Ch 3: Introduction to Interpolation

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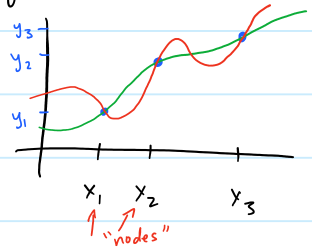
Topics:
- Interpolation definitions
- Interpolating polynomials: existence/uniqueness,
  - vis-a-vis Weierstrass
  - Vandermonde matrix
- Overfitting
  Fix: local / composite interpolation
- Runge phenomenon
  Fix: don't use equally spaced points!
  eg. Chebyshev nodes

to be discussed in more detail later in the course (or next semester)

Interpolation

Given a dataset $\{(x_i, y_i)\}$, an interpolant is a function g that agrees with the dataset, meaning $g(x_i) = y_i$

"nodes"



} both are interpolants

There is not necessarily a "true" function f though sometimes there is.

Usually we want the interpolating function to be nice, i.e., both easy to work with (ex: integrate) and have nice properties like differentiability, etc.

Traditionally, "interpolation" means filling in the blanks inside an interval

Ex: it was 73°F at 8 AM and 82°F at 11 AM.
What's a guess for the temperature at 9 AM?
(one approach: find an interpolant g such that $g(8) = 73$, $g(11) = 82$, then look at $g(9)$)

... in contrast to "extrapolation" which is about prediction outside your interval, i.e., the future.
In above example, extrapolation might ask what is the temperature at noon?

Interpolation isn't the only approach
- another class of methods is "least-squares" and many variants
  i.e. "best approximation"
- as you'll see, we'll focus on polynomials. The first thing you might think of is the Taylor polynomial but this is not a good idea usually, as these are only good very locally, and not on an interval

Polynomials

Often we require the interpolant to be a polynomial.

Why?
- Easy to work with, e.g. derivatives + integrals are very easy
- They are smooth ($C^\infty(\mathbb{R})$)
- Nice theory, and hundreds of years of experience

Ex: "interpolants" are not unique, but "polynomial interpolants" of low-degree are unique, as made precise below:

Refresher: Polynomials

Degree 0 polynomials: $p(x)=c$. If $c \neq 0$, no roots.
If $c=0$, ∞ roots.

"Fundamental Theorem of Algebra"

(1) Every polynomial h of degree 1 or higher has at least 1 (possibly complex) root

(2) (Corollary) A n^{th} degree polynomial has n (possibly complex) roots if you count with multiplicity *

* Except 0 polynomial has ∞ roots

i.e. at most n roots if you only count unique roots (i.e., not w/ multiplicity)

Def "0 polynomial" is $0(x)=0$ i.e., all coefficients are zero.

Ex: $f(x)=(x-1)^2(x-4)$ has "3" roots: $\{1,1,4\}$ if we count w/ multiplicity.

In particular, $f(x)=a_n(x-x_1)^{m_1}\cdot(x-x_2)^{m_2}\cdot\ldots\cdot(x-x_k)^{m_k}$

m_i unique, $\sum_{i=1}^{k} m_i = n$

(!) zero of a polynomial $\neq$ the zero polynomial

Corollary (2.18)

If $p(x)$ and $q(x)$ are polynomials, both of degree n or less, then if we have a set of k distinct points $\{x_1, x_2, \ldots, x_k\}$,

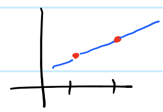
then if
$$p(x_i) = p(x_i) \quad \forall\, i=1,\ldots,k$$

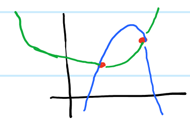
then $k>n \Rightarrow p=q$

Proof: If $k>n$, this means $p-q$ is a n degree polynomial with more than n roots, which is impossible unless $p-q=0$. $\square$

Theorem: (Uniqueness) If two polynomials $p(x)$ and $q(x)$ agree on k data points $\{x_i\}_{i=1}^{k}$ (distinct points) and both polynomials are degree n or less, then $p=q$ (i.e., the interpolating polynomial of degree n is $\boxed{\text{unique}}$) if $k>n$.

Ex: "$\underbrace{\text{two points}}_{k=2}$ determine a $\underbrace{\text{line}}_{n=1}$"





though an infinite number of quadratics interpolate 2 points.

