2nd-Order ODE Derivation (Using Lagrangian)
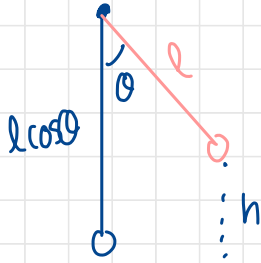
$$KE = \frac{1}{2}mv^2 = \frac{1}{2}ml^2\dot{\theta}^2$$

Arc length $s = l\theta$
$$v = \dot{s} = l\dot{\theta}$$

$U = 0$ at lowest point

$l\cos\theta$

θ

l

h

$$U = mgh = mgl(1-\cos\theta)$$

$$L = KE - U = \frac{1}{2}ml^2\dot{\theta}^2 - mgl(1-\cos\theta)$$

$$\Rightarrow \frac{1}{2}ml^2\dot{\theta}^2 + mgl\cos\theta = mgl$$

E_c

$$\frac{d}{dt}\left(\frac{\partial L}{\partial \dot{\theta}}\right) - \frac{\partial L}{\partial \theta} = 0$$

$$\frac{\partial L}{\partial \dot{\theta}} = ml^2\dot{\theta} \qquad \frac{d}{dt}\left(\frac{\partial L}{\partial \dot{\theta}}\right) = ml^2\ddot{\theta}$$

$$\frac{\partial L}{\partial \theta} = -mgl\sin\theta$$

$$\Rightarrow ml^2\ddot{\theta} + mgl\sin\theta = 0$$

$$\ddot{\theta} + \frac{g}{l}\sin\theta = 0 \qquad (*) \qquad\qquad (Eq1)$$

Deriving $\Theta(t)$ for large angles

Boundary Conditions: $\qquad \Theta(0) = \Theta_0$
$$\dot\Theta(0) = 0$$

$$W_0 = \sqrt{\frac{g}{\ell}} \quad , \quad T_0 = \frac{2\pi}{W} \quad \text{(SHM)}$$

Multiply (✱) by $\dot\Theta$:

$$\dot\Theta\ddot\Theta + W_0^2 \sin\Theta \; \dot\Theta = 0$$

$$\frac{d}{dt}\left[\frac{1}{2}\left(\frac{d\Theta}{dt}\right)^2 - W_0^2 \cos\Theta\right] = 0 \qquad \text{(conservation of mechanical energy)}$$

Integrate wrt

$$\int \frac{d}{dt}\left(\frac{1}{2}\dot\Theta^2 - W_0^2\cos\Theta\right)dt = \int 0 \, dt$$

$$\frac{1}{2}\dot\Theta^2 - W_0^2\cos\Theta = C \qquad \text{(law of conservation of Energy)}$$

$\times m\ell^2$

$$\Rightarrow \quad \underbrace{\frac{1}{2}m\ell^2\dot\Theta^2}_{T} - \underbrace{mg\ell\cos\Theta}_{V} = E_C$$

BC $\Rightarrow$ $\frac{1}{2}(0)^2 - W_0^2 \cos\Theta_0 = C$ $\Rightarrow$ $C = -\frac{g}{\ell}\cos\Theta_0$

$\Rightarrow$ Energy eqn

$$\frac{1}{2}\dot{\Theta}^2 - W_0^2 \cos\Theta = -W_0^2 \cos\Theta_0$$

$$\frac{d\Theta}{dt} = \dot{\Theta} = \sqrt{2W_0^2(\cos\Theta - \cos\Theta_0)}$$

using: $\cos\Theta = 1 - 2\sin^2\left(\frac{\Theta}{2}\right)$

$$\Rightarrow \frac{d\Theta}{dt} = \sqrt{4W_0^2\left(\sin^2\left(\frac{\Theta_0}{2}\right) - \sin^2\left(\frac{\Theta}{2}\right)\right)} \qquad (**)$$

let $y = \sin(\Theta/2)$ and $x = \sin^2(\Theta_0/2)$

$$\Rightarrow y(0) = \sqrt{x}$$

$$\frac{dy}{dt} = \frac{1}{2}\frac{d\Theta}{dt}\cos(\Theta/2)$$

$$\left(\frac{dy}{dt}\right)^2 = \frac{1}{4}\left[1 - \sin^2(\Theta/2)\right]\left(\frac{d\Theta}{dt}\right)^2 = \frac{1}{4}(1-y^2)\left(\frac{d\Theta}{dt}\right)^2$$

