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Inverting and noninverting H_∞ controllers

Urs Christen, Hans P. Geering*

Measurement and Control Laboratory, Swiss Federal Institute of Technology (ETH), CH-8092 Zurich, Switzerland

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Abstract

It is well known that two-block $S/KS/T$ H_∞ problems in which the plant is weighted at the output tend to invert the plant in the controller. This paper shows that even four-block $S/KS/T$ problems in which the plant is weighted at the input result in controllers which invert the plant. However, if a GS/T weighting scheme is used where the weight for the sensitivity includes the plant, the inversion is avoided. This GS/T scheme therefore is especially suited for ill-conditioned plants. An example confirms these results. © 1997 Elsevier Science B.V.

Keywords: H_∞ design; Pole/zero cancellation; Mixed sensitivity problem; Robustness; Ill-conditioned plants

1. Introduction

Sefton and Glover have shown in [5] that the popular $S/KS/T$ weighting scheme for the H_∞ design leads to controllers which have zeros at the asymptotically stable poles of the plant. As a remedy, they propose the use of four-block procedures. However, even if the sensitivity S and the complementary sensitivity T are weighted in a four-block scheme, pole/zero cancellations occur in the closed-loop transfer functions.

This property of the four-block $S/KS/T$ weighting scheme is derived in Section 2 of this paper. Section 3 introduces an alternative weighting scheme in which the weight for the sensitivity includes the plant. This GS/T scheme results in a controller which avoids the pole/zero cancellation and therefore is suitable whenever the inversion must be prevented, e.g., for ill-conditioned plants. This is demonstrated by an example in Section 4 and discussed in Section 5. The appendix contains the underlying mathematics of the H_∞ problem.

2. $S/KS/T$ loop shaping schemes

The best-known augmentation scheme for H_∞ controller designs is the so-called $S/KS/T$ scheme in which the H_∞ norm of

$$T_{zw} = \begin{bmatrix} W_e S_e \\ W_u K S_e \\ W_y T_e \end{bmatrix}$$

* Corresponding author.

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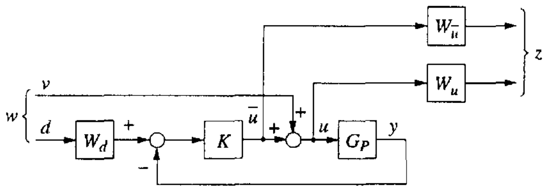

Fig. 1. Augmentation scheme for weighting at the plant input.

is minimized. The sensitivity $S_e = (I + G_P K)^{-1}$(G_P being the original plant) and the complementary sensitivity $T_e = I - S_e$ are weighted by W_e and W_y, respectively, which should be chosen to reflect the inverse of the desired shapes of S_e and T_e. The transfer function KS_e is weighted by W_u which is mainly used to avoid a singular H_∞ problem (cf. Assumption A.2 in the Appendix). W_u can be chosen to be constant and arbitrarily small such that it does not influence the H_∞ norm of T_{zw}.

H_∞ controllers based on this augmentation scheme cancel the asymptotically stable poles of the plant by zeros, as Sefton and Glover have shown in [5] by referring to a two-block H_∞ problem. In order to avoid the cancellation, they suggest to use a four-block weighting scheme, i.e., a scheme which results in an augmented plant with both feed-through matrices D_{12} and D_{21} being non-square. Such a scheme is, for instance, the $S/KS/T$ scheme in which the sensitivity $S_u = (I + KG_P)^{-1}$ at the input u of the plant and its complement $T_u = I - S_u$ are weighted. In this scheme, an external auxiliary signal v must be introduced which allows for the excitation at u (Fig. 1). The transfer function from v to u is the sensitivity S_u, which consequently is weighted by W_u. From v to $\bar{u}$, it is the complementary sensitivity T_u, which is weighted by $W_{\bar{u}}$. The input d must exist, again in order to avoid a singular problem. However, W_d can be chosen to be static and small such that the H_∞ norm of

$$T_{zw} = \begin{bmatrix} -W_{\bar{u}} T_u & W_{\bar{u}} K S_e W_d \\ W_u S_u & W_u K S_e W_d \end{bmatrix}$$

is not influenced. The considerations for choosing the weightings $W_{\bar{u}}$ and W_u are identical to those for choosing W_y and W_e above.

