Quantum Computing — Some Maths

February 11, 2019

- Complex Numbers
- Matrices
- Another Look at Circuits

Lesson Plan

Lesson Plan

- Week 17: Modelling classical circuits
  Some maths to prepare for quantum circuits
- Week 19: Quantum systems
  Modelling quantum properties (superposition, entanglement)
- Week 20: Quantum circuits

1: Complex Numbers

1.1: Imaginary and complex numbers

1.1.1: The square root of -1

The *imaginary number* $\mathbf{i}$ (or sometimes $\mathbf{j}$) is defined to be the square root of $-\mathbf{1}$. I.e. $\mathbf{i} \times \mathbf{i} = -\mathbf{1}$.

1.1.2: Complex numbers

A complex number consists of a *real part* and an *imaginary part*. For example, $\mathbf{3} + \mathbf{4i}$ is a complex number.

Unary Operators

1.1.3: Complex conjugate

E.g.

$$\overline{-3+4i} = -3-4i, \text{ and } \overline{2-6i} = 2+6i.$$

1.1.4: Magnitude

$$|a+bi| = \sqrt{a^2+b^2}$$

Arithmetic

1.2: Arithmetic

1.2.1: Addition and Subtraction

E.g.

$$\begin{aligned}(3+4i)+(2-6i) &= 5-2i,\\ (3+4i)-(2-6i) &= 1+10i.\end{aligned}$$

Arithmetic

1.2.2: Multiplication

E.g.

$$\begin{aligned} (3+4i)(2-6i) &= 3(2-6i)+4i(2-6i) \\ &= 6-18i+8i-24(i\times i) \\ &= 6-10i-24(-1) \\ &= 6-10i+24 \\ &= 30-10i. \end{aligned}$$

Arithmetic

1.2.3: Division

Multiply both numerator and denominator by the conjugate of the denominator E.g.

$$
\begin{aligned}
\frac{3+4i}{2-6i} &= \frac{3+2i}{2-6i} \times \frac{2+6i}{2+6i} \\
&= \frac{(3+4i)(2+6i)}{(2-6i)(2+6i)} \\
&= \frac{6+18i+8i+24i^2}{2^2+6^2} \\
&= \frac{6+26i-24}{4+16} \\
&= \frac{-18+24i}{20} \\
&= -0 \cdot 9 + 1 \cdot 2i
\end{aligned}
$$

Matrices

2: Matrices

A *matrix* (plural — matrices) is a two-dimensional array — e.g.:

$$\begin{bmatrix} \mathbf{2} & \mathbf{7} & \mathbf{-3} \\ \mathbf{-4} & \mathbf{0} & \mathbf{0} \end{bmatrix}$$

This is a $\mathbf{2} \times \mathbf{3}$ matrix

2.1: Addition and Subtraction

Matrix addition/subtraction is just piecewise addition/subtraction of the elements. E.g.

$$\begin{bmatrix} 3 & -5 & 7 \\ 0 & 2 & -1 \\ 12 & 0 & -4 \\ -1 & 2 & 0 \end{bmatrix} + \begin{bmatrix} 0 & 4 & 2 \\ -3 & -2 & 5 \\ -6 & 0 & -8 \\ -5 & 6 & 12 \end{bmatrix} = \begin{bmatrix} 3 & -1 & 9 \\ -3 & 0 & 4 \\ 6 & 0 & -12 \\ -6 & 8 & 12 \end{bmatrix}$$

$$\begin{bmatrix} 3 & -5 & 7 \\ 0 & 2 & -1 \\ 12 & 0 & -4 \\ -1 & 2 & 0 \end{bmatrix} - \begin{bmatrix} 0 & 4 & 2 \\ -3 & -2 & 5 \\ -6 & 0 & -8 \\ -5 & 6 & 12 \end{bmatrix} = \begin{bmatrix} 3 & -9 & 5 \\ 3 & 4 & -6 \\ 18 & 0 & 4 \\ 4 & -4 & -12 \end{bmatrix}$$

Scalar Multiplication

2.2: Multiplication

2.2.1: Scalar Multiplication

Each entry in the matrix is multiplied by a fixed number. E.g.:

$$3 \cdot \begin{bmatrix} 3 & -5 & 7 \\ 0 & 2 & -1 \\ 12 & 0 & -4 \\ -1 & 2 & 0 \end{bmatrix} = \begin{bmatrix} 9 & -15 & 21 \\ 0 & 6 & -3 \\ 36 & 0 & -12 \\ -3 & 6 & 0 \end{bmatrix}$$

