The Hurricane Evacuation Problem: Theoretical Bonus

Show that the single agent Hurricane Evacuation Problem (HEP) is NP-HARD.

Proof

We will show that $HAMILTON - CYCYLE \leq_p HEP$. Since $HAMILTON - CYCYLE$ is a known NP-hard problem, by the polynomial reduction below we will get that HEP is NP-hard as well.
Formally, we can denote

$HC = \{\langle V, E\rangle | G = \langle V, E\rangle \text{ is a graph that contains a Hamilton Cycle}\}$

$$HEP = \{\langle V, E, D, w, s\rangle | \langle V, E, w\rangle \text{ is a weighted graph with deadline function } D \text{ for } V, \\ \text{in which there exists a path } P, \text{starting with } s \in V, \text{that an agent can} \\ \text{follow to save all the people in the nodes}\}$$

This representation for HEP ignores details such as the notion of how many people are in each node, which nodes are shelters etc., but those are less relevant for this proof and can be easily added if necessary.

Let f be the following reduction function:
$\forall\langle x\rangle = \langle V, E\rangle, \quad f(\langle V, E\rangle) = \langle V, E, D, w, s\rangle$
Where w is a weight function defined such that: $\forall e \in E,\ w(e) = 1$.
Additionally, f sets the deadline function of all the vertices as follows: $\forall v \in V,\ D[v] = |V|$.
Finally, f sets a single arbitrary node, $s \in V$, to be a shelter node, and the rest are nodes with a positive number of people awaiting evacuation.
Let $N = |V|$.

(1) f is polynomial

f is obviously polynomial, as it simply adds weight and deadline functions to the input graph. Overall it operates in linear complexity w.r.t the input: $O(|V| + |E|) = O(|\langle x\rangle|)$

(2) $\langle x\rangle \in HC \Leftrightarrow f(\langle x\rangle) \in HEP$

a. $\langle x\rangle \in HC \Rightarrow f(\langle x\rangle) \in HEP$:
$\langle x\rangle \in HC$ implies $\langle x\rangle = \langle V, E\rangle$ represents a graph with a Hamilton cycle C.
$w.l.o.g,\ let\ C = \langle s = v_1, \dots, v_N, v_1 = s\rangle, v_i \neq v_j\ \forall i \neq j$. If an agent traverses C starting from the shelter s, it will visit each node v_i at $T_i = i \leq N = D[v_i]$, and end its journey in a shelter. In other words, it will save all the nodes, meeting their deadlines, so $f(\langle x\rangle) \in HEP$.

b. $f(\langle x\rangle) \in HEP \Rightarrow \langle x\rangle \in HC$:
$f(\langle x\rangle) \in HEP$ implies there is a path P beginning and ending in a shelter that the agent can follow to save all the people in the nodes. Since s is the only shelter node, this path must begin and end in s. Also, all the nodes have a positive number of people waiting for evacuation so the agent must visit all the nodes, so $(i) P$ visits each node at least once; N nodes in total. Since the weight function w is defined such that all the edges weigh 1 to traverse, the total cost of the journey is $N * w(e_i) = N$. It is then impossible that P passes through a node other that s more than once, as that would mean $w(P) > N > D[s]$, and P would defy the deadline of s, dooming all the carried survivors. Then (ii) P visits each node other than s at most once. By (i) and (ii), P visits each node excluding s exactly once. The journey starts and ends in $s \in V$, so in total $C = P$ is a Hamilton Cycle in $\langle V, E\rangle$. $\langle x\rangle = \langle V, E\rangle \in HC$

Then by (1) and (2), HEP is NP-hard.