The controller based on this weighting scheme inverts the plant, too. This may be inferred heuristically from T_{zw} by assuming that all its singular values are constant and roughly equal for all frequencies – to improve at one frequency, the system deteriorates at another which results in $\sigma_i(T_{zw}(j\omega)) \approx \gamma \ \forall\omega$. Thus, $W_u S_u \approx U$ (U being a stable all-pass) at those frequencies where W_u is the limiting weight, i.e., usually in the low-frequency range. In that frequency range, the loop gain $L_u = KG_P$ is much larger than I whence S_u may be approximated by

$$S_u = (I + L_u)^{-1} \approx L_u^{-1}.$$

Consequently,

$$\begin{aligned} W_u S_u &\approx W_u L_u^{-1} = W_u G_P^{-1} K^{-1} \approx U, \\ K &\approx U^{-1} W_u G_P^{-1}. \end{aligned} \tag{1}$$

A possible state-space representation for the augmented plant

$$G = \left[\begin{array}{cc|c} 0 & 0 & W_{\bar{u}} \\ W_u & 0 & W_u \\ \hline -G_P & W_d & -G_P \end{array}\right]$$

is given by

$$G = \left[\begin{array}{c|c:c} A & B_1 & B_2 \\ \hline C_1 & D_{11} & D_{12} \\ \hdashline C_2 & D_{21} & D_{22} \end{array}\right] = \left[\begin{array}{ccc|cc:c} A_{\bar{u}} & 0 & 0 & 0 & 0 & B_{\bar{u}} \\ 0 & A_u & 0 & B_u & 0 & B_u \\ 0 & 0 & A_P & B_P & 0 & B_P \\ \hline C_{\bar{u}} & 0 & 0 & 0 & 0 & D_{\bar{u}} \\ 0 & C_u & 0 & 0 & 0 & 0 \\ \hdashline 0 & 0 & -C_P & 0 & D_d & 0 \end{array}\right], \tag{2}$$

where the indices are the same as those of the systems forming G. It is assumed that W_d is static and that G_P and W_u have no feed-throughs. Due to the special structure of the augmented plant, the first Riccati equation (Eq. (A.1) in the Appendix) may be simplified. C_1 does not access the plant G_P, which causes its state not to be weighted by the constant term Q (A.3) of the Riccati equation. Hence, the elements of the solution X to the Riccati equation which correspond to the state of the plant are zero if the plant is asymptotically stable (more precisely, the kernel of X is equal to the subspace of $\mathbb{R}^n$ spanned by the unobservable, asymptotically stable modes of the pair (A,Q) [4, Theorem 2.2]).

This results in a pole/zero cancellation in the loop gain L_u as stated in the theorem below. It is assumed that the augmented plant G and the controller K have minimal realizations. The theorem shows that even under this "best case" assumption pole/zero cancellations occur.

Theorem 1. *S/KS/T scheme weighting at the plant input: The controller K has zeros at the asymptotically stable poles of G_P. Hence, there are pole/zero cancellations in $L_u = KG_P$.*

Proof. Partition the regulator gain matrix F (A.5) and the observer gain matrix H_2 (A.7) according to the systems forming the augmented plant (2):

$$F = \left[\begin{array}{ccc} F_{1v\bar{u}} & F_{1vu} & F_{1vP} \\ F_{1d\bar{u}} & F_{1du} & F_{1dP} \\ \hdashline F_{2\bar{u}} & F_{2u} & F_{2P} \end{array}\right] = \left[\begin{array}{c} -\gamma^{-2}B_1^{\mathrm{T}}X \\ \hdashline (D_{12}^{\mathrm{T}}D_{12})^{-1}(B_2^{\mathrm{T}}X + D_{12}^{\mathrm{T}}C_1) \end{array}\right],$$

$$H_2 = \begin{bmatrix} H_{2\bar{u}} \\ H_{2u} \\ H_{2P} \end{bmatrix} := \gamma^2 Y C_2^{\mathrm{T}} (D_{21}D_{21}^{\mathrm{T}})^{-1},$$

where $F_{1v\cdot} = 0$ due to the structure of B_1 (and $D_{21}F_1 = 0$ because $D_{21}B_1^{\mathrm{T}} = 0$). Moreover, $D_{12}^{\mathrm{T}}C_1$ only affects $F_{2\bar{u}}$ such that F_{2P} is zero for asymptotically stable modes, i.e., for modes in the kernel of X.

