Modeling Volatility Derivatives

The Mathematical Sciences Research Institute Undergraduate Program

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1. Financial Mathematics
2. Volatility Index (VIX) & VIX Derivatives
3. Futures

Abstract

The Chicago Board Options Exchange introduced the Market's Volatility Index (VIX) in 1993. The VIX is a financial instrument used to measure market volatility. The VIX cannot be directly traded; many investors invest in VIX derivatives to hedge against risk. The VIX has been a popular topic of finance and mathematical finance since its creation. Little is known about the behavior of Volatility Index derivatives. We explored a couple of stochastic processes to model these derivatives. We discovered a mean regressing behavior of VIX derivatives and used the Ornstein-Uhlenbeck process to model these derivatives. We used a couple methods to calculate the parameters for the Ornstein Method. We then compared the Ornstein-Uhlenbeck model using different parameter calculation methods and Geometric Brownian Motion, to historic VIX derivatives data. Preliminary results indicate that the Ornstein-Uhlenbeck model with parameters calculated using the Least Squares method most accurately modeled VIX derivatives.

We have shown it is possible to model VIX derivatives with reasonable accuracy in the short term. The results developed here can potentially aid investors interested in using volatility derivatives. Investors may be able to hedge against risk in unpredictable markets more effectively.

Acknowledgements

We would like to thank the National Science Foundation (grant No. DMS-0754872) and the National Security Agency (grant No. H98230-11-1-0213) for the support of the MSRI-UP program.

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Chapter 1

Background

1.1 Introduction

Addressing uncertainty and risk about the stock market is one of the oldest and greatest challenges investors face. Investors strategically hedging against risk must estimate the degree of risk in a position. In mathematical finance, risk and uncertainty is measured by volatility. The VIX is a financial instrument used to measure market volatility. Robert Whaley and the Chicago Board Options Exchange introduced the Market's Volatility Index (VIX) in 1993. The VIX cannot be directly traded, but many investors invest in VIX derivatives to hedge against risk. The VIX has been a popular topic of finance and mathematical finance since its creation, but little is known about the behavior of Volatility Index derivatives.

1.2 Implied Volatility

Implied volatility (IV) is a measure of risk and uncertainty of an underlying asset. IV reflects the magnitude of fluctuations of the option's price and not the direction of the movement in price. In finance and financial mathematics, IV is the volatility of an option on an asset implied by market participants. IV is forward looking and is an estimate of the stock's volatility in the next 30 days and is interpreted as markets expectation of future volatility. A high IV implies that over the next 30 days market participants expect the stock to move drastically in either direction. The Black-Scholes model is a common model for computing the price of an option. It uses several variables including volatility. IV treats the volatility variable as an unknown and works backward solving for volatility using the market price of the option and the Black Scholes model. Therefore, the factors that affect IV are the exercise price, maturity date, risk free rate and the price of the option.

1.3 The Volatility Index (VIX)

Knowing the implied volatility of the market, σ, is helpful to investors developing strategies to hedge against risk. The Market Volatility Index (VIX) was introduced in 1993 by Robert Whaley as a resource to measure minute to minute implied volatility of the market. By using the current prices of Standard & Poor's 500 Index options (SPX), the VIX represents the expected volatility of the market over the next 30 days.

1.3.1 How to calculate the VIX

The VIX is a weighted average of the implied volatility of the near term SPX options, σ_1^2, and the next term SPX options, σ_2^2. [1]The forward index level, F, is calculated by identifying the strike price with the smallest absolute difference between its call and put prices.

Only out-of-the-money options are used to calculate σ_1^2 and σ_2^2. Put options immediately lower than K_0 and between the first set of consecutive put options with bid prices of zero are selected. Call options immediately greater than K_0 and between the first set of consecutive call options with bid prices of zero are selected.

Depending on how far expiration the near- and next-term options are the relationship between σ_1^2 and σ_2^2 change. The equation below mathematically expresses the weighted average. When the near term options have less than 30 days until expiration and the next term options have more than 30 days until expiration each of their individual weights are less than or equal to 1, but their sum equals 1. Whenever both the near and next term options have more than 30 days until expiration the near term weight is greater than 1 and the next term weight is negative.

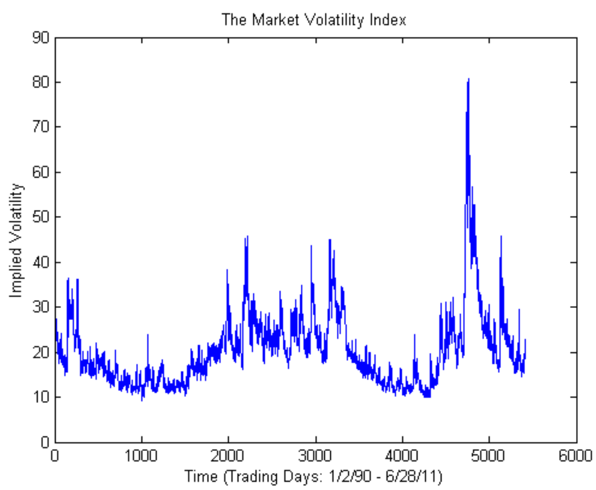


Figure 1.1: VIX values: 1/2/1990 - 6/28/11

[1]This formula was cited from the Chicago Board Options Exchange "The CBOE's Volatility Index - VIX."

1.4 VIX Derivatives

The VIX has two popular derivatives that are traded in the open market, the VXX and VXZ. The VXX is a derivative on the VIX that looks forward 1 month. The VXX is derived using 1 month and 2 month futures contracts (defined below) on the VIX and computed using a weighted average of these contracts. The VXZ, on the other hand, is a derivative that looks forward 5 months. The VXZ is derived using 4, 5, 6, and 7 month futures contracts on the VIX and is computed using a weighted average of these contracts. These derivatives can be traded in the market like any other security. Although these derivatives are derived from futures contracts that have maturity dates, these derivatives do not expire because, similar to the VIX, the contracts roll on to the next month as expiration approaches.

1.4.1 Futures

A Futures Contract is a contract between two parties who agree to exchange an asset at an agreed upon time in the future at an agreed upon price. Since there is an obligation to deliver the asset, future traders must deposit capital into margin accounts. By depositing capital into these accounts, as prices of these contracts fluctuate the difference is charged to the margin account. The ticker symbol for the VIX futures contracts is VX. The contract size is $1000 times the VIX and contracts are available for all 12 months of the year and expire on the third wednesday of the month. The VX has a minimum tick size of 0.01, which means that for each tick on the VX the margin accounts are credited or charged $10 per contract. For instance, if the VIX level is $22.5 then VX contract level would be $$22,500$. In the futures market, contango and backwardation are terms use to describe conditions that are going on in the futures market. The market is said to be in contango when the futures price of the commodity is higher than the expected spot price. The market is said to be in backwardation when the futures price of the commodity is lower than the expected spot price. When the futures market is in contango, as the contract approaches the settlement date we can expect the value of the contract to decline until the contract equals the expected spot price. When the futures market is in backwardation, as the contract approaches the settlement date we can expect the value of the contract to increase until the contract equals the expected spot price. Regardless of whether the market is in contango or backwardation, the expected futures spot price should converge to the futures price at settlement.

