Covariance Differencing and Higher Order Cumulant Methods for DOA Array Processing in the Presence of Colored Noise Fields

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***Abstract*— Subspace methods for direction of arrival estimation such as Multiple SIgnal Classification (MUSIC) are common due to their high accuracy and computational efficiency, however the standard assumptions of spatially and temporally white noise degrade the performance. Methods for mitigating colored noise such as covariance transformation differencing and higher order cumulants have been suggested and researched for the purpose of handling colored noise fields. In this paper, we investigate the performance differences in the presence of various levels of colored noise across SNRs.**

I. INTRODUCTION

The Multiple SIgnal Classification (MUSIC) algorithm, introduced by Schmidt [1], is a method for estimating the directions of arrival (DOA) from multiple emitters incident on a sensor array, and after its introduction quickly became one of the most popular forms of DOA estimation. MUSIC uses eigendecomposition of the sensor covariance matrix, separating the signal and noise into orthogonal subspaces in order to search the angle of incidence space for vectors which are maximally orthogonal to the noise subspace. As such, the algorithm becomes highly sensitive to non-idealities in the covariance matrix, namely highly or fully correlated (or *coherent*) signals and colored noise. Much research has been put into the handling of coherent signals, such as occurs in multipath environments, such as spatial smoothing with forward and backward subarray techniques [2], [3], and more recently with subspace and wavelet smoothing techniques [4], [5]. The focus of this paper is on methods of handling cases of correlated and spatially colored noise, namely covariance array differencing and higher order cumulant processing. While the authors of the relevant previous works on these techniques provide simulation examples, their tests typically choose a single value for SNR, and exclude performance in cases where their assumptions may not hold. In this paper we introduce the relevant DOA techniques and simulate their performance across varying noise fields against SNR in terms of steady state arrival angle resolution error and convergence time.

II. BACKGROUND

In the following section, II-A introduces the standard MUSIC algorithm applied to a linear uniform linear array, II-B discusses the effects of colored noise fields on the MUSIC algorithm, II-C introduces covariance differencing transformation techniques, and II-D introduces fourth order cumulants techniques.

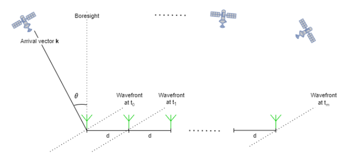

Fig. 1. 1D Uniform linear array receiving multiple signals

A. Standard MUSIC DOA

Consider a 1-dimensional uniform linear array (ULA) depicted in Fig. 1. The array consists of M identical isotropic sensors, each separated by a distance d, upon which there are N incident narrow-band signals approximated at frequency f_c arriving from N separate sources, each having an angle of arrival θ_k defined with reference to bore sight (90° from horizon). The signals are considered to be far-field, and are treated as plane waves. From the geometry of the system it is evident that the kth wavefront travels an additional $d\cos(\theta_k)$ between neighboring sensors, contributing a time difference of $d\cos(\theta_k)/c$, and a phase difference of $2\pi d\cos(\theta_k)/c$. For convenience, we define $a(\theta_k) = \exp\{-j2\pi\frac{d}{\lambda}\cos(\theta_k)k\}$ With reference to the first (leftmost) element of the array, the signal received by sensor k at time t is

$$x_k = \sum_{i=0}^{N} s_i(t)a(\theta_i) + n_k(t) \tag{1}$$

Where $s_i(t)$ is the signal transmitted by the ith source, λ is the source signal wavelength, and $n_k(t)$ is the additive noise on sensor k at time t. In vector notation, eqn. 1 becomes

$$x(t) = As(t) + n(t) \tag{2}$$

Under the assumption of signals which are uncorrelated to the noise, and the noise is spatially white, the array covariance matrix is

$$R_{xx} = E[X(t)X^\dagger(t)] \\ = AR_{ss}A^\dagger + \sigma_n^2 I \tag{3}$$

where R_{ss} the covariance matrix of the signal sources defined by $E[s(t)s^\dagger(t)]$.

