The Quillen-Suslin Theorem

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Department of Mathematics - Directed Reading Program 2025

Serre's Problem

In 1955, Jean-Pierre Serre asked whether every finitely generated projective module over the ring $k[x_1, \ldots, x_n]$, where k is a field, is free. The geometric motivation behind this question was that the affine scheme underlying $k[x_1, \ldots, x_n]$ is the affine n-space $\mathbb{A}^n_k$ and algebraic vector bundles over $\mathbb{A}^n_k$ correspond to finitely generated projective modules over $k[x_1, \ldots, x_n]$. Since the real affine space $\mathbb{R}^n$ is contractible, every topological (and even smooth) vector bundle over it is trivial. A similar argument shows that $\mathbb{C}^n$ admits no non-trivial holomorphic vector bundles. Thus, if Serre's question had an affirmative answer, it would imply the analogous claim in the algebraic setting. It took 21 years, but the statement was eventually proved by Daniel Quillen and Andrei Suslin, independently of each other.

Projective Modules

Definition: Let A be a commutative ring with identity. An A-module P is called **projective** if every short exact sequence

$$0 \longrightarrow K \longrightarrow V \overset{\varphi}{\longrightarrow} P \longrightarrow 0$$

splits. In other words, every surjective morphism $\varphi : V \twoheadrightarrow P$ admits a *section map*, i.e., a homomorphism $\psi : P \to V$ such that $\varphi \circ \psi$ is the identity map on P.

A consequence of this definition is that projective modules are precisely those that arise as direct summands of free modules. Indeed, if P is projective, the obvious surjection $\varphi : A^{\oplus x \in P} \twoheadrightarrow P$ induces the identification $A^{\oplus x \in P} \cong \ker(\varphi) \oplus P$. The converse is apparent.

Stably Free

In 1957, Serre took the first step toward answering his question by showing that every finitely generated projective module over $k[x_1, \ldots, x_n]$ is, in a sense, "almost" free.

Definition: An A-module U is **stably free** if $U \oplus A^n \cong A^m$ for some $n, m \in \mathbb{Z}_{\geq 0}$.

This definition resembles that of projective modules; however, it additionally requires that the complement in the direct sum decomposition of the free module is itself free. By definition, every stably free module is projective. To show that every projective module over $k[x_1, \ldots, x_n]$ is stably free, we need to define the notion of finite free resolution.

Definition: We say that an A-module M admits a **finite free resolution** if there exists an exact sequence of finite length

$$0 \to E_n \to \cdots \to E_0 \to M \to 0$$

such that each E_i is free of finite rank.

Proposition: *Let M be a projective A-module. Then M is stably free if and only if M admits a finite free resolution.*

The forward direction of this theorem is trivial. The idea of the converse is to induct on the length of the finite free resolution, taking the kernel of the map $E_1 \to E_0$ and constructing a new exact sequence where the kernel has a shorter finite free resolution than M.

Theorem: Let R be a commutative noetherian ring. If every finite R-module admits a finite free resolution, then every finite $R[x]$-module admits a finite free resolution.

This proof is long, but once we have this result we can use induction to show that every projective module over $k[x_1, \ldots, x_n]$ is stably free. See [1, Chapter XXI].

The question that now arises is when stably free modules are actually free. It turns out that if a ring satisfies certain elementary matrix-theoretic conditions, which we will elaborate on shortly, then all stably free modules over that ring are automatically free. Thus the challenge was to prove that the polynomial ring $k[x_1, \ldots, x_n]$ satisfied such matrix-theoretic properties. While such properties were known for polynomial rings over local domains, extending it to arbitrary rings required new ideas.

Acknowledgments

We would like to thank our mentor Waqar Ali Shah for his insight and guidance throughout this project. Additionally we would like to thank the DRP for organizing this project.

