Optimal sensor placement from empirical observability for flexible wings

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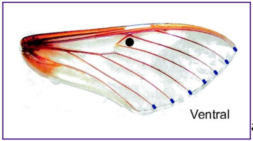
Motivation



[a]


[b]


[c]

[a] Bradley H. Dickerson, et al. (2014) Journal of Experimental Biology. doi:10.1242/jeb.103770

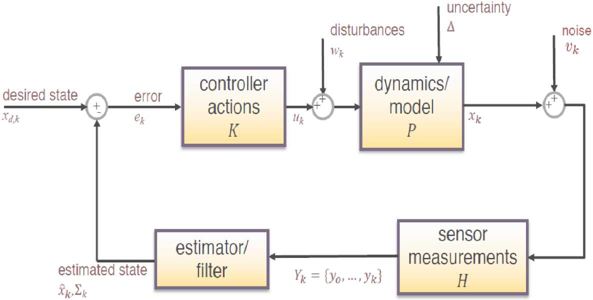
[b] Data Stephenson/Getty Images (2015) https://weather.com/news/news/tonga-volcano-eruption

[c] UW Applied Physics Lab, https://apl.uw.edu/project/project.php?id=seaglider

Motivation

Error covariance is bounded by observability[1]

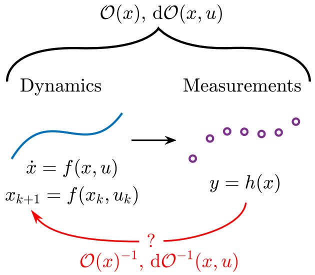
$$\mathcal{P} = \Sigma^{-1} \preceq FIM \preceq \sigma_{min}(R^{-1}) f(W_0)$$



[1] Powel and Morgansen (2015), Conf. on Decision and Control, doi:10.1109/CDC.2015.7403218

Introduction: Observability

- Can a systems' state be uniquely determined from the available measurements?
  - Rank conditions evaluate if a state $\mathbf{x}_0$ **can** be uniquely determined by the outputs
  - Observability Gramians can be used to investigate how **well** the states can be observed



Finite-Dimensional Linear Systems

- Linear system

$$\dot{x} = Ax + Bu, \qquad x \in \mathbb{R}^n$$
$$y = Cx + Du, \qquad y \in \mathbb{R}^p$$

- Observability matrix, $\mathcal{O}$ is

$$\mathcal{O} = \begin{bmatrix} C \\ CA \\ \vdots \\ CA^{n-1} \end{bmatrix}$$

- Observability Gramian, $W_o(t)$

$$W_o(t) = \int_0^t e^{A^\top \tau} C^\top C e^{A\tau} d\tau$$
$$= \sum_{\ell=1}^{p} \int_0^t \frac{\partial y_\ell(\tau)}{\partial \mathbf{x}_0} \left(\frac{\partial y_\ell(\tau)}{\partial \mathbf{x}_0} \right)^\top d\tau$$

- If $\mathcal{O}$ and $W_o(t)$ are full rank, the system is observable

Nonlinear Observability Analysis

- Control affine system:

$$\dot{\mathbf{x}} = \mathbf{f}(x,u) = \mathbf{f}_0(x) + \sum_{i=1}^{q} \mathbf{f}_i(x) u_i$$

$$y = h(\mathbf{x})$$

- The observability space, $\mathcal{O}_I$, is the span of the Lie derivatives, defined as $L_{\mathbf{f}_i} h = \frac{\partial h}{\partial \mathbf{x}} \mathbf{f}_i$

$$\begin{bmatrix} y \\ \dot{y} \\ \ddot{y} \\ \vdots \\ y^{(n)} \end{bmatrix} = \begin{bmatrix} h(\mathbf{x}) \\ \frac{\partial h}{\partial \mathbf{x}} \dot{\mathbf{x}} \\ \frac{\partial}{\partial \mathbf{x}} \left(\frac{\partial h}{\partial \mathbf{x}} \dot{\mathbf{x}} \right) \dot{\mathbf{x}} \\ \vdots \\ \frac{\partial}{\partial \mathbf{x}} \left(\frac{\partial}{\partial \mathbf{x}} \left(\cdots \left(\frac{\partial h}{\partial \mathbf{x}} \dot{\mathbf{x}} \right) \right) \right) \dot{\mathbf{x}} \end{bmatrix} = \begin{bmatrix} L_{\mathbf{f}}^0 h(x) \\ L_{\mathbf{f}}^1 h(x) \\ L_{\mathbf{f}}^2 h(x) \\ \vdots \\ L_{\mathbf{f}}^n h(x) \end{bmatrix} \neq \begin{bmatrix} L_{\mathbf{f}_0}^0 h(x) \\ L_{\mathbf{f}_0}^1 h(x) \\ L_{\mathbf{f}_0}^2 h(x) \\ \vdots \\ L_{\mathbf{f}_0}^n h(x) \end{bmatrix}$$

- If the observability codistribution, $d\mathcal{O}_I$, is full rank when evaluated at $\mathbf{x}_0$, the system is locally weakly observable at $\mathbf{x}_0$ [1]

[1] Hermann and Krener (1977) IEEE T Automatic Control. doi: 10.1109/TAC.1977.1101601

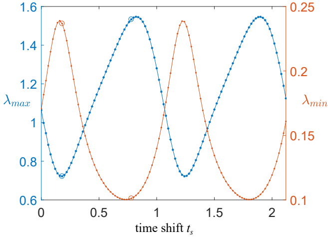
Empirical Observability Gramian

- Empirical observability Gramian

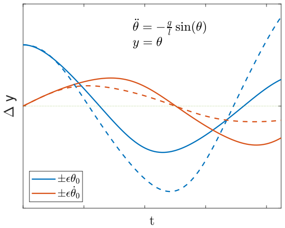
$$W_o^{\epsilon}(t) = \frac{1}{4\epsilon^2}\int_0^t \Delta Y^\top \Delta Y d\tau,$$
$$\Delta Y = \begin{bmatrix} \Delta y^{\pm 1}(\tau) & \Delta y^{\pm 2}(\tau) & \cdots & \Delta y^{\pm n}(\tau) \end{bmatrix}$$