The controller is given by (cf. Appendix)

$$K = \left[\begin{array}{ccc|c} A_{\bar{u}} - B_{\bar{u}}F_{2\bar{u}} & -B_{\bar{u}}F_{2u} & -B_{\bar{u}}F_{2P} + H_{2\bar{u}}C_P & H_{2\bar{u}} \\ -B_uF_{2\bar{u}} & A_u - B_uF_{2u} & -B_uF_{2P} + H_{2u}C_P & H_{2u} \\ -B_PF_{2\bar{u}} & -B_PF_{2u} & A_P + H_{2P}C_P - B_PF_{2P} & H_{2P} \\ \hline -F_{2\bar{u}} & -F_{2u} & -F_{2P} & 0 \end{array}\right].$$

Postmultiply K by G_P to form

$$L_u = \left[\begin{array}{cccc|c} A_{\bar{u}} - B_{\bar{u}}F_{2\bar{u}} & -B_{\bar{u}}F_{2u} & -B_{\bar{u}}F_{2P} + H_{2\bar{u}}C_P & H_{2\bar{u}}C_P & 0 \\ -B_uF_{2\bar{u}} & A_u - B_uF_{2u} & -B_uF_{2P} + H_{2u}C_P & H_{2u}C_P & 0 \\ -B_PF_{2\bar{u}} & -B_PF_{2u} & A_P + H_{2P}C_P - B_PF_{2P} & H_{2P}C_P & 0 \\ 0 & 0 & 0 & A_P & B_P \\ \hline -F_{2\bar{u}} & -F_{2u} & -F_{2P} & 0 & 0 \end{array}\right]$$

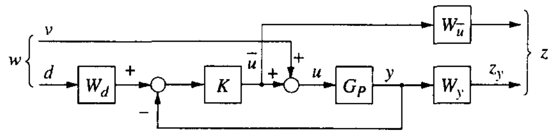

Fig. 2. GS/T augmentation scheme for noninverting controllers.

and apply a state transform to see that the asymptotically stable modes of G_P are not observable since F_{2P} is zero for these modes, as discussed above.

$$L_u = \left[\begin{array}{cccc|c} A_{\bar{u}} - B_{\bar{u}}F_{2\bar{u}} & -B_{\bar{u}}F_{2u} & -B_{\bar{u}}F_{2P} + H_{2\bar{u}}C_P & B_{\bar{u}}F_{2P} & 0 \\ -B_u F_{2\bar{u}} & A_u - B_u F_{2u} & -B_u F_{2P} + H_{2u}C_P & B_u F_{2P} & 0 \\ -B_P F_{2\bar{u}} & -B_P F_{2u} & A_P + H_{2P}C_P - B_P F_{2P} & B_P F_{2P} & B_P \\ 0 & 0 & 0 & A_P & B_P \\ \hline -F_{2\bar{u}} & -F_{2u} & -F_{2P} & F_{2P} & 0 \end{array}\right].$$

This proves the theorem because G_P and K are assumed to have minimal realizations. □

Thus, although this is a real four-block problem, the controller inverts the plant. The inversion could be avoided, however, by including the plant in the weight for the sensitivity, as implied by (1).

