Introduction to Interpolation

Tuesday, September 15, 2020 1:28 PM

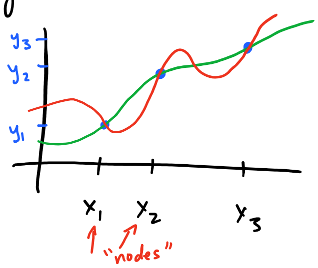
- Interpolation definitions
- Interpolating polynomials: existence / uniqueness,
  - vis-a-vis Weierstrass
  - Vandermonde matrix
- Overfitting
  - Fix: local / composite interpolation
- Runge phenomenon
  - Fix: don't use equally spaced points!
    - eg. Chebyshev nodes

to be discussed in more detail later in the course (or next semester)

Interpolation

Given a dataset $\{(x_i, y_i)\}$, an interpolant is a function g ("nodes": x_i)

that agrees with the dataset, meaning $g(x_i) = y_i$



} both are interpolants

There is not necessarily a "true" function f though sometimes there is.

Usually we want the interpolating function to be nice,
i.e., both easy to work with (ex: integrate)
and have nice properties like differentiability, etc.

Traditionally, "interpolation" means filling in the blanks inside an interval

Ex: it was 73°F at 8 AM
and 82°F at 11 AM.
What's a guess for the temperature at 9 AM?
(one approach: find an interpolant g such that $g(8) = 73$, $g(11) = 82$, then look at $g(9)$)

... in contrast to "extrapolation" which is about prediction outside your interval, i.e., the future.

In above example, extrapolation might ask what is the temperature at noon?

Interpolation isn't the only approach

- another class of methods is "least-squares" and many variants i.e. "best approximation"

- as you'll see, we'll focus on polynomials. The first thing you might think of is the Taylor polynomial but this is not a good idea usually, as these are only good very locally, and not on an interval

Polynomials

Often we require the interpolant to be a polynomial.

Why?
- Easy to work with, e.g. derivatives + integrals are very easy
- They are smooth ($C^\infty(\mathbb{R})$)
- Nice theory, and hundreds of years of experience

Ex: "interpolants" are not unique, but "polynomial interpolants" of low-degree are unique, as made precise below:

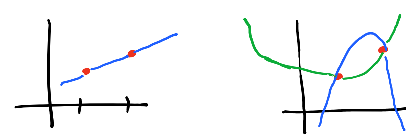
Theorem: uniqueness If two polynomials $p(x)$ and $q(x)$ agree on k data points $\{x_i\}_{i=1}^{k}$ and both polynomials are degree n or less,

(distinct points)

then $p=q$ (i.e., the interpolating polynomial of degree n is $\boxed{\text{unique}}$) (proof in last week's notes)

if $k > n$.

Ex: "$\underbrace{\text{two points}}_{k=2}$ determine a $\underbrace{\text{line}}_{n=1}$"



though an infinite number of quadratics interpolate 2 points.

Theorem: existence

If $\{x_i\}_{i=1}^{k}$ are distinct points, and $\{y_i\}_{i=1}^{k}$ are any y-values, then there exists a polynomial of degree $n = k-1$ that interpolates the data

proof: a $n=k-1$ degree polynomial looks like

$$p(x) = c_k x^{k-1} + c_{k-1} x^{k-2} + \dots + c_2 x^1 + c_1$$

so it has k degrees of freedom (parameters) $\{c_i\}_{i=1}^{k}$

Solve:

$$c_k x_1^{k-1} + \dots + c_2 x_1 + c_1 = y_1 \qquad \text{plug in } x = x_1$$
$$\vdots \qquad\qquad\qquad \vdots$$
$$c_k x_k^{k-1} + \dots + c_2 x_k + c_1 = y_k \qquad x = x_k$$

linear system

$$\begin{bmatrix} 1 & x_1 & x_1^2 & \cdots & x_1^{k-1} \\ 1 & x_2 & x_2^2 & & x_2^{k-1} \\ \vdots & \vdots & \vdots & & \vdots \\ 1 & x_k & x_k^2 & & x_k^{k-1} \end{bmatrix} \begin{bmatrix} c_1 \\ c_2 \\ \vdots \\ c_k \end{bmatrix} = \begin{bmatrix} y_1 \\ \vdots \\ y_k \end{bmatrix}, \quad \text{i.e. } \boxed{V \cdot \vec{c} = \vec{y}}$$