So long as $d < \lambda/2$, all of the columns of A are different, and because A is Vandermonde they must also be linearly independent. So long as R_{ss} is non-singular, which is true when the source signals are not coherent, the rank of $AR_{ss}A^\dagger$ must be equal to the rank of R_{ss} which is N. In the noiseless case, this means that the array covariance R_{xx} has N non-zero eigenvalues, whose eigenvectors $E_s = [v_1, \ldots, v_N]$ span the *signal subspace*, and $M - N$ zero eigenvalues, whose eigenvectors $E_n = [u_{N+1}, \ldots, u_M]$ span the *null subspace*. Notably, the columns of A also span the signal subspace, meaning that $A^\dagger u_i = 0 \ \forall u_i \in E_n$. When noise is added, all of the eigenvalues increase by the noise power and the arrival vectors remain orthogonal to the *noise subspace*. A more formal treatment of the above is shown by Schmidt [1].

The above provides the basis for the DOA estimation, which is performed by maximizing the MUSIC psuedospectrum:

$$P_{MUSIC}(\theta) = \frac{1}{a^\dagger(\theta)E_nE_n^\dagger a(\theta)} \tag{4}$$

which is equivalent to finding the angles of arrival which are maximally orthogonal to the noise subspace.

B. *Colored Noise Fields*

In the derivation of the MUSIC algorithm, the noise is assumed to be spatially white – zero mean, equal in power, and uncorrelated between elements. In real sensor arrays, equal noise power across channels is unrealistic, and the assumption of uncorrelated noise is questionable when mutual inductance between array elements are significant, or when the receivers have low noise figures. Torrieri and Bakhru [6] quantify the effects of spatially colored noise on the MUSIC and reduced MUSIC algorithms via simulation, finding that unequal noise power and correlated noise lead to steady state bias in resolved angles, fluctuations in said steady state, and a drastically increased convergence time, especially as SNR decreases beyond 10dB. As such, methods for handling spatially colored noise are highly desirable.

C. *Covariance Differencing*

The essence behind covariance differencing techniques is as follows. Consider the array covariance matrix R_{xx} in eqn. 3. If we did not assume spatially white noise, we instead have

$$R_{xx} = AR_{ss}A^\dagger + R_{nn} \tag{5}$$

where R_{nn} is likely unknown and difficult to estimate accurately. Covariance differencing circumvents this problem by providing two covariance matrices in which the signal subspace is meaningfully transformed, but the noise subspace is left constant, and then subsequently differencing the two covariance matrices to suppress the effects of noise without suppressing the signals.

1) Multiple Measurement Approach: As Paulraj and Kailath discussed in [7], two array covariance matrices can be obtained by taking two measurements with some transformation. Transformations can be spatial, such as rotations and translation of the array, or temporal. For example, noise is invariant to rotational transformations in the case where the noise is primarily contributed by the array platform, while the signal's arrival vectors will be altered, changing the signal covariance. The use of a temporal transformation depends on the noise being long-term stationary, while the signals are only short-term stationary.

The new array covariance is described by the difference of the array covariance of the two measurements, R_1,R_2

$$P = R_1 - R_2 \\ = \begin{bmatrix} A_1 & A_2 \end{bmatrix} \begin{bmatrix} R_{ss,1} & 0 \\ 0 & R_{ss,2} \end{bmatrix} \begin{bmatrix} A_1 & A2 \end{bmatrix}^\dagger \tag{6}$$

As shown in [7], in the treatment of transformations in which $A_1 \neq A_2$, for example in a rotational transformation, if the columns of $\begin{bmatrix} A_1 & A2 \end{bmatrix}$ are unique and $R_{ss,1}$, $R_{ss,2}$ are non-singular, P will have $M - 2N$ zero eigenvalues, which have corresponding eigenvectors orthogonal to the columns of $\begin{bmatrix} A_1 & A2 \end{bmatrix}$. The rank of P is

$$\rho(P) = \rho(\begin{bmatrix} A_1 & A_2 \end{bmatrix} \begin{bmatrix} R_{ss,1} & 0 \\ 0 & R_{ss,2} \end{bmatrix} \begin{bmatrix} A_1 & A2 \end{bmatrix}^\dagger) \\ = \rho(\begin{bmatrix} R_{ss,1} & 0 \\ 0 & R_{ss,2} \end{bmatrix} = 2N \tag{7}$$

Given eqn. 7, it is shown that the subspace spanned by the eigenvectors corresponding to the non-zero eigenvalues is the same subspace as is spanned by the direction vectors of A_1, A_2.

