Question 9:

A. Exercise 4.1.3 Sections B, C: Which of the following are functions from **R** to **R**? If f is a function, give its range.

   b. $f(x) = 1/(x^2 - 4)$

      b. Not a function from R to R because it is undefined for x = 2 or -2.

   c. $f(x) = \sqrt{x^2}$

      1. Function is defined from R to R, range is from 0 to positive infinity.

B. Exercise 4.1.5 Sections B, D, H, I, L: Express the range of each function using roster notation.

   b. Let A = {2, 3, 4, 5}. f: A → **Z** such that f(x) = x^2.

      i. f(2) = 2^2 = 4
      ii. f(3) = 3^2 = 9
      iii. f(4) = 4^2 = 16
      iv. f(5) = 5^2 = 25
      v. {4, 9, 16, 25}

   d. f: $\{0,1\}^5$ → **Z**. For x ∈ $\{0,1\}^5$, f(x) is the number of 1's that occur in x. For example f(01101) = 3, because there are three 1's in the string "01101".

      i. f(00000) = 0 (no 1's in the string)
      ii. f(00001) = 1 (one 1 in the string)
      iii. f(00011) = 2 (two 1's in the string)
      iv. f(00111) = 3 (three 1's in the string)
      v. f(01111) = 4 (four 1's in the string)
      vi. f(11111) = 5 (five 1's in the string)
      vii. {0, 1, 2, 3, 4, 5}

   h. Let A = {1, 2, 3}. f: A × A → **Z**×**Z**, where f(x,y) = (y, x).

      i. f(1, 1) = (1, 1)
      ii. f(1, 2) = (2, 1)
      iii. f(1, 3) = (3, 1)
      iv. f(2, 1) = (1, 2)
      v. f(2, 2) = (2, 2)
      vi. f(2, 3) = (3, 2)
      vii. f(3, 1) = (1, 3)
      viii. f(3, 2) = (2, 3)
      ix. f(3, 3) = (3, 3)
      x. {(1, 1), (2, 1), (3, 1), (1, 2), (2, 2), (3, 2), (1, 3), (2, 3), (3, 3)}

   i. Let A = {1, 2, 3}. f: A × A → **Z**×**Z**, where f(x,y) = (x,y+1).

      i. f(1, 1) = (1, 2)
      ii. f(1, 2) = (1, 3)
      iii. f(1, 3) = (1, 4)
      iv. f(2, 1) = (2, 2)
      v. f(2, 2) = (2, 3)
      vi. f(2, 3) = (2, 4)
      vii. f(3, 1) = (3, 2)

viii. $f(3, 2) = (3, 3)$
ix. $f(3, 3) = (3, 4)$
x. $\{(1, 2), (1, 3), (1, 4), (2, 2), (2, 3), (2, 4), (3, 2), (3, 3), (3, 4)\}$

l. Let $A = \{1, 2, 3\}$. $f: P(A) \to P(A)$. For $X \subseteq A$, $f(X) = X - \{1\}$.

i. $f(\varnothing) = \varnothing - \{1\} = \varnothing$
ii. $f(\{1\}) = \{1\} - \{1\} = \varnothing$
iii. $f(\{2\}) = \{2\} - \{1\} = \{2\}$
iv. $f(\{3\}) = \{3\} - \{1\} = \{3\}$
v. $f(\{1, 2\}) = \{1, 2\} - \{1\} = \{2\}$
vi. $f(\{1, 3\}) = \{1, 3\} - \{1\} = \{3\}$
vii. $f(\{2, 3\}) = \{2, 3\} - \{1\} = \{2, 3\}$
viii. $f(\{1, 2, 3\}) = \{1, 2, 3\} - \{1\} = \{2, 3\}$
ix. $\{\varnothing, \{2\}, \{3\}, \{2, 3\}\}$

Question 10:

