AMATH 585

Homework 2

Nate Whybra

1) The interpolation polynomial is

$$l_i(x) = \prod_{\substack{j=k-2 \\ j \neq i}}^{k} \frac{x - x_j}{x_i - x_j}, \quad i = k-2, k-1, k$$

So the quadratic is, where

$$p(x) = \sum_{i=k-2}^{k} f(x_i)\, l_i(x)$$

$$= f(x_{k-2})\left[\frac{x - x_{k-1}}{x_{k-2} - x_{k-1}} \cdot \frac{x - x_k}{x_{k-2} - x_k}\right] + f(x_{k-1})\left[\frac{x - x_{k-2}}{x_{k-1} - x_{k-2}} \cdot \frac{x - x_k}{x_{k-1} - x_k}\right]$$

$$+ f(x_k)\left[\frac{x - x_{k-2}}{x_k - x_{k-2}} \cdot \frac{x - x_{k-1}}{x_k - x_{k-1}}\right]$$

Now suppose $f(x) = x^3 - 2$ w/ $x_0 = 0$, $x_1 = 1$, $x_2 = 2$,
Then, as $f(0) = -2$, $f(1) = -1$, and $f(2) = 6$,

$$p(x) = -2\left[\frac{(x-1)(x-2)}{-1 \cdot -2}\right] - \left[\frac{x(x-2)}{1 \cdot -1}\right] + 6\left[\frac{x(x-1)}{2 \cdot 1}\right]$$

$$= -(x-1)(x-2) + x(x-2) + 3x(x-1)$$

To find x_3, we solve $p(x) = 0$

$$\rightarrow \quad x(x-2) + 3x(x-1) = (x-1)(x-2)$$

$$\rightarrow \quad x(x - 2 + 3x - 3) = x^2 - 3x + 2$$

$$\rightarrow \quad 4x^2 - 5x = x^2 - 3x + 2$$

$$\rightarrow \quad 3x^2 - 2x - 2 = 0$$

$$\rightarrow \quad x = \frac{2}{6} \pm \frac{1}{6}\sqrt{4 + 4 \cdot 3 \cdot 2}$$

$$= \frac{1}{6}\left[2 \pm \sqrt{28}\right]$$

$$= \frac{1}{6}\left[2 \pm 2\sqrt{7}\right]$$

$$= \frac{1}{3}\left[1 \pm \sqrt{7}\right] \approx 4.74, \; -2.14$$

we take the value closest to x_2, so

$$x_3 = \frac{1}{3}\left[1 + \sqrt{7}\right].$$

2) we want

$$\ell(x) = c_1 x + c_2$$

$$\ell(-1) = -c_1 + c_2 = a$$

$$\ell(1) = c_1 + c_2 = b$$

$$\longleftrightarrow \quad \begin{bmatrix} -1 & 1 \\ 1 & 1 \end{bmatrix} \begin{bmatrix} c_1 \\ c_2 \end{bmatrix} = \begin{bmatrix} a \\ b \end{bmatrix}$$

$$\longleftrightarrow \quad c_1 = \frac{b-a}{2}, \quad c_2 = \frac{a+b}{2}$$

Hence, $\ell(x) = \frac{b-a}{2} \cdot x + \frac{a+b}{2}$

$$= \frac{1}{2}\left[(b-a)x + (a+b)\right]$$

If we have $x_j = \cos\left(\frac{j\pi}{n}\right)$, $j = 0, 1, \dots, n$ on $[-1,1]$, and instead want our interpolation points on $[a,b]$, we can just apply our function $\ell(x)$ on the x_j's

$$x_j' = \ell(x_j) = \frac{1}{2}\left[(b-a)\cos\left(\frac{j\pi}{n}\right) + (a+b)\right]$$

These (x_j')'s are the desired solution.