don't confuse with the Wronskian

a matrix of this form is called a Vandermonde matrix

and you can prove $\det(V) = \prod_{1 \le i < j \le k} (x_i - x_j)$

notation: "$\prod$" means product like "Σ" for sum

ex $\prod_{i=1}^{3} 2^i = 2^1 \cdot 2^2 \cdot 2^3$

so $\{x_i\}$ distinct $\Rightarrow \det(V) \neq 0 \Rightarrow V$ is invertible

$\Rightarrow \vec{c} = V^{-1}\vec{y}$ is the (unique) sol'n. $\square$

We actually proved existence and uniqueness.

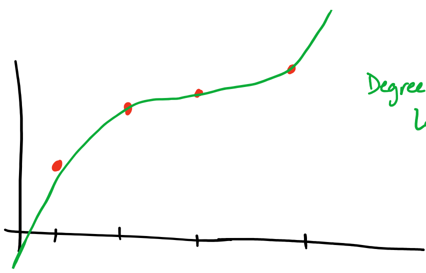
Construction

i.e. how to find coefficients $\{c_i\}$.

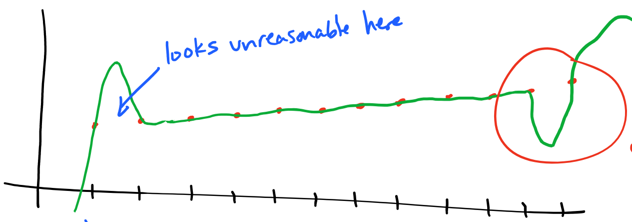
Do not invert the Vandermonde matrix, since this is ill-conditioned

Instead, see the other videos (ch 3.1, 3.3), so we'll come back to this

Overfitting



Degree 3 polynomial. Looks ok



and very unreasonable for extrapolation

High degree polynomial

these large values near the boundary of our interval is what we call the RUNGE PHENOMENON.

What to do? (it's a fundamental problem, independent of the algorithm)

① With many nodes, we can't interpolate globally with a low-degree polynomial. But, we could interpolate locally w/ a low-degree polynomial, and then stitch these together (i.e., create a piecewise polynomial)

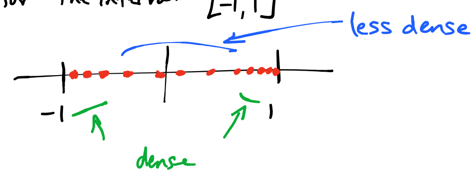
We'll later discuss splines and composite quadrature, which are both techniques based on this general idea

(2) If we have the luxury of choosing the nodes $\{x_i\}$,
(sometimes this is reasonable, like for numerical integration, though othertimes it's not reasonable, like if it's a measurement someone gives you).

then choose the nodes *wisely*. In particular, don't choose them equispaced, but rather they should concentrate near the boundaries (ex. Chebyshev nodes of the second kind)

$$x_i = -\cos\left(\frac{i \cdot \pi}{n}\right), \quad i = 0, 1, \ldots, n$$

for the interval $[-1, 1]$



More on this later
(ex., ch. 4.7 Gaussian Quadrature, 2nd semester
ch 8.3 Chebyshev polynomials)

Last bit of theory:
Weierstrass Approximation theorem

Thm: if $f \in C([a,b])$ then $(\forall \varepsilon > 0)$ $\exists$ a polynomial p such that $|f(x) - p(x)| < \varepsilon$ $(\forall x \in [a,b])$

proof via Bernstein polynomials.

Remark Can we find a polynomial p such that

$$|f(x) - p(x)| < \varepsilon \quad \forall x \in \mathbb{R} \; ?$$ i.e., change $[a,b]$ to $\mathbb{R}$?

No! All interesting polynomials (meaning all polynomials that are not identically constant) must go to $\pm\infty$ as $|x| \to \infty$.

So if $f(x) = \sin(x)$, we cannot hope to find a global (i.e., all of $\mathbb{R}$) polynomial approximation. Must settle for local approximation.

This is another reason for the piecewise / composite approach.