1.4.2 Options

An option is a financial contract between two parties; the writer of the contract and the holder of the contract. Options allow the holder of the contract the right, but not the obligation, to follow through with the contact. There are two different categories of options; call options and put options. A call option gives an investor the right to buy the underlying asset at an agreed upon price at an agreed upon date. A put option, on the other hand, gives an investor the right to sell a specified amount at an agreed upon price at an agreed upon date. The two main types of options are American options and European options. American options can be exercised at any time prior to expiration. European options can only be exercised at expiration. The contract size for an option on the VIX is \$100 times the actual VIX value. The VIX options are European options since they can only be exercised at expiration. VIX options expire on the third Wednesday of the month and are cash settled. Since the VIX can not be bought or sold, option holders on the VIX exchange cash at expiry, hence the term cash settled. Since the VIX is mean reverting, at times when the VIX spikes call options on the VIX seem to trade at a discount. The reason for this is that in a mean reverting processes when an asset is above its long run mean, the probability of the asset going down is higher than the probability of it going up. Hence, call options trade at a discount to account for this. If the VIX, on the other hand, declines substantially from its long run mean, put options on the VIX trade at a discount because the probability of the asset increasing is higher than the probability of it decreasing.

1.4.3 How to calculate VIX derivatives

The Short-Term VIX Futures Contract Index (VXX) and the VIX Mid-Term Futures Contract Index (VXZ) are based on previous VIX Futures. They are found using the equation:

$$IndexTR_t = IndexTR_{t-1} * (1 + CDR_t + TBR_t) \tag{1.1}$$

where

- $IndexTR_{t-1}$ is the IndexTR (index total return for VXX and VXZ) for the previous business day and is defined whenever the Index is calculated
- CDR_t is the Contract Daily Return
- TBR_t is the Treasury Bill Return

In short, the Short-Term VIX Futures Contract and Mid-Term Futures Contract are derived based on the index in previous days, a summation of the weighted daily contract reference prices and the treasury bill return rates.

1.4.4 Exchange Traded Funds & Exchange Traded Notes

A security that tracks an index that can be traded in the open market and trades just like a stock is called an Exchange Traded Fund (ETF). An Exchange Traded Note (ETN) is an unsecured debt security, much like a bond, in which returns are based on the market return of an index yet can be traded just like a stock. The major difference between ETF's and ETN's is that ETF's are subject to market risk while EFN's are subject to market risk and the risk of default by the issuing bank. While this is a disadvantage, ETN's provide hard to reach markets such as commodities and volatility index's that ETF's can not. The VIX derivatives are ETN's. These ETN's are based on holding a long position in futures on the VIX index. An increase in the VIX is not sufficient for an increase on the VIX derivatives because an increase in the VIX will not necessarily cause an increase in the VIX futures. As stated previously, the VIX is derived using mathematical equations and cannot be traded. The prices on the VIX futures are based on these mathematical calculations. The ETN's on the VIX are linked to its performance. SInce ETN's are unsecured debt security, they are not rated and are backed buy the credit of the underwriting bank. The underwriting bank that provides ETN's on the VIX is Barclays Capital. The value of an ETN depends on the performance of the index which it tracks and the credit rating of the issuing bank.

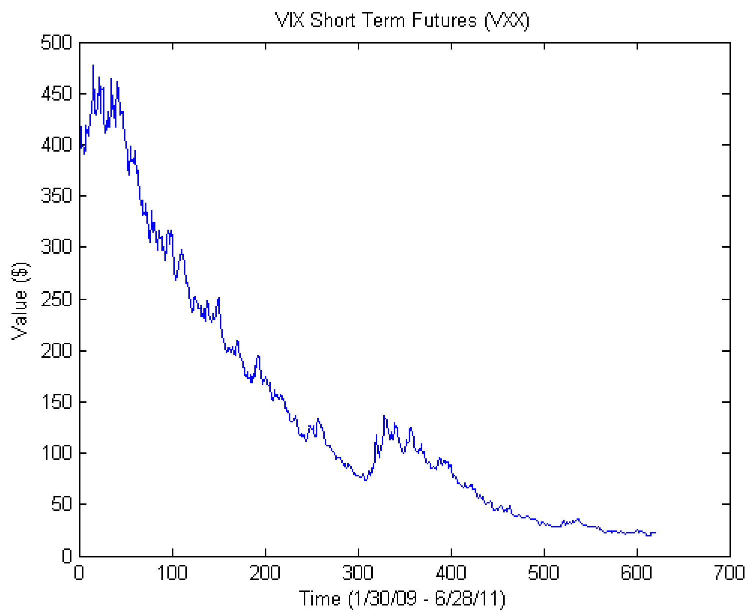


Figure 1.2: VXX values: 1/29/09 - 6/28/11

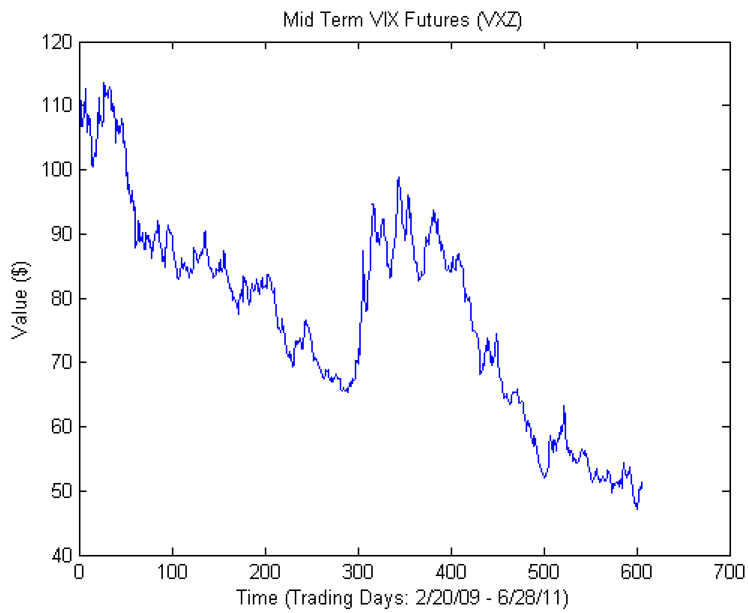


Figure 1.3: VXZ values: 2/20/09 - 6/28/11

Chapter 2

Observations of VIX Derivatives

The VIX follows a mean reverting process. Therefore whenever the VIX is higher than its long run mean, it typically reverts back to its long run mean, and, when it is lower than the long run mean, it typically reverts upward toward its long run mean. This behavior is mathematically expressed in the equations below.

$$\mu \leq E[X_2|X_1 = c] < c \tag{2.1}$$

$$c < E[X_2|X_1 = c] \leq \mu \tag{2.2}$$

It is natural to suspect that VIX derivatives share the same mean reverting characteristics as the VIX. The VIX and VXX have both fallen drastically since January 29, 2010, the inception date of the and VXX, as though returning to a particular level. The VXZ, which was introduced February 20, 2010, has exhibited the same behavior.