5)

(a) Firstly,

$$\lambda_1(x) = \sum_{i=0}^{1} |l_i(x)|$$

$$= \left| \frac{x - x_1}{x_0 - x_1} \right| + \left| \frac{x - x_0}{x_1 - x_0} \right|$$

$$= \left| \frac{x - \frac{3}{2}}{1 - \frac{3}{2}} \right| + \left| \frac{x - 1}{\frac{3}{2} - 1} \right|$$

$$= 2\left|x - \tfrac{3}{2}\right| + 2|x-1| = 2\left(\left|x - \tfrac{3}{2}\right| + |x-1|\right)$$

Secondly,

$$\Lambda_1 = \max_{x \in [1,2]} \overbrace{2\left(\left|x - \tfrac{3}{2}\right| + |x-1|\right)}^{\lambda_1(x)} \quad (1)$$

As $x \geq 1$, we have $|x-1| = x-1$ and $|x - \frac{3}{2}| = x - \frac{3}{2}$ when $x \geq 3/2$ and $|x - \frac{3}{2}| = \frac{3}{2} - x$ when $x < \frac{3}{2}$, so

$$(1) = \max_{x \in [1,2]} \begin{cases} 2\left(x-1 + \frac{3}{2} - x\right) = 1, & \text{if } 1 \leq x \leq 3/2 \\ 2\left(x-1 + x - \frac{3}{2}\right) = 4x - 5, & \text{if } x > 3/2 \end{cases}$$

$4x-5$ is monotonically increasing on $[1,2]$, so it achieves its maximum when $x=2$, $\to 4(2)-5=3$. 3 is always larger than 1, so it must be that $\boxed{\lambda_1 = 3}$

b) Let $y_1 = f(1)$ and $y_2 = f(\frac{3}{2})$, then we can write (with $x_1 = 1$ and $x_2 = 3/2$)

$$p_1(x) = \frac{y_2 - y_1}{x_2 - x_1}(x - x_1) + y_1$$

$$= 2(y_2 - y_1)(x-1) + y_1$$

Now, we'd like a continuous function f on $[1,2]$ such that

$$\max_{x\in[1,2]} |p_1(x)| = 3$$

and

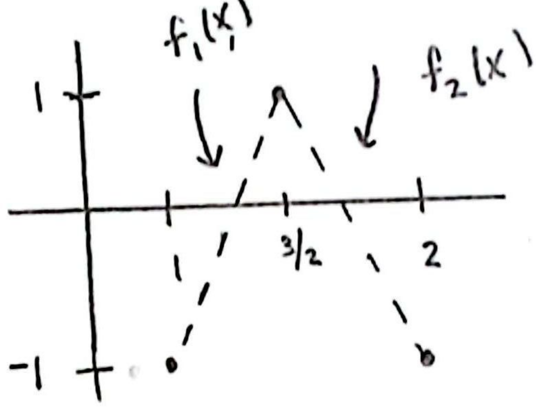
$$\max_{x\in[1,2]} |f(x)| = 1$$

Let's simplify the problem a bit, choose $y_1 = -1$ and $y_2 = 1$, so that

$$p_1(x) = 4x - 5$$

which has $\max_{[a,b]} |f(x)| = 3$ at $x = 2$.