Quillen-Suslin & $k[x_1, \ldots, x_n]$

The jump from stably free to free took significantly more time and is by no means straightforward. However, a completely elementary version of the proof of the Quillen–Suslin Theorem was given by Leonid Vaseršteĭn, and it is this proof that we will discuss. We follow the exposition given in [1, Chapter XXI].

Definition: Let A be a commutative ring with identity. We call a vector $(f_1, \ldots, f_n) \in A^n$ **unimodular** if its elements generate the unit ideal in A. Additionally, we say that the vector has the **unimodular extension property** if there exists an invertible square matrix with entries in A whose first column is $(f_1, \ldots, f_n)^T$. We say that two unimodular vectors f and g are **equivalent** if there exists an invertible matrix M such that $f = Mg$ and we write $f \sim g$ to denote this.

The first step in Vaseršteĭn's proof is to use a specific result about the unimodular extension property for a polynomial ring over local domains. This is known as Horrocks' theorem.

Theorem: *Let ω be a local ring and let f be a unimodular vector in $\omega[x]^n$ such that some component of f is monic. Then f has the unimodular extension property.*

The proof uses the relation $\sum g_i f_i = 1$ where $g_i \in \omega[x]$ and elementary row operations to induct down on the highest degree of a monic entry in f to show that any unimodular vector in $\omega[x]^n$ is equivalent to the first standard basis vector. An immediate corollary of this is that $f \sim f(0)$ over $\omega[x]$. The next step is to globalize Horrocks' result.

Proposition: *Let R be an integral domain and let f be a unimodular vector in $R[x]^n$ such that some component of f is monic. Then $f \sim f(0)$ over $R[x]$.*

This proof uses a lemma in which we go from the local result of Horrocks to two variables by shifting $x \mapsto x + cy$, where the set of all possible c's which we can shift turns out to be the whole of R. We use this fact to get a global result that $f \sim f(0)$ over $R[x]$. Finally, we have the full Quillen-Suslin theorem.

Theorem: *Let k be a field. Then every finitely generated projective module over the polynomial ring $k[x_1, \ldots, x_n]$ is free.*

This result follows by establishing that $k[x_1, \ldots, x_n]$ has the unimodular column extension property for all unimodular vectors, not just those with some component monic. This is achieved via a clever substitution of variables. More specifically, we make the substitution $x_n = y_n$, and $x_i = y_i - y_n^{r_i}$ for some collection of sufficiently large r_i. This coupled with the fact that any finitely generated projective module is stably free gives us the full result. To elaborate, every finitely generated projective module M over $A := k[x_1, \ldots, x_n]$ is stably free by the work of Serre. One now appeals to the unimodular extension property of A to show that if $M \oplus A^n \cong A^m$ for some $m, n \in \mathbb{Z}_{\geq 0}$, then $m \geq n$ and $M \cong A^{m-n}$.

Spec$(k[x_1, \ldots, x_n])$ & Affine Space

Suppose k is an algebraically closed field. Then the prime spectrum of the ring $A := k[x_1, \ldots, x_n]$ can be identified with a topological space that is essentially the affine space $\mathbb{A}^n_k$, equipped with the so-called *Zariski topology*. This follows from **Hilbert's Nullstellensatz**, which shows that the maximal ideals of $k[x_1, \ldots, x_n]$ are all of the form $(x_1 - a_1, \ldots, x_n - a_n)$ for some $a_1, \ldots, a_n \in k$. See [2, §15.3] for details. We can now identify the maximal ideal $(x_1 - a_1, \ldots, x_n - a_n)$ with the point $(a_1, \ldots, a_n)$ in the affine n-space, i.e.,

$$k^n \xrightarrow{\sim} \operatorname{MaxSpec}(k[x_1, \ldots, x_n]) \subseteq \operatorname{Spec}(k[x_1, \ldots, x_n]).$$

It can be shown that any finitely generated projective module M over A is finite locally free; that is, for each $\mathfrak{p} \in \operatorname{Spec}(A)$, there exists an element $f \in A \setminus \mathfrak{p}$ such that M_f is free over A_f of fixed finite rank. This fact admits a geometric interpretation: M corresponds to an *algebraic vector bundle* in the Zariski topology, and the freeness of M_f over A_f reflects the property of *local triviality*, which we now explain.

