**Question 5**

A.

1. Exercise 1.12.2, Sections B,E - *Use the rules of inference and the laws of propositional logic to prove that each argument is valid. Number each line of your argument and label each line of your proof "Hypothesis" or with the name of the rule of inference used at that line. If a rule of inference is used, then include the numbers of the previous lines to which the rule is applied.*

   p → (q ∧ r)
   ¬q
   _________
   ∴ ¬p

   1. $p \rightarrow (q \wedge r)$ - Hypothesis
   2. $\neg q$ - Hypothesis
   3. $p \rightarrow q$ - Simplification, 1
   4. $\neg p$ - Modus Tollens, 2,3

   p ∨ q
   ¬p ∨ r
   ¬q
   _______
   ∴ r

   1. $p \vee q$ - Hypothesis
   2. $\neg p \vee r$ - Hypothesis
   3. $\neg q$ - Hypothesis
   4. $p$ - Disjunctive Syllogism, 1,3
   5. $r$ - Disjunctive Syllogism, 2,4

2. Exercise 1.12.3, Section C - *Some of the rules of inference can be proven using the other rules of inference and the laws of propositional logic. One of the rules of inference is Disjunctive syllogism:*

   *p* ∨ *q*
   ¬*p*
   ______
   ∴ *q*

   *Prove that Disjunctive syllogism is valid using the laws of propositional logic and any of the other rules of inference besides Disjunctive syllogism.*

1. $p \lor q$ - Hypothesis
2. $\neg p$ - Hypothesis
3. $\neg p \rightarrow q$ - Conditional Identity, 1
4. $q$ - Modus Ponens, 2,3

3. Exercise 1.12.5, Sections C,D - *Give the form of each argument. Then prove whether the argument is valid or invalid. For valid arguments, use the rules of inference to prove validity.*
   c. I will buy a new car and a new house only if I get a job.
   I am not going to get a job.

   ---

   ∴ I will not buy a new car.

   p: I will buy a new car.
   q: I will buy a new house.
   r: I will get a job.

   Prove $\neg p$

   1. $r \rightarrow (p \land q)$ - Hypothesis
   2. $\neg r$ - Hypothesis
   3. $\neg(p \land q)$ - Modus Tollens, 1,2
   4. $\neg p \lor \neg q$ - De Morgan's, 3
   5. $\neg p$ - Simplification, 4

   d. I will buy a new car and a new house only if I get a job.
   I am not going to get a job.
   I will buy a new house.

   ---

   ∴ I will not buy a new car.

   p: I will buy a new car.
   q: I will buy a new house.
   r: I will get a job.

   Prove $\neg p$

   1. $r \rightarrow (p \land q)$ - Hypothesis
   2. $\neg r$ - Hypothesis
   3. $q$ - Hypothesis
   4. $\neg r \lor (p \land q)$ - Conditional Identity, 1
   5. $\neg r \land \neg p \land \neg q$ - De Morgan's, 4
   6. $\neg r \land \neg p$ - Disjunctive Syllogism, 3,5

7. $\neg p$ - Disjunctive Syllogism, 2,6

B.

1. Exercise 1.13.3, Section B - *Show that the given argument is invalid by giving values for the predicates P and Q over the domain {a, b}.*

   ∃x (P(x) ∨ Q(x))
   ∃x ¬Q(x)
   ________________
   ∴ ∃x P(x)

| | P | Q |
|---|---|---|
| **A** | F | T |
| **B** | F | F |

   1. ∃x (P(x) ∨ Q(x)) - Q(a) = True fulfills hypothesis 1
   2. ∃x ¬Q(x) - Q(b) = False fulfills hypothesis 2
   3. ∴ ∃x P(x) - P(a) & P(b) = False, invalidating conclusion

2. Exercise 1.13.5 Sections D, E - *Prove whether each argument is valid or invalid. First find the form of the argument by defining predicates and expressing the hypotheses and the conclusion using the predicates. If the argument is valid, then use the rules of inference to prove that the form is valid. If the argument is invalid, give values for the predicates you defined for a small domain that demonstrate the argument is invalid.*

   d.

   Every student who missed class got detention.
   Penelope is a student in the class.
   Penelope did not miss class.
   ________________________________________
   Penelope did not get detention.

