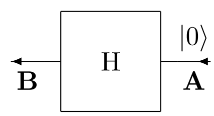
Quantum Computers

Last updated: March 4$^{\text{th}}$ 2019, at 11.21am

1. A Hadamard gate has the matrix

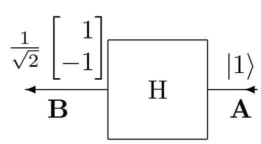
$$\frac{1}{\sqrt{2}}\begin{bmatrix}1 & 1\\ 1 & -1\end{bmatrix}$$

   (a) **Model question.** A Hadamard gate is given input $|0\rangle$ — i.e. the line at position **A** below carries the qubit $|0\rangle$,as shown:

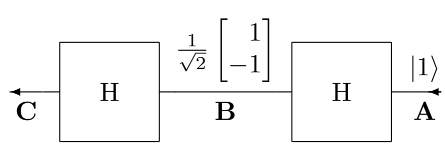


   What is the output — i.e. what value is on the line at position **B**?

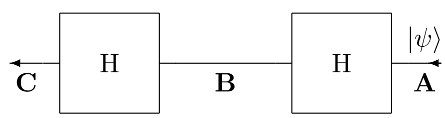
   (b) **Model question.** The output of a Hadamard gate, given input $|1\rangle$ is $\frac{1}{\sqrt{2}}[1,-1]^T$:



   What is the output if this output is input to a second Hadamard gate? I.e. what qubit is on the line at position **C**?



   (c) **Logbook question.** If $|\psi\rangle$ is a pure state (i.e. $|\psi\rangle = |0\rangle$ or $|\psi\rangle = |1\rangle$, what happens if $|\psi\rangle$ is passed as input to a circuit consisting of two Hadamard gates in sequence.



I.e. describe the state of the qubit at points **A**, **B** and **C**. How does this demonstrate that we need the "ket" (or the vector) representation of qubits, rather than just describing them in terms of probabilities (e.g. "this is a qubit with a 50% probability of being a 1").

2. The matrix for a controlled not is:

$$^{C}\mathrm{NOT} = \begin{bmatrix} 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 0 & 1 \\ 0 & 0 & 1 & 0 \end{bmatrix}$$

   Show that the $^{C}\mathrm{NOT}$ gate is reversible. Hint: $^{C}\mathrm{NOT}$ is its own inverse. Let $|c_0, c_1\rangle = \begin{bmatrix} q_0 & q_1 & q_2 & q_3 \end{bmatrix}^T$ be some qubit pair. Apply $^{C}\mathrm{NOT}$ to this twice and show that the result is $|c_0, c_1\rangle$.

3. Verify that $^{C}\mathrm{NOT}\,|0, y\rangle = |0, y\rangle$ and that $^{C}\mathrm{NOT}\,|1, y\rangle = |1, \text{not } y\rangle$, for $y = |0\rangle$ and $y = |1\rangle$.

4. What is $^{C}\mathrm{NOT}\left|\frac{|0\rangle+|1\rangle}{\sqrt{2}}, 1\right\rangle$?

5. What is $^{C}\mathrm{NOT}\left|\frac{|0\rangle+|1\rangle}{\sqrt{2}}, \frac{|0\rangle+|1\rangle}{\sqrt{2}}\right\rangle$?

6. Let U be a single qubit gate with matrix

$$\begin{bmatrix} a & b \\ c & d \end{bmatrix}$$

   and let y be the qubit $\begin{bmatrix} y_0 & y_1 \end{bmatrix}^T$. What is $U(y)$?

   If gate U is as defined above then a controlled U gate has matrix

$$^{C}U = \begin{bmatrix} 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & a & b \\ 0 & 0 & c & d \end{bmatrix}$$

   Show that this controlled U gate does implement:

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if (x == 0) {
  0,y;
} else {
  1,U(y);
}
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7. Show that the $R_{\overrightarrow{x}}$, $R_{\overrightarrow{y}}$, and $R_{\overrightarrow{z}}$ gates from the lecture are reversible.

   Hint: each of these rotates the Bloch sphere through θ radians. How can you "reverse" a rotation through θ radians?

   Note that $\sin^2\theta + \cos^2\theta = 1$, for any θ.

8. Have a look at the Toffoli gate from the lecture, and show that it is a ${}^{C}\left({}^{C}\mathrm{NOT}\right)$ gate.

   Hint: show that a ${}^{C}\left({}^{C}\mathrm{NOT}\right)$ gate exhibits the behaviour required of a Toffoli gate.