free precession experiment :

initial conditions: $M_x(0) = M_0$, $M_y(0) = 0$, $M_z(0) = 0$
The equations for M_x and M_y are coupled, ignoring relaxation T, the equations describe a circle, adding the $-\frac{1}{T}$ term causes the radius of that circle to shrink exponentially over time.

$$\frac{d\vec{M}}{dt} = -\gamma(\vec{B} \times \vec{M}) - \frac{\vec{M} - \vec{M}_0}{T} \qquad \text{(eqn 3)}$$

magnetic field : $\vec{B} = (0, 0, B_0)$
equilibrium magnetisation : $\vec{M}_0 = (0, 0, M_0)$

the cross product $-\gamma \vec{B} \times \vec{M}$:

$$-\gamma \begin{vmatrix} \hat{i} & \hat{j} & \hat{k} \\ 0 & 0 & B_0 \\ M_x & M_y & M_z \end{vmatrix} = -\gamma\left(-B_0 M_y \underline{\hat{i}} + B_0 M_x \hat{j} + 0 \underline{k}\right)$$

$\therefore$ forms 3 differential equations :

$$\frac{dM_x}{dt} = \gamma B_0 M_y - \frac{M_x}{T}$$

$$\frac{dM_y}{dt} = -\gamma B_0 M_x - \frac{M_y}{T}$$

$$\frac{dM_z}{dt} = -\frac{M_z - M_0}{T}$$

larmor equation: $\omega_0 = \gamma B_0$

$$\Rightarrow \frac{dM_x}{dt} = \omega_0 M_y - \frac{M_x}{T}$$

$$\frac{dM_y}{dt} = -\omega_0 M_x - \frac{M_y}{T}$$

define a new complex function: $M_{xy}(t) = M_x(t) + i M_y(t)$

$$\frac{dM_{xy}}{dt} = \frac{dM_x}{dt} + i\frac{dM_y}{dt}$$

sub $\frac{dM_x}{dt}$ and $\frac{dM_y}{dt}$ in

$$\frac{dM_{xy}}{dt} = \left(\omega_0 M_y - \frac{M_x}{T}\right) + i\left(-\omega_0 M_x - \frac{M_y}{T}\right)$$

$$= -\frac{1}{T}\left(M_x + iM_y\right) + \omega_0\left(M_y - iM_x\right)$$

$(M_y - iM_x)$ is same as $-i(M_x + iM_y)$ because $-i \cdot i = 1$

$$\Rightarrow \frac{dM_{xy}}{dt} = \left(-\frac{1}{T} - i\omega_0\right) M_{xy}$$

standard 1st order eqn of form $y' = ky$

$$M_{xy}(t) = M_{xy}(0)\exp\left(-\left(\frac{1}{T} + i\omega_0\right)t\right)$$

as initial condition: $M(0) = (M_0, 0, 0)$

$$M_{xy}(0) = M_0 + i(0) = M_0$$

$$M_{xy}(t) = M_0 e^{-\frac{t}{T}} e^{-i\omega_0 t}$$

using euler's: $e^{-i\theta} = \cos\theta - i\sin\theta$

$$M_x + iM_y = M_0 e^{-\frac{t}{T}}\left(\cos(\omega_0 t) - i\sin(\omega_0 t)\right)$$

comparing real and imaginary parts:

$$M_x(t) = M_0 e^{-\frac{t}{T}}\cos(\omega_0 t) \qquad \text{(real)}$$

$$M_y(t) = -M_0 e^{-\frac{t}{T}}\sin(\omega_0 t) \qquad \text{(imaginary)}$$

M_z equation is independent of others $\qquad \frac{dM_z}{dt} = \frac{M_0 - M_z}{T}$

linear, non-homogeneous eqn, general sol:

$$M_z(t) = M_0 + Ce^{-t/T}$$

for $M(0) = (M_0, 0, 0)$

$M_z = 0 \quad \Rightarrow \quad 0 = M_0 + C \quad \Rightarrow \quad C = -M_0$

$$M_z(t) = M_0 (1 - e^{-\frac{t}{T}})$$

<u>Inverse recovery experiment</u>

$M(0) = (0, 0, -M_0)$

$M_z = -M_0 \quad \Rightarrow \quad -M_0 = M_0 + C \quad \Rightarrow \quad C = -2M_0$

$$M_z(t) = M_0 (1 - 2e^{-\frac{t}{T}})$$

Since initial values of $M_x(0)$ and $M_y(0)$ are zero and there is no driving force, the cross product of 2 parallel vectors is zero, stays zero for all time.

$$\Rightarrow \quad M_x(t) = 0 \quad \text{and} \quad M_y(t) = 0$$