I. Exercise 4.2.2: Sections C, G, K - For each of the functions below, indicate whether the function is onto, one-to-one, neither or both. If the function is not onto or not one-to-one, give an example showing why.
   C. h: **Z** → **Z**. $h(x) = x^3$
      1. It is one-to-one because every distinct integer x will result in a distinct cube x^3. That means, if $x_1 \neq x_2$ then $h(x_1) = x_1^3 \neq x_2^3 = h(x_2)$.
      2. It is onto because for every integer y, we can find an integer x (specifically, the cube root of y) such that h(x) = y.
   G. f: **Z** × **Z** → **Z** × **Z**, f(x, y) = (x+1, 2y)
      1. In this equation, different pairs of integers (x, y) will yield different pairs (x+1, 2y). For instance, if we have two different pairs (x1, y1) and (x2, y2) such that (x1, y1) ≠ (x2, y2), then f(x1, y1) = (x1+1, 2*y1) ≠ (x2+1,* 2y2) = f(x2, y2). Hence, the function is one-to-one.
      2. Can't find a pair of integers (x, y) that we can put into the function f(x, y) = (x+1, 2y) to get the pair (0, 1). This is why the function is not onto.
   A. f: $\mathbf{Z}^+ \times \mathbf{Z}^+ \to \mathbf{Z}^+$, $f(x, y) = 2^x + y$.
      1. Take $(x_1, y_1) = (1, 2)$ and $(x_2, y_2) = (2, 0)$. Even though the input pairs are different, f(x1, y1) = f(1, 2) = 2*1 + 2 = 4 and f(x2, y2) = f(2, 0)* = 22 + 0 = 4 are equal. Hence, this function is not one-to-one.
      2. for z = 1 or z = 2, there is not a pair (x, y) in Z+ × Z+ such that f(x, y) = z, because all outputs of the function f(x, y) = 2x + y for x and y in Z+ are greater than or equal to 3 (because x and y are positive, so 2x + y is at least 3).

II. Give an example of a function from the set of integers to the set of positive integers that is:
   A. One-to-one, but not onto: $f(x) = 2x$
   B. Onto, but not one-to-one: $f(x) = |x| + 1$
   C. One-to-one and onto: A function from the set of integers to the set of positive integers that is both one-to-one and onto is not possible because there are more integers (which include negative numbers, zero, and positive numbers) than there are positive integers. Thus, there's no way to pair every integer with a unique positive integer and cover all positive integers without duplication.
   D. Neither one-to-one nor onto: $f(x) = |x \% 10| + 1$

Question 11:

A. Exercise 4.3.2 Sections C, D, G, I - For each of the following functions, indicate whether the function has a well-defined inverse. If the inverse is well-defined, give the input/output relationship of f^{-1}

   c. **R → R.** $f(x) = 2x + 3$
      i. f(x) = y
      ii. 2x + 3 = y
      iii. 2x = y - 3
      iv. x = (y - 3)/2
      v. Well defined - f^{-1}(y) = (y-3)/2

   d. For $X \subseteq A$, $f(x) = |x|$. Recall that for a finite set A, P(A) denotes the power set of A which is the set of all subsets of A.
      i. The function is onto, however it's not one-to-one because there are multiple different subsets X of A that have the same number of elements. For example, the subsets {1} and {2} both have 1 element, so f({1}) = f({2}) = 1.
      ii. Since the function f is not one-to-one, it does not have a well-defined inverse.

   g. $f: \{0, 1\}^3 \rightarrow \{0, 1\}^3$. The output of f is obtained by taking the input string and reversing the bits. For example, f(011) = 110
      i. This function is one-to-one because each input string in $\{0,1\}^3$ maps to a unique output string in $\{0,1\}^3$. No two distinct inputs will yield the same output.
      ii. This function is onto because every possible string in $\{0,1\}^3$ can be obtained from some input string in $\{0,1\}^3$.
      iii. **The inverse function f^-1: $\{0,1\}^3$ --> $\{0,1\}^3$ is the same as the original function f.** This is because if you take a string and reverse its bits, then reversing the bits again will give you back the original string.The input/output relationship of f^-1 is that for each string y in $\{0,1\}^3$, f^-1 outputs the string obtained by reversing the bits of y.

   i. $f: Z \times Z \rightarrow Z \times Z$, $f(x, y) = (x + 5, y - 2)$
      i. The function is one-to-one because for each different input (x,y), the output (x+5, y-2) will be different. No two different inputs will map to the same output.
      ii. The function is onto because for any possible output pair (a,b) in Z x Z, there is a pair (x,y) such that f(x,y) = (a,b). Specifically, (x,y) = (a-5, b+2).
      iii. The inverse function f^-1: Z x Z --> Z x Z is then given by f^-1(a,b) = (a-5, b+2). This means, for each ordered pair (a,b) in Z x Z, f^-1 outputs the ordered pair (a-5, b+2).