- Perturb each initial condition by $\pm\epsilon$ to get $\Delta y^{\pm i}(\tau) = y_i^+(\tau) - y_i^-(\tau)$, which is the difference in the scalar outputs based on trajectories generated by $\mathbf{x}_0 \pm \epsilon\mathbf{e}_i$
- Works for systems that can be simulated or even those that can be appropriately perturbed experimentally

Empirical Observability Gramian - Pendulum Example



(a) Max and Min Eigenvalues



(b) Magnitude of $\Delta y^{\pm i}$

- Perturbations made at different times within swing produce different Δy

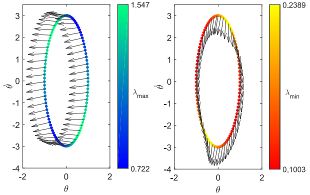


Figure: Phase Portrait

Continuum Empirical Observability Gramian

The empirical observability Gramian for a cantilever beam, W_∞^ϵ, is given by $W_\infty^\epsilon(t, x_\ell) = \int_0^t \Delta Y_\infty^\top \Delta Y_\infty d\tau \in \mathbb{R}^{n_\eta \times n_\eta 2}$, where

$$\Delta Y_\infty^\top = \begin{bmatrix} \sum_{j=1}^\infty y^{\eta_{j,0}+\epsilon} - y^{\eta_{j,0}-\epsilon} \\ \sum_{j=1}^\infty y^{\dot{\eta}_{j,0}+\epsilon} - y^{\dot{\eta}_{j,0}-\epsilon} \end{bmatrix}$$

Augmenting with the rigid body components

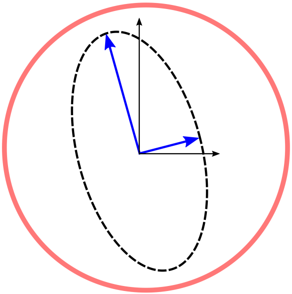
$$\Delta Y_{\infty,RB}^\top = \begin{bmatrix} \Delta Y_\infty^\top \\ y^{U_{b,0}+\epsilon} - y^{U_{b,0}-\epsilon} \\ y^{V_{b,0}+\epsilon} - y^{V_{b,0}-\epsilon} \\ y^{R_{b,0}+\epsilon} - y^{R_{b,0}-\epsilon} \end{bmatrix}$$

where $y^{x_j \pm \epsilon}$ indicates the output due to perturbing the initial condition of state x_j

[2] Brace et al (2022), IEE Conf. Decision and Control, doi:10.1109/CDC51059.2022.9992639

Measures of the Observability Gramian : Condition Number

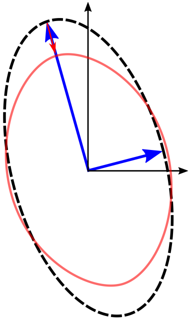
- Condition number: $\kappa(W) = \frac{\bar{\lambda}(W)}{\underline{\lambda}(W)}$
- Characterizes the equity of energy output between states
- Significance: **shape** of the uncertainty ellipse
- Caveat: May prioritize minimizing λ_{max}



Error covariance ellipse

Measures of the Observability Gramian : Eigenvalues [3]

- Maximum eigenvalue: $\bar{\lambda}(W)$ or $\underline{\nu}(W) = \underline{\lambda}(W)^{-1}$
  Characterizes output energy of the most observable system mode
  Significance: maximum estimation uncertainty
- Minimum eigenvalue: $\underline{\lambda}(W)$ or $\bar{\nu}(W) = \bar{\lambda}(W)^{-1}$
  Characterizes output energy of the least observable system mode
  Significance: minimum estimation uncertainty

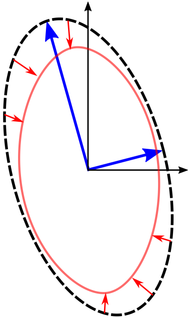


Error covariance ellipse
Shown increasing $\underline{\lambda}(W)$

[1] A. J. Krener and K. Ide, "Measures of unobservability," in Proc. 48th IEEE Conf. on Decision and Control (CDC), pp. 6401–6406, IEEE, Dec 2009

Measures of the Observability Gramian : Log-Determinant

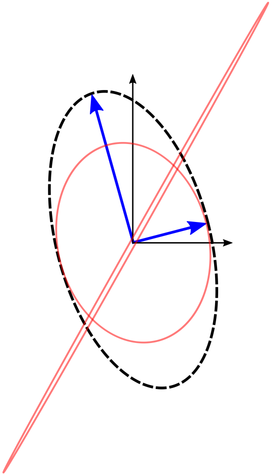
- Log Determinant: $\log(\det((W))$
  Significance: **volume** of uncertainty ellipse



Error covariance ellipse

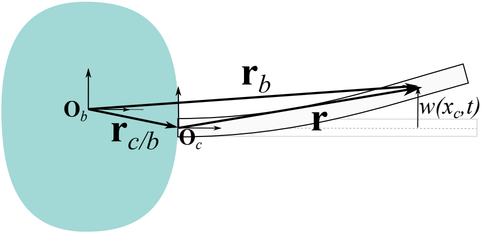
Measures of the Observability Gramian : Trace

- Trace: trace(W),
  Characterizes average of the diagonal of W
  Significance: **average** estimation uncertainty
- Caveat: allows minimum eigenvalue(s) to be zero



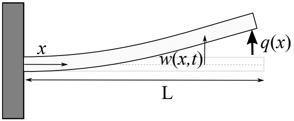
Error covariance ellipse

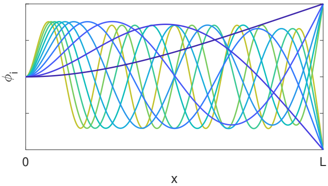
Cantilever Beam Attached to a Rigid Body



- Point $\mathbf{O}_b$ is the center of rotation and can translate in $\hat{\imath}_b$ and $\hat{\jmath}_b$
- Offset of beam base $\mathbf{O}_c$ from center of rotation is $\mathbf{r}_{c/b} = \begin{bmatrix} r_x & r_y \end{bmatrix}$

