$$P(k_{\text{eq}}) \propto \left(\frac{k_{\text{eq}}}{k_{\text{in}}}\right) P(k_{\text{in}}). \tag{18}$$

In other words, in the matter-dominated era, modes that enter the horizon evolve such that $P(k) \propto k$.

Another way to describe this is as imagining some pre-existing power spectrum which is evolved by a transfer function of some kind (Lazarides 1999; Duran et al. 2012):

$$P(k) = P_0(k)T^2(k). \tag{19}$$

In this case, $P_0(k) \propto k$, and $T(k) = k^{-2}$ in the radiation-dominated era, and $T(k) = 1$ in the matter-dominated era. More generally, we can describe $P_0(k)$ as a sort of 'primordial' power spectrum, which has the form $P(k) \propto k^n$, where in our case we take n to be approximately 1.

III. THE HARRISON-ZEL'DOVICH-PEEBLES POWER SPECTRUM

This brings us the Harrison-Zel'dovich-Peebles power spectrum, and the source of much confusion. The Harrison-Zel'dovich power spectrum is described as being a **flat** power spectrum, with equal power in all modes. But we have just described a primordial power spectrum that is

$$P_0(k) \propto k^n. \tag{20}$$

Furthermore, HZP power spectra that are flat are described as having $n = 1$, and what we have is most definitely not flat in that case—it's linear. The answer is that we are talking about two different things. Recall that in section I we said that all density perturbation modes have the same amplitude **at the point at which they enter the horizon** (Duran et al. 2012). This means that the power spectrum is flat along the horizon's worldline. Consider Figure 1 for a visual representation. As time goes on, the horizon radius grows as ct, and ever-smaller k corresponding to larger wavelengths become included within the horizon, at which point their evolution is determined by the density properties of the universe at that time. The power spectrum along the solid black line is constant. However, outside the horizon, modes are continually growing. If they are all the same amplitude when they hit the solid line, then modes that have not yet hit the solid line must have less amplitude than modes that are just hitting the solid line. That means that if you were to sample the amplitudes of modes that are not yet contained within the horizon *at a given point in time*, you would expect the power to decrease with decreasing k (or increasing wavelength). That is exactly what we see in a $P(k) \propto k$ power spectrum. In the diagram, the red triangles represent modes at the same point in time–their amplitudes will be different, as the longer-wavelength mode has longer to grow before entering the horizon. Conversely, the purple dots lie along the horizon's world-line. They represent two different modes entering the horizon at different times—but they have the same amplitude. Sticking with the n we described for Equation 20, we can describe the power spectrum along the horizon as

$$P(k) \propto k^{n-1}. \tag{21}$$

This is the **Harrison-Zel'dovich-Peebles power spectrum**. For $n = 1$, it describes a flat power spectrum, and its physical properties are why we can derive the k^{-3} and k^1 behavior of the evolved matter power spectrum during radiation and matter dominance respectively.

This means there is an important distinction we can make. For an observer in the universe during these post-inflation moments, hoping to observe and record the primordial power spectrum, untouched by universe-dependent effects, perturbation modes only become observable when they enter the horizon. This observer then records that every mode in the primordial power spectrum has the same amplitude. Our astute observer however reasons that since the larger modes took longer to become observable, if modes could grow prior to entering the horizon, then they might not have always all had the same amplitude. The observer reasons that a power spectrum of all modes at the initial instant, the primordial power spectrum, would evolve with the universe and present an observable evolved power

