Efficient Bus Allocation

XI Iberian Modeling Week

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Problem Statement

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For public transport companies, it is important to determine the hourly number of buses required to meet passenger demand throughout a typical day.

Given a forecast of daily demand patterns and origin-destination data, the challenge involves estimating trip durations by modeling factors such as:

- Bus movement
- Route alignment
- Boarding times
- Stop frequency

To ensure efficient and eco-conscious operations, participants must also consider synchronization strategies between buses to reduce waiting and travel times, aiming to minimize fleet size, fuel consumption, and environmental impact.

Model Considerations

Model Considerations - Building Complexity

We develop our model by progressively adding real-world complexity:

1. **Free Movement** - Theoretical baseline with constant speed
2. **Bus Stops** - Adding acceleration/deceleration dynamics
3. **Dwell Time** - Passenger boarding and alighting
4. **Traffic Lights** - Urban infrastructure delays
5. **Traffic** - Vehicle interactions and congestion

Approach: From theoretical foundations to practical implementation, transitioning from stochastic modeling to deterministic simulation using expected values.

Free Movement - Baseline

Simplest Case: Constant velocity along the entire route

Route Characteristics:

- Total distance: 11.5 km
- Maximum legal speed: 50 km/h
- Almost straight road (no turning delays)

Travel Time Calculation:

$$T_{\text{free}} = \frac{D}{v_{\text{max}}} = \frac{11.5 \text{ km}}{50 \text{ km/h}} = 0.23 \text{ h} = 13.8 \text{ min} \tag{1}$$

This provides our theoretical minimum travel time, serving as a baseline for measuring the impact of real-world factors.

Bus Stops - Braking and Speeding Up

Physical Constraints:

- 34 bus stops along the route
- Smooth acceleration and deceleration profiles
- Constant braking/acceleration distances

Key Parameters:

$$\begin{aligned} t_a &= \text{Acceleration time} \\ t_d &= \text{Deceleration time} \\ d_p &= \text{Distance lost per stop (braking + acceleration)} \\ n_p &= \text{Number of stops} = 34 \end{aligned}$$

Modified Travel Time:

$$T_{\text{stops}} = \frac{D - (n_p - 1)d_p}{v_m} + (n_p - 1)(t_a + t_d) \tag{2}$$

Dwell Time - Passenger Interactions

Definition: Time spent at each stop for passenger boarding/alighting

Stochastic Model:

$$\begin{aligned} t_0 &= \text{Base door opening and closing time} \\ \alpha &= \text{Time per passenger (boarding/alighting)} \\ N_{ip} &\sim \mathcal{N}(\mu_{ip}, \sigma_{ip}^2) \\ N_{ip} &= \text{Number of passengers at stop } i \\ W_i &= t_0 + N_{ip} \cdot \alpha \end{aligned}$$

Total Travel Time:

$$T_{\text{total}} = \frac{D - (n_p - 1)d_p}{v_m} + (n_p - 1)(t_a + t_d) + \sum_{i=1}^{n_p} W_i \qquad (3)$$

Traffic Lights - Urban Infrastructure

Synchronization Model:

- Traffic lights are synchronized when encountered consecutively
- Synchronization breaks at bus stops
- Each inter-stop segment has probability $\frac{1}{2}$ of encountering a red light

Random Variables:

$$S_i \sim \text{Bernoulli}(p = 0.5) \quad \text{(encounter red light)}$$
$$t_{tl} \sim \mathcal{U}(0, 60) \quad \text{(waiting time at red light)}$$
$$n_{tl} \sim \text{Binomial}(33, 0.5) \quad \text{(total red lights encountered)}$$

Extended Model:

$$T_{\text{lightstotal}} = T_{\text{total}} + \sum_{i=1}^{n_p} S_i \cdot t_{tl,i} \tag{4}$$

Traffic - Vehicle Interactions

Congestion Effects:

- Reduced average speed during peak hours
- Probability of encountering slower vehicles
- Speed depends on passenger density and time of day

Stochastic Speed Model:

$$
\begin{aligned}
C_i &\sim \text{Bernoulli}\left(\frac{\sum_j N_{p_j}(h)}{\max_{\hat{h}}(\sum_j N_{p_j}(\hat{h}))} \cdot \frac{N_{p_i}}{\max_j(N_{p_j}(h))}\right) \\
\hat{v}_i &\sim \mathcal{N}(\mu(h), \sigma^2) \\
v_i &= C_i \cdot \hat{v}_i + (1 - C_i) \cdot v_M \\
\mu(h) &= v_M - \gamma N_p(h)
\end{aligned}
$$

Where C_i represents encountering congestion, $\hat{v}_i$ is the reduced speed, and $\mu(h)$ is the hour-dependent expected speed.