Cantilever Beam Modal Approximation





- Transverse deflection of an Euler-Bernoulli beam

$$\frac{\partial^2}{\partial x^2}\left(E(x)I(x)\frac{\partial^2 w(x,t)}{\partial x^2}\right) = -\mu\frac{\partial^2 w(x,t)}{\partial t^2}$$

$$w(0,t) = w_x(0,t) = w_{xx}(L,t) = w_{xxx}(L,t) = 0,$$

- Modal approximation: $w(x,t) = \sum_{i=1}^{\infty} \phi_i(x)\eta_i(t)$
- Assumption:
  flexural rigidity, EI, and material density, μ, are constants

Equations of Motion

The equations of motion are found using Lagrange's equations

$$\frac{d}{dt}\begin{bmatrix}\boldsymbol{\eta}\\ \dot{\boldsymbol{\eta}}\\ U_b\\ V_b\\ R_b\end{bmatrix} = \begin{bmatrix}\dot{\boldsymbol{\eta}}\\ K\boldsymbol{\eta} - M_C\begin{bmatrix}R_bU_b\\ R_b\end{bmatrix}\\ 0\\ 0\\ 0\end{bmatrix} + \begin{bmatrix}0 & \mathbf{0}\\ 0 & -M_A\\ 1 & \mathbf{0}\\ \mathbf{0} & I^{2\times 2}\end{bmatrix}\begin{bmatrix}\dot{U}_b\\ \dot{V}_b\\ \dot{R}_b\end{bmatrix}$$

where $K = R_b^2 I_{n_\phi} - \bar{U}$,

$$\bar{M}_{a_i} = \frac{1}{2}\int_0^L \phi_i(x)\begin{bmatrix}1 & r_x + x\end{bmatrix}dx,\ \ \bar{M}_{c_i} = \frac{1}{2}\int_0^L \phi_i(x)\begin{bmatrix}-1 & r_y\end{bmatrix}dx$$

$$\bar{U}_{ii} = \frac{EI_z}{\rho h}\int_0^L\left(\frac{\partial^2\phi_i}{\partial x^2}\right)^2 dx,\ \ \bar{U}_{ij} = 0 \text{ if } i \neq j,\ I_z = \frac{bh^3}{12}$$

[4]

[4] Hinson and Morgansen (2015), Bioinspr. Biomim., doi:10.1088/1748-3190/10/5/056013

Sensor Models

- Strain

$$\begin{aligned}\mathbf{h}_\varepsilon(\mathbf{x}) &= C_\varepsilon \mathbf{x} = \begin{bmatrix} C_{\Phi''} & \mathbf{0} \end{bmatrix} \\ c_{\varepsilon,x_\ell} &= h_\ell \begin{bmatrix} \frac{\partial^2 \phi_1(x_\ell)}{\partial x^2} & \frac{\partial^2 \phi_2(x_\ell)}{\partial x^2} & \dots & \frac{\partial^2 \phi_{n_\phi}(x_\ell)}{\partial x^2} & \mathbf{0}_{1\times n_\phi} & \mathbf{0}_{1\times 3} \end{bmatrix} \mathbf{x} \\ &= h_\ell \begin{bmatrix} \frac{\partial^2 \Phi}{\partial x^2} & \mathbf{0}_{1\times n_\phi} & \mathbf{0}_{1\times 3} \end{bmatrix} \mathbf{x}\end{aligned}$$

- Acceleration

$$\begin{aligned}\mathbf{h}_a(\mathbf{x}) &= C_\Phi \ddot{\boldsymbol{\eta}}, \quad c_{\phi,x_\ell} = \begin{bmatrix} \phi_1(x_\ell) & \phi_2(x_\ell) & \dots & \phi_{n_\phi}(x_\ell) \end{bmatrix} \\ &= C_\Phi \Big(\underbrace{(R_b^2 I_{n_\phi} - U) - M_1 R_b U_b + M_{C2} R_b}_{\mathbf{f}_0} - \underbrace{M_1 \dot{V}_b}_{u_V} - \underbrace{M_{A2} \dot{R}_b}_{u_R} \Big)\end{aligned}$$

Analytical Observability - Strain Sensors

$$d\mathcal{O}_\epsilon = \begin{bmatrix} C_{\Phi''} & \mathbf{0} & \vdots & 0 & 0 & 0 \\ \mathbf{0} & C_{\Phi''} & \vdots & 0 & 0 & 0 \\ KC_{\Phi''} & \mathbf{0} & \vdots & -R_b C_{\Phi''} M_1 & 0 & C_{\Phi''}(M_{C2} - U_b M_1) \\ \mathbf{0} & KC_{\Phi''} & \vdots & 0 & 0 & 0 \\ K^2 C_{\Phi''} & \mathbf{0} & \vdots & -KR_b C_{\Phi''} M_1 & 0 & KC_{\Phi''}(M_{C2} - U_b M_1) \end{bmatrix}$$

- V_b is not directly observable
- The determinant of a full rank subset includes R_b and $\bar{M}_c$, indicating rotation and vertical offset r_y are required for observability without control
- Two strain sensors seem to be required

Analytical Observability - Accelerometers

$$
d\mathcal{O}_a = \begin{bmatrix} K & 0 & \vdots & -R_b M_1 & 0 & M_{C2} - U_b M_1 \\ 0 & K & \vdots & 0 & 0 & 0 \\ 0 & 0 & \vdots & 0 & 0 & -KM_1 \\ 0 & 0 & \vdots & -KM_1 & 0 & 0 \end{bmatrix}
\begin{matrix} \left.\vphantom{\begin{matrix} a \\ a \end{matrix}}\right\} \text{Drift only} \\ \} \text{With control } u_U \\ \} \text{With control } u_R \end{matrix}
$$

- V_b is not directly observable
- Control (acceleration) in U_b or R_b is required for observability

