Stat 3202: Homework 3

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Due Monday, February 27 by 11:59 pm

Instructions

- Replace "FirstName LastName (name.n)" above with your information.
- Provide your solutions below in the spaces marked "Solution:".
- Include any R code that you use to answer the questions; if a numeric answer is required, show how you calculated it in R.
- Knit this document to **pdf** and upload both the **pfd** file and your completed **Rmd** file to Carmen
- Make sure your solutions are clean and easy-to-read by
  - formatting all plots to be appropriately sized, with appropriate axis labels.
  - only including R code that is necessary to answer the questions below.
  - only including R output that is necessary to answer the questions below (avoiding lengthy output).
  - providing short written answers explaining your work, and writing in complete sentences.

Due on Carmen Monday, February 27 by 11:59 pm. All uploads must be .pdf. Submissions will be accepted for 24 hours past this deadline, with a deduction of 1% per hour. Absolutely no submissions will be accepted after this grace period.

Concepts & Application

In this assignment, you will

- find consistent estimators of several probability distributions.
- find sufficient statistics of several probability distributions.
- Finding Method of Moment (MOM) estimators.
- Finding Maximum likelihood estimators (MLE).

This homework is worth 40 points.

Question 1

Consider two samples $x_1, x_2, \cdots, x_{n_x} \overset{\text{iid}}{\sim} f_x$ with $E(X) = \mu_x$ and finite $V(X) = \sigma_x^2$, and $y_1, y_2, \cdots, y_{n_y} \overset{\text{iid}}{\sim} f_y$ with $E(Y) = \mu_y$ and finite $V(Y) = \sigma_y^2$.

Assuming that the two samples are independent, show that $\bar{x} - \bar{y}$ is consistent estimator for $\mu_x - \mu_y$.

Solution to Question 1 $E(\bar{x} - \bar{y}) = E(x) - E(y) = \mu_x - \mu_y$. Since $E(\bar{x} - \bar{y})$ equals the parameter, it is unbiased. $V(\bar{x} - \bar{y}) = V(\bar{x}) + V(\bar{y}) + 2cov(\bar{x}, \bar{y}) = \frac{V(x)}{n} + \frac{V(y)}{n}$ As n approaches infinity, $V(\bar{x} - \bar{y})$ approaches 0 so it is consistent.

Question 2

(a) Consider a sample $x_1, x_2, \cdots, x_n \overset{\text{iid}}{\sim} Unif(\theta, \theta + 1)$. Show that $\hat{\theta}_{MOM} = \bar{x} - \frac{1}{2}$. You may use known facts about $E(X)$.

(b) Prove that $\hat{\theta}_{MOM}$ is consistent estimator for θ. You may use known facts about $E(X), E(\bar{X}), V(X), V(\bar{X})$.

Solution to Question 2

Your answers go here.

Part a: $E(X) = \bar{x} = \frac{2\theta+1}{2}$ Solve for $\hat{\theta}$ and you get $\hat{\theta}_{MOM} = \bar{x} - \frac{1}{2}$.

Part b: $E(\hat{\theta}) = E(x) - \frac{1}{2} = \theta$. Since $E(\hat{\theta})$ is equal to the parameter, it is unbiased. $V(\hat{\theta}) = \frac{V(X)}{n} = \frac{\frac{1}{12}}{n}$. As n approaches infinity, $V(\hat{\theta})$ will approach 0 so it is consistent.

Question 3

(a) Let $x_1, x_2, \cdots, x_n \overset{\text{iid}}{\sim} Gamma(\alpha, \beta)$ where β **is known**. Show that $\sum_{1=1}^{n} ln(x_i)$ is sufficient for α.

(b) Let $x_1, x_2, \cdots, x_n \overset{\text{iid}}{\sim} \chi_k^2$. Show that $\sum_{1=1}^{n} ln(x_i)$ is sufficient for k.

(c) Let $x_1, x_2, \cdots, x_n \overset{\text{iid}}{\sim} N(\mu, \sigma^2)$ where μ **is known**. Show that $\sum_{1=1}^{n} (x_i - \mu)^2$ is sufficient for σ^2.

Solution to Question 3

Your answers go here.

Part a: $L(\alpha, \beta; \sum_{i=1}^{n} ln(x_i)) = \prod_{i=1}^{n} \frac{(\alpha-1)\beta^{\alpha}}{(\alpha-1)!} \sum_{i=1}^{n} ln(x_i) e^{-\beta \sum_{i=1}^{n} ln(x_i)}$. Everything before the sum is $g(u, \alpha)$ and everything after and including the sum is $h(\sum_{i=1}^{n} ln(x_i))$. So it is sufficient. **Part b:** $L(k; \sum_{i=1}^{n} ln(x_i)) = \prod_{i=1}^{n} (\frac{\frac{k}{2}-1}{2^{\frac{k}{2}} (\frac{k}{2}-1)!}) \sum_{i=1}^{n} ln(x_i) e^{\frac{-\sum_{i=1}^{n} ln(x_i)}{2}}$ Everything before the sum is $g(u, k)$ and everything after and including the sum is $h(\sum_{i=1}^{n} ln(x_i))$. So it is sufficient. **Part c:** $L(\sigma^2; \sum_{i=1}^{n} (x_i - \mu)^2) = \prod_{i=1}^{n} \frac{1}{\sigma\sqrt{2\pi}} e^{\frac{-1}{2} \frac{(\sum_{i=1}^{n} (x_i-\mu)^2 - \mu)}{\sigma}^2}$. σ can be rewritten as $\sqrt{\frac{\sum_{i=1}^{n} (x_i-\mu)^2}{n-1}}$ which n and mu are known so $g(u, \sigma) = 1$ and $h(\sum_{i=1}^{n} (x_i - \mu)^2) = \prod_{i=1}^{n} \frac{1}{\sigma\sqrt{2\pi}} e^{\frac{-1}{2} \frac{(\sum_{i=1}^{n} (x_i-\mu)^2 - \mu)}{\sigma}^2}$.