References

[1] Serge Lang. *Algebra*. New York, NY: Springer, 2012.
[2] David S. Dummit and Richard M. Foote. *Abstract Algebra*. Wiley, NJ, 2003.
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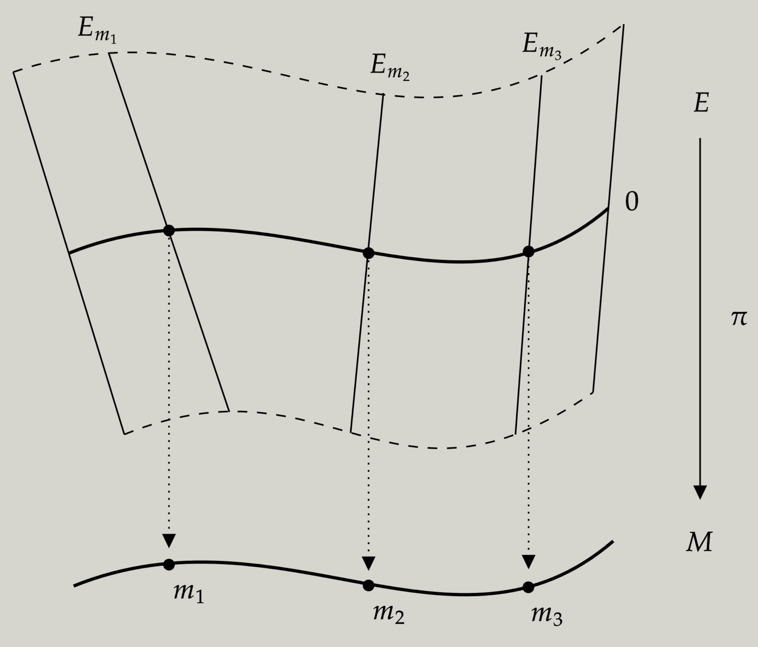
[4] Karen E. Smith et al. *An Invitation to Algebraic Geometry*. Springer, NY, 2003.

Vector Bundles

Vector bundles generalize the idea of families of vector spaces. Formally, a vector bundle is a map $\pi : E \to M$, where E is the "total space", and M is a topological space (e.g., the prime spectrum of a ring), called the "base space". The fiber $\pi^{-1}(\{m\})$ over a point $m \in M$ is the vector space associated with m, denoted E_m. The fibers over points in M vary continuously. These collections are *locally trivial*, which means that around every point $m \in M$, there is some open set U containing m such that the fibers $\pi^{-1}(U)$ look locally like $U \times k^n$, for some field k and natural number n. See [3, §14.1] for details.