2.1 Stochastic Process

A stochastic process is a collection of random variable with respect to the probability space, (Ω, γ, ρ), usually indexed by time.

2.2 Daily Rate of Change

We observed many characteristics of VIX derivatives. We observed a general decrease in the daily rate of change of the VXX from its inception date. The daily rate of change strongly implied mean reversion as if the further away from a long run mean the faster the derivatives return to it. The decay of daily rate of change can be seen in the figures below.

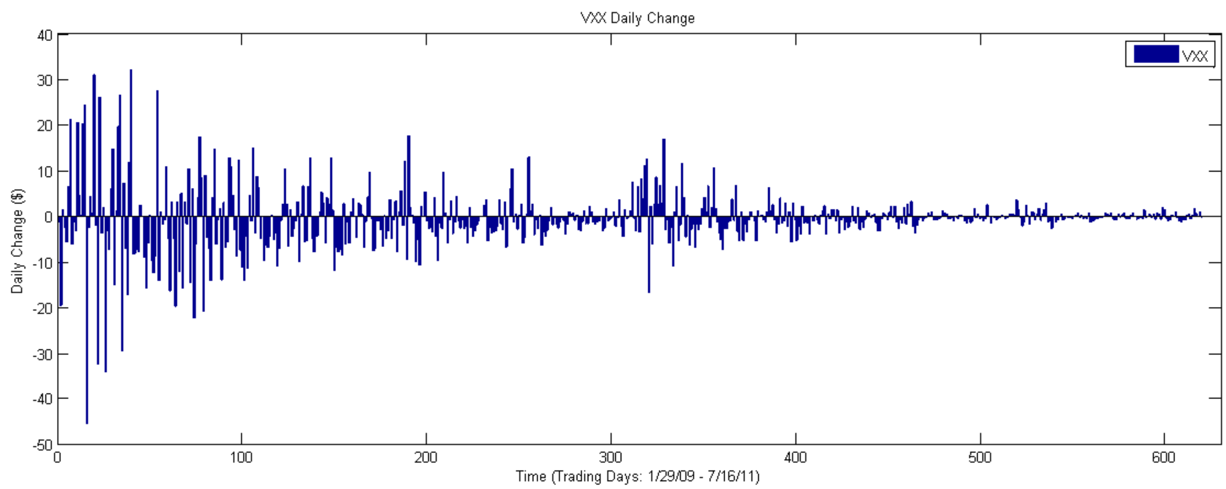


Figure 2.1: Daily Change of VXX

Chapter 3

Geometric Brownian Motion

3.1 Introduction to Geometric Brownian Motion

The Geometric Brownian Motion (GBM) is a continuous time stochastic process. In the model r is the drift parameter, σ is the volatility, W(t) is the wiener process and S(0) is the initial value.

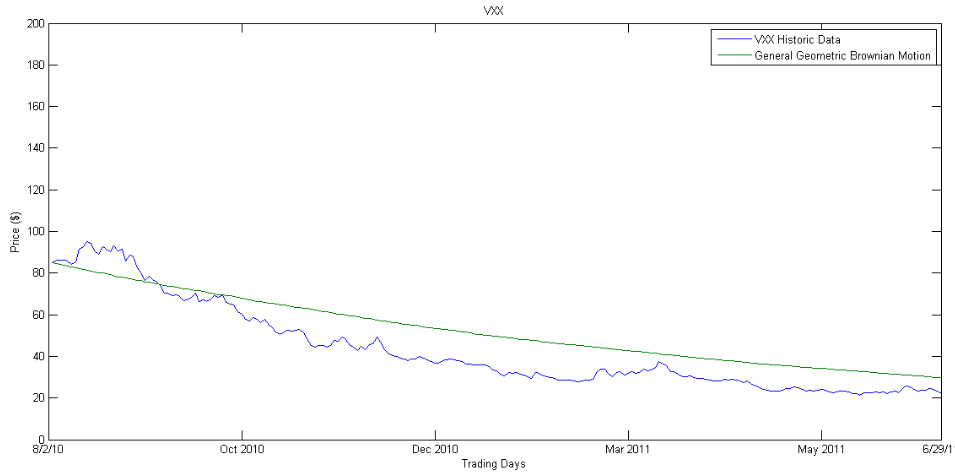
For $0 < t \leq T$:

$$S(t) = S(0)e^{(r-\frac{1}{2}\sigma^2)t+\sigma W(t)}$$

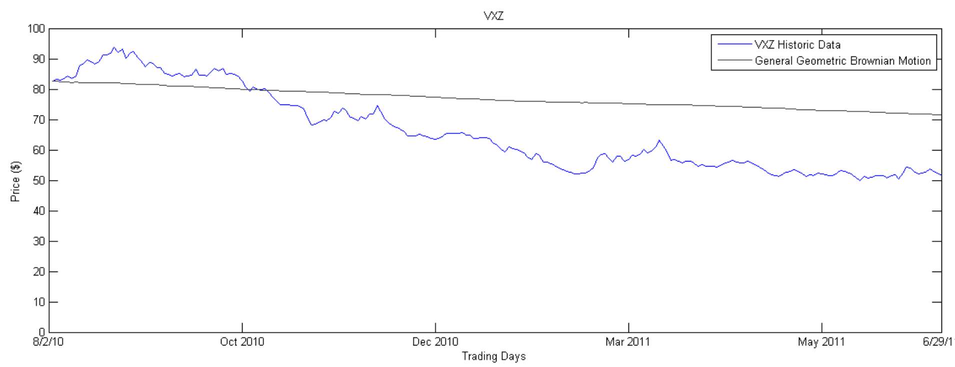
In financial mathematics, GBM is commonly used to model stock prices. This is mainly because the expected return on GBM are independently identically distributed as stock is suspected to be. In addition, GBM will never go below zero and show a similar jaggedness much like stocks. However, the general GBM model falls short when modeling the VIX since general GBM models assume a constant volatility and drift. In reality, the VIX volatility and drift change continually. In our attempt to model the VIX, we changed the volatility and drift to be a function that is continually updated.

In our model we used what we call a moving average. We took data from t-25 through day t to calculate the expected volatility by the GBM model at day t+1. The parameters, σ and r, where continually updated as we moved from day t to day t+1. The general GBM refers to a GBM with none updating parameters.