Forming the MUSIC psuedospectrum requires application specific information. Consider the case of rotational transformation, such that every arrival vector $a(\theta_k) \in A_1$ has a corresponding arrival vector $a(\theta_{k+\alpha}) \in A_2$, where α is the rotation applied to the array plane between covariance measurements. Here the psuedospectrum is

$$\frac{1}{a^\dagger(\theta)E_nE_n^\dagger a(\theta) + a^\dagger(\theta+\alpha)E_nE_n^\dagger a(\theta+\alpha)} \tag{8}$$

where E_n are the noise subspace eigenvectors of the differenced array covariance matrix P.

2) Linear-Transformation Approach: The multiple measurement approach depends temporal stationarity, and typically a rotation or translation which may not be practical. Prasad *et al* [8] proposed a method using linear transformations on a single measurement, given *a priori* information about the structure of the noise covariance matrix R_{nn}. Prasad et al focus on a structure which assumes spatially stationary, but correlated noise – ie a Hermitian symmetric Teoplitz matrix.

Symmetric Teoplitz matrices are invariant under the tranformation

$$\begin{bmatrix} 0 & \cdots & & 1 \\ & & 1 & \\ & \cdots & & \\ & 1 & & \\ 1 & \cdots & & 0 \end{bmatrix} R_{nn} \begin{bmatrix} 0 & \cdots & & 1 \\ & & 1 & \\ & \cdots & & \\ & 1 & & \\ 1 & \cdots & & 0 \end{bmatrix} \tag{9}$$

leading to the array covariance difference

$$\begin{aligned} P &= R_{xx} - JR_{xx}J \\ &= AR_{ss}A^\dagger - JAR_{ss}A^\dagger J \\ &= \begin{bmatrix} A & JA \end{bmatrix} \begin{bmatrix} R_{ss} & 0 \\ 0 & -R_{ss} \end{bmatrix} \begin{bmatrix} A & JA \end{bmatrix}^\dagger \end{aligned} \tag{10}$$

Proof that the new covariance upholds signal/noise subspace orthogonality is covered in [8]. Using the new covariance, the arrival angles are estimated with the MUSIC psuedospectrum as before.

Moghaddamjoo [9] discusses a similar process, now in the case where the structure is not Toeplitz, but rather non-uniform diagonal, ie the uncorrelated case with unequal power noise across sensors.

D. Higher Order Cumulants

1) Gaussian Noise: Porat and Friedlander [10] introduce a method of MUSIC which uses fourth order cumulants in the place of covariance. Cumulants of degree higher than 2 have the benefit that Gaussian processes are zero, meaning that Gaussian noise is completely suppressed in the DOA estimate with no *a priori* knowledge of the noise covariance. It is important to note that the signals must be non-Gaussian, or else they too will be suppressed.

For the second order moment (covariance) defined by

$$\mu_2(k,l) = E[y_k(t)y_l^*(t)] \tag{11}$$

and fourth order moment (kurtosis) defined by

$$\mu_4(k_1,k_2,l_1,l_2) = E[y_{k_1}(t)y_{k_2}(t)y_{l_1}^*(t)y_{l_2}^*(t)] \tag{12}$$

the fourth order cumulant can be given by

$$\begin{aligned} \kappa_4(k_1,k_2,l_1,l_2) = \mu_4(k_1,k_2,l_1,l_2) \\ - \mu_2(k_1,l_1)\mu_2(k_2,l_2) \\ - \mu_2(k_1,l_2)\mu_2(k_2,l_1) \end{aligned} \tag{13}$$