A trivial vector bundle

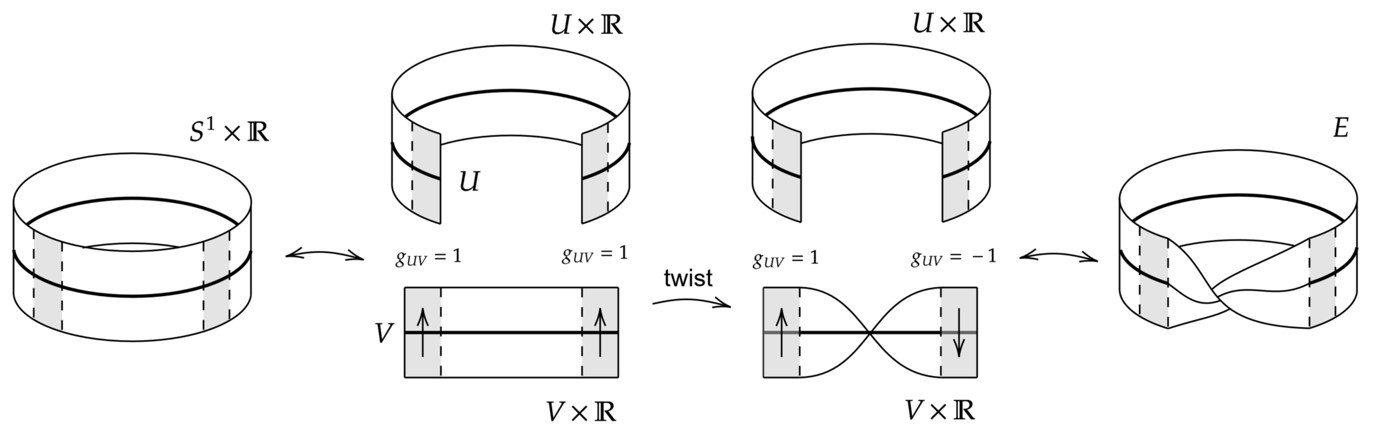
However, a vector bundle is not necessarily globally trivial. Typically, it is constructed by taking locally trivial collections of fibers and gluing them together via *transition functions* along overlaps of local trivializations. Take two trivial collections of fibers $V \times k^n$, and $U \times k^n$. We define a transition function as a map $g_{UV} : U \cap V \to GL_n(k)$. This function "glues" the two collections together by enforcing agreement on their overlap. Moreover, this agreement is consistent with any third such collection, provided the transition functions satisfy the so-called *cocycle condition*. If the base space is an algebraic variety, then an algebraic vector bundle is one whose local trivializations are glued together using algebraic transition functions.

Over affine spaces, all algebraic vector bundles are trivial, as classified by the Quillen–Suslin theorem. The classification of algebraic vector bundles over other algebraic varieties, especially over *projective spaces*, remains an active area of research. A **projective space** of dimension n over k is the set of one dimensional subspaces of $\mathbb{A}^{n+1}_k$. Intuitively, this is just the set of lines through the origin of the vector space k^{n+1}. A geometric interpretation of projective space is just taking affine space and adding a "point at infinity".

One obvious way to define a one dimensional vector bundle (also called a *line bundle*) on projective space is that of the **tautological line bundle**, $\mathcal{O}_{\mathbb{P}^n}(-1)$. This bundle assigns to each point in $\mathbb{P}^n$ the one-dimensional vector space it is associated with in $\mathbb{A}^{n+1}$. See [4, §8.4]

Classifying general vector bundles over projective n-space is quite difficult. For projective space of dimension one, a celebrated theorem of Alexander Grothendieck asserts that every vector bundle splits as a direct sum of line bundles obtained by "twisting" the tautological bundle. However, as the dimension increases, vector bundles become more intricate, and there exist non-split vector bundles on $\mathbb{P}^n$ for all $n \geq 2$. On the other hand, a famous conjecture of Hartshorne from the 1970s asserts that all rank two vector bundles over $\mathbb{P}^n$ for $n \geq 7$ split as the direct sum of two line bundles. As of 2025, this conjecture remains open.

Example: To illustrate how non-trivial vector bundles can arise, let us consider the **Möbius strip** as a line bundle on the real projective space $\mathbb{RP}^1$. This can be constructed by taking an affine line, gluing its ends to shape it into a circle and inserting a twist in the middle.
More specifically, we take two local trivializations $U \times \mathbb{R}$ and $V \times \mathbb{R}$. The globally trivial line bundle is obtained by gluing these two charts using the identity transition function $g_{UV} = 1$.
The Möbius strip, on the other hand, arises by using the transition function $g_{UV} = -1$. This construction is essentially that of the tautological line bundle $\mathcal{O}(-1)$ on $\mathbb{RP}^1$.



Construction of the Möbius strip