1) (4.1.2)

a) The function $f(z) = \frac{z^2+1}{z^2-a^2} = \frac{z^2+1}{(z-a)(z+a)}$

has simple poles at $z=a$ and $z=-a$. So by the residue theorem ( and $a^2<1 \rightarrow -1<a<1$ ) so we are in C

$$\frac{1}{2\pi i}\int_C \frac{z^2+1}{(z-a)(z+a)}\,dz = \frac{\cancel{2\pi i}}{\cancel{2\pi i}}\Big[\underbrace{\mathrm{Res}(f,a)}_{①} + \underbrace{\mathrm{Res}(f,-a)}_{②}\Big]$$

$$① = \lim_{z\to a} \frac{\cancel{(z-a)}\, z^2+1}{\cancel{(z-a)}(z+a)} = \frac{a^2+1}{2a}$$

$$② = \lim_{z\to -a} \frac{\cancel{(z+a)}\, z^2+1}{\cancel{(z+a)}(z-a)} = \frac{a^2+1}{-2a}$$

So our integral

$= ① + ② = 0.$

b) $$\frac{1}{2\pi i}\int_C \frac{z^2+1}{z^3}\,dz = I$$

our function f has a pole of order 3 at $z=0$, so by the residue theorem

$$I = \frac{1}{2!}\frac{\cancel{2\pi i}}{\cancel{2\pi i}}\lim_{z\to 0}\frac{d^2}{d^2z}\left(\cancel{z^3}\cdot\frac{z^2+1}{\cancel{z^3}}\right)$$

$$= \frac{1}{2}\lim_{z\to 0} 2$$

$$= \frac{1}{2}2$$

$$= 1$$

c) we have

$$f(z) = z^2 e^{-1/z} = z^2 \sum_{n=0}^{\infty} \frac{\left(-\frac{1}{z}\right)^n}{n!}$$

$$= z^2 \sum_{n=0}^{\infty} \frac{(-1)^n}{n!\, z^n}$$

$$= \sum_{n=0}^{\infty} \frac{(-1)^n}{n!} z^{2-n}$$

$$= \frac{(-1)^0}{0!} z^2 + \frac{(-1)^1}{1!} z + \frac{(-1)^0}{2!} z^0$$

$$+ \frac{(-1)^3}{3!} z^{-1} + \dots$$

Consider

$$I = \frac{1}{2\pi i} \int_C f(z)\, dz$$

By definition of residue, the integral will be $2\pi i$ times the coefficient of the $\frac{1}{z}$ term in the Laurent expansion, hence

$$I = \frac{\cancel{2\pi i}}{\cancel{2\pi i}} \left(\frac{-1}{6}\right) = \frac{-1}{6}$$

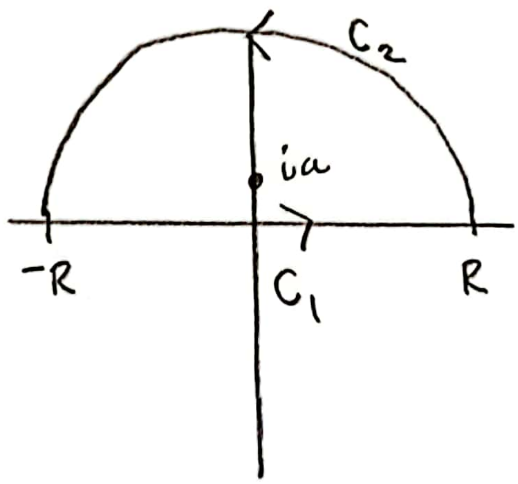
2) (4.2.1) (b)

Notice the function $f(x) = \dfrac{1}{(x^2+a^2)^2}$ is even, ie $f(-x) = f(x)$. So that

$$\int_0^\infty f(x)\,dz = \frac{1}{2}\int_{-\infty}^{\infty} f(x)\,dx$$

Also notice $f(x) = \dfrac{1}{(x-ia)^2(x+ia)^2}$.