Rearrange $\left(\frac{d\Theta}{dt}\right)^2 = \frac{4}{1-y^2}\left(\frac{dy}{dt}\right)^2$

Sub y, x and $\left(\dfrac{d\theta}{dt}\right)^2$ into (**)

$$\frac{4}{1-y^2}\left(\frac{dy}{dt}\right)^2 = 4w_0^2\,(x-y^2)$$

$$\Rightarrow \left(\frac{dy}{dt}\right)^2 = w_0^2\, x\,(1-y^2)\left(1-\frac{y^2}{x}\right)$$

New variables: $\tau = w_0 t \qquad z = \dfrac{y}{\sqrt{x}}$

$$\left(\frac{dz}{dt}\right)^2 = (1-z^2)(1-xz^2)$$

where $\quad 0 < x < 1 \quad$ and $\quad z(0)=1$

$$\left(\frac{dz}{d\tau}\right)_{\tau=0} = 0$$

$\therefore \quad d\tau = \pm \dfrac{dz}{\sqrt{(1-z^2)(1-xz^2)}}$

the time t to go from $(1,0)$ to $(z, \, dz/dt)$ in the lower half plane of graph $dz/d\tau$ as a func of z is:

$$\tau = -\int_1^2 \frac{d\xi}{\sqrt{(1-\xi^2)(1-x\xi)^2}}$$

Obtain τ as a func of z and x

$$\tau(z) = K(x) - F(\arcsin z\,;\,x)$$

where $K(m)$ and $F(\varphi; m)$ are the complete and incomplete elliptical integral of the 1st kind, defined as:

$$K(m) = \int_0^1 \frac{dz}{\sqrt{(1-z^2)(1-mz^2)}}$$

$$F(\varphi; m) = \int_0^\varphi \frac{dz}{\sqrt{(1-z^2)(1-mz^2)}} \qquad \text{and} \quad z = \sin\phi$$

Period $T = 4 \times \frac{1}{4}$ cycle   where $\frac{1}{4}$ cycle is $\Theta = \Theta_0 \to \Theta = 0$

$$T = 4t(0) = \frac{4\,\tau(0)}{\omega_0} = \frac{4}{\omega_0} K(\varkappa) = \frac{2}{\pi} T_0 K(\varkappa)$$

$\Rightarrow$ $F(\arcsin z; \varkappa) = K(\varkappa) - \tau$

$\qquad z = sn(K(\varkappa) - \tau; \varkappa)$ Jacobi elliptical func

$z = \sin\left(\frac{\Theta}{2}\right)$, $\tau = \omega_0 t$

$\Rightarrow$ $\Theta(t) = 2\arcsin\left(\sin\frac{\Theta_0}{2} sn\left[K(\varkappa) - \omega_0 t); \varkappa\right]\right)$ (Eq 4)

Time Period T:

from (**)

$$dt = \frac{1}{2w_0} \left[\sin^2 \frac{\Theta_0}{2} - \sin^2 \frac{\Theta}{2} \right]^{-1/2}$$

$T = 4 \times \frac{1}{4}$ cycle

$$\frac{T}{4} = \frac{1}{2w_0} \int_0^{\Theta_0} \left[\sin^2 \frac{\Theta_0}{2} - \sin^2 \frac{\Theta}{2} \right]^{-1/2} d\Theta$$

$$\Rightarrow T = \frac{1}{w_0} \int_0^{\Theta_0} \left[\sin^2 \frac{\Theta_0}{2} - \sin^2 \frac{\Theta}{2} \right]^{-1/2} d\Theta \qquad (\text{fg 5})$$

Let $0 \leqslant \varphi \leqslant 2\pi$ be a variable that changes during one full oscillation of pendulum