3. Loop shaping without inversion: *GS/T* scheme

We proceed from the previous scheme for loop-shaping at the plant input. Instead of the sensitivity being weighted by W_u at u, it is weighted at y to include G_P in its weight (Fig. 2). The transfer function from w to z is

$$T_{zw} = \begin{bmatrix} -W_{\bar{u}}T_u & W_{\bar{u}}KS_eW_d \\ W_yG_PS_u & W_yT_eW_d \end{bmatrix}.$$

$W_{\bar{u}}$ weights the complementary sensitivity T_u as in the previous section. The sensitivity is part of the transfer function from v to z_y: $W_yG_PS_u = W_yS_eG_P$. Thus, W_y must reflect that part of the inverse of the desired sensitivity which is not yet taken into account by G_P. Formally, one could write $W_y = G_P^{-1}\hat{S}^{-1}$ ($\hat{S}$ being the desired sensitivity). But it is only the envelope of the plant's singular values (i.e., mainly its static gain) which should be taken into account (cf. Section 4). If the inverted plant indeed were included, then the controller would contain it again.

The controller designed using this GS/T scheme does not invert the plant and at low frequencies is directly shaped by W_y as can be derived heuristically from T_{zw}. Assuming again that the singular values of T_{zw} are constant and equal and approximating the sensitivity with the loop gain, the transfer function from v to z_y is

$$W_yS_eG_P \approx W_yL_e^{-1}G_P = W_yK^{-1}G_P^{-1}G_P = W_yK^{-1} \approx U$$

or

$$W_yG_PS_u \approx W_yG_PL_u^{-1} = W_yG_PG_P^{-1}K^{-1} = W_yK^{-1} \approx U.$$

Hence

$$K \approx U^{-1}W_y. \tag{3}$$

Under the assumptions that G_P has no feed-through and that W_d is static, a state-space representation of the augmented plant

$$G = \left[\begin{array}{cc|c} 0 & 0 & W_{\tilde{u}} \\ W_y G_P & 0 & W_y G_P \\ \hline -G_P & W_d & -G_P \end{array}\right]$$

is given by

$$G = \left[\begin{array}{c|c|c} A & B_1 & B_2 \\ \hline C_1 & D_{11} & D_{12} \\ \hline C_2 & D_{21} & D_{22} \end{array}\right] = \left[\begin{array}{ccc|cc|c} A_{\tilde{u}} & 0 & 0 & 0 & 0 & B_{\tilde{u}} \\ 0 & A_y & B_y C_P & 0 & 0 & 0 \\ 0 & 0 & A_P & B_P & 0 & B_P \\ \hline C_{\tilde{u}} & 0 & 0 & 0 & 0 & D_{\tilde{u}} \\ 0 & C_y & D_y C_P & 0 & 0 & 0 \\ \hline 0 & 0 & -C_P & 0 & D_d & 0 \end{array}\right].$$

Its structure does not allow for any further simplifications. The solutions X and Y of the Riccati equations cannot be zero because those blocks of the constant terms Q (A.3) and $\tilde{Q}$ (A.4) which correspond to the plant are not zero.

In the following theorem, it is again assumed that the augmented plant and the controller are minimal.

Theorem 2. *GS/T weighting scheme: The controller K contains no zeros at the poles of G_P. Hence, there is no pole/zero cancellation in $L_e = G_P K$ or $L_u = K G_P$.*

Proof. Since both of the constant terms

$$Q = \begin{bmatrix} \star & 0 & 0 \\ 0 & \star & \star \\ 0 & \star & \star \end{bmatrix} \quad \text{and} \quad \tilde{Q} = \gamma^{-2} \begin{bmatrix} 0 & 0 & 0 \\ 0 & 0 & 0 \\ 0 & 0 & B_P B_P^{\mathrm{T}} \end{bmatrix}$$

of the Riccati equations are nonzero for the plant's block and the plant is assumed to be controllable and observable, the solutions X and Y of the Riccati equations are nonzero for all the modes of the plant [4]. Hence, it is not possible to cancel anything in L_e or L_u. □

4. Example

A simple multivariable example shall be used to illustrate the properties mentioned in the previous sections. The plant is given by

$$G_P = \frac{s + 0.4}{s + 0.07} \begin{bmatrix} 10 & 13 \\ 3.5 & 5 \end{bmatrix}.$$

With a condition number of $\kappa(G_P) = 68$, it is ill-conditioned, i.e., its gain strongly depends on the direction of the input vector. The design specifications include static disturbance rejection and tracking error better than 1% at low frequencies, a minimum bandwidth of the closed-loop system of 0.4 rad/s, and a maximum bandwidth of 4 rad/s. Thus, the sensitivity must have singular values smaller than 0.01 at low frequencies. This requirement and the bandwidth constraints translate to weights for the sensitivity and the complementary sensitivity.