Consider the following contour $C(R) = C_1(R) + C_2(R)$




Then

$$\frac{1}{2}\int_{-\infty}^{\infty} f(x)\,dx$$

$$= \lim_{R\to\infty} \frac{1}{2}\int_{-R}^{R} f(x)\,dx = \frac{1}{2}\lim_{R\to\infty}\left[\overset{①}{\int_{C(R)} f(z)\,dz} - \overset{②}{\int_{C_2(R)} f(z)\,dz}\right]$$

we can compute ① w/ the residue theorem
Since there is a 2nd order pole at $z = ia$.

So

$$① = \frac{1}{2}\left(2\pi i \lim_{z \to ia} \frac{d}{dz}\left((z-ia)^2 \frac{1}{(z-ia)^2(z+ia)^2}\right)\right)$$

$$= \frac{1}{2} 2\pi i \lim_{z \to ia} \frac{d}{dz} (z+ia)^{-2}$$

$$= \frac{1}{2} 2\pi i \lim_{z \to ia} -2(z+ia)^{-3}$$

$$= \cancel{\tfrac{1}{2}}\cancel{2}\pi i(-2)\frac{1}{(2ia)^3} = \frac{\pi i(-2)\, 1}{+8 i^3 a^3} = \frac{\pi}{4a^3}$$

we'd like to show ② $\to 0$ as $R \to \infty$, so

put $z = Re^{i\theta}$ $0 \leq \theta \leq \pi$, then

$$② = \frac{1}{2} \lim_{R \to \infty} \int_0^{\pi} \frac{iRe^{i\theta}}{((Re^{i\theta})^2 + a^2)^2} d\theta$$

$$= \frac{1}{2} \lim_{R \to \infty} \int_0^{\pi} \frac{iRe^{i\theta}}{(R^2e^{i2\theta} + a^2)^2} d\theta$$

$$= \frac{1}{2} \lim_{R \to \infty} \int_0^{\pi} \frac{iRe^{i\theta}}{R^4 e^{i4\theta} + a^2 2R^2 e^{i2\theta} + a^4} d\theta$$

$$= \frac{1}{2} \lim_{R \to \infty} \frac{iR}{R^4} \int_0^{\pi} \frac{e^{i\theta}}{\underbrace{e^{i4\theta} + \frac{2a^2}{R^2} e^{i2\theta} + \frac{a^4}{R^4}}_{\text{this is bounded}}} \, d\theta \, \Bigg| \quad (*)$$

$$\hookrightarrow \left| e^{i4\theta} + \frac{2}{R^2} e^{i2\theta} + \frac{a^4}{R^4} \right|$$

$$= \left| \left( e^{i2\theta} + \frac{a^2}{R^2} \right)^2 \right|$$

$$= \left| e^{i2\theta} + \frac{a^2}{R^2} \right|^2 \geq \left| |e^{i2\theta}| - \left| \frac{a^2}{R^2} \right| \right|^2$$

$$= \left| 1 - \frac{a^2}{R^2} \right|^2$$

So the above $|(*)|$

$$\leq \frac{1}{2} \lim_{R \to \infty} \frac{1}{R^3} \int_0^{\pi} \frac{1}{\left(1 - \frac{a^2}{R^2}\right)^2} \, d\theta$$

$$= \frac{1}{2} \lim_{R \to \infty} \frac{1}{R^3} \frac{\pi}{\left(1 - \frac{a^2}{R^2}\right)^2}$$

$$= 0$$

So we get as $R \to \infty$

$$\frac{1}{2} \int_{-\infty}^{\infty} f(x) \, dx = \int_0^{\infty} f(x) \, dx = \frac{\pi}{4a^3}$$

(4.2.1 b) (continued)

Checking my work, put $x = a\tan\theta \rightarrow dx = a\sec^2\theta\, d\theta$

$$x^2 = a^2\tan^2\theta$$

$$x^2 + a^2 = a^2(\tan^2 + 1)$$

$$= a^2\sec^2\theta$$

So $$\int_0^\infty \frac{1}{(x^2+a^2)^2}\,dx = \int_0^{\pi/2} \frac{1}{\cancel{a^4}^{\,3}\sec^{\cancel{4}\,2}\theta} \cdot \cancel{a}\sec^{\cancel{2}}\theta\, d\theta$$