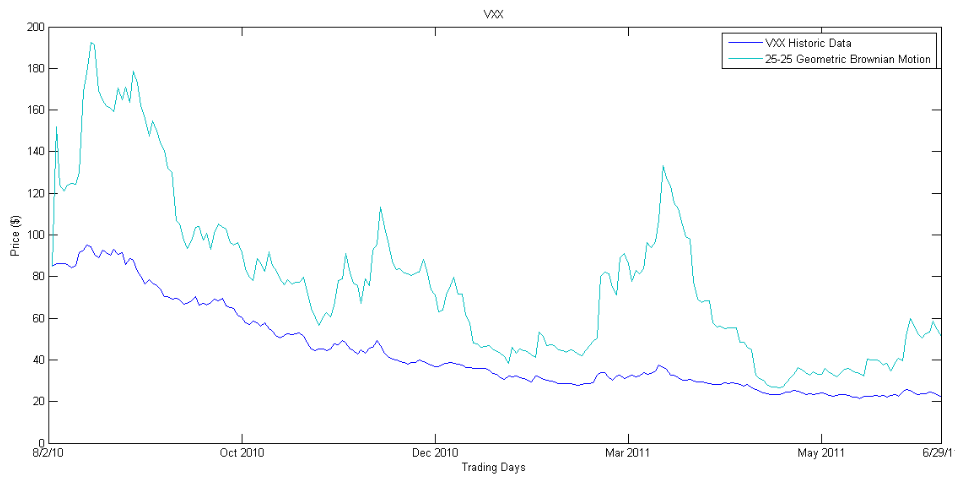
The plot below shows a General Geometric Brownian Model. The data used to calculate the parameters ranged from January 30th 2009 to July 30th 2010. The green path shows the Monte Carlo path of 1000 GBM paths. The blue line shows the VXX historical data from August 2^{nd}, 2010 to June 29^{th}, 2011.



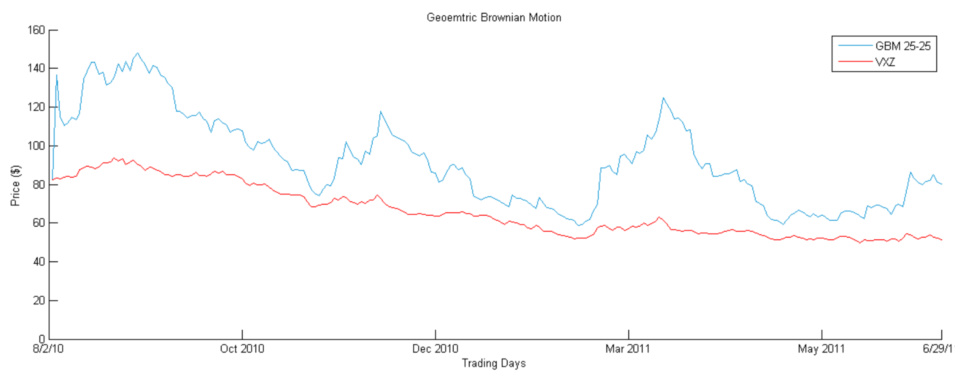
The plot below shows a General Geometric Brownian Model. The data used to calculate the parameters ranged from February 20th 2009 to July 30th 2010. The black path shows the Monte Carlo path of 1000 GBM paths. The blue line shows the VXZ historical data from August 2^{nd}, 2010 to June 29^{th}, 2011.



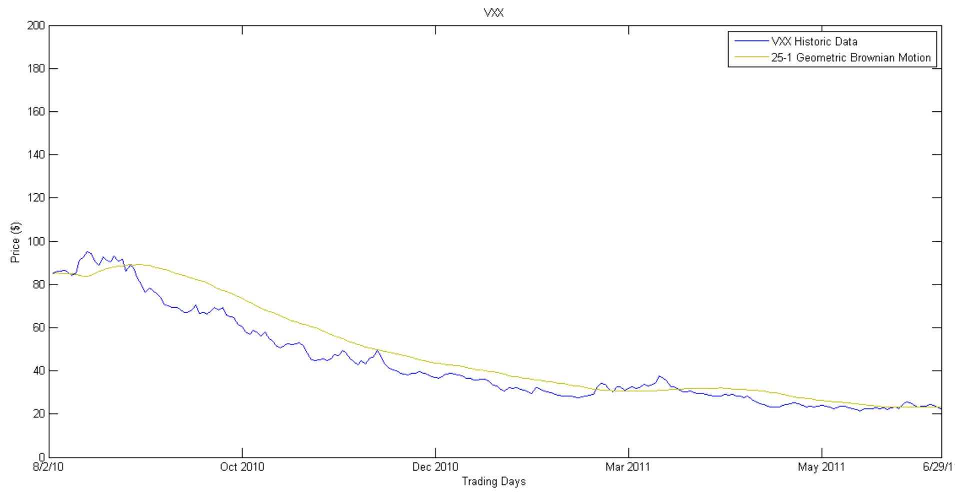
The plot below shows a Geometric Brownian Model looking back 25 days and modeling 25 day's ahead. The light blue path shows the Monte Carlo path of 1000 GBM paths. The blue path shows the VXX historical data from August 2^{nd}, 2010 to June 29^{th}, 2011.



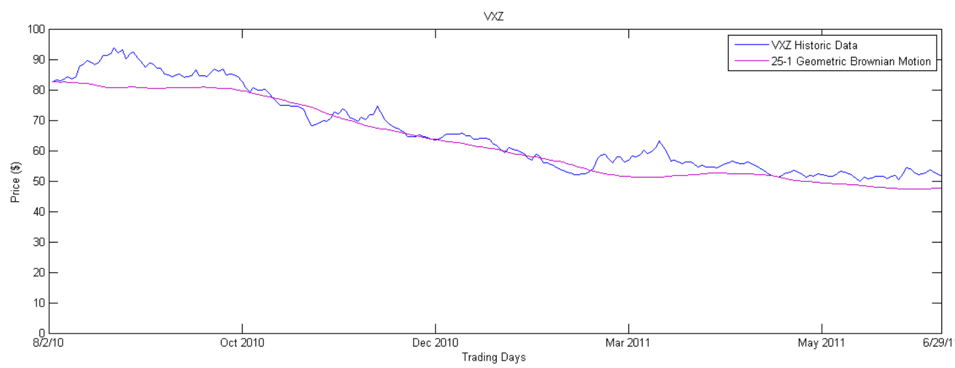
The plot below shows a Geometric Brownian Model looking back 25 days and modeling 25 day's ahead. The light blue path shows the Monte Carlo path of 1000 GBM paths. The red path shows the VXZ historical data from August 2^{nd}, 2010 to June 29^{th}, 2011.



The plot below shows a Geometric Brownian Model looking back 25 days and modeling 1 day ahead. The yellow path shows the Monte Carlo path of 1000 GBM paths. The blue line shows the VIX from August 2^{nd}, 2010 to June 29^{th}, 2011. Although the GBM had minor drifts to the downside around the beginning, towards the end the GBM seems to have done a good job modeling the VXX. Since the GBM is continually updated with new information and only looking forward 1 day, it made sense for the GBM to model the VXX well.



The plot below shows a Geometric Brownian Model looking back 25 days and modeling 1 day ahead. The purple path shows the Monte Carlo path of 1000 GBM paths. The blue path shows the VXZ historical data from August 2^{nd}, 2010 to June 29^{th}, 2011.