Sensor Placement

- Error covariance is bounded by observability Gramain
- Introduce binary sensor selection variables $\alpha_l \in \{0, 1\}$
- Total observability Gramian

$$\tilde{W}_o(\boldsymbol{\alpha}, t) = \sum_{\ell=1}^{n_p} W_o^{\ell} \alpha_{\ell}$$

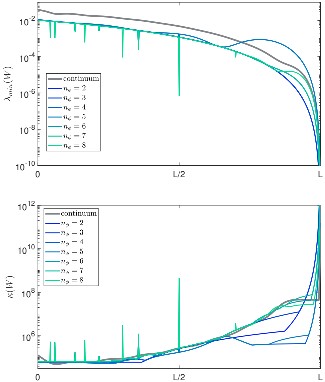
- Convex functions of the observability Gramian are also convex with respect to the sensor selection variables

Optimization problem

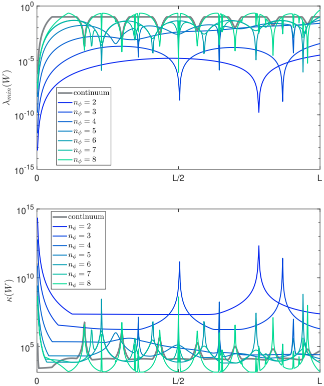
$$\begin{aligned} \min_{\mathbf{a}} \quad & J(W) \\ \text{subject to} \quad & \textstyle\sum_{\ell=1}^{n_p} a_{\ell} \leq p \\ & 0 \leq a_{\ell} \leq 1 \ \forall \ell \end{aligned}$$

- Relax the binary requirement to define a sub-optimal but convex optimization problem to place p sensors

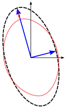
Objective Functions: Min. Eigenvalue and Condition Number

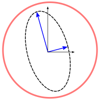


Strain

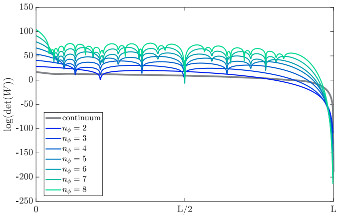


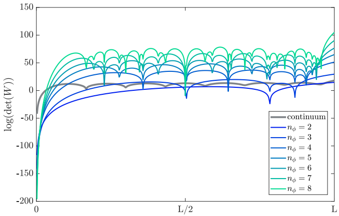
Acceleration

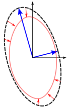


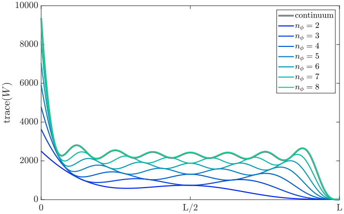


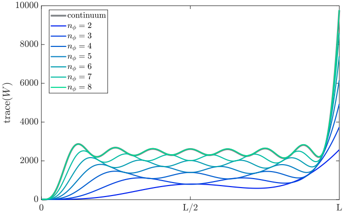
Objective Functions: Log-Determinant and Trace

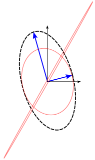








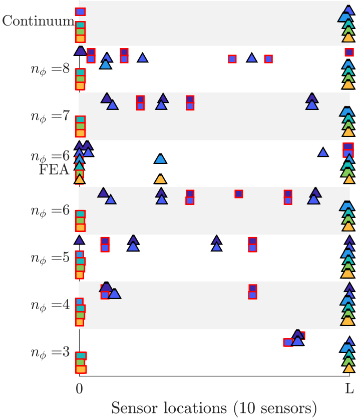


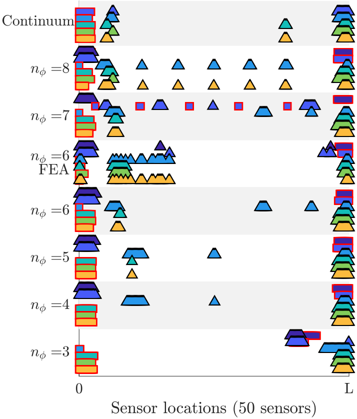


Strain

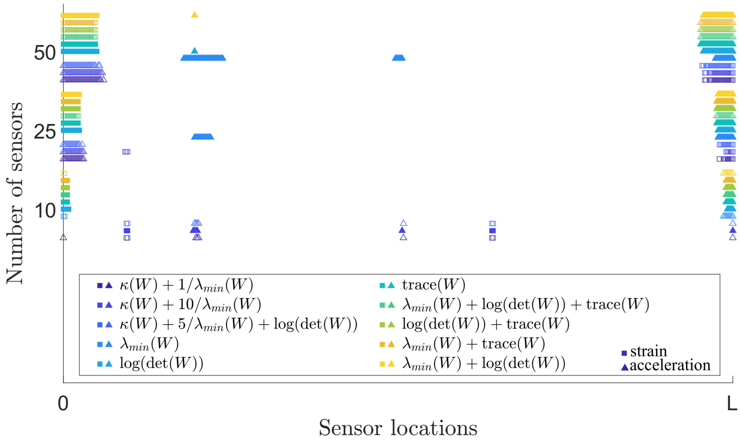
Acceleration

ACC Reults

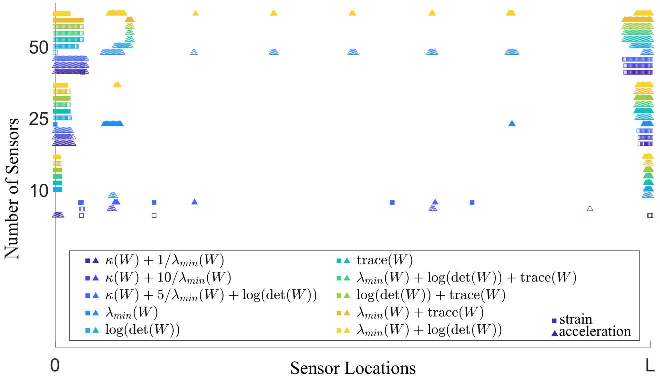

Continuum
n_ϕ =8
n_ϕ =7
n_ϕ =6
FEA
n_ϕ =6
n_ϕ =5
n_ϕ =4
n_ϕ =3
0
L
Sensor locations (10 sensors)