$$= \frac{1}{a^3}\int_0^{\pi/2} \frac{1}{\sec^2\theta}\, d\theta$$

$$= \frac{1}{a^3}\int_0^{\pi/2} \cos^2\theta\, d\theta$$

$$= \frac{1}{a^3}\int_0^{\pi/2} \frac{1+\cos(2\theta)}{2}\, d\theta$$

$$= \frac{1}{2a^3}\int_0^{\pi/2} 1+\cos(2\theta)\, d\theta$$

$$= \frac{1}{2a^3}\left[\frac{\pi}{2} + \frac{1}{2}\sin(2\theta)\Big|_0^{\pi/2}\right]$$

$$= \frac{1}{2a^3}\left[\frac{\pi}{2}\right] = \frac{\pi}{4a^3}$$ as desired.

3) a) Suppose there are at least 2 polynomials that are degree $n-1$ ($p_1(z)$ and $p_2(z)$) that satisfy $p_j(z_i) = f_i \quad \forall \quad 1 \le i \le n$. Consider $v(z) = \prod_{i=1}^{n} (z - z_j)$, then

$$g(z) = \frac{p_1(z) - p_2(z)}{v(z)}$$

has a degree $n-1$ polynomial in the numerator and a degree $n$ polynomial in the denominator, so as $z \to \infty$, $g(z) \to 0$. As $p_1 - p_2$ is $0$ at each $z_i$, $p_1 - p_2$ can be factored such that the singularities of $g$ are removable, meaning $g(z)$ is entire. As $g$ is entire and $0$ at $\infty$, it is bounded. So Liouville's theorem tells us $g(z)$ is constant. Since $g(\infty) = 0$, this constant is $0$, so

$$g(z) = \frac{p_1(z) - p_2(z)}{v(z)} = 0$$

$$\rightarrow p_1(z) = p_2(z)$$

we have shown that if there are 2 such polynomials, that they must be the same. So we are done

3) (b) Suppose that $p(z)$ is an interpolant. Define $v(z) = \prod_{j=1}^{n} (z - z_j)$. Now define

$$g(z) = \frac{p(z)}{v(z)}$$

$g(z)$ is a meromorphic function with simple poles at $z = z_j$ for $1 \leq j \leq n$. We'd like to remove the singularities of $g$. To do so define $R_j = \operatorname{Res}(g, z_j)$. Then define

$$h(z) = g(z) - \sum_{j=1}^{n} \frac{R_j}{z - z_j} = \frac{p(z)}{v(z)} - \sum_{j=1}^{n} \frac{R_j}{z - z_j}$$

This function $h$ is entire as we have removed the singularities of $g$. Now we compute $R_j$,

$$R_j = \lim_{z \to z_j} \frac{(z - z_j)\, p(z)}{\prod_{i=1}^{n} (z - z_i)}$$

$$= \lim_{z \to z_j} \frac{p(z)}{\prod_{i \neq j}^{n} (z - z_i)}$$

$$= \frac{p(z_j)}{\prod_{i \neq j}^{n} (z_j - z_i)} = \frac{f_j}{\prod_{i \neq j}^{n} (z_j - z_i)}$$

So,

$$h(z) = \frac{p(z)}{v(z)} - \sum_{j=1}^{n} \frac{f_j}{(z-z_j)\prod_{i\neq j}^{n}(z_j - z_i)}$$

For now, let's keep using $R_j$ for simplicity, so

$$h(z) = \frac{p(z)}{v(z)} - \left(\frac{R_1}{z-z_1} + \frac{R_2}{z-z_2} + \dots + \frac{R_n}{z-z_n}\right)$$

$$= \frac{p(z)}{v(z)} - \left(\frac{R_1 \frac{v(z)}{z-z_1} + R_2 \frac{v(z)}{z-z_2} + \dots + R_n \frac{v(z)}{z-z_n}}{v(z)}\right)$$

$$= \frac{p(z) - \overbrace{\sum_{j=1}^{n} R_j \frac{v(z)}{z-z_j}}^{*}}{v(z)}$$

$p(z)$ is a degree $n-1$ polynomial. $v(z)$ is a degree $n$ polynomial where for every $j$, $(z-z_j)$ is a factor, hence $\frac{v(z)}{z-z_j}$ is a degree $n-1$ polynomial for every $j$. So the sum $*$ is a sum of $n$ degree $n-1$ polynomials, which is also a degree $n-1$ polynomial. So $h(z)$ is an entire function that is the ratio of a degree $n-1$ polynomial and a degree $n$ polynomial, so that as $z \to \infty$, $h(z) \to 0$. This means that $h$ must be bounded.