The fourth order cumulant matrix can be formed using Kronecker products ($\otimes$) as

$$C = (A \otimes A^*)S(A \otimes A^*)^\dagger \tag{14}$$

which is an M^2xM^2 matrix for the 4th order cumulant matrix of $s(t)$:

$$\begin{aligned} S = E[(s(t) \otimes s^*(t))(s(t) \otimes s^*(t))^\dagger] \\ -E[s(t) \otimes s^*(t)]E[(s(t) \otimes s^*(t))^\dagger] \\ -E[s(t)s^\dagger(t)] \otimes E[s(t)s^\dagger(t)] \end{aligned} \tag{15}$$

For statistically independent (incoherent) sources, the matrix C will have $M^2 - N^2$ zero eigenvalues, with the rest being non-zero, but potentially negative. With singular value decomposition,

$$C = \begin{bmatrix} U_1 & U_2 \end{bmatrix} \begin{bmatrix} \Lambda & 0 \\ 0 & 0 \end{bmatrix} \begin{bmatrix} V_1^\dagger \\ V_2^\dagger \end{bmatrix} \tag{16}$$

The columns of matrix $(A \otimes A^*)$ are orthogonal to the columns of U_2, ie the vectors which span the null subspace, meaning we can define the psuedospectrum as

$$P_{MUSIC} = \frac{1}{\|(a(\theta) \otimes a^*(\theta))^\dagger U_2\|^2} \tag{17}$$

2) Non-Gaussian Noise: The primary benefit of 4th order statistics over the traditional second order is Gaussian noise suppression. Dogan and Mendel [11] introduce a study of the effects of non-Gaussian noise on 4th order cumulant methods in DOA estimation and propose methods of non-Gaussian noise suppression with 4th order cumulants.

TODO!!

III. METHODS

The goal of this study is to investigate the performance of the aforementioned modified MUSIC algorithms against SNR for varying noise fields.

1) Algorithms: The algorithms which are studied are

- Standard MUSIC [1]
- Covariance differencing under Toeplitz and diagonal assumptions [8][9]
- 4th order cumulants method [10]

2) Noise Covariance Structures: The noise fields which are studied are defined by the structure of the noise covariance matrices. The relevant structures used are

- **Uniform diagonal**, corresponding to the ideal case of uncorrelated noise of equal power across sensors.
- **Non-Uniform diagonal**, corresponding to the case of uncorrelated noise of unequal power across sensors.
- **Block diagonal**, corresponding to the case of correlated noise such that closer sensors are correlated, as in the case of correlated noise caused by coupling in the receiver.
- **Symmetric Toeplitz**, corresponding to the case of correlated noise of equal power across sensors, as may occur when receiver noise figures are very low, and noise sources may be off-platform.
- **Symmetric Non-Toeplitz**, corresponding to the case of correlated noise of unequal power across sensors, as may occur when a mix of the above situations are equally relevant.

For non-diagonal symmetric noise covariances, it is meaningful to evaluate correlation ratios, meaning the ratio of the cross-covariance to the auto-covariance or power. We can construct a correlation ratio matrix CR such that

$$CR_{i,j} = \frac{R_{i,j}}{\min\{R_{i,i}, R_{j,j}\}} \tag{18}$$

which can be used to depict "how correlated" the noise is between given sensors.

The prescribed covariance matrices are obtained by eigen-decomposition of the desired matrix and a set of normally distributed random variables [12]:

$$\begin{gathered} R_{nn} = E\Lambda E^\dagger \\ x(t) = E(\Lambda)^{\frac{1}{2}} \begin{bmatrix} n_1(t) \\ n_2(t) \\ \vdots \\ n_M(t) \end{bmatrix} \end{gathered} \tag{19}$$

where n_m are normally distributed random processes, E are the eigenvectors and Λ the eigenvalues of the noise covariance matrix, and $x(t) \in \mathbb{C}^M$ the noise random process having prescribed covariance.