Continuum
n_ϕ =8
n_ϕ =7
n_ϕ =6
FEA
n_ϕ =6
n_ϕ =5
n_ϕ =4
n_ϕ =3
0
L
Sensor locations (50 sensors)

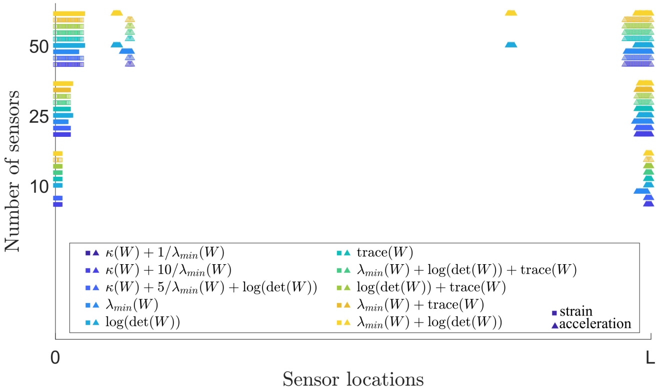
Sensor Placement for $n_\phi = 5$



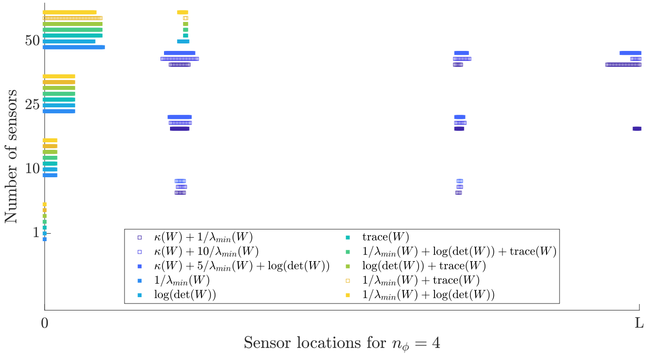
Sensor Placement for $n_\phi = 8$



Sensor Placement for $n_\phi = 8$ with Continuum EOG

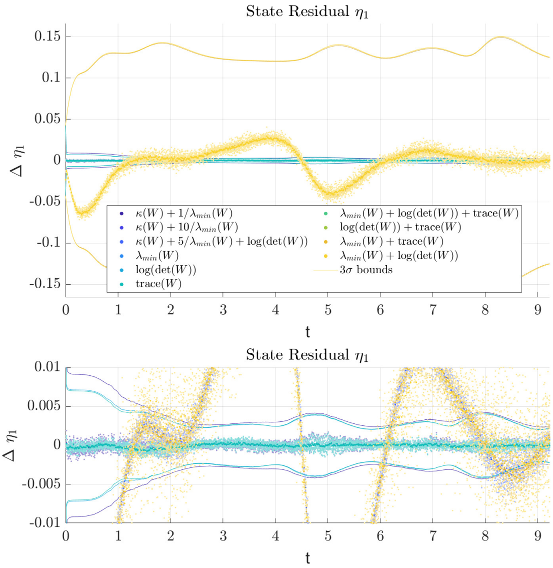


Sensor Placement for $n_\phi = 4$ from FEA

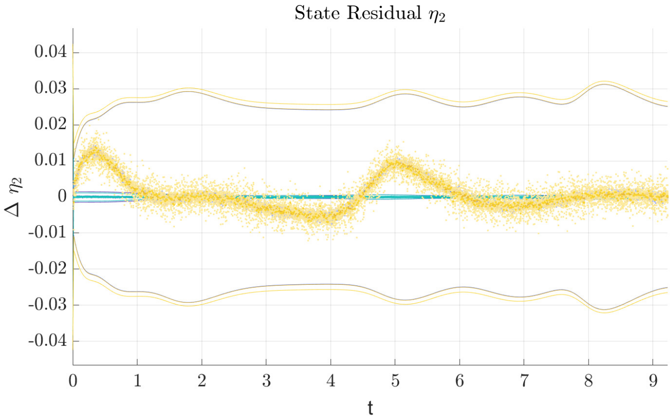


Note: this optimization was done with only strain sensor data

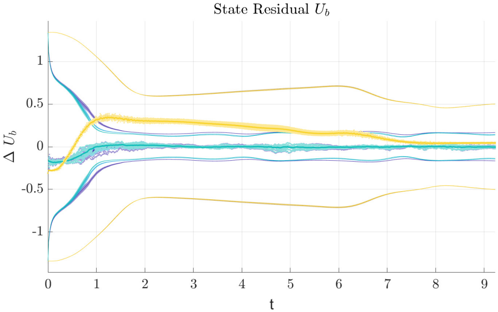
UKF Results for $n_\phi = 5$: η_1

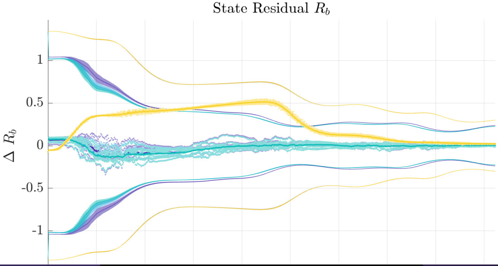


UKF Results for $n_\phi = 5$: η_2

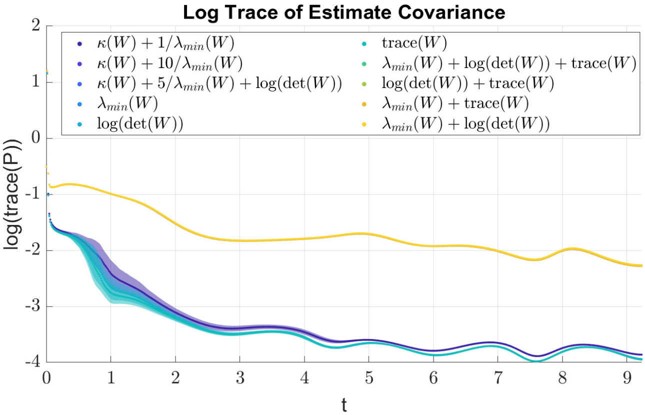


UKF Results for $n_\phi = 5$: U_b and R_b

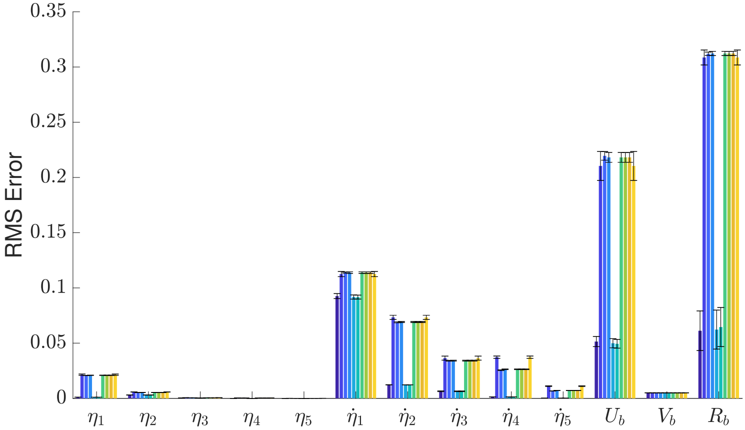




UKF Results for $n_\phi = 5$: Log Trace of Error Covariance



UKF Results for $n_\phi = 5$: Error RMS



Conclusions

- Considered a very simplified model of an airplane wing: a cantilever beam connected to a translating and rotating rigid body
- Presented sensor placement results for strain gauges and accelerometers and the resulting UKF output for different cost functions
- Working to verify results of a similar FEA model and continue with flat plate analysis (analytical and FEA) before modeling an aircraft structure

Thank you!

Questions?

Cantilever Beam Solution

- Mode shapes [5]

$$\phi_i(x) = \cosh b_i x - \cos b_i x + f_i(L)(\sin b_i x - \sinh b_i x),$$

$$f_i(L) = \frac{\cos b_i L + \cosh b_i L}{\sin b_i L + \sinh b_i L},$$

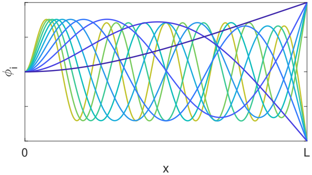
where b_i satisfies $\cos(bi_L)\cosh(b_i L) = -1$

- Modal coefficients

$$\eta_i(t) = \alpha_{1,i}\cos(\omega_i t) + \alpha_{2,i}\frac{\sin(\omega_i t)}{\omega_i},$$

$$\omega_i = b_i^2\sqrt{\frac{EI}{\mu}}, \quad c_i = \int_0^L \phi_j(x)^2 dx,$$

$$\alpha_{1,i} = \frac{1}{c_i}\int_0^L w_0(x)\phi_i(x)dx, \quad \alpha_{2,i} = \frac{1}{c_i}\int_0^L \dot{w}_0(x)\phi_i(x)dx$$



[5] P. L. Gatti, *Applied structural and mechanical vibrations: theory and methods*. CRC Press, second ed., 2014

Optimization Problem