So by Liouville's theorem $h$ is a constant, and this constant must be $0$ ( plugging $z=\infty$ returns $0$), hence

$$h(z) = \frac{p(z) - v(z)\sum_{j=1}^{\tilde{n}} \frac{R_j}{z-z_j}}{v(z)} = 0, \quad \forall z \in \mathbb{C} \cup \{\infty\}$$

This means

$$p(z) = v(z)\sum_{j=1}^{\tilde{n}} \frac{R_j}{z-z_j}$$

where $R_j$ is defined from earlier computation,

4) we'd like to show,

$$f(x) - p(x) = \frac{v(x)}{2\pi i} \underbrace{\int_C \frac{f(z)}{z - x} \frac{dz}{v(z)}}_{\circledast}, \quad x \in [-1,1]$$

Firstly, notice that if $x = x_j$ for any $1 \le j \le n$, we have $v(x_j) = 0$, and $f(x_j) - p(x_j) = 0$ so that the LHS and RHS above are equal.

Now for $x \neq x_j$, we want to compute $\circledast$ with the residue theorem. To do so, we realize the integrand has simple poles at $z = x$ and also at $z = \underbrace{x_j}_{\text{which I will denote as } z_j}$ for $1 \le j \le n$.

So (with the integrand being $g$)

$$\circledast = 2\pi i \left[ \operatorname{Res}(g, x) + \sum_{j=1}^{n} \operatorname{Res}(g, z_j) \right]$$

$$= 2\pi i \left[ \lim_{z \to x} \frac{\cancel{(z-x)}}{\cancel{(z-x)}} \frac{f(z)}{v(z)} + \sum_{j=1}^{n} \lim_{z \to z_j} \frac{(z - z_j) f(z)}{(z - x) v(z)} \right]$$

$$= 2\pi i \left[ \frac{f(x)}{v(x)} + \sum_{j=1}^{n} \lim_{z \to z_j} \frac{f(z)}{(z-x) \prod_{i \neq j} (z - z_i)} \right]$$

$$= 2\pi i \left[ \frac{f(x)}{v(x)} + \sum_{j=1}^{n} \frac{R_j}{z_j - x} \right]$$

where $R_j$ is defined in problem (3b) with $f(z_j)$ here instead is $f_j$

$$= 2\pi i \left[ \frac{f(x)}{v(x)} - \sum_{j=1}^{n} \frac{R_j}{x - z_j} \right]$$

$$= 2\pi i \left[ \frac{f(x) - v(x) \sum_{j=1}^{n} \frac{R_j}{x - z_j}}{v(x)} \right] \quad \text{( combining fractions )}$$

$$= 2\pi i \left[ \frac{f(x) - p(x)}{v(x)} \right] \quad \text{( from 3b )}$$

So $$\frac{v(x)}{2\pi i} \cdot (*) = \frac{\cancel{v(x)}}{\cancel{2\pi i}} \cdot \cancel{2\pi i} \left[ \frac{f(x) - p(x)}{\cancel{v(x)}} \right]$$

$$= f(x) - p(x)$$

as desired.

Important note ) As all the $z_j$'s ($x_j$'s) are on the line $[-1, 1]$, our contour $C$ will not pass through any of the poles of the integrand.

