Serre's Problem - An Overview

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Abstract

This paper presents a review of Serre's problem, first proved by Daniel Quillen and Andrei Suslin. All proofs are taken from 'Algebra' by Serge Lang. The aim of this paper is to reproduce those proofs with as much detail added in order to make them clear and accessible to the average undergraduate. We assume that the reader has a basic understanding of rings and modules. We define necessary concepts, state the main theorems, and investigate their rigorous proofs. Further implications and applications are discussed.

1 Preliminary Definitions

Let A be a commutative ring with unity.

Definition 1.1 (Unimodular). Let $f = (f_1, \dots, f_n)$ be a vector in $A^{(n)}$. We call f unimodular if the components of f generate the unit ideal of A. That is for some elements $a_i \in A$, $\sum_i a_i f_i = 1$.

Definition 1.2 (Unimodular Extension Property). Let $f = (f_1, \dots, f_n)$ be a unimodular vector in $A^{(n)}$. We say that f has the unimodular extension property if there exists a matrix in $GL_n(A)$ with first column $(f_1, \dots, f_n)^T$.

Note that f has the unimodular extension property if and only if any permutation or transformation of the components of f has the unimodular extension property. Specifically, we can do row operations to change f without taking the matrix out of $GL_n(A)$.

Definition 1.3 (Equivalence). We say that two column vectors f and g are equivalent over A, or more succinctly $f \sim g$ if there exists a matrix $M \in GL_n(A)$ such that $f = Mg$. For example, Horrocks' theorem claims that a unimodular vector with leading coefficient one is equivalent to the unit vector e_1 (the first unit vector) over $\omega[x]$.

2 Horrocks' Theorem

Theorem 2.1 (Horrocks). *Let (ω, m) be a local ring and let $A = \omega[x]$ be the polynomial ring in one variable over ω. Let f be a unimodular vector in $A^{(n)}$ such that some component has leading coefficient 1. Then f has the unimodular extension property.*

Proof. We proceed by an induction on the smallest degree d of a component of f with leading coefficient 1. Let n denote the length of the unimodular vector f. Note that the theorem does not require induction when $n = 1, 2$, as when it is one then f must consist of a unit, and if it is two then we know that $r_1 f_1 + r_2 f_2 = 1$ for some $r_1, r_2 \in A$, and we can set up the following matrix in $GL_n(A)$:

$$\begin{bmatrix} f_1 & -r_2 \\ f_2 & r_1 \end{bmatrix}$$

So we assume that $n \geq 3$. Before we proceed with the proof, note that without loss of generality we can assume that f_1 has leading coefficent 1, as some component of f has leading coefficient 1 and we can move whichever one it is to the top of the matrix. Further, since f_1 has degree d (by definition of d) and leading coefficient 1, we can use row operations to get that all other f_i must have degree less than d.

Now we will show that not all coefficients of $f_2, \ldots, f_n$ lie in the maximal ideal m. To do this, consider the relation

$$\sum_i g_i f_i = 1$$

given by definition of unimodularity, where $g_1, \ldots, g_n \in \omega[x]$. Assume on the contrary that all coefficients of $f_2, \ldots, f_n$ lie in the maximal ideal m. We may rewrite the relation as the following:

$$g_1 f_1 = 1 - \sum_{i=2}^{n} g_i f_i$$

Since each coefficient of $f_2, \ldots, f_n$ are in m, we observe the previous relation modulo $m[x]$. Recall that $1 \notin m[x]$

$$\{g_1 f_1\} \equiv \{1\} \mod m[x]$$

This shows us that f_1 is a unit in the ring $\omega[x]/m[x]$, and this can only happen when f_1 is a constant polynomial, but by assumption f_1 is of degree d, a contradiction. (Note in the case that d is actually zero the theorem is proved as 1 is a component of f).

Thus we can assume that some component of some $f_2, ..., f_n$ has a coefficient which is a unit (if it is not in the maximal ideal it must be a unit). Now observe that through row operations, we can assume that f_2 is the component with some coefficient which is a unit. For clarity, lets write out the structure of $f_1(x)$ and $f_2(x)$.

$$f_1(x) = x^d + a_{d-1}x^{d-1} + \cdots + a_0$$

$$f_2(x) = b_s x^s + \cdots + b_0$$

where $a_i, b_i \in \omega$ for all a_i and with at least one b_i which is a unit. Let α be the ideal generated by all leading coefficients of polynomials $g_1 f_1 + g_2 f_2$ of degree less than d. Now we will prove that α contains all the coefficients b_i, and since at least one is a unit, α is the unit ideal. We will do this by descending strong induction on the coefficients b_i.