So let's choose $f(x)$ to be a linear piecewise function

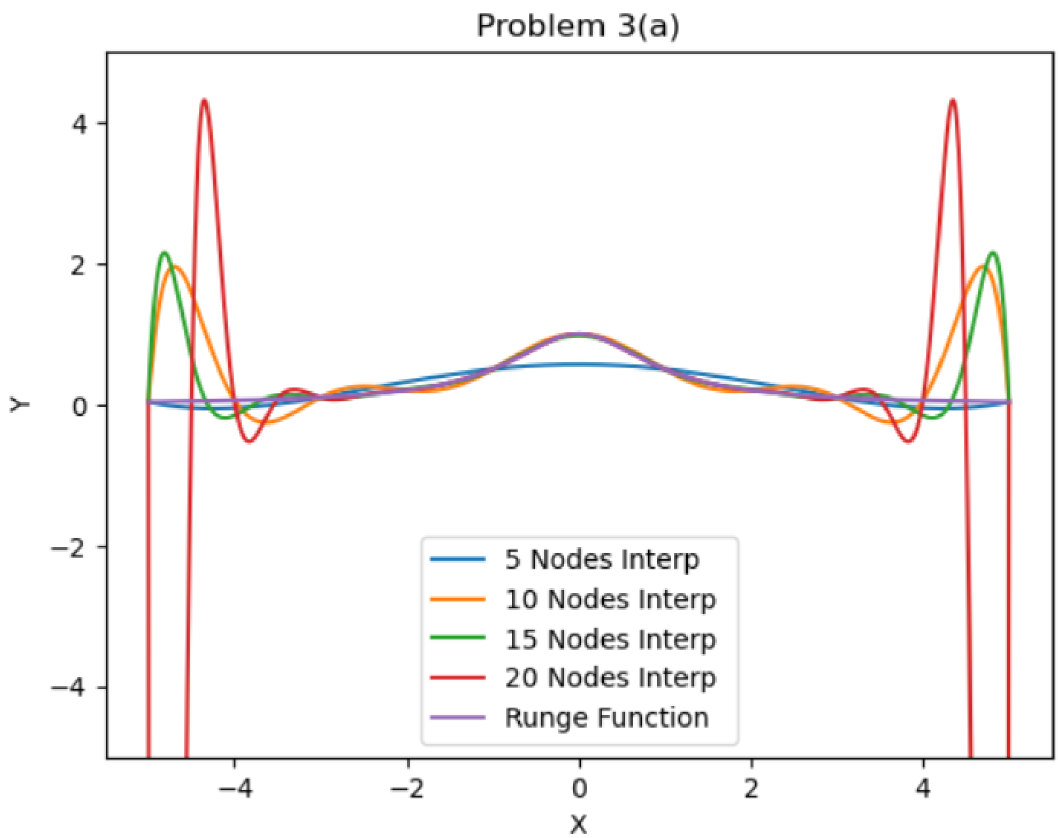


$$f(x) = \begin{cases} f_1(x), & x \in [1, 3/2] \\ f_2(x), & x \in [3/2, 1] \end{cases}$$

$$= \begin{cases} 4x - 5, & x \in [1, 3/2] \\ -4x + 7, & x \in [3/2, 1] \end{cases}$$

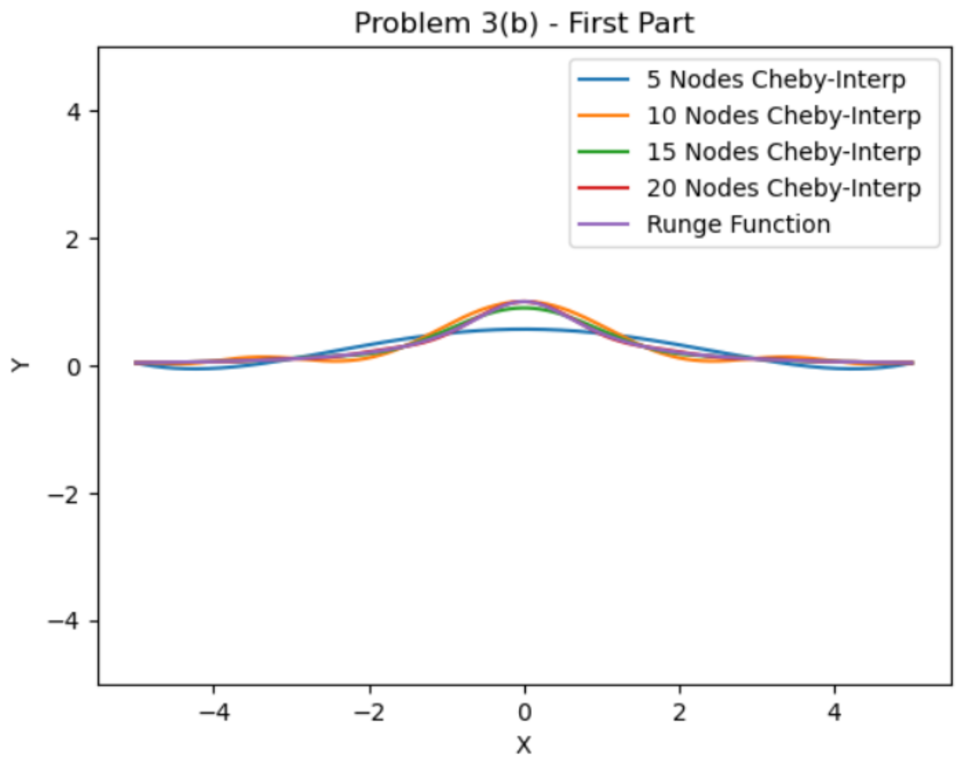
Clearly f is continuous and it meets the required specifications.