5) a) From the solutions of homework 3, if $z = x + iy = \cos\theta + i\sin(\theta)$, then

$$T_n(x) = T_n(\cos\theta) = \cos(n\theta)$$

So $T_n(x)$ is $0$ when $\cos(n\theta) = 0$. This happens when $n\theta = (2K+1)\frac{\pi}{2}$, or $\theta = \frac{(2K+1)}{n} \cdot \frac{\pi}{2}$, where $0 \le K \le n-1$ to ensure $\arg(z) \in [-\pi, \pi)$. So $x_K = \cos\left(\frac{2K+1}{n} \cdot \frac{\pi}{2}\right)$ for $0 \le K \le n-1$. In homework 3 we also showed that $T_n$ is a degree $n$ polynomial that can have at most $n$ distinct roots. we already found $n$ of them, so the $x_K$'s as defined above must be all of them. As $\cos(\theta) \in [-1, 1]$ for any values of $\theta \in \mathbb{R}$, all of the roots satisfy

$-1 \le x_0 < x_1 < x_2 \dots < x_n \le 1$ as desired.

5) b) Put $w = pe^{i\theta}$ with $p > 1$. Then

$$\begin{aligned} J(w) &= J(pe^{i\theta}) \\ &= \frac{1}{2}\left(pe^{i\theta} + \frac{1}{p}e^{-i\theta}\right) \\ &= \frac{1}{2}\left(p\cos\theta + ip\sin\theta + \frac{1}{p}\cos\theta - \frac{i}{p}\sin\theta\right) \\ &= \frac{1}{2}\left(\left[p + \frac{1}{p}\right]\cos\theta + i\left(p - \frac{1}{p}\right)\sin\theta\right) \\ &= \frac{p+p^{-1}}{2}\cos\theta + i\frac{(p-p^{-1})}{2}\sin\theta \\ &= a\cos\theta + ib\sin\theta \end{aligned}$$

which is the parametrization of the ellipse in the complex plane. To show this ellipse contains $[-1,1]$, all we must do is show $a > \frac{1}{2}$ and $b > 0$ which amounts to showing that $p + p^{-1} > 1$ (which is obvious as $p > 1$ and $p^{-1} > 0$) and that $p - p^{-1} > 0$, (again this is obvious as $p > 1$ and $0 < p^{-1} < 1$).

(5b) (continued)

Now,

$$\phi(J(w)) = \frac{w}{2} + \frac{1}{2w} + \sqrt{\frac{w}{2} + \frac{1}{2w} - 1} \cdot \sqrt{\frac{w}{2} + \frac{1}{2w} + 1}$$

$$= \frac{w}{2} + \frac{1}{2w} + \sqrt{\frac{w^2 - 2w + 1}{2w}} \cdot \sqrt{\frac{w^2 + 2w + 1}{2w}}$$

$$= \frac{w^2+1}{2w} + \sqrt{\frac{(w-1)^2}{2w}} \cdot \sqrt{\frac{(w+1)^2}{2w}}$$

$$= \frac{w^2+1}{2w} + \frac{w-1}{\sqrt{2w}} \cdot \frac{w+1}{\sqrt{2w}}$$

$$= \frac{w^2+1 + (w-1)(w+1)}{2w}$$

$$= \frac{w^2+1+w^2-1}{2w}$$

$$= \frac{2w^2}{2w}$$

$$= w$$

as desired. (P.S. we never said $\sqrt{z-1}\sqrt{z+1} = \sqrt{z^2-1}$. !!)

(5c) we can apply the formula from problem 4 to say

$$|f(x) - p(x)| = \left| \frac{T_n(x)}{2\pi i} \int_{B_p} \frac{f(z)}{z-x} \frac{dz}{T_n(z)} \right|$$

From homework 3, $\max_{-1 \leq x \leq 1} |T_n(x)| = 1$

So the RHS is

$$(\star) \quad \leq \frac{1}{2\pi} \int_{B_p} \frac{|f(z)|}{|z-x|} \frac{|dz|}{|T_n(z)|} \qquad \text{(by triangle inequality)}$$

we have that $|f(z)| \leq M$. But we must find lower bounds $|z-x|$ and $|T_n(z)|$.