For the *S/KS/T* weighting scheme, the following weights guarantee the specifications if $\|T_{zw}\|_\infty \leqslant 1$ is achieved:

$$W_u = \frac{100}{s/0.01 + 1} I, \qquad W_{\tilde{u}} = 0.003 \frac{s/0.01 + 1}{s/100 + 1} I, \qquad W_d = 0.002 I.$$

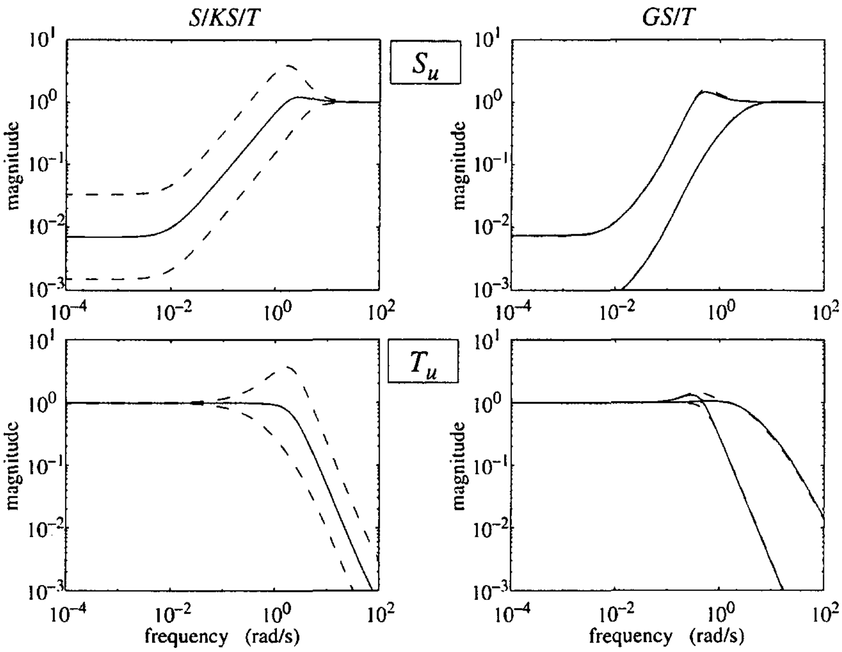

Fig. 3. Singular values of the sensitivities S_u and complementary sensitivities T_u for the nominal (—) and the disturbed (- - -) plant. The controllers are designed using the $S/KS/T$ (left) and the GS/T (right) weighting schemes.

The resulting controller

$$K = \begin{bmatrix} 10 & 13 \\ 3.5 & 5 \end{bmatrix}^{-1} \frac{s+0.07}{s+0.4} \frac{0.0471(s+100)}{(s+0.01)(s+3.314)}$$

clearly contains the inverted plant.

For the GS/T scheme, the weights are chosen to be

$$W_y = \frac{20}{s/0.008+1} I, \qquad W_{\tilde{u}} = 0.003 \frac{s/0.01+1}{s/100+1} I, \qquad W_d = 0.002 I.$$

The plant, which has quite a large gain, is included in the weight for the sensitivity. W_y therefore is smaller than the weight W_u for the sensitivity in the $S/KS/T$ case. The weight $W_{\tilde{u}}$ for the complementary sensitivity is the same in both cases. This controller cannot be written as compactly as the previous one. However, its zeros $\{-0.3595, -0.2015, -100, -100\}$ show that only the weight $W_{\tilde{u}}$ is inverted, but the plant is not.