Theorem: existence

If $\{x_i\}_{i=1}^{k}$ are distinct points, and $\{y_i\}_{i=1}^{k}$ are any y-values, then there exists a polynomial of degree $n=k-1$ that interpolates the data

proof: a $n=k-1$ degree polynomial looks like

$$p(x) = c_k x^{k-1} + c_{k-1}x^{k-2} + \dots + c_2 x^1 + c_1$$

so it has k degrees of freedom (parameters) $\{c_i\}_{i=1}^{k}$

Solve:

$$\begin{aligned} c_k x_1^{k-1} + \dots + c_2 x_1 + c_1 &= y_1 \qquad \text{plug in } x=x_1 \\ &\vdots \\ c_k x_k^{k-1} + \dots + c_2 x_k + c_1 &= y_k \qquad x=x_k \end{aligned}$$

linear system

$$\underbrace{\begin{bmatrix} 1 & x_1 & x_1^2 & \dots & x_1^{k-1} \\ 1 & x_2 & x_2^2 & & x_2^{k-1} \\ \vdots & \vdots & \vdots & & \vdots \\ 1 & x_k & x_k^2 & & x_k^{k-1} \end{bmatrix}}_{\text{a matrix of this form is called a Vandermonde matrix}} \begin{bmatrix} c_1 \\ c_2 \\ \vdots \\ c_k \end{bmatrix} = \begin{bmatrix} y_1 \\ \vdots \\ y_k \end{bmatrix}, \quad \text{i.e. } \boxed{V\cdot\vec{c}=\vec{y}}$$

don't confuse with the Wronskian

and you can prove $\det(V) = \prod_{1\le i<j\le k}(x_i - x_j)$

notation: "$\prod$" means product like "$\sum$" for sum

ex $\prod_{i=1}^{3} 2^i = 2^1\cdot 2^2\cdot 2^3$

so $\{x_i\}$ distinct $\Rightarrow \det(V)\neq 0 \Rightarrow V$ is invertible

$\Rightarrow \vec{c} = V^{-1}\vec{y}$ is the (unique) sol'n. $\square$

↳ we actually proved existence and uniqueness.

Construction

i.e. how to find coefficients $\{c_i\}$.

Do not invert the Vandermonde matrix, since this is ill-conditioned

Instead, see the other videos (ch 3.1, 3.3), so we'll come back to this

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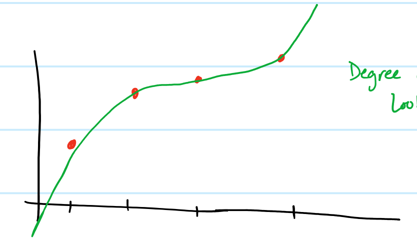
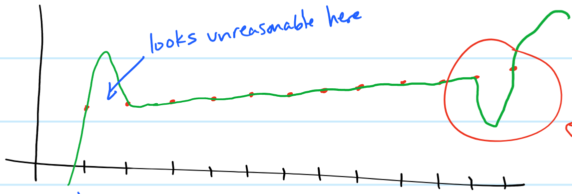
Overfitting



Degree 3 polynomial.
Looks ok



these large values near the boundary of our interval is what we call the RUNGE PHENOMENON.

and very unreasonable for extrapolation

What to do? (it's a fundamental problem, independent of the algorithm)

① With many nodes, we can't interpolate globally with a low-degree polynomial. But, we could interpolate locally w/ a low-degree polynomial, and then stitch these together (ie., create a piecewise polynomial)

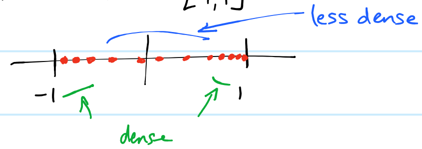
We'll later discuss splines and composite quadrature, which are both techniques based on this general idea

② If we have the luxury of choosing the nodes $\{x_i\}$, (sometimes this is reasonable, like for numerical integration, though othertimes it's not reasonable, like if it's a measurement someone gives you).

then choose the nodes wisely. In particular, don't choose them equispaced, but rather they should concentrate near the boundaries (ex. Chebyshev nodes of the second kind)

$$x_i = -\cos\left(\frac{i \cdot \pi}{n}\right), \quad i = 0, 1, \ldots, n$$

for the interval $[-1, 1]$



More on this later
(ex., ch. 4.7 Gaussian Quadrature, ← 2^{nd} semester
ch 8.3 Chebyshev polynomials)

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Last bit of theory:

Weierstrass Approximation theorem

Thm: if $f \in C([a,b])$ then $(\forall \varepsilon > 0)$ $\exists$ a polynomial p such that $|f(x) - p(x)| < \varepsilon$ $(\forall x \in [a,b])$

proof via Bernstein polynomials.

Remark Can we find a polynomial p such that

$$|f(x) - p(x)| < \varepsilon \quad \forall x \in \mathbb{R} \;?$$

ie., change $[a,b]$ to $\mathbb{R}$?

No! All interesting polynomials (meaning all polynomials that are not identically constant) must go to $\pm\infty$ as $|x| \to \infty$.

So if $f(x) = \sin(x)$, we cannot hope to find a global (i.e., all of $\mathbb{R}$) polynomial approximation. Must settle for local approximation.

This is another reason for the piecewise / composite approach.