Question 4

a) Take an iid sample of size n from exponential distribution, $Exp(\lambda)$. Show the sum of the observations is sufficient for λ.
b) Compute a method of moments estimator for λ. You can use known facts about $Exp(\lambda)$.
c) Compute a maximum likelihood estimator for λ.

Solution to Question 4

Your answers go here.

Part a: $g(u, \lambda) = 1.h(\sum_{i=1}^{n} x_i) = \prod_{i=1}^{n} \lambda e^{-\lambda \sum_{i=1}^{n} x_i}$. Since it can be split up, it is sufficient. **Part b:** $\bar{x} = \frac{1}{\lambda}$ so $\lambda_{MOM} = \frac{1}{\bar{x}}$. **Part c:** $L(\lambda; x_1, ..., x_n) = \lambda^n e^{-n\lambda \sum_{i=1}^{n} x_i}$ $\frac{d(ln(L(\lambda;x_1,...,x_n))}{d\lambda} = \frac{n}{\lambda} - n\sum_{i=1}^{n} x_i$ Set the derivative equal to 0 and solve for λ and you get $\lambda = \frac{1}{\sum_{i=1}^{n} x_i}$

Question 5

a) Let $x_1, x_2, \cdots, x_n \overset{\text{iid}}{\sim}$ Laplace$(\mu = 0, \sigma)$ where $f(x \mid \mu = 0, \sigma) = \frac{1}{2\sigma} e^{-\frac{|x|}{\sigma}}$. Compute $\hat{\sigma}_{MLE}$.

b) Let $x_1, x_2, \cdots, x_n \overset{\text{iid}}{\sim}$ Bernoulli(p). Compute $\hat{p}_{MLE}$.

c) Let $x_1, x_2, \cdots, x_n \overset{\text{iid}}{\sim}$ N(μ, σ^2). Compute $\hat{\mu}_{MLE}$.
Is $\hat{\mu}_{MLE}$ biased or unbiased?

d) Let $x_1, x_2, \cdots, x_n \overset{\text{iid}}{\sim}$ N(μ, σ^2). Compute $\hat{\sigma}^2_{MLE}$. Hint: let $v = \sigma^2$. This makes the calculus and algebra a bit easier.
Is $\hat{\sigma}^2_{MLE}$ biased or unbiased?

Solution to Question 5

Your answers go here.

Part a: $L(\sigma; x_1, ..., x_n) = \frac{1}{(2\sigma)^n} e^{-\frac{\sum_{i=1}^{n} |x_i|}{\sigma}}$ Then take the derivative of $ln(L(\sigma; x_1, ..., x_n))$ and solve for σ at 0 and you get: $\hat{\sigma}_{MLE} = \frac{\sum_{i=1}^{n} |x_i|}{n}$. **Part b:** $L(p; x_1, ..., x_n) = p^{\sum_{i=1}^{n} x_i}(1-p)^{n-\sum_{i=1}^{n} x_i}$ Then take the derivative of $ln(L(p; x_1, ..., x_n))$ and solve for p at 0 and you get: $\hat{p}_{MLE} = \frac{\sum_{i=1}^{n} x_i}{n}$ **Part c:** $L(\mu, \sigma^2; x_1, ...x_n) = (2\pi\sigma^2)^{\frac{-n}{2}} e^{-\frac{1}{2\sigma^2}\sum_{i=1}^{n}(x_i-\mu)^2}$. Take derivative of $ln(L(\mu, \sigma^2; x_1, ...x_n))$ and solve for μ at 0 and you get: $\hat{\mu}_{MLE} = \frac{\sum_{i=1}^{n} x_i}{2n\sigma^2}$. $E(\frac{\sum_{i=1}^{n} x_i}{2n\sigma^2}) = \frac{1}{2n\sigma^2}\sum_{i=1}^{n} E(x_i) = \frac{\mu}{2\sigma^2}$. Since $\hat{\mu}_{MLE}$ does not equal μ, it is biased. **Part d:** As taken from above: $L(\mu, \sigma^2; x_1, ...x_n) = (2\pi\sigma^2)^{\frac{-n}{2}} e^{-\frac{1}{2\sigma^2}\sum_{i=1}^{n}(x_i-\mu)^2}$. Take derivative of $ln(L(\mu, \sigma^2; x_1, ...x_n))$ and solve for $v = \sigma^2$ at 0 and you get: $\hat{\sigma}^2_{MLE} = \frac{\sum_{i=1}^{n}(x_i-\mu)^2}{n}$. $E(\frac{\sum_{i=1}^{n}(x_i-\mu)^2}{n}) = \frac{1}{n}\sum_{i=1}^{n} E((x_i-\mu)^2) = \sigma^2 + \mu^2$ which is not σ^2 so it is biased.