For the base case, we want to show that $b_s \in \alpha$. We see this by taking $g_1 = 0$ and $g_2 = 1$, giving us the relation $(0)f_1 + (1)f_2 = f_2$, which has leading coefficient b_s and degree $s < d$. So $b_s \in \alpha$. Now for the inductive step assume that $b_s, b_{s-1}, \ldots, b_j \in \alpha$, we will show that $b_{j-1} \in \alpha$. To do this, lets take the relation

$$h = x^{d-j} f_2 - b_j f_1$$

which gives us the following polynomial:

$$h = b_s x^{d-j+s} + b_{s-1}x^{d-j+s-1} + \cdots + b_j x^{d-j+j} - b_j x^d + (b_{j-1} - b_j a_{d-1})x^{d-1} + \cdots + (-b_j a_0)$$

Observe that in this polynomial the x^d terms cancel, but since $s < d$ this polynomial is of degree greater than $d-1$. However, observe that we can subtract multiples of f_1 (multiplied by a suitable x and coefficients b_i) to cancel any terms with degree $\geq d$. For example to cancel out the $b_s x^{d-j+s}$ term in h, we can take the relation $h - b_s x^{s-j} f_1$. It is important to acknowledge that this may change the coefficient of x^{d-1}, however it will only do so by a factor of $b_k a_i$, with $k \geq j$ (we will soon show why this isn't an issue). Let p be the total polynomial which was needed to cancel out all of the terms of degree greater than $d-1$ in h. Then $h - pf_1$ has leading coefficient

$$b_{s-1} - b_j a_{d-1} - (\text{terms in the form } b_k a_i), \text{with } k \geq j$$

Additionally, since $h - pf_1 = g_1 f_1 + g_2 f_2 - pf_1 = (g_1 - p)f_1 + g_2 f_2$, then $h - pf_1$ is also a polynomial of degree less than d which is a $\omega[x]$-linear combination of f_1, f_2, so its leading coefficient is in α. Also, sincen each b_i with $k \geq j$ was in α (by strong induction hypothesis), then each $b_i a_k$ is also in α (because it is an ideal). So now we have that

$$b_{s-1} - b_j a_{d-1} - (\text{terms in the form } b_k a_k i) \in \alpha, \text{with } k \geq j,\, b_k \in \alpha \text{ for } k \geq j$$

From this we can subtract the terms we know are already in the ideal, the $b_i a_k$, to get that $b_{s-1} \in \alpha$, completing the descending induction. Finally, because some b_i is a unit, we must have that α is the unit ideal.

Before we complete the proof of Horrocks Theorem, we need to prove a short Lemma.

Lemma 2.2. *If $q(x), r(x)$ are polynomials such that r is monic, then there exists another polynomial σ such that $q - \sigma r$ is monic of degree: $max\{deg(q), deq(r)\}$.*

Proof. To prove this statement, we split into cases. Let $\deg(q) = m$ and $\deg(r) = \ell$

(1) $m < \ell$. In this case, take $\sigma = -1$, then $q - \sigma r$ has leading coefficient the same as r, which is one, and the same degree as r.

(2) $m > \ell$. If $q = c_m x^m + c_{m-1}x^{m-1} + \cdots + c_0$, and $r = x^\ell + e_{\ell-1}x^{\ell-1} \cdots + e_0$, then we choose $\sigma = (c_m - 1)x^{m-\ell}$. So now we have that $q - \sigma r = (c_m - c_m + 1)x^m + \cdots + c_0 = x^m + \cdots + c_0$, which is a monic polynomial of degree $m < d$

(3) $m = \ell$. If $q = c_m x^m + c_{m-1}x^{m-1} + \cdots + c_0$, and $r = x^\ell + e_{\ell-1}x^{\ell-1} \cdots + e_0$, then we choose $\sigma = (c_m - 1)$, so $q - \sigma r = (c_m - c_m + 1)x^m + \cdots + c_0 = x^m + \cdots + c_0$, which is again a monic polynomial of degree $m < d$.