Problem 3(a):

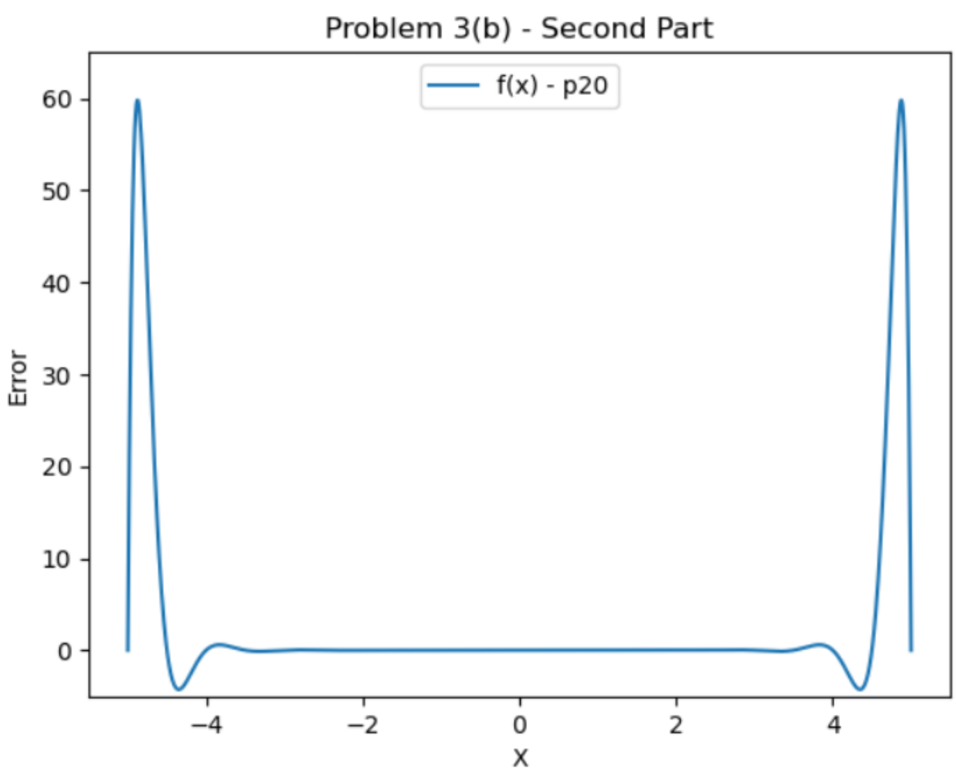


Problem 3(a)
4
2
0
−2
−4
Y
−4
−2
0
2
4
X
5 Nodes Interp
10 Nodes Interp
15 Nodes Interp
20 Nodes Interp
Runge Function

Problem 3(b):