With both controllers, the specifications are fulfilled for the nominal plant, as shown in Fig. 3. However, with the $S/KS/T$ scheme, the performance achieved is not robust. For a disturbance of 10% in the plant output, i.e.,

$$\tilde{G}_P = \begin{bmatrix} 0.9 & 0 \\ 0 & 1.1 \end{bmatrix} G_P,$$

the sensitivity S_u and the complementary sensitivity T_u deteriorate significantly, whereas for the GS/T design, the differences are nearly indiscernible.

5. Conclusions

The result of Section 4 is compatible with the results of Freudenberg [2]. He pointed out that in order to be robust, the controller for an ill-conditioned plant must not invert it. If the plant is inverted, the control system may be sensitive at one location (in the example: the plant input) to uncertainty at another (the plant output). Freudenberg concluded in [2] that the controller should have a small condition number. Hence, for an ill-conditioned plant the *S/KS/T* schemes are not suitable because the controllers are as badly conditioned as the plant itself since they invert it: Theorem 1 shows that the controller cancels the asymptotically stable poles of the plant, and expression (1) indicates that the plant indeed is inverted, which is confirmed by the example. The *GS/T* scheme, however, yields a controller which has a small condition number if the weight W_y is chosen as a scalar transfer function times identity such that its condition number is one (cf. (3)). In fact, the *S/KS/T* controller for the example plant has a condition number of 68, while that of the *GS/T* controller is around five. However, the controller being well-conditioned, the closed-loop system has to be as ill-conditioned as the plant (Fig. 3). Thus, there is a trade-off between nice nominal closed-loop transfer functions and robustness.

The use of the *GS/T* weighting scheme often offers an additional advantage. Since the plant is included in the weight of the sensitivity, the order of the weight W_y may be smaller than for the sensitivity weights in the *S/KS/T* schemes. In fact, W_y can be chosen to be static if the plant G_P has integrating behavior. This actually sounds familiar, since the weighting scheme of LQG/LTR is exactly the one shown in Fig. 2, except for all the weights being static (including one in the input v). The advantage of the *GS/T* scheme over normalized coprime factor designs (proposed in [5] to avoid pole/zero cancellations) lies in the possibility to shape the closed-loop transfer functions. Coprime factor designs, on the other hand, require open-loop shaping, which leads to a higher order of the controller.

A more realistic application of the *GS/T* scheme can be found in [1] where an H_∞ controller is designed for an ill-conditioned industrial distillation column. The *GS/T* controller performs as well as a controller designed with μ synthesis. Other applications of the *GS/T* weighting scheme include plants with a resonance which due to uncertainty ought not to be inverted.

Appendix: The mathematical H_∞ problem

For the generalized plant

$$G = \left[\begin{array}{c|cc} A & B_1 & B_2 \\ \hline C_1 & D_{11} & D_{12} \\ C_2 & D_{21} & D_{22} \end{array}\right], \qquad \begin{array}{ll} A \in \mathbb{R}^{n\times n}, & B_i \in \mathbb{R}^{n\times m_i}, \\ C_i \in \mathbb{R}^{p_i \times n}, & D_{ij} \in \mathbb{R}^{p_i \times m_j}, \end{array}$$

satisfying the assumptions

A1: (A, B_2, C_2) is stabilizable and detectable,

A2: $\operatorname{rank}(D_{12}) = m_2$, $\operatorname{rank}(D_{21}) = p_2$,

A3: For all ω, the two matrices

$$\begin{bmatrix} A - \mathrm{j}\omega I & B_2 \\ C_1 & D_{12} \end{bmatrix} \quad \text{and} \quad \begin{bmatrix} A - \mathrm{j}\omega I & B_1 \\ C_2 & D_{21} \end{bmatrix}$$

have full column and row rank, respectively,

A4: $D_{11} = 0, D_{22} = 0$,

a controller K is sought which stabilizes the closed-loop system

$$T_{zw} = \mathscr{F}_{\mathscr{C}}(G, K) = G_{11} + G_{12}K(I - G_{22}K)^{-1}G_{21}$$

and constrains its H_∞ norm:

$$\min_K \|T_{zw}\|_\infty < \gamma.$$

Such a compensator K exists if and only if
(a) a stabilizing solution $X \geqslant 0$ exists for

$$0 = (A - B_2(D_{12}^{\mathrm{T}}D_{12})^{-1}D_{12}^{\mathrm{T}}C_1)^{\mathrm{T}}X + X(A - B_2(D_{12}^{\mathrm{T}}D_{12})^{-1}D_{12}^{\mathrm{T}}C_1) \\ -XB_2(D_{12}^{\mathrm{T}}D_{12})^{-1}B_2^{\mathrm{T}}X + \gamma^{-2}XB_1B_1^{\mathrm{T}}X + Q, \tag{A.1}$$