This proves the lemma. □

Now here is where we use descending induction on the degree d of the polynomial with leading coefficient 1 and minimal degree among f. By our earlier conclusion that α was the unit ideal, we must have that there exists a polynomial of the form $g_1 f_1 + g_2 f_2$ which has leading coefficient 1 and degree less than d, lets call it $y = g_1 f_1 + g_2 f_2$ for some g_1, g_2. Now we can use elementary row operations along with Lemma 2.2 to construct an equivalent unimodular vector with a component of degree less than d with leading coefficient one. Visually this corresponds to the row operation shown below of Row 3=

Row 3 - σy, where we use Lemma 2.2 with $q = f_3$ and $r = y$.

$$\begin{bmatrix} f_1 \\ f_2 \\ f_3 \\ \vdots \\ f_n \end{bmatrix} \xrightarrow{\text{row operations}} \begin{bmatrix} f_1 \\ f_2 \\ f_3 - \sigma y \\ \vdots \\ f_n \end{bmatrix}$$

Thus Row 3 now has a polynomial of degree less than d with leading coefficient 1. Now we can move this entry to row one and repeat this process, inducting down on the minimal degree of a polynomial with leading coefficient 1 eventually getting to $d = 0$. In this case, the theorem is proved as we have a unimodular vector in which one component is 1, and we can use row operations to cancel out all the other components and show that this vector is equivalent to e_1 (the first standard basis vector). $\square$

3 Corollaries - Horrocks

Before we continue, lets recall the definition of equivalence for two unimodular vectors. Let A be a commutative ring.

Definition 3.1 (Equivalence)**.** We say that two column vectors f and g are equivalent over A, or more succinctly $f \sim g$ if there exists a matrix $M \in GL_n(A)$ such that $f = Mg$.

Corollary 3.2. *Let ω be a local ring. Let f be a unimodular vector in $\omega[x]^{(n)}$ such that some component has leading coefficient one. Then $f \sim f(0)$ over $\omega[x]$.*

Proof. By Horrocks, we know that $f(0)$, a member of $\omega^{(n)}$, has some component which is a unit. Now we can use row operations to show that any element of $\omega^{(n)}$ which contains a unit is equivalent to e_1 over ω. Now we can use the one unit and row operations (which are all invertible) to form an M such that $f(0) = Mf$. $\square$

Lemma 3.3 (Two Variables)**.** *Let R be an integral domain, with a multiplicative subset S. Let x, y be independent variables. If $f(x) \sim f(0)$ over $S^{-1}R[x]$, then there exists $c \in S$ such that $f(x + cy) \sim f(x)$ over $R[x, y]$.*

Proof. Let $M \in GL_n(S^{-1}R[x])$ be a matrix such that $f(x) = M(x)f(0)$. By this construction, we know that $M^{-1}(x)f(x) = f(0)$ is a constant, so it does not depend on x. Thus we can introduce the translation $x \mapsto x + y$ without affecting this definition. Now we define another matrix,

$$G(x, y) = M(x)M(x + y)^{-1}$$

Where we similarly define $M(x + y)^{-1}f(x + y) = f(0)$. Now observe that we can obtain the following expression through substitution.

$$G(x, y)f(x + y) = M(x)M(x + y)^{-1}f(x + y) = M(x)f(0) = f(x)$$

Additionally, by definition of $G(x, y)$, we have that $G(x, 0) = M(x)M(x + 0)^{-1} = I$. Expanding on this, we have that $G(x, y) = I + yH(x, y)$ for some $H(x, y) \in GL_n(S^{-1}R[x, y])$. Since the coefficients of $H(x, y)$ are from $S^{-1}R$, we know that there exists some $c \in S$ such that $cH(x, y)$ has coefficients in R (here we are essentially clearing denominators). From this we can deduce that $G(x, cy)$'s entries must have coefficients in R.

Since we know that $M(x)^{-1}$ exists we must have that $det(M(x))$ is a unit in $S^{-1}R$. From the definition of $M(x)$, we must have that $det(M(x+cy))$ is also equal to that same unit, and since $G(x, cy) = M(x)M(x+cy)^{-1}$, we must have that the determinant of $G(x, cy)$ is 1, so $G(x, cy)$ is invertible over $GL_n(R[x, y])$. Thus we have that $f(x) = G(x, cy)f(x + cy)$, and thus $f(x + cy) \sim f(x)$ over $R[x, y]$. □

Theorem 3.4. *Let R be an integral domain, let $f \in R[x]^{(n)}$ such that some component is monic. Then $f \sim f(0)$ over $R[x]$.*