- Minimize a convex objective function of the total observability Gramian to place p sensors

$$\begin{aligned} \min_{\alpha} \quad & J(\tilde{W}_o(\boldsymbol{\alpha}, t)) \\ \text{subject to} \quad & \sum_{l=1}^{n_p} \alpha_l \leq p \\ & \alpha_l \in \{0,1\} \ \forall l \end{aligned} \quad \rightarrow \quad \begin{aligned} \min_{\mathbf{a},\kappa,\nu} \quad & \kappa + w\nu \\ \text{subject to} \quad & \tilde{W}(\bar{\mathbf{a}}) - I \succeq 0 \\ & \kappa I - \tilde{W}(\bar{\mathbf{a}}) \succeq 0 \\ & 0 \leq \bar{a}_l \leq \nu \ \forall l \\ & \sum_{l=1}^{n_p} \bar{a}_l \leq p\nu \end{aligned}$$

Functional Definition

- A *functional*, K, is a map between functions: $K : z(t) \to K[z](t)$
- The *first variation*[6] of a functional $K[z](t)$ is a functional that maps the perturbation function, h, to

$$\delta K[z; h] = \lim_{\epsilon \to 0} \frac{K[z + \epsilon h] - K[z]}{\epsilon}$$

[6] I. Gelfand and S. Fomin, *Calculus of Variations*. Prentice-Hall, 1963

Analytical Continuum Gramian

Perturbed measurements:

$$y_\ell(t)^{+f} = h_\ell \sum_{i=1}^{\infty} \phi_{i,xx}(x_\ell) \left(\eta_i(t) + \epsilon C_i^f \cos(\omega_i t) \right)$$

$$y_\ell(t)^{+g} = h_\ell \sum_{i=1}^{\infty} \phi_{i,xx}(x_\ell) \left(\eta_i(t) + \epsilon C_i^g \frac{\sin(\omega_i t)}{\omega_i} \right).$$

First variations of the measurement $y_\ell[w_0, \dot{w}_0](t)$ with respect to the functions $f(x)$ and $g(x)$ are

$$\delta y_\ell[w_0, \dot{w}_0; f](t) = \lim_{\epsilon \to 0} \frac{y_\ell^{+f} - y_\ell}{\epsilon} = h_\ell \sum_{i=1}^{\infty} \phi_{i,xx}(x_\ell) C_i^f \cos(\omega_i t) \tag{1}$$

$$\delta y_\ell[w_0, \dot{w}_0; g](t) = \lim_{\epsilon \to 0} \frac{y_\ell^{+g} - y_\ell}{\epsilon} = h_\ell \sum_{i=1}^{\infty} \phi_{i,xx}(x_\ell) C_i^g \frac{\sin(\omega_i t)}{\omega_i}. \tag{2}$$

Analytical Continuum Gramian

Perturbed measurements:

$$y_\ell(t)^{+f} = h_\ell \sum_{i=1}^{\infty} \phi_{i,xx}(x_\ell) \left(\eta_i(t) + \epsilon C_i^f \cos(\omega_i t) \right)$$

$$y_\ell(t)^{+g} = h_\ell \sum_{i=1}^{\infty} \phi_{i,xx}(x_\ell) \left(\eta_i(t) + \epsilon C_i^g \frac{\sin(\omega_i t)}{\omega_i} \right).$$