Firstly as we are on $B_p$,

$$|z-x| \geq \frac{1}{2}(p+p^{-1}) - 1 \quad \Big] \text{ minimum distance from the ellipse to the line}$$

$$= \frac{1}{2}p + \frac{1}{2}p^{-1} - \frac{2}{2}$$

$$= \frac{p + p^{-1} - 2}{2}$$

Now

$$|T_n(x)| = \frac{1}{2}|\phi^n + \phi^{-n}|$$

$$\geq \frac{1}{2}\big||\phi|^n - |\phi|^{-n}\big|$$

$$\geq \frac{1}{2}(p^n - p^{-n}) \qquad \text{put } \phi = pe^{i\theta}$$

So

$$\circledast \le \frac{1}{2\pi} M \left(p+p^{-1}-2\right)^{-1}\left(p^{n}-p^{-n}\right)^{-1} \int_{B_p} |dz|$$

$$= \frac{4}{2\pi} M \left(p+p^{-1}-2\right)^{-1}\left(p^{n}-p^{-n}\right)^{-1} |B_p|$$

$$= \frac{2}{\pi} M \left(p+p^{-1}-2\right)^{-1}\left(p^{n}-p^{-n}\right)^{-1} |B_p|$$

continuing, consider

$$\left(p^{n}-p^{-n}\right)^{-1}\left(p+p^{-1}-2\right)^{-1} = \frac{1}{p^{n}-\frac{1}{p^{n}}}\,\frac{1}{\left(p+\frac{1}{p}-2\right)}$$

$$= \frac{1}{\left(p^{n+1}+p^{n-1}-2p^{n}-\frac{1}{p^{n-1}}-\frac{1}{p^{n+1}}-\frac{2}{p^{n}}\right)}$$

$$= \frac{p^{2-n}}{p^{3}+p-2p^{2}-p^{3-2n}-p^{1-2n}+2p^{2-2n}}$$

$$\le \frac{p^{2-n}}{p^{3}-3p^{2}-3p-1}$$

] the denominator above can be bounded below by this

$$= \frac{p^{2-n}}{(p-1)^{3}}$$

So $\textcircled{\star} \leq \dfrac{2M|B_p|}{\pi} \dfrac{p^{2-n}}{(p-1)^3}$

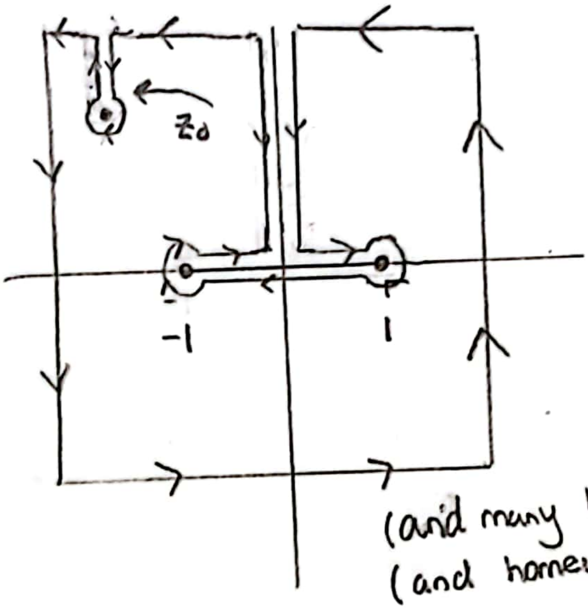
$$\leq c_p \, p^{-n}$$

where $c_p = \dfrac{2M|B_p|p^2}{\pi(p-1)^3}$

so we are done.

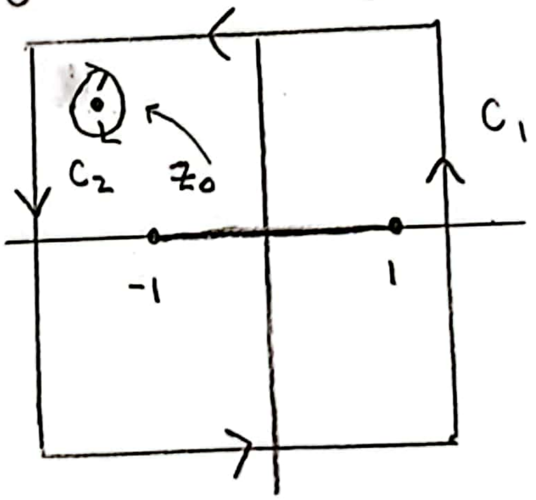
(6) Consider the following contour $C(\varepsilon)$

(a)




(and many book examples)
(and homework 5)