Traffic - Vehicle Interactions

Integrating factors:

- Distance covered when accelerating
- Traffic stop time as a limit in velocity
- Factors occur independently in-between stops

Stochastic complete model:

$$T_t = (n_p - 1)(t_a + t_d) + \sum_{i=1}^{n_p} [W_i + d_i / v_i + S_i \cdot (t_{s_i} + \max(t_a + t_d - d_p / v_i, 0))] \tag{5}$$

From Stochastic to Deterministic

Transition to Deterministic Model

Motivation: Simplify complex stochastic simulation using expected values

Key Simplifications:

- **Traffic Lights Duration**: $E[t_{tl}] = 30$ seconds (uniform distribution)
- **Red Lights Encounters**: $E[n_{tl}] = 32 \times 0.5 = 16$ lights
- **Traffic Speed**: $E[v_i] = P(\text{traffic}) \cdot E[\hat{v}_i] + P(\text{no traffic}) \cdot v_M$

Expected Travel Time:

$$E[T_{\text{total}}] = (n_p - 1)(t_a + t_d) + n_p t_0 + \alpha \sum_{i=1}^{n_p} E[n_{ip}] + \sum_{i=1}^{n_p} \frac{d_i}{E[v_i]} + E[n_{tl}] \cdot E[t_{tl}] \quad (6)$$

Practical Implementation

Deterministic Simulation: Using expected values eliminates randomness while maintaining model accuracy

Implementation Benefits:

- Reproducible results
- Faster computation
- Easier parameter sensitivity analysis

Model Parameters:

$$\begin{aligned} t_0 &= 5 \text{ seconds (door opening)} \\ \alpha &= 2 \text{ seconds per passenger} \\ t_a = 2t_d &= 10 \text{ seconds (acceleration/deceleration time)} \\ E[t_{tl}] &= 30 \text{ seconds (traffic light delay)} \\ \gamma &= 1/120 \text{ m}\cdot\text{s}^{-1}\cdot\text{passenger}^{-1} \text{ (passengers coefficient)} \end{aligned}$$

Smooth Acceleration Model

Goal: Interpolate a smooth velocity profile during acceleration.

Conditions:

- Initial position: $x = 0$, with $v = 0$, $a = 0$
- Final time: $t = t_a$ with $x = x_f$, $v = v_{\max}$, $a = 0$
- Use a degree-3 polynomial for velocity:

$$v(t) = a_3 t^3 + a_2 t^2 + a_1 t + a_0$$

Why cubic?

- Enough degrees of freedom to match position, velocity, and acceleration constraints
- Ensures continuous acceleration $\rightarrow$ smoother gas supply and realistic dynamics

Results

Dwell Time Analysis

Using the dwell time formula $W_i = t_0 + n_{ip}\alpha$:

Peak Hour (17:00 - 980 passengers):

$$\begin{aligned} \text{Total dwell time} &= 34 \times 4 + 980 \times 2 \\ &= 136 + 1960 \\ &= 2096 \text{ seconds} \\ &= 34.93 \text{ minutes} \end{aligned}$$

Off-Peak Hour (23:00 - 9 passengers):

$$\begin{aligned} \text{Total dwell time} &= 34 \times 4 + 9 \times 2 \\ &= 136 + 18 \\ &= 154 \text{ seconds} \\ &= 2.57 \text{ minutes} \end{aligned}$$

Total Trip Time (one way)

Peak Hour (17:00 - 980 passengers)

Traffic	Traffic Lights	Time (min)	Increase
		53.35	+0.00
	✓	61.35	+8.00
✓		58.98	+5.63
✓	✓	66.98	+13.63