First variations of the measurement $y_\ell[w_0, \dot{w}_0](t)$ with respect to the functions $f(x)$ and $g(x)$ are

$$\delta y_\ell[w_0, \dot{w}_0; f](t) = \lim_{\epsilon \to 0} \frac{y_\ell^{+f} - y_\ell}{\epsilon} = h_\ell \sum_{i=1}^{\infty} \phi_{i,xx}(x_\ell) C_i^f \cos(\omega_i t)$$

$$\delta y_\ell[w_0, \dot{w}_0; g](t) = \lim_{\epsilon \to 0} \frac{y_\ell^{+g} - y_\ell}{\epsilon} = h_\ell \sum_{i=1}^{\infty} \phi_{i,xx}(x_\ell) C_i^g \frac{\sin(\omega_i t)}{\omega_i}.$$

Analytical Continuum Gramian

For displacement, define $f(x) = \phi_j(x)$

$$C_i^f = \frac{1}{c_i}\int_0^L \phi_j(x)\phi_i(x)dx = \begin{cases} 1 & i = j \\ 0 & i \neq j, \end{cases}$$

variations δy_ℓ with respect to $f(x)$ and $g(x)$ simplify to

$$\delta y_\ell^f[w_0, \dot{w}_0; \phi_j](t) = h_\ell \phi_{j,xx}(x_\ell)\cos(\omega_j t)$$
$$\delta y_\ell^g[w_0, \dot{w}_0; \phi_j](t) = h_\ell \phi_{j,xx}(x_\ell)\frac{\sin(\omega_j t)}{\omega_j}.$$

Then sum to get the full solution

$$\delta Y_1 = \sum_{j=1}^{\infty} \delta y_\ell^f[w_0, \dot{w}_0; \phi_j](t), \quad \delta Y_2 = \sum_{j=1}^{\infty} \delta y_\ell^g[w_0, \dot{w}_0; \phi_j](t).$$

Empirical Continuum Gramian

Proof of Theorem 2:

- Perturb the initial mode shapes: $$\begin{aligned} w_0^{\pm j}(x) &= w_0(x) \pm \epsilon_{1,j}\phi_j(x) \\ \dot{w}_0^{\pm j}(x) &= \dot{w}_0(x) \pm \epsilon_{2,j}\phi_j(x) \end{aligned}$$
- Calculate $\alpha_{1,i}$ and $\alpha_{2,i}$ for the perturbed shapes

Due to the orthogonality of the mode shapes, the $\alpha_{k,i}$ components of the modal coefficients will in turn be perturbed by $\epsilon_{k,j}$

$$\alpha_{1,i}^{\pm j} = \frac{1}{c_i}\int_0^L \big(w_0(x) + \epsilon_{1,j}\phi_j(x)\big)\phi_i(x)dx = \begin{cases} \alpha_{1,i} & i \neq j \\ \alpha_{1,j} \pm \epsilon_{1,j} & i = j \end{cases}$$

$$\alpha_{2,i}^{\pm j} = \frac{1}{c_i}\int_0^L \big(\dot{w}_0(x) + \epsilon_{2,j}\phi_j(x)\big)\phi_i(x)dx = \begin{cases} \alpha_{2,i} & i \neq j \\ \alpha_{2,j} \pm \epsilon_{2,j} & i = j, \end{cases}$$

Empirical Continuum Gramian

- Modal coefficients
$$\eta_i(t)^{\pm\epsilon_{1,j}} = \begin{cases} \eta_i(t) & i \neq j \\ \eta_j(t) + \epsilon_{1,j}\cos(\omega_j t) & i = j \end{cases}$$
$$\eta_i(t)^{\pm\epsilon_{2,j}} = \begin{cases} \eta_i(t) & i \neq j \\ \eta_j(t) + \epsilon_{2,j}\frac{\sin(\omega_j t)}{\omega_j} & i = j. \end{cases}$$
- Measurement functions
$$y(x_\ell, t)^{\pm\epsilon_{kj}} = \left(\sum_{i=1}^{\infty} \phi_{i,xx}(x_\ell)\eta_i(t)\right) \pm \epsilon_{kj}\phi_{j,xx}(x_\ell)\beta_{kj}(t),$$
$$\beta_{kj(t)} = \begin{cases} \cos(\omega_j t) & k = 1 \\ \frac{\sin(\omega_j t)}{\omega_j} & k = 2 \end{cases}$$
- Then $\Delta Y_\infty = \sum_{j=1}^{\infty} \frac{1}{2\epsilon_{k,j}} \Delta y(x_\ell, t)^{+\epsilon_{k,j}}$
$$\Delta y(x_\ell, t)^{\pm\epsilon_{k,j}} = 2\epsilon_{k,j} h_\ell \phi_{j,xx}(x_\ell)\beta_{kj}(t).$$

Single Sensor Observability

Proof of Theorem 3:

- Linear observability matrix

$$\mathcal{O} = \begin{bmatrix} c' & \mathbf{0} \\ \mathbf{0} & c' \\ \langle c', \bar{\boldsymbol{\omega}} \rangle & \mathbf{0} \\ \mathbf{0} & \langle c', \bar{\boldsymbol{\omega}} \rangle \\ \vdots & \vdots \\ \langle c', \bar{\boldsymbol{\omega}}.^{n-1} \rangle & \mathbf{0} \\ \mathbf{0} & \langle c', \bar{\boldsymbol{\omega}}.^{n-1} \rangle \end{bmatrix},$$

$$y = \begin{bmatrix} c' & \mathbf{0} \end{bmatrix},$$

$$\boldsymbol{\omega}.^{a} = \begin{bmatrix} (-\omega_1^2)^a & (-\omega_2^2)^a & \cdots & (-\omega_n^2)^a \end{bmatrix}$$