   S(x): x is a student.
   M(x): x missed class.
   D(x): x got detention.

   Prove: $\neg D(Penelope)$

   1. $\forall x[(S(x) \wedge M(x)) \rightarrow D(x)]$ - Hypothesis
   2. $S(Penelope)$ - Hypothesis
   3. $\neg M(Penelope)$ - Hypothesis

4. $(S(Penelope) \wedge M(Penelope)) \rightarrow D(Penelope)$ - Universal Instantiation, 1
5. $\neg D(Penelope) \vee (S(Penelope) \wedge M(Penelope))$ - Conditional Identity, 4
6. $\neg D(Penelope) \vee (True \wedge False)$ - Substitution 2, 3, 5
7. $\neg D(Penelope) \vee False$ - Simplification, 6
8. $\neg D(Penelope)$ - Identity, 7

e.

Every student who missed class or got a detention did not get an A.
Penelope is a student in the class.
Penelope got an A.

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Penelope did not get detention.

S(x): x is a student.
M(x): x missed class.
D(x): x got detention.
A(x): x got an A.

Prove : $\neg D(Penelope)$

1. $\forall x[(S(x) \wedge (M(x) \vee D(x))) \rightarrow \neg A(x)]$ - Hypothesis
2. $S(Penelope)$ - Hypothesis
3. $A(Penelope)$ - Hypothesis
4. $[S(Penelope) \wedge (M(Penelope) \vee D(Penelope))] \rightarrow \neg A(Penelope)$ - Universal Instantiation, 1
5. $\neg(S(Penelope) \wedge (M(Penelope) \vee D(Penelope))) \vee \neg A(Penelope)$ - Conditional Identity, 4
6. $(\neg S(Penelope) \vee \neg(M(Penelope) \vee D(Penelope))) \vee \neg A(Penelope)$ - De Morgan's, 5
7. $(\neg S(Penelope) \vee (\neg M(Penelope) \wedge \neg D(Penelope))) \vee \neg A(Penelope)$ - De Morgan's, 6
8. $\neg M(Penelope) \wedge \neg D(Penelope) \vee \neg S(Penelope) \vee \neg A(Penelope)$- Associative - 7
9. $\neg D(Penelope)$ - Disjunctive syllogism, 2,3,8

**Question 6**

1. Exercise 2.4.1 Section D - Each statement below involves odd and even integers. An odd integer is an integer that can be expressed as $2k + 1$ where $k$ is an integer. An even integer is an integer that can be expressed as $2k$, where $k$ is an integer. Prove each of the following statements using a direct proof.

    - The product of two odd integers is an odd integer.
        i. The product of two odd integers, $x$ and $y$, can be expressed as $xy = (2n + 1)(2m + 1)$
        ii. $4nm + 2n + 2m + 1$
        iii. $2(2nm + n + m) + 1$
        iv. $(2nm + n + m)$ simplifies to an integer, therefore the product of two odd integers, x and y: $xy = 2k + 1$
2. Exercise 2.4.3 Section B - Prove each of the following statements using a direct proof.
    - Premises: X is a real number, X is less than or equal to 3.
    - If $x$ is a real number and $x \leq 3$, then $12 - 7x + x^2 \geq 0$.
    - Plug the value 3 in to the formula for x: $12 - 21 + 9 \geq 0$.
    - $0 \geq 0$.