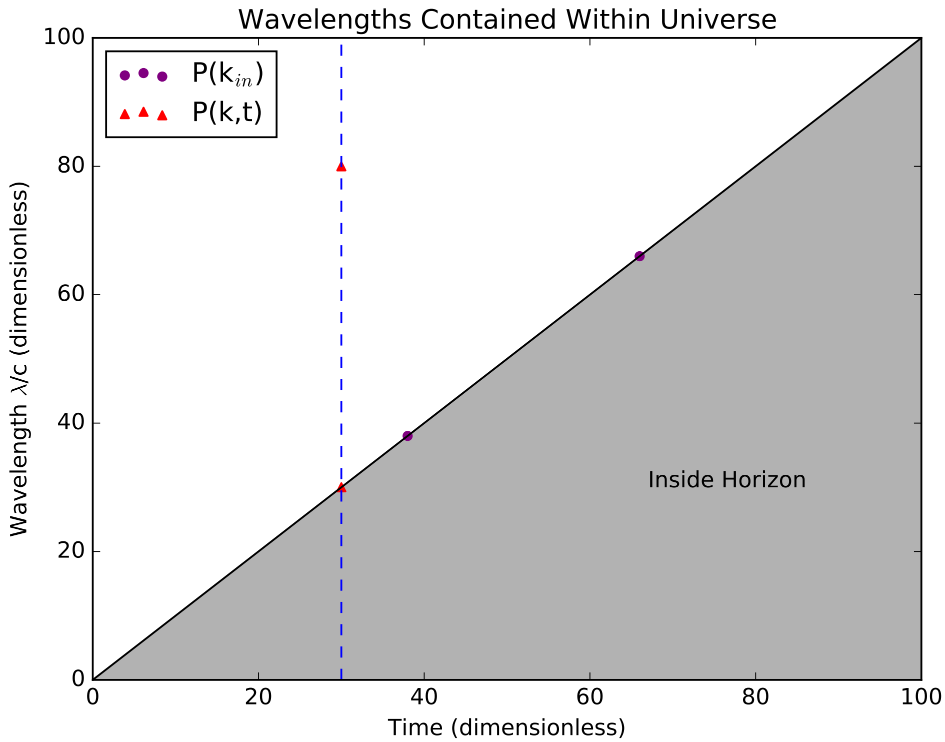

Figure 1. The event horizon of the observable universe in terms of wavelength and as a function of time. The solid line represents the world-line of the horizon, such that at a given time, all wavelengths less than the solid line are contained within the observable universe and can evolve according to the universe's density properties. All perturbation modes along this line have the same amplitude. Modes outside the horizon are continually growing, so for a given time, i.e. along the dashed blue line, larger modes will have a lower amplitude as they have longer to grow before reaching the horizon, at which point they have the same amplitude as a smaller mode that entered the horizon at the blue dashed line. The k^{n-1} Harrizon-Zel'dovich power spectrum describes the power of each mode along the horizon. The k^n power spectrum describes the power of each mode outside the horizon at a given instant in time.

spectrum. If the observer could determine from other means how modes would evolve at different epochs in the universe's history, then the observed flat power spectrum at the horizon could be combined with that growth function to give the form of the present-day matter power spectrum—and it would reveal the form of the power spectrum that describes the power in each mode at a given instant in time. In our case, we found Equation 19 this exact way, and found it had the form

$$P(k) \propto k\,T^2(k), \tag{22}$$

suggesting $P_0(k) \propto k$.

IV. OBSERVING THE PRIMORDIAL POWER SPECTRUM AND THE BAO

The evolved matter power spectrum is actually observable today. After recombination, matter roughly traced these perturbations, and they continue to evolve to the present day. We can measure the preferred modes by taking a two-point correlation function of galaxies (or other matter observables) on the sky. This features a peak at the smallest modes, corresponding to universe-sized global modes and a gently sloping tail, with the exception of a bump at 150 comoving Mpc, or 105 h^{-1} Mpc. That bump gives the characteristic diameter of baryon shells at recombination, when photons free-streamed and perturbations more or less froze-in (the amplitudes continued to evolve, but for the most part their modes stayed constant past this point, since photons had been carrying all the momentum). The matter power spectrum is obtained by taking a fourier transform of this correlation function. An example of what that looks like is shown in Figure 2, which shows the matter power spectra of several models of the universe that treat neutrinos slightly differently (Wong 2011).