- Rearrange to get

$$\mathcal{O} = \begin{bmatrix} \mathcal{O}_C & \mathbf{0} \\ \mathbf{0} & \mathcal{O}_C \end{bmatrix}$$

- Let $d_j = -\omega_j^2$ and $p_k = \frac{\partial^2 \phi_k(x_\ell)}{\partial x^2}$; then

$$\mathcal{O}_C = h_\ell \begin{bmatrix} p_1 & p_2 & \cdots & p_{n_\phi} \\ p_1 d_1 & p_2 d_2 & \cdots & p_{n_\phi} d_{n_\phi} \\ p_1 d_1^2 & p_2 d_2^2 & \cdots & p_{n_\phi} d_{n_\phi}^2 \\ \vdots & & & \\ p_1 d_1^{n_\phi - 1} & p_2 d_2^{n_\phi - 1} & \cdots & p_{n_\phi} d_n^{n_\phi - 1} \end{bmatrix}.$$

- Note that $\mathcal{O}_C = h_\ell V^T P$, where $P = \operatorname{diag}(p_1, p_2, \ldots, p_{n_\phi})$ and V is the Vandermonde matrix in the d_j

$$\det(\mathcal{O}_C) = h_\ell \prod_{1 \le i \le j \le n_\phi} (\omega_i^2 - \omega_j^2) \prod_{k=1}^{n_\phi} \frac{\partial^2 \phi_k(x_\ell)}{\partial x^2}$$

- Since $\omega_i \neq \omega_j$, $\det(\mathcal{O}_C) \neq 0$ if $\frac{\partial^2 \phi_k(x_\ell)}{\partial x^2}$

Lie Derivatives - Strain

$$h_{\varepsilon @ x_\ell} = \frac{\partial^2 \Phi(x_\ell)}{\partial x^2} \boldsymbol{\eta}$$

$$L_{\mathbf{f}_0} h_{\varepsilon @ x_\ell} = \frac{\partial^2 \Phi(x_\ell)}{\partial x^2} \dot{\boldsymbol{\eta}}$$

$$L_{\mathbf{f}_0^2} h_{\varepsilon @ x_\ell} = \frac{\partial^2 \Phi(x_\ell)}{\partial x^2} (K\boldsymbol{\eta} - R_b U_b M_1 + R_b M_{C2})$$

$$L_{\mathbf{f}_0^3} h_{\varepsilon @ x_\ell} = K \frac{\partial^2 \Phi(x_\ell)}{\partial x^2} \dot{\boldsymbol{\eta}}$$

$$L_{\mathbf{f}_0^4} h_{\varepsilon @ x_\ell} = K \frac{\partial^2 \Phi(x_\ell)}{\partial x^2} (K\boldsymbol{\eta} - R_b U_b M_1 + R_b M_{C2})$$

Lie Derivatives - Acceleration

$$L_{\mathbf{f}_U} L_{\mathbf{f}_0} \mathbf{h} = 0$$
$$L_{\mathbf{f}_U} L_{\mathbf{f}_0^2} \mathbf{h} = -R_b K M_1$$
$$L_{\mathbf{f}_V} L_{\mathbf{f}_0} \mathbf{h} = -K M_1$$
$$L_{\mathbf{f}_V} L_{\mathbf{f}_0^2} \mathbf{h} = 0$$
$$L_{\mathbf{f}_R} L_{\mathbf{f}_0} \mathbf{h} = -K M_{A2}$$
$$L_{\mathbf{f}_R} L_{\mathbf{f}_0^2} \mathbf{h} = K(M_{C2} - U_b M_1)$$

Analytical Continuum Gramian

Theorem 1

The continuum observability Gramian, W_∞, for system Σ_∞ is given by $W_\infty(t, x_\ell) = \int_0^t \delta W_\infty^\top \delta W_\infty d\tau \in \mathbb{R}^{n_\eta \times n_\eta}$, where

$$\delta W_\infty^\top = \begin{bmatrix} \sum_{j=1}^\infty h_\ell \phi_{j,xx}(x_\ell) \cos(\omega_j \tau) \\ \sum_{j=1}^\infty h_\ell \phi_{j,xx}(x_\ell) \frac{\sin(\omega_j \tau)}{\omega_j} \end{bmatrix}$$

- Proof outline: replace the derivative in $W_o^\ell(t) = \int_0^t \frac{\partial y_\ell(\tau)}{\partial \mathbf{x}_0} \frac{\partial y_\ell(\tau)}{\partial \mathbf{x}_0}^\top d\tau$ with the first variation and consider $y_\ell[w_0(x), \dot{w}_0(x)]$ as a functional that get perturbed by the mode shapes ϕ_i
- Agrees with Georges'[7] observability Gramian for an advection-diffusion model

[7] D. Georges, "Optimal sensor location and mobile sensor crowd modeling for environmental monitoring," *IFAC-PapersOnLine*, vol. 50, pp. 7076–7081, Jul 2017.

Empirical Continuum Gramian

Theorem 2

The empirical observability Gramian, W_∞^ϵ, for system Σ_∞ is given by $W_\infty^\epsilon(t, x_\ell) = \int_0^t \Delta Y_\infty^\top \Delta Y_\infty d\tau \in \mathbb{R}^{n_\eta \times n_\eta}$, where

$$\Delta Y_\infty^\top = \begin{bmatrix} \sum_{j=1}^{\infty} h_\ell \phi_{j,xx}(x_\ell)\cos(\omega_j \tau) \\ \sum_{j=1}^{\infty} h_\ell \phi_{j,xx}(x_\ell)\frac{\sin(\omega_j \tau)}{\omega_j} \end{bmatrix}$$

- Proof outline: Perturb the initial conditions by each mode shape and calculate the resulting changes in measurements
- Result is the same as for the analytical system

Single Sensor Observability

Theorem 3

The truncated linear system Σ_n is observable with a single sensor measurement if and only if the sensor is not located at a zero of $\frac{\partial^2 \phi_i}{\partial x^2}$ for all i.

- Proof outline: Calculate the linear observability matrix, $\mathcal{O}$, and use its structure to determine when the rank condition will be satisfied
- Indicates the sensor placement optimization is viable for a single sensor