AMATH 567 FALL 2024
HOMEWORK 8 — DUE NOVEMBER 18 ON GRADESCOPE BY 1:30PM

All solutions must include significant justification to receive full credit. If you handwrite your assignment you must either do so digitally or if it is written on paper you must *scan* your work. A standard photo is not sufficient.

If you work with others on the homework, you must name your collaborators.

1: The Korteweg-de Vries (KdV) equation arises whenever long waves of moderate amplitude in dispersive media are considered. For instance, it describes waves in shallow water, and ion-acoustic waves in plasmas. The equation is given by

$$u_t = 6uu_x + u_{xxx},$$

where indices denote partial differentiation.

(a) By looking for solutions $u(x,t) = U(x)$, derive a first-order ordinary differential equation for $U(x)$. Introduce integration constants as required.

(b) Let $U = U_0 \wp(x - x_0)$. Determine U_0 so that $u = U(x)$ solves the KdV equation.

2: From A&F: 3.6.5

3: Here's a way to evaluate

$$\sum_{k=1}^{\infty} \frac{1}{k^2},$$

due to Euler. We've seen that

$$\frac{\sin \pi z}{\pi z} = \prod_{j=1}^{\infty} \left(1 - \frac{z^2}{j^2}\right).$$

(a) Equate the coefficients of z^2 on both sides, to recover the desired sum.

(b) Equate the coefficients of z^4 on both sides to recover a different sum.

By equating coefficients of higher powers of z, one can recover other identities too.

4: For the following, suppose that $f(z)$ is analytic in an open set Ω that contains $[-1,1]$.

(a) Show that there exists a contour C, encircling $[-1,1]$, such that

$$\int_{-1}^{1} \frac{f(x)\mathrm{d}x}{\sqrt{1-x}\sqrt{1+x}} = \frac{1}{2i} \oint_C \frac{f(z)\mathrm{d}z}{\sqrt{z-1}\sqrt{z+1}}.$$

(b) Use this to evaluate

$$I_1 = \int_{-1}^{1} \frac{\mathrm{d}x}{\sqrt{1-x}\sqrt{1+x}}, \quad I_2 = \int_{-1}^{1} \sqrt{1-x}\sqrt{1+x}\,\mathrm{d}x,$$
$$I_3 = \int_{-1}^{1} \frac{\sqrt{1-x}}{\sqrt{1+x}}\,\mathrm{d}x, \quad I_4 = \int_{-1}^{1} \frac{\sqrt{1+x}}{\sqrt{1-x}}\mathrm{d}x,$$

without using any changes of variable (e.g., no trig subs!).

5: Suppose, for $|z| = 1$, that the series

$$f(z) = \sum_{n=-\infty}^{\infty} f_n z^n,$$

converges uniformly.

(a) Compute series representations for

$$F(z) := \frac{1}{2\pi i} \oint_C \frac{f(\xi)}{\xi - z}\mathrm{d}\xi, \quad |z| \neq 1, \quad C = \partial B_1(0).$$

(b) For $|z| = 1$, compute

$$\lim_{\epsilon \to 0^+} F(z(1-\epsilon)) - \lim_{\epsilon \to 0^+} F(z(1+\epsilon)).$$