At small k, corresponding to large modes that entered the horizon later, the underlying shape scales linearly with k, peaks at $k_{\rm eq}$ where radiation and matter had equal contributions to density, and then scales as k^{-3} at larger k, corresponding to modes that entered the horizon earlier during the radiation-dominated era. At this time, photons expanding outward from perturbations severely depleted the depth of potential wells, reducing the amplitude of dark matter density perturbations, which were tracing photon density perturbations. This rapid photon diffusion of sorts smoothed and damped perturbations during the era of radiation dominance. During the matter-dominated era, dark matter density perturbations could largely continue to grow linearly. The wiggles in the power spectrum correspond to the fourier transform of the 105 h^{-1} peak in the correlation function—the fourier transform of a single peak results in an oscillating power spectrum. When the smooth underlying evolved matter power spectrum is divided out, the resulting oscillations about a horizontal axis are called the **baryon acoustic oscillation**. So taking our observed power spectrum and its trends, along with our knowledge of how perturbations would have grown during the matter- and radiation-dominated eras, we can obtain the primordial power spectrum. In our case, we find that $n \approx 0.96$, which suggest a power spectrum that was very nearly flat along the horizon.

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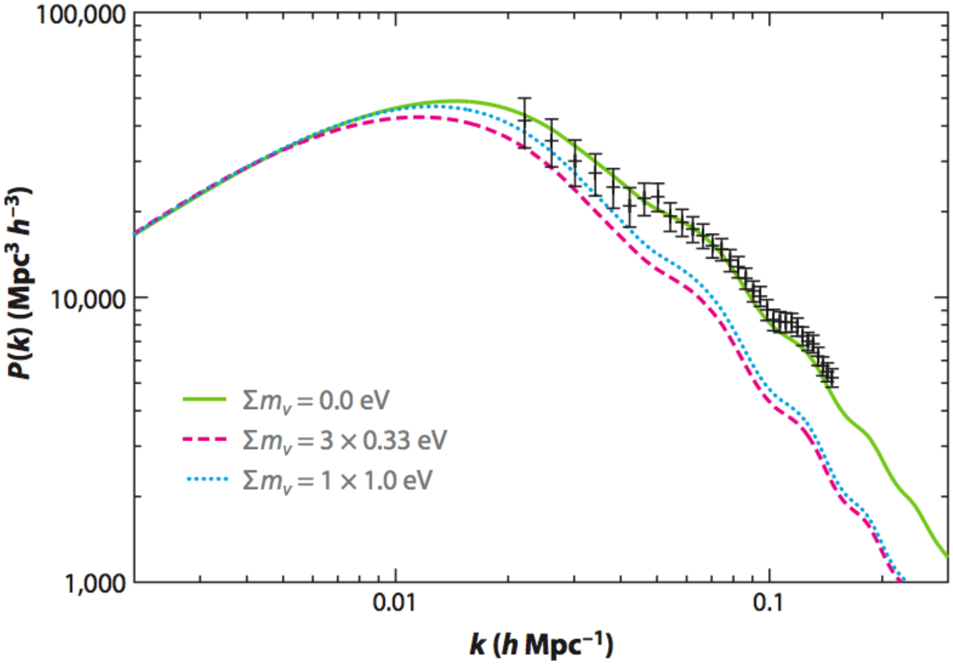

Figure 2. Matter power spectra for models that treat neutrinos in various different ways. Note that the underlying shape is $P(k) \propto k$ for small k (large modes) and $P(k) \propto k^{-3}$ for large k (small modes). The wiggles are the result of the 105 h^{-1} Mpc bump in the correlation function, which is itself due to the expanding baryon shells at recombination. The wiggles themselves are called the **Baryon Acoustic Oscillation**. When studied on their own, the smooth underlying power spectrum is usually divided out so that the oscillations are about a horizontal axis. Taken from Wong (2011).