Proof. Let J be the set of elements $c \in R$ such that $f(x + cy) \sim f(x)$ over $R[x, y]$. We claim that J is an ideal, and further that $J = R$. First observe that if $c, c' \in J$, then $f(x + cy) \sim f(x) \sim f(x + c'y)$ implies that $f(x + (c + c')y) \sim f(x + cy) \sim f(x)$, so $f(x+(c+c')y) \sim f(x)$ (essentially we shift $f(x+cy)$ by $x \mapsto x+c'y$). Similarly if $a \in R$, $c \in J$, then $f(x + cay) \sim f(x)$ over $R[x, ay]$, and since $R[x, ay] \subseteq R[x, y]$, we have that $f(x + cay) \sim f(x)$ over $R[x, y]$. Thus J is an ideal. Now to show $J = R$ we let $\mathfrak{p}$ be a prime ideal of R. Then by corollary 3.2 we have that $f \sim f(0)$ over $R_{\mathfrak{p}}[x]$. Similarly by Lemma 3.3 we have that there exists some $c \in R$ with $c \notin \mathfrak{p}$ such that $f(x + cy) \sim f(x)$ over $R[x, y]$. Since $\mathfrak{p}$ was arbitrary, we have that $J \not\subset \mathfrak{p}$ for all $\mathfrak{p} \subset R$. Thus J is the unit ideal, and we can say that $f(x+y) \sim f(x)$ over $R[x, y]$, or there exists an invertible matrix $M(x, y)$ over $R[x, y]$ such that $f(x + y) = M(x, y)f(x)$.

Now to conclude this proof we take the ring homomorphism $\varphi : R[x, y] \to R[y]$ which sends x to 0. Applying φ component-wise to $M(x, y)$ in the equality $f(x + y) = M(x, y)f(x)$ gives us another invertible matrix $M(0, y)$ such that $f(y) = M(0, y)f(0)$, or $f(y) \sim f(0)$ over $R[y]$. □

4 Quillen-Suslin & UCEP

Theorem 4.1. *Let k be a field and f be a unimodular vector in $k[x_1, \cdots, x_n]^{(n)}$. Then f has the unimodular extension property.*

Proof. We proceed by induction on the number of variables n in $k[x_1, \ldots, x_n]$. Observe that for the base case $n = 1$ we take any unimodular vector $f \in k[x]^{(m)}$. Then we take any nonzero component of f (it must have at least one), and multiply that component by the inverse of the leading coefficient of that component. Note that this is an invertible row operation because k is a field. Thus we obtain a unimodular vector such that at least one component is monic, and we apply Theorem 3.4 to conclude that $f \sim f(0)$, and by the same reasoning as Corollary 3.2 we have $f(0) \sim e_1$ (the first standard basis vector) over $k[x]$.

Now suppose that any unimodular vector in $k[x_1, \ldots, x_{n-1}]^{(m)}$ has the unimodular extension property. Let $f \in k[x_1, \ldots, x_n]^{(m)}$ be a unimodular vector. We view f as a vector of polynomials in the last variable x_n and aim to apply Theorem 3.4, but to do this we need some component of f to be monic in the variable x_n. To do this lets introduce an invertible substitution of variables $x_n = y_n$ and $x_i = y_i - y_n^{r_i}$ for some scalars $r_1 >> \cdots >> r_{n-1}$. In particular observe that if we take a nonzero component of f, call it f_i. It has the form

$$f_i = \sum c x_1^{\alpha_1} x_2^{\alpha_2} \ldots x_n^{\alpha_n}$$

for some exponents c and exponents α_i, where we sum terms of the polynomial in the component of f. Let $cx_1^{\alpha_1} x_2^{\alpha_2} \ldots x_n^{\alpha_n}$ be the term of the polynomial such that $r_1\alpha_1 + \cdots +$

$r_{n-1}\alpha_{n-1} + \alpha_n$ is maximal (Note we can choose $r_1, \ldots, r_{n-1}$ sufficiently different so that this maximal sum is unique). Then after the substitution we get

$$f_i = \sum c(y_1 - y_n^{r_1})^{\alpha_1} \ldots (y_{n-1} - y_n^{r_{n-1}})^{\alpha_{r-1}}(y_n)^{\alpha_n}$$