Chapter 4

Mean Regression

4.1 Introduction to Mean Regression

Geometric Brownian Motion is based on the belief that a certain event has a tendency to either increase or decrease depending on the drift. Since the VIX derivatives are based on the VIX which is constantly changing, we know that these derivatives should not continuously grow to infinity or decrease to negative infinity. Therefore another process had to be devised that would better reflect the behavior of the derivatives; mean regression is that process. Mean regression is a procedure that is based on the notion that a certain data set has a ”long-term mean”. The process is then based on the fact that the data will always try to revert back to this long run mean. In other words the data is more likely to move towards the mean. As a result, large moments of increasing and decreasing will only be temporary. Mean regression is determined by the following equation:

$$S_{t+1} - S_t = \theta\left(\mu - S_t\right) + \sigma S_t \tag{4.1}$$

where

- μ is the mean reversion level or long equilibrium price
- S_t is the spot price
- θ is the mean reversion rate
- σ is the volatility

The equation explains that the difference between today’s spot price and tomorrow’s predicted spot price is equivalent to the distance between today’s spot price and the long run mean times the mean regressing rate θ plus some random term. The equation shows that the prices are gravitating towards the long-term mean by the θ rate.

4.2 Mean Regressing Behavior of the VIX

Robert Whaley, the creator of the VIX, observed that volatility is a mean reverting process. Therefore whenever the VIX is higher than its long run mean, it eventually reverts back to its long run mean, and, when it is lower than its long run mean, it typically reverts upward toward its long run mean. The VIX has shown this characteristic over its history. During very volatile periods of history, such as the bursting of the United States housing bubble in 2008, the VIX reaches very high values. Once the VIX reached its peak it then began to fall with very large rates of change until it begins approach its long run mean. Notice on the graph below how all prominent increases of the VIX seem shortly lived.

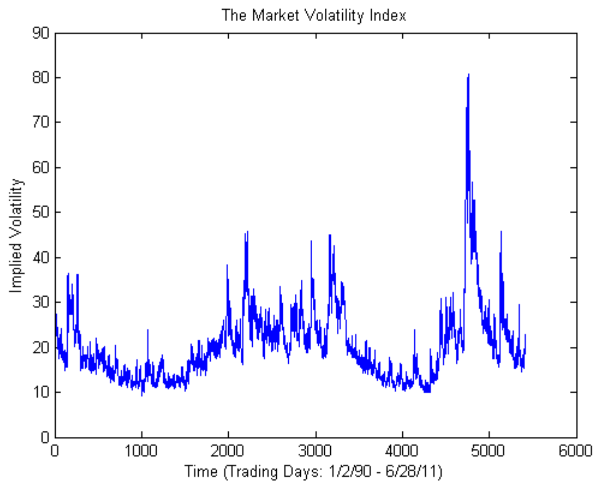


Figure 4.1: VIX values: 1/2/1990 - 6/28/11

4.3 Mean Regressing Behavior of VIX derivatives

It is a natural conjecture that VIX derivatives share the same mean reverting characteristics as the VIX. The VIX, VXX, and VXZ have all fallen drastically since, the inception dates of the VXZ and VXX. The VIX derivatives the VXX and the VXZ were introduced in the midst of global economic turmoil, a period when the VIX was extremely high. If the VXZ and VXX are mean reverting it is to be expected that the VXX and VXZ would have large rates of change until they began to approach particular values. Though the period in which VXX and VXZ were introduced was extremely volatile, the VXX and VXZ have responded to their high initial positions as mean reverting processes typically do. This further supported our hypothesis that the VXX and VXZ are mean reverting processes.

Chapter 5

Ornstein Uhlenbeck Model

5.1 Introduction to the Ornstein-Uhlenbeck Model

We selected the Ornstein-Uhlenbeck model as the mean regressing process with which to model the VXX and VXZ. This model was developed by Leonard Ornstein and George Eugene Uhlenbeck. It is simply a modification of the general mean regression model. The Ornstein-Uhlenbeck equation is as follows:

$$dS_t = \theta\left(\mu - S_t\right) + \sigma dW_t \tag{5.1}$$

where

- θ is the mean reversion rate
- μ is the long-term mean
- σ is the volatility

5.2 Calculating Parameters

5.2.1 Speed of Reversion to mean

The Ornstein-Uhlenbeck process uses linear regression to fit a linear function to the relationship between correlating today's spot price and tomorrow's spot price. This graph an be viewed in figure 5.1. The slope and intercept from the linear equation is then used to determine the long-term mean, μ, and the rate at which the data regresses to this mean, θ. Therefore the slope and intercept tells information about how the graph will behave.

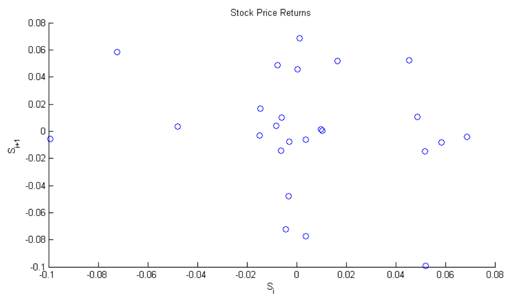


Figure 5.1: This graph shows the plot of today's spot price to tomorrow's spot price for $0 < t \leq 25$.

$$S_{i+1} = aS_i + b \tag{5.2}$$

For $0 < a < 1$ we have $\theta = -\ln a$ and $\mu = \frac{b}{1-a}$.

The slope of our linear function, a, tells a great deal about the movement of the data. If $a = 1$ then $S_{i+1} = S_i + b\ \forall i$ which implies that the data is not moving towards the mean at all and is not mean regressing, so the Ornstein-Uhlenbeck process would not work for the data. If $a = 0$ then $S_{i+1} = b\ \forall i$ creating a constant sequence. If $a < -1$ then the data oscillates between large positive numbers and large negative numbers. This does not represent mean regressing behavior either. If $a > 1$ then the data will move away from the long-term mean. This is why θ is desired for $0 < a < 1$.

We want to show that S_i tends to μ. Let $\varepsilon > 0$ be given. We need to find an $N \in \mathbb{N}$ such that $|S_{i+1} - \mu| < \varepsilon. So\, |S_{i+1} - \mu| = |(aS_i + b) - \mu| = \left|e^{-\theta} S_i + \mu\left(1 - e^{-\theta}\right) - \mu\right|$
$= \left|e^{-\theta} S_i + \mu - \mu e^{-\theta} - \mu\right|$
$= \left|e^{-\theta}(S_i - \mu)\right|$
$= e^{-\theta}|S_i - \mu| < e^{-\theta}\varepsilon < \varepsilon$
Therefore S_{i+1} will converge to the long-term mean, μ.

5.3 Calculating Parameters

A disadvantages of mean reverting models compared to Geometric Brownian Motion is it is more complicated to calculate parameters. Mean reverting models contain three parameters: volatility (σ), the long run mean (μ), and the speed at which the process reverts to the mean (θ).