Matrix Product

2.2.2: Matrix Product

If **A** and **B** are matrices the entry in the $\mathbf{j^{th}}$ column of the $\mathbf{i^{th}}$ row of $\mathbf{A} * \mathbf{B}$ is the sum of pairwise products from **A**'s $\mathbf{i^{th}}$ row and **B**'s $\mathbf{j^{th}}$ column. E.g.:

$$\begin{bmatrix} \cdot & \cdot & \cdot \\ -2 & 6 & 5 \\ \cdot & \cdot & \cdot \\ \cdot & \cdot & \cdot \\ \cdot & \cdot & \cdot \end{bmatrix} * \begin{bmatrix} \cdot & \cdot & 3 & \cdot \\ \cdot & \cdot & 7 & \cdot \\ \cdot & \cdot & 1 & \cdot \end{bmatrix} = \begin{bmatrix} \cdot & \cdot & \cdot & \cdot \\ \cdot & \cdot & 41 & \cdot \\ \cdot & \cdot & \cdot & \cdot \\ \cdot & \cdot & \cdot & \cdot \\ \cdot & \cdot & \cdot & \cdot \end{bmatrix}$$

since $-2 \times 3 + 6 \times 7 + 5 \times 1 = 41$

If **A** is a $\mathbf{k} \times \mathbf{m}$ matrix, then **B** must be a $\mathbf{m} \times \mathbf{n}$ matrix (and vice versa), and the result will be a $\mathbf{k} \times \mathbf{n}$ matrix

Tensor Product

2.2.3: Tensor Product

The tensor product is the scalar product of the second matrix with each of the entries of the first matrix E.g.:

$$\begin{bmatrix} 2 & 0 \\ -1 & -2 \\ 0 & 1 \end{bmatrix} \otimes \begin{bmatrix} 2 & -1 \\ 3 & 0 \end{bmatrix} = \begin{bmatrix} 2 \cdot \begin{bmatrix} 2 & -1 \\ 3 & 0 \end{bmatrix} & 0 \cdot \begin{bmatrix} 2 & -1 \\ 3 & 0 \end{bmatrix} \\ -1 \cdot \begin{bmatrix} 2 & -1 \\ 3 & 0 \end{bmatrix} & -2 \cdot \begin{bmatrix} 2 & -1 \\ 3 & 0 \end{bmatrix} \\ 0 \cdot \begin{bmatrix} 2 & -1 \\ 3 & 0 \end{bmatrix} & 1 \cdot \begin{bmatrix} 2 & -1 \\ 3 & 0 \end{bmatrix} \end{bmatrix}$$

$$= \begin{bmatrix} 4 & -2 & 0 & 0 \\ 6 & 0 & 0 & 0 \\ -2 & 1 & -4 & 2 \\ -3 & 0 & -6 & 0 \\ 0 & 0 & 2 & -1 \\ 0 & 0 & 3 & 0 \end{bmatrix}$$

Transpose

2.3: Unary Operators

2.3.1: Transpose

"Flip" rows and columns E.g.

$$\begin{bmatrix} 3 & 8 & -2 & 0 \\ 6 & -5 & 0 & 7 \end{bmatrix}^T = \begin{bmatrix} 3 & 6 \\ 8 & -5 \\ -2 & 0 \\ 0 & 7 \end{bmatrix}$$

Conjugate

2.3.2: Conjugate

The piecewise conjugate of the elements E.g.

$$\overline{\begin{bmatrix} 2+i & -3+2i & 4 \\ 0 & 2-3i & i \end{bmatrix}} = \begin{bmatrix} 2-i & -3-2i & 4 \\ 0 & 2+3i & -i \end{bmatrix}$$

Adjoint

2.3.3: Adjoint

The transpose of the conjugate (or vice versa)

$$\mathbf{A}^{\dagger} = \overline{\mathbf{A}^{\mathsf{T}}} = \overline{\mathbf{A}}^{\mathsf{T}}$$

E.g.

$$\begin{bmatrix} 2+i & -3+2i & 4 \\ 0 & 2-3i & i \end{bmatrix}^{\dagger} = \begin{bmatrix} 2-i & 0 \\ -3-2i & 2+3i \\ 4 & -i \end{bmatrix}$$

Bits

3: Another Look at Circuits

3.1: Bits

We can represent bits as matrices:

$$\textbf{false} \equiv \begin{bmatrix} \mathbf{1} \\ \mathbf{0} \end{bmatrix}, \textbf{true} \equiv \begin{bmatrix} \mathbf{0} \\ \mathbf{1} \end{bmatrix}.$$

A '$|\mathbf{b}\rangle$' notation is often used to represent these bits:

$$\textbf{false} \equiv |\mathbf{0}\rangle \equiv \begin{bmatrix} \mathbf{1} \\ \mathbf{0} \end{bmatrix}, \textbf{true} \equiv |\mathbf{1}\rangle \equiv \begin{bmatrix} \mathbf{0} \\ \mathbf{1} \end{bmatrix}.$$