Now observe that because we chose $r_1, \ldots, r_{n-1}$ so that one term dominates in the sum of exponents we have that f_i has the form

$$f_i = cy_n^{r_1\alpha_1+\cdots+r_{n-1}\alpha_{n-1}+\alpha_n} + \sum \text{terms with } y_n \text{ exponent} < r_1\alpha_1 + \cdots + r_{n-1}\alpha_{n-1} + \alpha_n$$

Now since k is a field and $c \neq 0$ we multiply the row f_i by c^{-1} (this corresponds to an invertible row operation on the column vecotr f) to get a monic polynomial in y_r. Then we can apply Theorem 3.4 to get that there exists an invertible matrix $M(y_1, \ldots, y_n)$ such that

$$f(y_1, \ldots, y_n) = M(y_1, \ldots, y_n)f(y_1, \ldots, y_{n-1}, 0)$$

Now by our induction hypothesis $f(y_1, \ldots, y_{n-1}, 0)$ has the unimodular extension property, and there exists an invertible matrix E in $GL_m(k[y_1, \ldots, y_{n-1}])$ with first column $f(y_1, \ldots, y_{n-1}, 0)$. Now undoing our previous substitution by using $y_n = x_n$ and $y_i = x_i + x_n^{r_i}$ we get a vector which has the unimodular extension property (since E is invertible its determinant is constant and so its elements invariant under translation) which is equivalent to our original unimodular column f, completing the proof. □

Definition 4.2 (Stably Free)**.** We say that a module M over a commutative ring R is stably free if there exists a finite free module F such that $M \oplus F$ is finite free.

Theorem 4.3. *Let R be a commutative ring which has the unimodular extension property. Then every stably free module over R is free.*

Proof. Let E be stably free. We proceed by induction on the rank of free modules F such that $E \oplus F$ is free, starting with the base case $F \cong R$. Then $E \oplus R \cong R^n$ for some $n \in \mathbb{N}$. Let $p : R^n \to R$ be the projection map. Let u_1 be a basis of R over itself. Then viewing R as a summand in $E \oplus R \cong R^n$ we have $u_1 = (a_{11}, \ldots, a_{n1})^T$ with $a_{i1} \in R$. Now since p is a linear map, it has a matrix representation $[p_1, \ldots, p_n]$ for some $p_i \in R$. Then since $p(u_1) = 1$ we have that

$$\begin{bmatrix} p_1 & p_2 & \cdots & p_n \end{bmatrix} \begin{bmatrix} a_{11} \\ a_{21} \\ \vdots \\ a_{n1} \end{bmatrix} = p_1a_{11} + \cdots + p_na_{n1} = 1$$

So by definition u_1 is unimodular and by hypothesis there exists an invertible matrix $M = (a_{ij})$. Let the jth column of M be $u_j = Me_j$ for $j = 1, \ldots, n$, where e_j is the j-th standard basis vector of R^n. Then we wish to change M in such a way that $p(u_k) = 0$ for $k \neq 1$, meaning that for $k = 2, \ldots, n$ we would have $u_k \in E$. To do this note that if $p(u_k) = c$, then use elementary column operations to change the $k - th$ column to $u_k - cu_1$. Then $p(u_k - cu_1) = p(u_k) - cp(u_1) = c - c = 0$. By repeating this process, we get that $u_2, \ldots, u_n \in E$, and since M corresponds to an automorphism of R^n we have that M maps the free module with basis $(e_2, \ldots, e_n)$ onto E, so $E \cong R^{n-1}$, completing the proof.

Now for the inductive step suppose that if $E \oplus R^{n-1}$ is free then E is free. Suppose that $E \oplus R^n$ is free. Observe that $E \oplus R^n = (E \oplus R) \oplus R^{n-1}$, and by the induction hypothesis we have that $E \oplus R$ is free. But by our work in the base case we see that this implies that E is free, completing the inductive step. □

5 Projective over $k[x_1, \dots, x_n] \implies$ Stably Free

In this section we aim to show that every projective module over $k[x_1, \dots, x_n]$ is stably free. Combining this with work done in the Quillen-Suslin Theorem we will have that every projective module over $k[x_1, \dots, x_n]$ is free.

Definition 5.1 (Projective)**.** We say that a module M is projective if it is the direct summand of a free module. Observe that every free and stably free module is projective.

