AMATH 561 Autumn 2024
Problem Set 3

Due: Wed 10/23 at 10am

Note: Submit electronically to Canvas.

1. Give an example of a probability space $(\Omega, \mathcal{F}, P)$, a random variable X and a function f such that $\sigma(f(X))$ is strictly smaller than $\sigma(X)$ but $\sigma(f(X)) \neq \{\emptyset, \Omega\}$. Give a function g such that $\sigma(g(X)) = \{\emptyset, \Omega\}$. Hint: Look at finite sample spaces with a small number of elements.

2. Give an example of events A, B, and C, each of probability strictly between 0 and 1, such that $P(A \cap B) = P(A)P(B), P(A \cap C) = P(A)P(C)$, and $P(A \cap B \cap C) = P(A)P(B)P(C)$ but $P(B \cap C) \neq P(B)P(C)$. Are A, B and C independent? Hint: You can let Ω be a set of eight equally likely points.

3. Let $(\Omega, \mathcal{F}, P)$ be a probability space such that Ω is countably infinite, and $\mathcal{F} = 2^{\Omega}$. Show that it is impossible for there to exist a countable collection of events $A_1, A_2, ... \in \mathcal{F}$ which are independent, such that $P(A_i) = 1/2$ for each i. Hint: First show that for each $\omega \in \Omega$ and each $n \in \mathbb{N}$, we have $P(\omega) \leq 1/2^n$. Then derive a contradiction.

4. (a) Let $X \geq 0$ and $Y \geq 0$ be independent random variables with distribution functions F and G. Find the distribution function of XY.

(b) If $X \geq 0$ and $Y \geq 0$ are independent continuous random variables with density functions f and g, find the density function of XY.

(c) If X and Y are independent exponentially distributed random variables with parameter λ, find the density function of XY.