3) Metrics: Convergence time is defined as the number of samples required to reach a steady state, which occurs when the resolved angles are within 2° of the true angles for 10 samples. The bias error is defined by the difference between the true angles and the resolved angle in the steady state. The standard deviation of the bias error in the steady state is also investigated. The metrics are evaluated against varying SNR in the range [-20, 20] dB, which is defined by the *minimum* ratio of signal power to noise power across the sensors, as SNR may not be equal across the array or across signal sources depending on the noise covariance structure.

4) Array Configuration: The array is configured as a 16 isotropic element uniform linear array with quarter wave-length separation, with no steering or weighting applied. The array will be used to detect arrival angles from 4 non-coherent sources which are generated using random symbols in a 16-QAM modulation scheme.

IV. RESULTS

V. DISCUSSION

VI. CONCLUSION

REFERENCES

[1] R. Schmidt, "Multiple emitter location and signal parameter estimation," in IEEE Transactions on Antennas and Propagation, vol. 34, no. 3, pp. 276-280, March 1986, doi: 10.1109/TAP.1986.1143830.

[2] Tie-Jun Shan, M. Wax and T. Kailath, "On spatial smoothing for direction-of-arrival estimation of coherent signals," in IEEE Transactions on Acoustics, Speech, and Signal Processing, vol. 33, no. 4, pp. 806-811, August 1985, doi: 10.1109/TASSP.1985.1164649.

[3] S. U. Pillai and B. H. Kwon, "Forward/backward spatial smoothing techniques for coherent signal identification," in IEEE Transactions on Acoustics, Speech, and Signal Processing, vol. 37, no. 1, pp. 8-15, Jan. 1989, doi: 10.1109/29.17496

[4] Dominic Grenier, Éloi Bossé, A new spatial smoothing scheme for direction-of-arrivals of correlated sources, Signal Processing, Volume 54, Issue 2, 1996, Pages 153-160, ISSN 0165-1684, https://doi.org/10.1016/S0165-1684(96)00104-1.

[5] Yanbo Xue, Jinkuan Wang and Zhigang Liu, "A novel improved MUSIC algorithm by wavelet denoising in spatially correlated noises," IEEE International Symposium on Communications and Information Technology, 2005. ISCIT 2005., Beijing, China, 2005, pp. 515-518, doi: 10.1109/ISCIT.2005.1566906.

[6] D. Torrieri and K. Bakhru, "The effects of nonuniform and correlated noise on superresolution algorithms," in IEEE Transactions on Antennas and Propagation, vol. 45, no. 8, pp. 1214-1218, Aug. 1997, doi: 10.1109/8.611239.

[7] A. Paulraj and T. Kailath, "Eigenstructure methods for direction of arrival estimation in the presence of unknown noise fields," in IEEE Transactions on Acoustics, Speech, and Signal Processing, vol. 34, no. 1, pp. 13-20, February 1986, doi: 10.1109/TASSP.1986.1164776

[8] S. Prasad, R. T. Williams, A. K. Mahalanabis and L. H. Sibul, "A transform-based covariance differencing approach for some classes of parameter estimation problems," in IEEE Transactions on Acoustics, Speech, and Signal Processing, vol. 36, no. 5, pp. 631-641, May 1988, doi: 10.1109/29.1573.

[9] A. Moghaddamjoo, "Transform-based covariance differencing approach to the array with spatially nonstationary noise," in IEEE Transactions on Signal Processing, vol. 39, no. 1, pp. 219-221, Jan. 1991, doi: 10.1109/78.80789.

[10] B. Porat and B. Friedlander, "Direction finding algorithms based on high-order statistics," in IEEE Transactions on Signal Processing, vol. 39, no. 9, pp. 2016-2024, Sept. 1991, doi: 10.1109/78.134434.

[11] M. C. Dogan and J. M. Mendel, "Applications of cumulants to array processing. II. Non-Gaussian noise suppression," in IEEE Transactions on Signal Processing, vol. 43, no. 7, pp. 1663-1676, July 1995, doi: 10.1109/78.398727.

[12] N. Bienert, "Lecture 6", Stochastic and Environmental Signal Processing, ECEN 5244, University of Colorado at Boulder, pp. 4-8, September 2024.