Definition 5.2 (Finite Free Resolution)**.** We say that a module M has finite free resolution if there exists an exact sequence of finite length

$$0 \to E_n \to \cdots \to E_0 \to M \to 0$$

such that each E_i is finite free. Note an exact sequence of modules is just a sequence of module homomorphisms such that the image of the previous homomorphism is the kernel of the next homomorphism.

Another property of projective modules is that every short exact sequence ending with a projective module splits, i.e. if M is projective and

$$0 \to E_1 \to E_0 \to M \to 0$$

is a short exact sequence, then $E_0 \cong E_1 \oplus M$.

Theorem 5.3. *Let M be a projective module over a commutative ring R. Then M is stably free if and only if M admits a finite free resolution.*

Proof. For the forward direction observe that if M is stably free then there exists a finite free module F such that $M \oplus F \cong R^n$. By definition of a projective module we have the following finite free resolution for M.

$$0 \to F \to R^n \to M \to 0$$

For the backwards direction we proceed by induction on the length n of finite free resolutions of M. For the base case $n = 0$ we have that there exists a finite free module E_0 such that

$$0 \to E_0 \to M \to 0$$

is exact, or $E_0 \cong M$. So M is free, and thus stably free. Now suppose that any projective module which has finite free resolution of length $n-1$ or less is stably free. Suppose now that M has finite free resolution of length n. Then there exist free modules $E_n, \dots, E_0$ such that

$$0 \to E_n \to \cdots \to E_0 \to M \to 0$$

is exact. Let $M_1 := \ker(E_0 \to M)$. Then we have the short exact sequence

$$0 \to M_1 \to E_0 \to M \to 0$$

. Now since M is projective, this exact sequence splits, and $E_0 \cong M_1 \oplus M$. Observe that M_1 has finite free resolution

$$0 \to E_n \to \cdots \to E_1 \to M_1 \to 0$$

which is of length $n-1$, and since M_1 is projective (it is a direct summand of the free module E_0) we can apply the induction hypothesis to get that M_1 is stably free, i.e. there exists a free module F such that $M_1 \oplus F$ is free. Now observe that $E_0 \oplus F \cong M_1 \oplus F \oplus M$, and since $M_1 \oplus F$ and $E_0 \oplus F$ are free we must have that M is stably free, completing the proof. □

Lemma 5.4. *Let $0 \to M_1 \to M_2 \to M_3 \to 0$ be an exact sequence. Then if any two of these modules have a finite free resolution, then so does the third.*

Theorem 5.5. *Let R be a commutative Noetherian ring. If every finite R-module has finite free resolution, then every finite $R[x]$-module has finite free resolution*

Proof. For proof of Lemma 5.4 and Theorem 5.5, see Lang's Algebra Chapter XXI Theorem 2.7 and Chapter XX Lemma 3.8. The proofs are not long, however too much background knowledge for the average undergraduate is needed to prove them. □

Theorem 5.6 (Serre)**.** *If k is a field then every finite projective module over $k[x_1, \dots, x_n]$ is stably free.*

Proof. By Theorem 5.3, it suffices to show that every finite projective module over $k[x_1, \dots, x_n]$ admits a finite free resolution. We proceed by induction on the number of variables n. If $n = 0$ then k is a field, and every finite projective module over k is a finite dimensional vector space which admits a basis, and is thus free. Suppose that every finite projective module over $k[x_1, \dots, x_{n-1}]$ admits a finite free resolution, then by Theorem 5.5, every finite projective module over $k[x_1, \dots, x_{n-1}][x_n] = k[x_1, \dots, x_n]$ admits a finite free resolution, completing the inductive step. Now by Theorem 5.3, every finite projective module over $k[x_1, \dots, x_n]$ is stably free. □

6 Conclusion

Theorem 6.1. *Let k be a field. Then every finite projective module over the polynomial ring $k[x_1, \dots, x_n]$ is free.*

Proof. This follows from Theorem 5.6, which establishes that every projective module over this ring is stably free, and Theorem 4.1 & 4.3, which establishes that all stably free modules over this ring are free. □

The Quillen-Suslin theorem is a foundational result in algebraic geometry, establishing a general result about vector bundles over algebraic affine space. If you want to learn more about the connection made by the Quillen-Suslin theorem between vector bundles and projective modules, we produced a poster about this topic for the directed reading program 2025 at UC Santa Barbara. Link: `https://natehurst.github.io/DRPposter`.

References

[1] Lang, *Algebra*, Springer, 2002.