5.3.1 Linear Regression

We used linear regression to estimate the parameters for our Ornstein-Uhlenbeck model. By plotting the relationship between the values of VIX derivatives on day i and day $i+1$ it is possible to observe a linear relationship.

$$S_{i+1} = aS_i + b + \varepsilon \tag{5.3}$$

Once the coefficient a, b, and ε have been identified then parameters for the Ornstein-Uhlenbeck model can be calculated. The equations below describe the relationship between the coefficients of the linear relationship the values of VIX derivatives on day i and day $i+1$ and the parameters of the Ornstein-Uhlenbeck model. The length of each time step is represented by δ. For our simulations $\delta = 1$, because we only interday changes one day at a time.

$$\sigma = sd(\varepsilon)\sqrt{\frac{-2\ln a}{\delta(1-a^2)}} \tag{5.4}$$

$$\mu = \frac{b}{1-a} \tag{5.5}$$

$$\theta = -\frac{\ln a}{\delta} \tag{5.6}$$

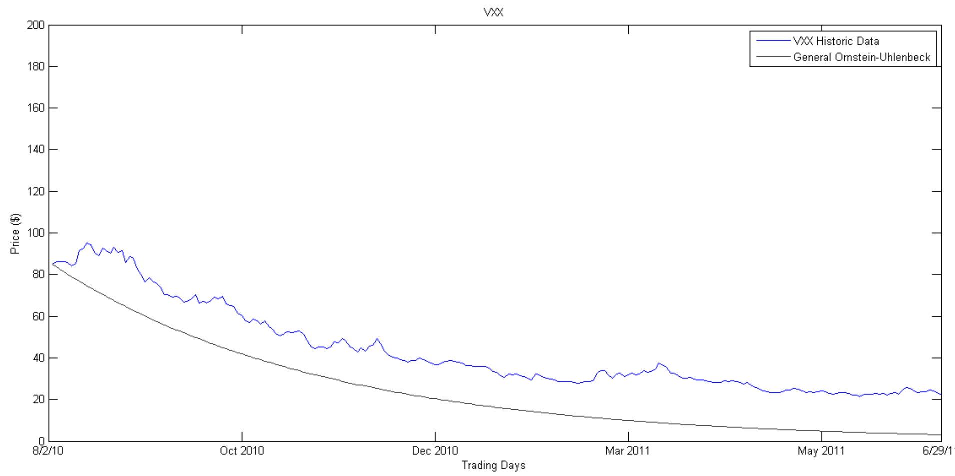
5.3.2 Least Squares

There are several methods to calculate the relationship between the values of VIX derivatives on day i and day $i+1$. One method used to calculate parameters for Ornstein-Uhlenbeck models is the Least Squares method. The Least Squares method is frequently used to approximate solutions for linear systems with more equations than unknowns. This method finds the solution with the smallest sum of errors between the solution and every equation of the linear system.

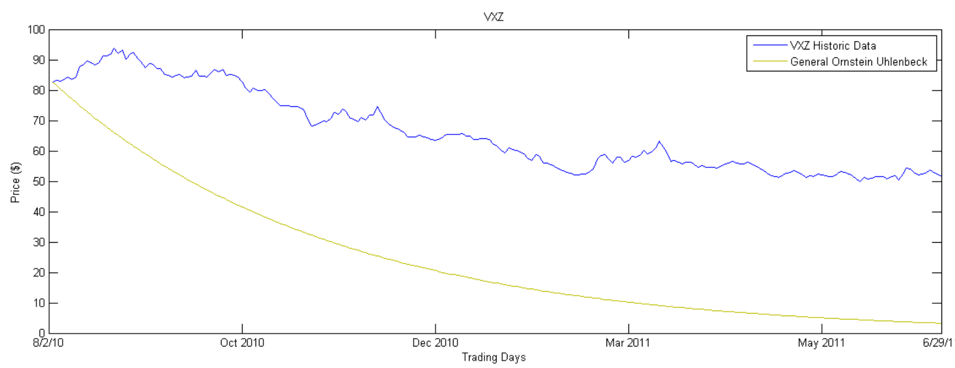
5.3.3 Maximum Likelihood Estimate

Maximum likelihood is another method to calculate parameters of the Ornstein-Uhlenbeck model. Maximum likelihood uses historic price data and selects parameters for the model that produce the greatest probability. We used historic data from the VIX derivatives' prices to estimate parameters.

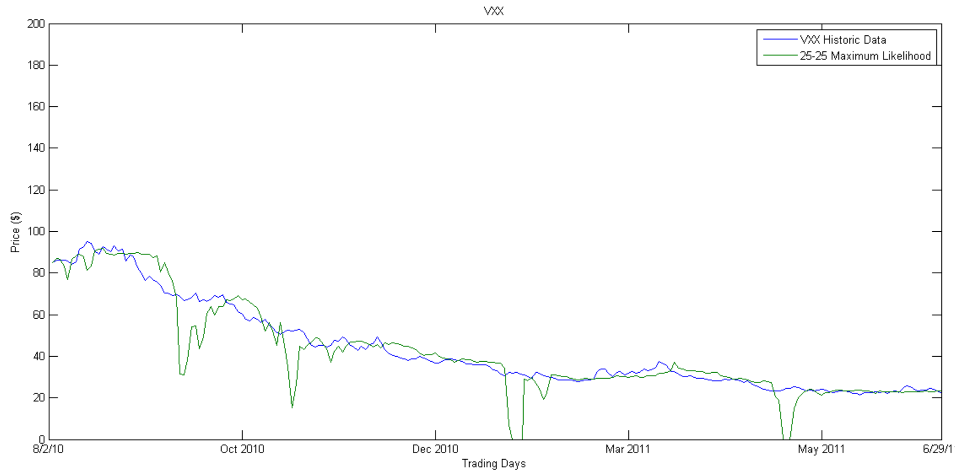
The plot below shows a General Ornstein-Uhlenbeck Model. The data used to calculate the parameters ranged from January 30th 2009 to July 30th 2010. The brown path shows the Monte Carlo path of 1000 Ornstein-Uhlenbeck paths. The blue line shows the VXX historical data from August 2^{nd}, 2010 to June 29^{th}, 2011.



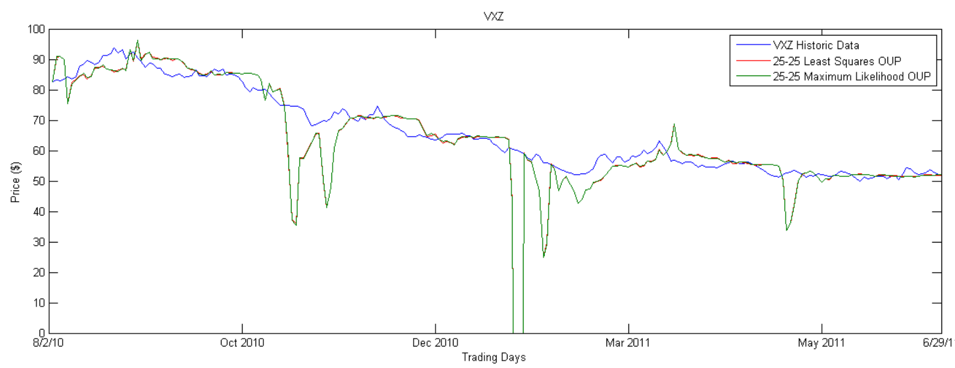
The plot below shows a General Ornstein-Uhlenbeck Model. The data used to calculate the parameters ranged from February 20th 2009 to July 30th 2010. The yellow path shows the Monte Carlo path of 1000 Ornstein-Uhlenbeck paths. The blue line shows the VXZ historical data from August 2^{nd}, 2010 to June 29^{th}, 2011.



