AMATH 483 / 583 (Roche) - Homework Set 8

Due Friday June 6, 5pm PT

May 30, 2025

Homework 8 (80 points)

1. (+20) **Fourier transforms**. Evaluate the Fourier transform of the following functions by hand. Use the definitions I provided (includes $\frac{1}{\sqrt{2\pi}}$, this is common in physics but also now the default used in WolframAlpha - a powerful math AI tool) as well as the definition for Dirac delta I used if needed.
   (a) $f(x) = \frac{1}{\sigma\sqrt{2\pi}} e^{-\frac{1}{2\sigma^2}(x-\mu)^2}$
   (b) $f(t) = sin(\omega_0 t)$,ω_0 constant
   (c) $f(x) = e^{-a|x|}$ and $a > 0$
   (d) (distribution) $f(t) = \delta(t)$
2. (+10) **Correlation**. By definition, *correlation* is $p \odot q = \frac{1}{\sqrt{2\pi}} \int_{-\infty}^{\infty} p^*(\tau) q(t+\tau) d\tau$, and measures how similar one signal or data function is to another. Let $p(\tau) = \langle p \rangle + \delta_p(\tau)$ and $q(\tau) = \langle q \rangle + \delta_q(\tau)$, where $\langle\rangle$ and $\delta()$ denote the mean values and fluctuation functions (deviations about the mean). Two functions are defined to be *uncorrelated* when $p \odot q = \langle p \rangle \langle q \rangle$. Evaluate $p \odot q$ of the following functions:
$$p(t) = \begin{cases} 0 & t < 0 \\ 1 & 0 < t < 1 \\ 0 & t > 1 \end{cases} \quad , \; q(t) = \begin{cases} 0 & t < 0 \\ 1-t & 0 < t < 1 \\ 0 & t > 1 \end{cases}$$
3. (+10) **Autocorrelation**. Aside, periodic functions exhibit pronounced *autocorrelations* as shifting such functions by their period puts the function directly on itself. Alternatively, random functions or noise is characterized as being uncorrelated. Evaluate the autocorrelation $p \odot p$ of the following function:
$$p(t) = \begin{cases} 0 & t < 0 \\ 1 & 0 < t < 1 \\ 0 & t > 1 \end{cases}$$
4. (+20) **Fourier transform diffusion equation solve**. Consider the diffusion equation $\frac{\partial T}{\partial t} = \kappa \frac{\partial^2 T}{\partial x^2}$ where $T(x,t)$ describes the temperature profile of a long metal rod.
   (a) Assume you know $T(x,0)$ and define the Fourier transform of $T(x,t)$ to be $\tau(k,t)$. Transform the original equation and initial conditions into k-space. Solve the resulting equation. Inverse transform the result to obtain the solution in terms of the original variables.
   (b) Find the temperature in the rod given initial conditions $\kappa = 10^3 \frac{m^2}{s}$ and
$$T(x,0) = \begin{cases} 0 & |x| > 1m \\ 100^o \text{ C} & |x| \leq 1m \end{cases} .$$
5. (+20) **Compare FFTW to CUFFT on HYAK**. Measure and plot the performance of calculating the gradient of a 3D double complex plane wave defined on cubic lattices of dimension n^3 from 16^3 to $n = 256^3$, stride $n* = 2$ for both the FFTW and CUDA FFT (CUFFT) implementations on HYAK. Let each n be measured *ntrial* times and plot the average performance for each case versus n, $ntrial \geq 3$. Submit your performance plot and C++ test code. Your plot will have 'flops' on the y-axis (or some appropriate unit of FLOPs) and the dimension of the cubic lattices (n) on the x-axis. You will need to estimate the operation count of computing the derivative using FFT on a lattice.