Bit Sequences

Bit sequences are calculated as the *tensor product* of bit matrices:

$$|\mathbf{01}\rangle \equiv |\mathbf{0}\rangle \otimes |\mathbf{1}\rangle \equiv \begin{bmatrix} 1 \\ 0 \end{bmatrix} \otimes \begin{bmatrix} 0 \\ 1 \end{bmatrix} = \begin{bmatrix} 1 \begin{bmatrix} 0 \\ 1 \end{bmatrix} \\ 0 \begin{bmatrix} 0 \\ 1 \end{bmatrix} \end{bmatrix} = \begin{bmatrix} 0 \\ 1 \\ 0 \\ 0 \end{bmatrix}$$

Note: a byte (e.g. $|\mathbf{0110\ 1001}\rangle$) would have 256 rows.

Not

3.2: Gates

Gates are also defined as matrices.

3.2.1: Not

$$\mathbf{not} = \begin{bmatrix} 0 & 1 \\ 1 & 0 \end{bmatrix}.$$

and application of a gate to a boolean value is modelled by matrix multiplication. So

$$\mathbf{not\ true} \equiv \mathbf{not}\ |1\rangle \equiv \begin{bmatrix} 0 & 1 \\ 1 & 0 \end{bmatrix} * \begin{bmatrix} 0 \\ 1 \end{bmatrix} = \begin{bmatrix} 1 \\ 0 \end{bmatrix} \equiv |0\rangle \equiv \mathbf{false}$$

And

3.2.2: And

$$\mathbf{and} = \begin{bmatrix} 1 & 1 & 1 & 0 \\ 0 & 0 & 0 & 1 \end{bmatrix}$$

with

$$\mathbf{and}\,|01\rangle \equiv \begin{bmatrix} 1 & 1 & 1 & 0 \\ 0 & 0 & 0 & 1 \end{bmatrix} * \begin{bmatrix} 0 \\ 1 \\ 0 \\ 0 \end{bmatrix} = \begin{bmatrix} 1 \\ 0 \end{bmatrix} \equiv |0\rangle$$

Truth Table

Note: can derive matrix from truth table

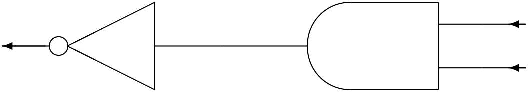
$\mathbf{x}$	**0**	**0**	**1**	**1**	input
$\mathbf{y}$	**0**	**1**	**0**	**1**	input
$\mathbf{x} \wedge \mathbf{y}$	**0**	**0**	**0**	**1**	output
$\lvert \mathbf{x} \wedge \mathbf{y} \rangle$	$\lvert \mathbf{0} \rangle$	$\lvert \mathbf{0} \rangle$	$\lvert \mathbf{0} \rangle$	$\lvert \mathbf{1} \rangle$	output as matrix
	1	**1**	**1**	**0**	matrix expanded
	0	**0**	**0**	**1**	”

Sequential Circuits

3.3: Circuits

3.3.1: Sequential circuits

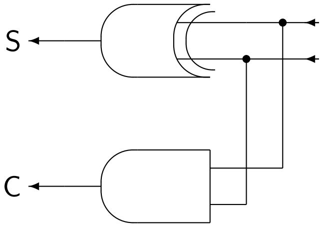
Sequential circuits use the matrix product. For example a **nand** gate



$$\textbf{nand} = \textbf{not} * \textbf{and} = \begin{bmatrix} 0 & 1 \\ 1 & 0 \end{bmatrix} * \begin{bmatrix} 1 & 1 & 1 & 0 \\ 0 & 0 & 0 & 1 \end{bmatrix} = \begin{bmatrix} 0 & 0 & 0 & 1 \\ 1 & 1 & 1 & 0 \end{bmatrix}$$

3.3.2: Parallel Circuits

Parallel circuits use the tensor product. For example a half-adder



So we take **xor** $\otimes$ **and**.

Note: Identify the "most significant bit" — here top to bottom.

Parallel Circuits

$$
\begin{aligned}
& \textbf{xor} \otimes \textbf{and} \\
= & \begin{bmatrix} 1 & 0 & 0 & 1 \\ 0 & 1 & 1 & 0 \end{bmatrix} \otimes \begin{bmatrix} 1 & 1 & 1 & 0 \\ 0 & 0 & 0 & 1 \end{bmatrix} \\
= & \begin{bmatrix} 1 & 1 & 1 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 1 & 1 & 1 & 0 \\ 0 & 0 & 0 & 1 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 1 \\ 0 & 0 & 0 & 0 & 1 & 1 & 1 & 0 & 1 & 1 & 1 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & 1 & 0 & 0 & 0 & 1 & 0 & 0 & 0 & 0 \end{bmatrix}
\end{aligned}
$$