Off-Peak Hour (23:00 - 9 passengers)

Traffic	Traffic Lights	Time (min)	Increase
		21.05	+0.00
	✓	29.05	+8.00
✓		21.05	+0.00
✓	✓	29.05	+8.00

Plots - Velocity Profile at 17:00

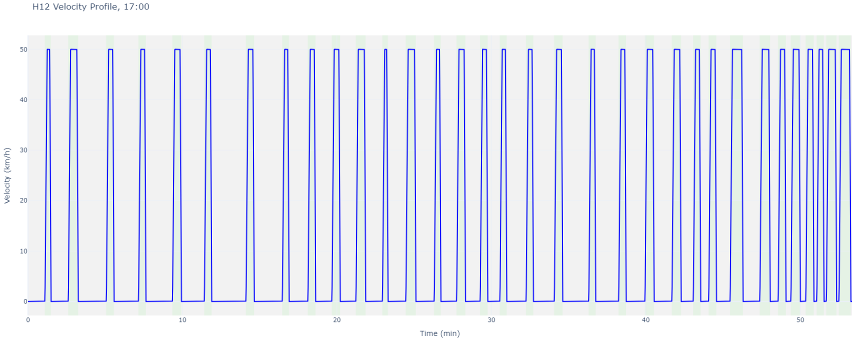

Figure: bus stops + dwell time

Plots - Velocity Profile Shape

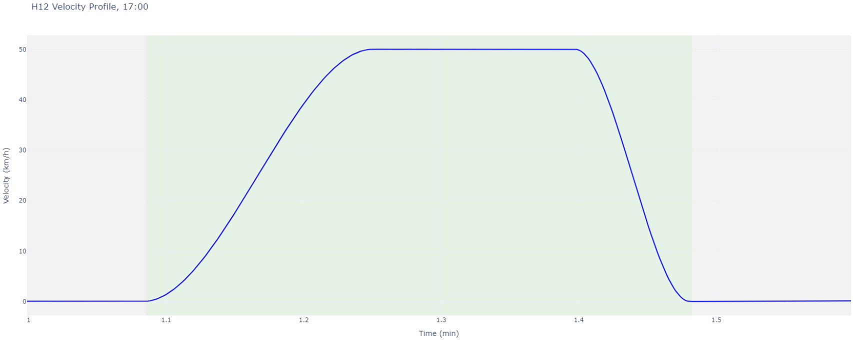

Figure: speeding up and braking

Plots - Velocity Profile at 17:00

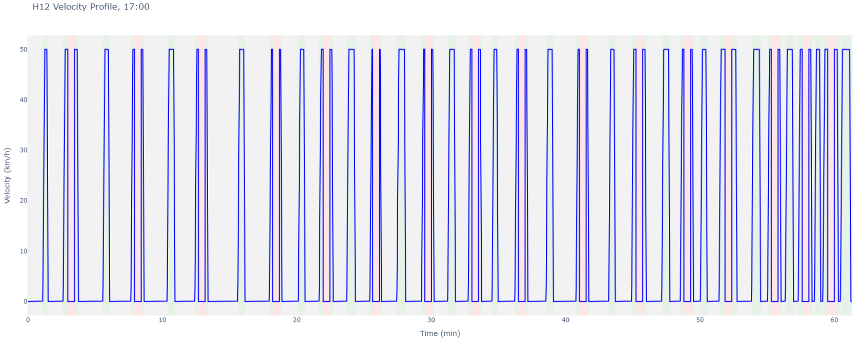

Figure: bus stops + dwell time + traffic lights

Plots - Velocity Profile Shape

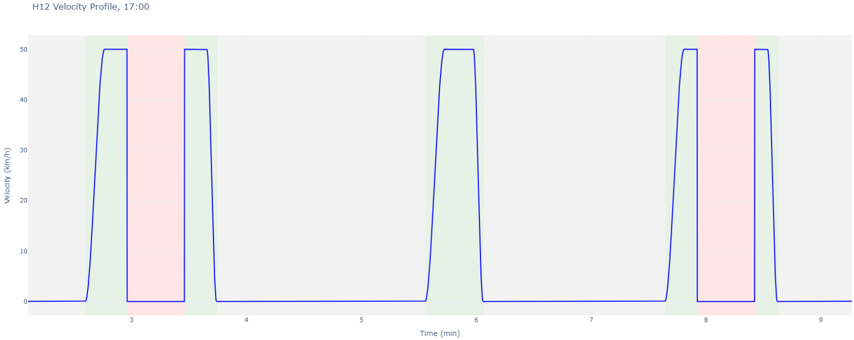

Figure: traffic lights

Plots - Velocity Profile at 17:00

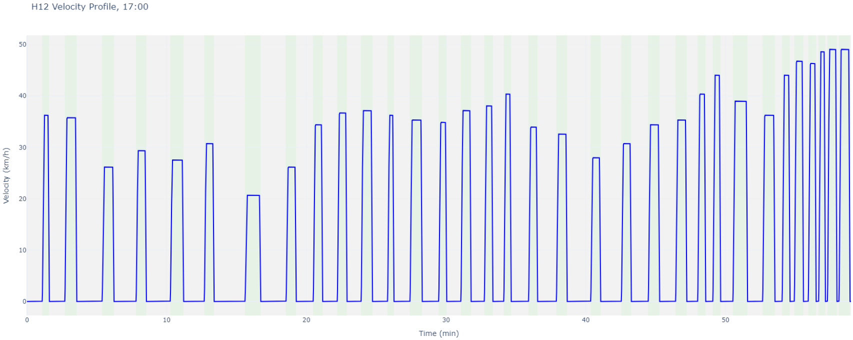

Figure: bus stops + dwell time + traffic

Plots - Velocity Profile at 17:00

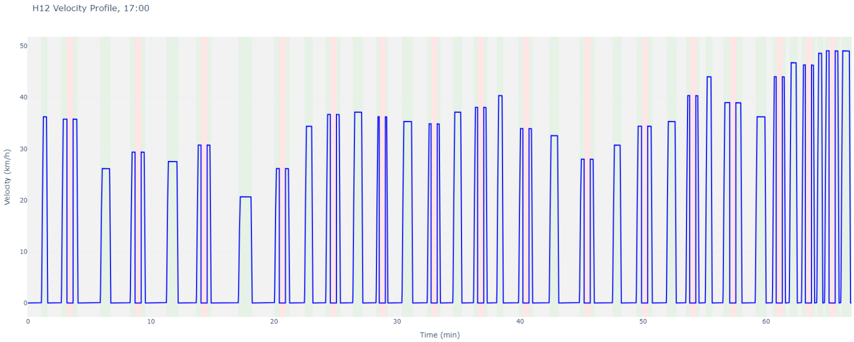

Figure: bus stops + dwell time + traffic lights + traffic

Bus Capacity and Frequency

Bus Capacity and Frequency

Given:

- N_{p_j}: known number of passengers boarding at stop j
- Drop-off delay for each passenger follows Poisson(μ)

Expected onboard passengers at stop i:

$$S_i = \sum_{j=1}^{i} N_{p_j} \cdot P(\text{Poisson}(\mu) \geq i - j) \tag{7}$$

Interpretation:

- Each term counts the expected passengers still on the bus from earlier stops
- Poisson tail gives probability that a boarding passenger hasn't gotten off yet

Bus Capacity and Frequency

Goal: Determine the number of buses required to respect onboard capacity constraints

Steps:

1. Use the onboard model to compute S_i: expected passengers on the bus at each stop
2. Find the maximum occupancy:

$$S_{\max} = \max_i S_i$$

3. Impose a load constraint (e.g., 70% of 90 seats):

$$\frac{S_{\max}}{k} < 90 \times 0.7 \Rightarrow k > \frac{S_{\max}}{63} \tag{8}$$

4. Choose minimal integer k satisfying the inequality: number of buses in circulation

Limitations and Future Research

Limitations and Future Research

Current Limitations:

- Simplified linear speed-density relationship
- Uniform passenger distribution assumptions
- Static traffic light synchronization model
- Limited validation with real-world data

Future Research Directions:

- Integration with real-time traffic data
- Dynamic route optimization
- Multi-modal transport coordination
- Machine learning for demand prediction
- Environmental impact optimization

The End

Thank you!