**Question 7**

1. Exercise 2.5.1 Section D - Prove each statement by contrapositive.
    d. For every integer n, if $n^2 - 2n + 7$ is even, then n is odd.
    Contrapositive: For every integer n, if n is not odd then $n^2 - 2n + 7$ is not even.
        i. Assume n is even. Therefore, n = 2k.
        ii. Plug 2k into $n^2 - 2n + 7 = (2k)^2 - 2(2k) + 7 = 4k^2 - 4k + 7$
        iii. Simplify to $2(2k^2 - 2k + 3) + 1$
        iv. $2k^2 - 2k + 3$ is an integer because it is formed from arithmetic operations on integers. Therefore, simplify $2k^2 - 2k + 3$ to an odd integer k: 2(k) + 1, which is the form of an odd integer.
        v. Therefore, if n is even, $n^2 - 2n + 7$ is odd. And if n is odd, $n^2 - 2n + 7$ is even.
2. Exercise 2.5.4 Sections A,B - Prove each statement by contrapositive
    a. For every pair of real numbers x and y, if $x^3 + xy^2 \leq x^2y + y^3$, then $x \leq y$.
    Contrapositive: For every pair of real numbers x and y, if x < y, then $x^3 + xy^2 < x^2y + y^3$.
        i. Assume x and y are real numbers and x < y.
        ii. $x^3 + xy^2 = x^2 * x + y^2 * x$
        iii. $x^2y + y^3 = x^2 * y + y^2 * y$
        iv. For every permutation, if x is less than y, line 3 is greater than line 2.
        v. Therefore, the contrapositive if x is less than y, then $x^3 + xy^2 < x^2y + y^3$ is true, and by extension the original statement.

3. Exercise 2.5.5 Section C - Prove each statement using a direct proof or proof by contrapositive. One method may be much easier than the other.
    a. For every non-zero real number x, if x is irrational then 1/x is also irrational.
        i. Assume that x is a non-zero real number and x is irrational. This means that x can't be expressed as a ratio of two integers.
        ii. The property of rational numbers is such that the inverse of a rational number is also rational. In other words, if a/b is rational (where a and b are integers and b ≠ 0), then its inverse b/a is also rational.
        iii. However, x is given as irrational. Therefore, by the property in step 2, the inverse of x, which is 1/x, must also not be rational (since the inverse of a rational number is rational, the inverse of an irrational number must be irrational).
        iv. Therefore, if x is an irrational number, then 1/x is also an irrational number.

**Question 8**

1. Exercise 2.6.6 Section C, D - Give a proof for each statement.
    c. The average of three real numbers is greater than or equal to at least one of the numbers.
        i. For the sake of contradiction assume the average of a, b, and c is less than all three numbers:
            - $(a + b + c) / 3 < a$
            - $(a + b + c) / 3 < b$
            - $(a + b + c) / 3 < c$
        ii. Simplify
            - $a + b + c < 3a$
            - $a + b + c < 3b$
            - $a + b + c < 3c$
        iii. Simplify
            - $b + c < 2a$
            - $a + c < 2b$
            - $a + b < 2c$
        iv. Consolidate
            - $2a + 2b + 2c < 2(a + b + c)$
        v. Reduce
            - $0 < 0$
        vi. 0 is not less than 0, therefore the initial contradiction in step i must be false. the average of the three real numbers must be greater than or equal to at least one of the numbers.
    d. There is no smallest integer.
        i. Assume for the sake of contradiction that there is a smallest integer, and let's call this integer 'n'.
        ii. Subtracting 1 from any integer results in another integer. Thus, we can create the integer n-1, which is an integer and smaller than n.
        iii. This contradicts our original assumption that 'n' is the smallest integer, because we have found an integer (n-1) that is even smaller.
        iv. The contradiction implies that our original assumption is incorrect. Therefore, there is no smallest integer.

**Question 9**

1. Exercise 2.7.2 Section B - Prove each statement.
    b. If integers x and y have the same parity, then x+y is even. The parity of a number tells whether the number is odd or even. If x and y have the same parity, they are either both even or both odd.
        i. Assume even numbers are 2n and odd numbers are 2n + 1 where n is an integer.
        ii. **Case 1:** Both x and y are even:
            1. If x and y are both even, then they can be written as x = 2a and y = 2b for integers a and b.
            2. Adding x and y gives x + y = 2a + 2b = 2(a + b).
            3. Since a + b is an integer (the sum of two integers is always an integer), x + y is of the form 2n where n is an integer.
            4. Therefore, x + y is even.
        **iii.** **Case 2:** X and Y are both odd.
            1. If x and y are both odd, then they can be written as x = 2a + 1 and y = 2b + 1 for integers a and b.
            2. Adding x and y gives x + y = 2a + 1 + 2b + 1 = 2(a + b + 1).
            3. Again, a + b + 1 is an integer (the sum of two integers plus one is always an integer), so x + y is of the form 2n where n is an integer.
            4. Therefore, x + y is even.