B. Exercise 4.4.8 Sections C, D - The domain and target set of functions f, g, and h are **Z**. The functions are defined as:

   f(x) = 2x + 3

$g(x) = 5x + 7$

$h(x) = x^2 + 1$

Give an explicit formula for each function given below.

c. f o h
    i. $f(x) = 2x + 3$
    ii. $h(x) = x^2 + 1$
    iii. $f(h(x)) = f(x^2 + 1)$
    iv. $f(h(x)) = 2(x^2 + 1) + 3$
    v. $f(h(x)) = 2x^2 + 5$
d. h o f
    i. $h(f(x)) = h(2x + 3)$
    ii. $h(f(x)) = (2x + 3)^2 + 1$
    iii. $h(f(x)) = 4x^2 + 12x + 9 + 1$
    iv. h(f(x)) = 4x^2 + 12x + 10

C. Exercise 4.4.2 - Sections B, C, D - Consider three functions f, g, and h, whose domain and target are **Z**. Let $f(x) = x^2$ $g(x) = 2^x$ h(x) = ceiling x/5
    c. Evaluate (f o h)(52)
        i. h(52) = ceil(52/5) = ceil(10.4) = 11
        ii. $f(h(52)) = f(11) = 11^2 = 121$
        iii. (f o h)(52) = 121
    d. Evaluate (g o h o f)(4)
        i. $f(4) = 4^2 = 16$
        ii. h(f(4)) = h(16) = ceil(16/5) = ceil(3.2) = 4
        iii. g(h(f(4))) = g(4) = 2*4 = 8
        iv. (g o h o f)(4) = 8
    e. Give a mathematical expression for h o f
        i. $h(f(x)) = h(x^2)$
        ii. h(f(x)) = ceil(x^2/5)

D. Exercise 4.4.6 Section C, D, E
    - Define the following functions f, g, and h:
        i. f: $\{0, 1\}^3 \to \{0, 1\}^3$. The output of f is obtained by taking the input string and replacing the first bit by 1, regardless of whether the first bit is a 0 or 1. For example, f(001) = 101 and f(110) = 110.
        ii. g: $\{0, 1\}^3 \to \{0, 1\}^3$. The output of g is obtained by taking the input string and reversing the bits. For example, g(011) = 110.
        iii. h: $\{0, 1\}^3 \to \{0, 1\}^3$. The output of h is obtained by taking the input string x, and replacing the last bit with a copy of the first bit. For example, h(011) = 010.
            c. What is (h o f)(010)?

        i. f(010) = 110
        ii. h(f(010)) = h(110) = 110
        iii. (h o f)(010) = 110
    d. What is the range of h o f?
        i. First, apply f which replaces the first bit of the input by 1
        ii. Then apply h, which replaces the last bit of the result with a copy of the first bit.
        iii. The first bit of the output is always 1 (due to f), and the last bit of the output is also always 1 (due to h copying the first bit). The middle bit is the only bit that can vary based on the input, so it can be either 0 or 1. **The range of h o f is {100, 110}.**
    e. What is the range of g o f?
        i. First, apply f, which replaces the first bit of the input by 1
        ii. Then apply g, which reverses the bits of the result.
        iii. The last bit is always 1, and the other bits depend on the input.
        iv. Therefore, the range of g o f is {001, 011, 101, 111}.

E. Exercise 4.4.4, Section C, D:
    c. Is it possible that f is not one-to-one and g o f is one-to-one? Justify your answer. If the answer is "yes", give a specific example for f and g.
        i. No
        ii. f is not one-to-one, this means there exist x1 and x2 in X such that f(x1) = f(x2) but x1 ≠ x2. If g o f were one-to-one, then applying g to both sides of f(x1) = f(x2) would give g(f(x1)) = g(f(x2)). But because g o f is supposed to be one-to-one, this would mean x1 = x2, which contradicts our assumption that x1 ≠ x2.
    d. Is it possible that g is not one-to-one and g o f is one-to-one? Justify your answer. If the answer is "yes", give a specific example for f and g.
        i. Yes
        ii. X = {1}, Y = {2, 3}, and Z = {4}
        iii. f: X → Y and g: Y → Z = f(1) = 2, g(2) = 4, g(3) = 4
        iv. g is not one-to-one as it maps 2 and 3 both to 4.
        v. The composite function g o f is one-to-one, as f only maps to 2, and g o f(1) = g(2) = 4. No matter what value you choose in X, g o f will always map it to 4, thus it is still one-to-one