(b) a stabilizing solution $Y \geqslant 0$ exists for

$$0 = (A - B_1F_1 - B_1D_{21}^{\mathrm{T}}(D_{21}D_{21}^{\mathrm{T}})^{-1}(C_2 - D_{21}F_1))Y \\ +Y(A - B_1F_1 - B_1D_{21}^{\mathrm{T}}(D_{21}D_{21}^{\mathrm{T}})^{-1}(C_2 - D_{21}F_1))^{\mathrm{T}} \\ -\gamma^2 Y(C_2 - D_{21}F_1)^{\mathrm{T}}(D_{21}D_{21}^{\mathrm{T}})^{-1}(C_2 - D_{21}F_1)Y + YF_2^{\mathrm{T}}D_{12}^{\mathrm{T}}D_{12}F_2Y + \tilde{Q}, \tag{A.2}$$

where

$$Q = C_1^{\mathrm{T}}C_1 - C_1^{\mathrm{T}}D_{12}(D_{12}^{\mathrm{T}}D_{12})^{-1}D_{12}^{\mathrm{T}}C_1, \tag{A.3}$$

$$\tilde{Q} = \gamma^{-2}(B_1B_1^{\mathrm{T}} - B_1D_{21}^{\mathrm{T}}(D_{21}D_{21}^{\mathrm{T}})^{-1}D_{21}B_1^{\mathrm{T}}), \tag{A.4}$$

$$F = \begin{bmatrix} F_1 \\ F_2 \end{bmatrix} = \begin{bmatrix} -\gamma^{-2}B_1^{\mathrm{T}}X \\ (D_{12}^{\mathrm{T}}D_{12})^{-1}(B_2^{\mathrm{T}}X + D_{12}^{\mathrm{T}}C_1) \end{bmatrix}. \tag{A.5}$$

If the conditions (a) and (b) are fulfilled, the central controller [5] is given by

$$K = \left[\begin{array}{c|c} A - B_1F_1 - B_2F_2 - H_2(C_2 - D_{21}F_1) & H_2 \\ \hline -F_2 & 0 \end{array}\right], \tag{A.6}$$

where

$$H_2 = (Y\gamma^2(C_2^{\mathrm{T}} - F_1^{\mathrm{T}}D_{21}^{\mathrm{T}}) + \gamma B_1D_{21}^{\mathrm{T}})(D_{21}D_{21}^{\mathrm{T}})^{-1}. \tag{A.7}$$

Remark. "Stabilizing solution" means that

$$(A - B_2(D_{12}^{\mathrm{T}}D_{12})^{-1}D_{12}^{\mathrm{T}}C_1) - B_2(D_{12}^{\mathrm{T}}D_{12})^{-1}B_2^{\mathrm{T}}X + \gamma^{-2}B_1B_1^{\mathrm{T}}X$$

and

$$(A - B_1F_1 - B_1D_{21}(D_{21}D_{21}^{\mathrm{T}})^{-1}(C_2 - D_{21}F_1)) \\ -\gamma^2 Y(C_2 - D_{21}F_1)^{\mathrm{T}}(D_{21}D_{21}^{\mathrm{T}})^{-1}(C_2 - D_{21}F_1) + YF_2^{\mathrm{T}}D_{12}^{\mathrm{T}}D_{12}F_2$$

are stabilized by X and Y, respectively.

The derivation of this solution is similar to that of Glover and Doyle [3], except that their last step is skipped where the second Riccati equation is reformulated to be independent of the solution of the first.

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