Define φ s.t $\quad \sin \varphi = \dfrac{\sin \Theta/2}{\sin \Theta_0/2} \quad$ and let $k = \sin \dfrac{\Theta_0}{2}$

$$\frac{T}{4} = \frac{1}{2w_0} \int_0^{\Theta_0} \left[\sin^2 \frac{\Theta_0}{2} - \sin^2 \frac{\Theta}{2} \right]^{-1/2} d\Theta$$

$$\Rightarrow \frac{T}{4} = \frac{1}{w_0} \int_0^{\pi/2} \left[1 - k^2 \sin^2 \varphi \right]^{-1/2} d\varphi$$

$$\therefore T = \frac{2 T_0}{\pi} \int_0^{\pi/2} \left[1 - k^2 \sin^2 \varphi \right]^{-1/2} d\varphi$$

As $\theta \to 0$ $(\varphi \to 0)$ $\Rightarrow$ $T \to T_0$ $\left(\dfrac{T}{T_0} \to 1\right)$

$\theta_0 > 0$ $\Rightarrow$ $\ast > 0$ $\Rightarrow$ $T \geqslant T_0$

for $k < 1$ can do binomial

$$(1-x)^{-1/2} = \sum_{n=0}^{\infty} \frac{(2n)!}{4^n (n!)^2} x^n$$

$$(1-k^2 \sin^2 \varphi)^{-1/2} = \sum_{n=0}^{\infty} \frac{(2n)!}{4^n (n!)^2} k^{2n} \sin^{2n} \varphi$$

$$\Rightarrow \frac{T}{T_0} = \frac{2}{\pi} \sum_{n=0}^{\infty} \frac{(2n)!}{4^n (n!)^2} k^{2n} \int_0^{\pi/2} \sin^{2n} \varphi \, d\varphi$$

$$\frac{(2n-1)!!}{(2n)!!} \cdot \frac{\pi}{2} = \frac{(2n)!}{2^{2n}(n!)^2} \cdot \frac{\pi}{2}$$

$$\Rightarrow \frac{T}{T_0} = \sum_{n=0}^{\infty} \frac{[(2n)!]^2}{2^{4n}(n!)^4} k^{2n}$$

$\ast\alpha\ast$

$n=0$ $\ast\ast\ast$ $= \dfrac{1}{1} = 1$

$n=1$ $\ast\ast\ast$ $= \dfrac{(2!)^2}{2^4 (1!)^4} = \dfrac{4}{16} = \dfrac{1}{4}$

$n=2$ $\ast\ast\ast$ $= \dfrac{(4!)^2}{2^8 (2!)^4} = \dfrac{576}{4096} = \dfrac{9}{64}$

$n=3$ $\ast\ast\ast$ $= \dfrac{(6!)^2}{2^{12}(3!)^4} = \dfrac{25}{256}$

$$\frac{T}{T_0} = 1 + \frac{1}{4}k^2 + \frac{9}{64}k^4 + \frac{25}{256}k^6 + \dots$$

amplitude corrections to the approx of SHM

$$K = \sin\frac{\theta_0}{2}$$

$$= \frac{\theta_0}{2} - \frac{\theta_0^3}{48} + \frac{\theta_0^5}{3840} - \dots$$

$$k^2 = \left(\frac{\theta_0}{2} - \frac{\theta_0^3}{48} + \dots\right)^2 \qquad \text{Ignore higher order - don't contribute as much}$$

$$= \frac{\theta_0^2}{4} - \frac{\theta_0^4}{48} + O(\theta_0^6)$$

$$k^4 = \left(\frac{\theta_0^2}{4} - \frac{\theta_0^4}{48}\right)^2$$

$$= \frac{\theta_0^4}{16} + O(\theta_0^6)$$

$$\Rightarrow \frac{T}{T_0} = 1 + \frac{1}{4}\left(\frac{\theta_0^2}{4} - \frac{\theta_0^4}{48}\right) + \frac{9}{64}\left(\frac{\theta_0^4}{16}\right)$$

$$= 1 + \frac{1}{16}\theta_0^2 - \frac{\theta_0^4}{192} + \frac{9}{1024}\theta_0^4 + \dots$$

$$= 1 + \frac{1}{16}\theta_0^2 + \frac{11}{3072}\theta_0^4 + \dots \qquad (\text{Eq } 6)$$