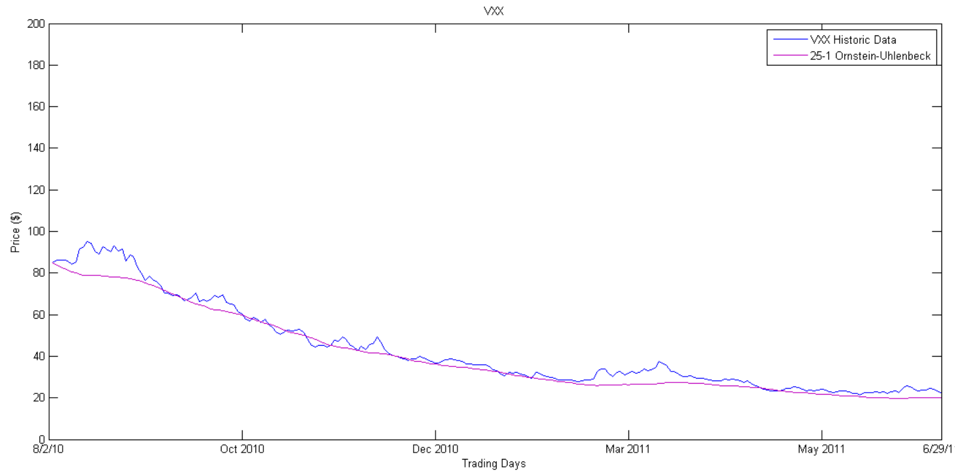
The plot below shows an Ornstein Uhlenbeck Model looking back 25 days and modeling 25 day's ahead. The green path shows the Monte Carlo path of 1000 Ornstein-Uhlenbeck paths. The blue path shows the VXX historical data from August 2^{nd}, 2010 to June 29^{th}, 2011.



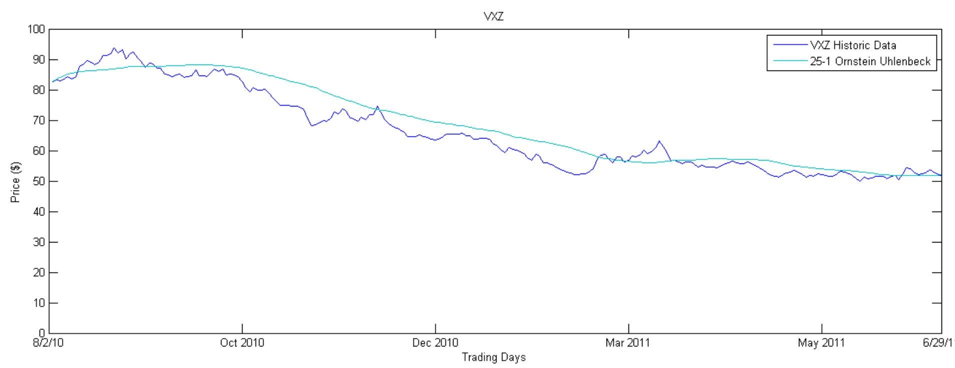
The plot below shows an Ornstein-Uhlenbeck Model looking back 25 days and modeling 25 day's ahead. The red path shows the Monte Carlo path of 1000 Ornstein-Uhlenbeck paths with parameters estimated using the Least Squares method. The green path shows the Monte Carlo path of 1000 Ornstein-Uhlenbeck paths with parameters estimated with the Maximum Likelihood method. The blue path shows the VXZ historical data from August 2^{nd}, 2010 to June 29^{th}, 2011.



The plot below shows an Ornstein-Uhlenbeck looking back 25 days and modeling 1 day ahead. The magenta path shows the Monte Carlo path of 1000 Ornstein-Uhlenbeck paths. The blue line shows the VIX from August 2^{nd}, 2010 to June 29^{th}, 2011.



The plot below shows an Ornstein-Uhlenbeck Model looking back 25 days and modeling 1 day ahead. The light blue path shows the Monte Carlo path of 1000 Ornstein-Uhlenbeck paths. The blue path shows the VXZ historical data from August 2^{nd}, 2010 to June 29^{th}, 2011.



Chapter 6

Results

6.1 Error Estimates

As stated earlier Geometric Brownian Motion and the Ornstein-Uhlenbeck mean regressing processes were used to model the Short-Term VIX Futures (VXX) and the Long-Term VIX Futures (VXZ). The Monte Carlo and sum of squares methods were used to test the accuracy of each process.

6.1.1 Monte Carlo Method

The Monte Carlo method was used to ensure that the path we compared to the actual VXX and VXZ was the best prediction. For both the Geometric Brownian Motion and the Ornstein-Uhlenbeck models we ran 1000 simulations for each of the different variations of both models. When the models were ran, the programs produced a matrix $X_{i,j}$ where the i^{th} row is the day and the j^{th} column represented a certain path. X was then transposed to create $\widehat{X_{j,i}}$ so that the i^{th} column would represent a day and the j^{th} row what the different paths foresaw. Then $\widetilde{X_{1,i}} = \sum_{k=1}^{j}(\widehat{X_k})$. $\widetilde{X}$ was then divided by 1000 to compensate for the paths. This produced $X_{1,i} = \frac{1}{1000} * \widetilde{X_{1,i}}$ where $\overline{X}$ is the Monte Carlo path. It was this new Monte Carlo path that was compared to the VXX and VXZ.

6.1.2 Sum of Squares Method

The accuracy of the estimate was measured using the sum of squares (SS) method. First matrices $VXX_{1,i}$ and $VXZ_{1,i}$ where they both have 1 colunm with the actual data for the i^{th} day. Each of the different variations made to the Geometric Brownian Motion and Ornstein-Uhlenbeck models had its own $\overline{X}$ Monte Carlo path. $Y = VXX_{1,i} - \overline{X_{1,i}}$ and $Z = VXZ_{1,i} - \overline{X_{1,i}}$ to determine how far the models predictions were from the known data. To account for negative distances, $\widetilde{Y_{1,i}} = \left(VXX_{1,i} - \overline{X_{1,i}}\right)^2$ and $\widetilde{Z_{1,i}} = \left(VXZ_{1,i} - \overline{X_{1,i}}\right)^2$. (The Sum of Squares process squares the distances instead of using absolute values so that the equation is differentiable.) Finally $SS_Y = \frac{1}{i}\sum_{k=1}^{i}\widetilde{Y_{1,k}}$

and $SS_Z = \frac{1}{k}\sum_{k=1}^{i}\widetilde{Z_{1,k}}$. Because the higher the number of days included in the sum of squares the higher the sum will be, the sum was divided by the number of days. This number then indicated how close the various Monte Carlo paths were to the known data. The smaller the number the more accurate the path is.