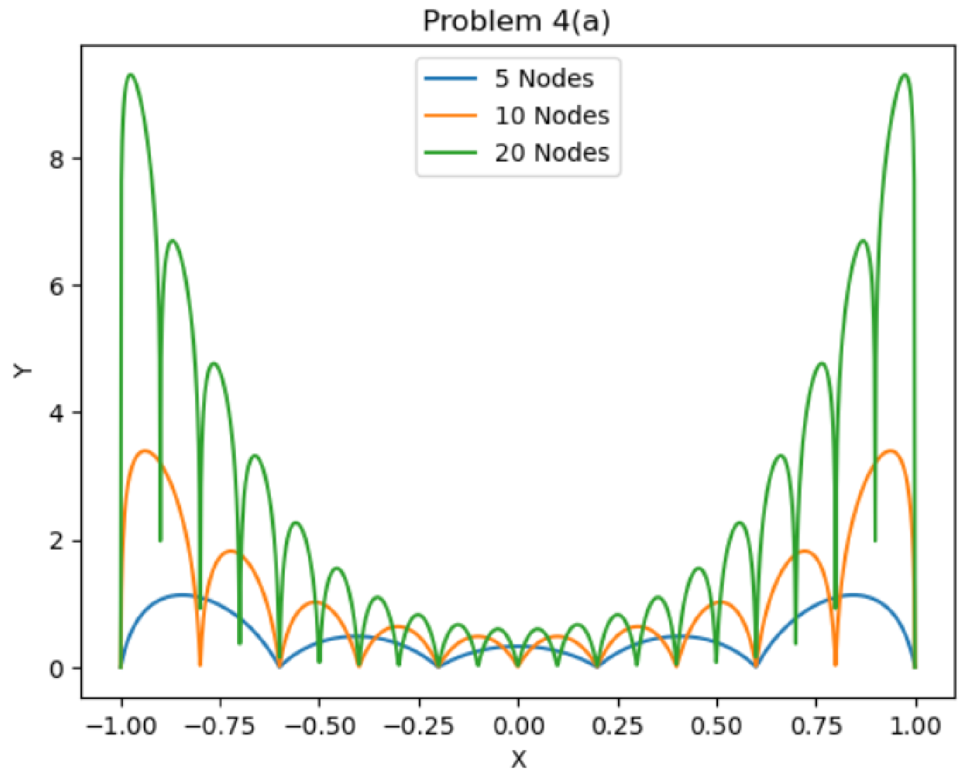


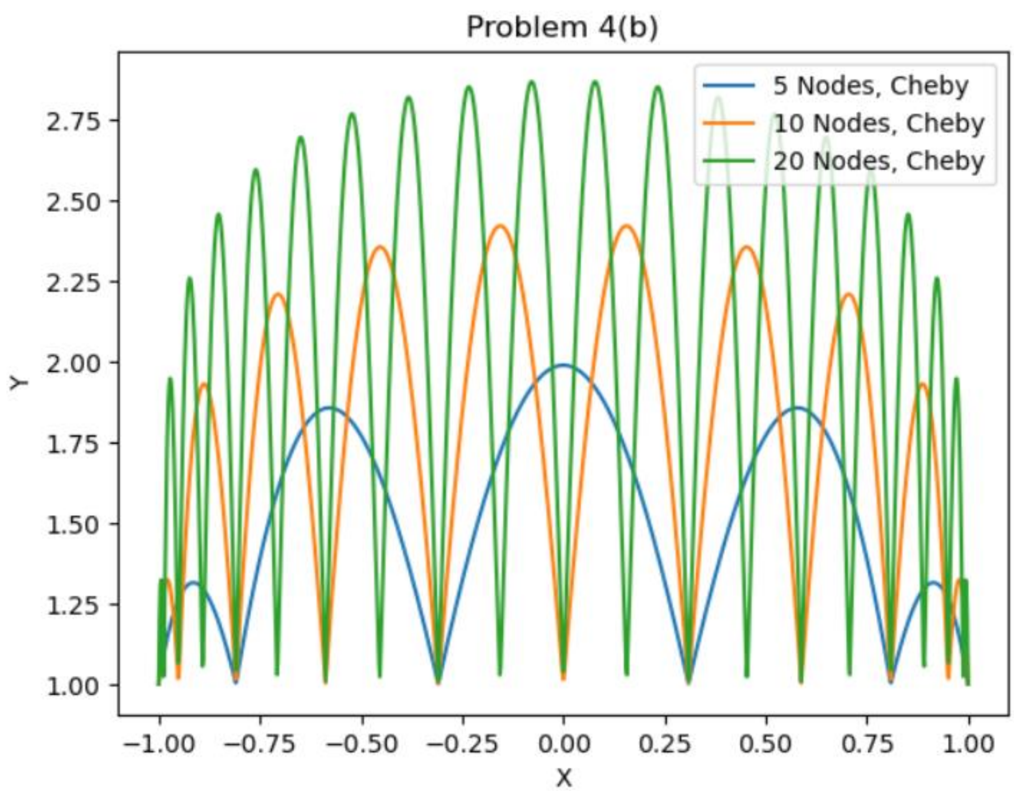
Problem 3(b) - First Part
5 Nodes Cheby-Interp
10 Nodes Cheby-Interp
15 Nodes Cheby-Interp
20 Nodes Cheby-Interp
Runge Function
Y
X
4
2
0
−2
−4



Problem 3(b) - Second Part
f(x) - p20
Error
X
60
50
40
30
20
10
0
−4
−2
2
4

Problem 4:


Problem 4(a)
5 Nodes
10 Nodes
20 Nodes
Y
X
0
2
4
6
8
−1.00
−0.75
−0.50
−0.25
0.00
0.25
0.50
0.75
1.00


Problem 4(b)
5 Nodes, Cheby
10 Nodes, Cheby
20 Nodes, Cheby
Y
X
1.00
1.25
1.50
1.75
2.00
2.25
2.50
2.75
−1.00
−0.75
−0.50
−0.25
0.00
0.25
0.50
0.75
1.00

Problem 4 (Comments):

This is interesting. Noting that the plot for Problem 4(a) is the logarithm of the actual values (they are way larger before taking the logarithm), the values of the Lebesgue interpolating function really depend on the choice of interpolation nodes. In each case, you can see a bump corresponding to each node. If there are 5 nodes, then there are 5 bumps, and so on. Also in either case, the function seems to be generally larger when we add more nodes, which makes sense as we are summing more terms when we use more nodes.