6.2 Geometric Brownian Motion

Geometric Brownian Motion was the first process used to try and model the VXX and VXZ. As stated previously, the data was split into two sections; what would be considered the past and what would be considered the future. The past data was then used to determine the parameters. Different variations was then made on which information would be included and which days would be forecasted. The first attempt was the general Geometric Brownian Motion (GGBM). The simulations produced from the GGBM were then used to make the Monte Carlo method and compared to the VXX and VXZ. The graph and sum of squares are as shown in figure 6.1.The alteration made was to retrieve data from the past twenty-five days to predict twenty-five days in the future. This model is the $25-25$ Geometric Brownian Motion (25-25GBM). Again the Monte Carlo path was found and compared to the VXX and VXZ using the sum of squares. They results are shown below in figure **??**. The final modification was then made to only collect data from the past twenty-five days to predict the next day. This model is called the twenty-five day Geometric Brownian Motion (25-1GBM). The Monte Carlo path was then found and compared to the VXX and VXZ. The results are shown in figure **??**. After conjecturing that the VIX is mean regressing, the decision was made to produce simulations with a mean regressing model.

Geometric Brownian Motion Figures

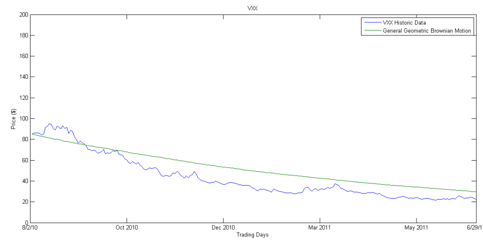


Figure 6.1: This graph shows the relationship between the VXX and GGBM where SS is 135.09.

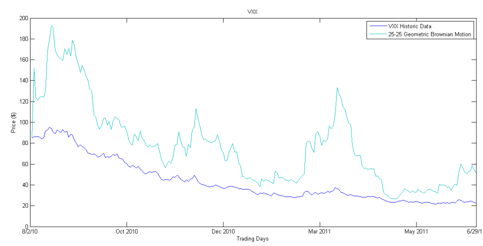


Figure 6.2: This graph shows the comparison of the VXX and 25/25GBM where SS is 1,729.

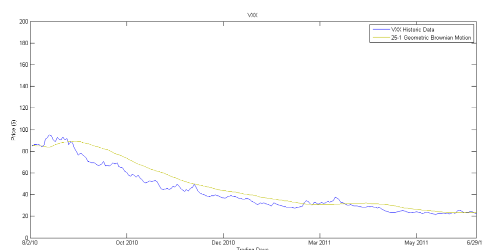


Figure 6.3: This graph shows the relationship between the VXX and 25/1GBM where SS is 51.32.

6.3 Ornstein-Uhlenbeck Model

Ornstein-Uhlenbeck (OU) model was the mean regressing process selected. The same modifications were made to the OU process as were made to the GBM with one difference. The twenty-five day Ornstein-Uhlenbeck process has been evaluated using two different methods to calculate the parameters; one using a least squares method and the other maximum likelihood. As with before, the first simulations were made using the general Ornstein-Uhlenbeck process (GOUP). The graph and sum of squares are shown in figure 6.4. The Ornstein-Uhlenbeck model was changed to gather data from the past twenty-five days to model twenty-five trading days in the future. As stated previously, this model created parameters using two different methods. The first using a least squares method and therefore called the $25-25$ Ornstein-Uhlenbeck least squares model (25-25OUPLS). The other using a maximum likelihood method and consequently named twenty-five Ornstein-Uhlenbeck maximum likelihood (25-25OUPML). The results for both processes are shown in figures 6.5 and 6.6. The final alteration was made to gather data for the past thirty days to predict the VXX and VXZ for the next trading day. This process is called the twenty-five day Ornstein-Uhlenbeck process (25-1OUP). Figure 6.7 shows the Monte Carlo path generated from the 1000 simulations next to the known VXX data. Now that both the Geometric Brownian Motion and the Ornstein-Uhlenbeck process have been used to model the VXX and VXZ it is time to determine which model was most accurate.

Ornstein-Uhlenbeck Figures

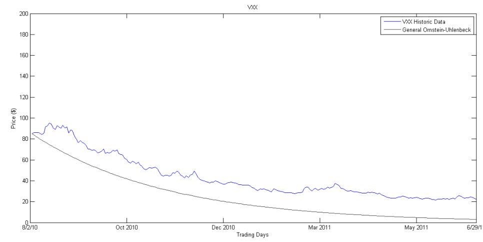


Figure 6.4: This graph shows the VXX and the GOUP where SS is 363.66.

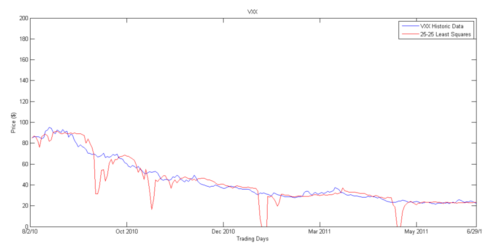


Figure 6.5: This graph depicts the 25-25OUPLS with the VXX where SS is 31.81.

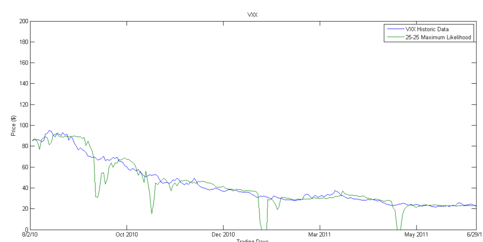


Figure 6.6: This graph shows 25-25OUPML next to the VXX where SS is 32.09.

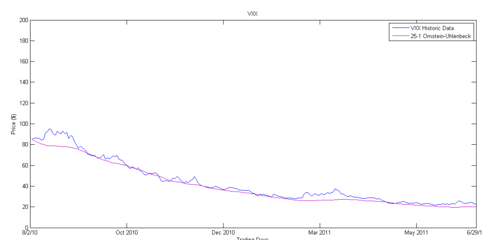


Figure 6.7: This graph compares the 25-1OUP to the VXX where SS is 21.67.

Chapter 7

Conclusion

The sum of squares process was used to determine how accurate the different processes were to modeling the VXX and VXZ. It can be seen in the Ornstein-Uhlenbeck models that certain spikes are shown throughout the Monte Carlo paths. Aside from these particular spikes, the Ornstein-Uhlenbeck models are more in line with the actual VXX data. Therefore these spikes were removed when calculating the sum of squares. To maintain a sense of uniformity, the same days were removed from all the different models' sum of squares. As a result, all the sum of squares reflect only 96 percent of the interval being modeled. From observing the sum of squares alone, the twenty-five day Geometric Brownian Motion came closest to modeling the VXX with the least squares twenty-five day Ornstein-Uhlenbeck process next. Since these two models do not predict the same days both will be considered. 25-1GBM is best for predicting the next trading day while 25-25OUPLS is best for predicting twenty-five trading days away. Both are fairly accurate in predicting the VXX. The following graphs depict the general models on the same coordinate system with the VXX as well as with the moving averages.

Model	Sums of Squares
GGBM	135.09
GOUP	363.66
25-25GBM	1,730
25-25OUPLS	31.81
25-25OUPML	32.09
25-1GBM	51.32
25-1OUP	21.67

Table 7.1: This graph shows the sum of squares of the various models.

On a grand scale the Ornstein-Uhlenbeck process generally came closer to forecasting the actual VXX than the Geometric Brownian Motion. This supports our conjecture that the VXX is mean regressing.

Chapter 8

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Chapter 9

Bibliography