By the same argument as homework 8, this contour converges uniformly to a closed contour $C$ (as $\varepsilon = \frac{1}{n} \to 0$)




So that

(with the numerator function 1 is analytic)

$$\circledast = \frac{1}{\pi} \int_{-1}^{1} \frac{1}{\sqrt{1-x}\,\sqrt{1+x}} \cdot \frac{1}{x - z_0}\, dx = \frac{1}{2\pi i} \int_{\substack{C \\ (ccw)}} \frac{1}{\sqrt{z-1}\sqrt{z+1}} \frac{1}{z - z_0}\, dz$$

where $C = C_1 - C_2$ with $C_1 =$ the box and $C_2 =$ the circle (ccw) where the additional pole at $z_0$ introduces $C_2$ but doesn't change our derivation relating the real and complex integrals. So we have

$$\frac{1}{2\pi i}\int_C \overbrace{\frac{1}{\sqrt{z-1}\,\sqrt{z+1}}\,\frac{1}{z-z_0}}^{f(z)}\,dz = \frac{1}{2\pi i}\left[\overset{1}{\int_{C_1} f(z)\,dz} + \overset{2}{\int_{C_2} f(z)\,dz}\right]$$

For 1 we compute the residue at $\infty$,

$$\begin{aligned} 1 &= 2\pi i \operatorname{Res}(f,\infty) \\ &= 2\pi i \operatorname{Res}\left(\frac{f(\frac{1}{z})}{z^2}, 0\right) \end{aligned}$$

we have from homework 6

$$\sqrt{z-1}\,\sqrt{z+1} = z - \frac{1}{2}z^{-1} + O(z^{-3})$$

So $$\sqrt{1/z-1}\,\sqrt{1/z+1} = \frac{1}{z} - \frac{1}{2}z + O(z^3)$$

And
$$\begin{aligned} \frac{1}{z^2} f\left(\frac{1}{z}\right) &= \frac{1}{z^2}\,\frac{1}{\frac{1}{z}-\frac{1}{2}z+O(z^3)}\,\frac{1}{\frac{1}{z}-z_0} \\ &= \frac{1}{z+O(z^3)}\cdot\frac{1}{\frac{1}{z}-z_0} \\ &= \frac{1}{z+O(z^3)}\cdot\frac{z}{1-z_0 z} \\ &= \frac{z}{z(1-z_0 z)+O(z^3)} \\ &= \frac{z}{(z-z_0 z^2)+O(z^3)} = \frac{1}{1-z_0 z+O(z^3)} \end{aligned}$$

which by long division, takes the form

$$1 + z_0 + O(z^2)$$

which has no $\frac{1}{z}$ term, hence the residue at $\infty$ is 0. Now for the residue at $z_0$ (to compute ②)

$$\mathrm{Res}(f, z_0) = \lim_{z \to z_0} \cancel{(z - z_0)} \frac{1}{\sqrt{z-1}\sqrt{z+1}\,\cancel{(z - z_0)}}$$

$$= \frac{1}{\sqrt{z_0 - 1}\sqrt{z_0 + 1}}$$

(where the limit is permissable as we are away from the branch cut on $[-1, 1]$)

So that

$$\text{(*)} \quad \frac{1}{\cancel{2\pi i}} \cancel{2\pi i} \left[ \mathrm{Res}(f, \infty) - \mathrm{Res}(f, z_0) \right]$$

$$= \boxed{-\frac{1}{\sqrt{z_0 - 1}\sqrt{z_0 + 1}}}$$

$$= \frac{1 - \frac{1}{z^2}}{1 - \frac{1}{2}z^2 + o(z^4) - z_0 z - \frac{1}{2}z_0 z^3 - o(z^5)}$$

$$= \frac{1 - \frac{1}{z^2}}{1 - z_0 z + o(z^2)}$$

which by long division

$$= 1 - \frac{z_0}{z} - (\text{other terms we don't care about})$$

So the coefficient of $\frac{1}{z}$ is $-z_0$, so $\mathrm{Res}(f, \infty) = -z_0$. Now,

$$\mathrm{Res}(f, z_0) = \lim_{z \to z_0} \frac{(z-\cancel{z_0})}{(\cancel{z}-z_0)} \frac{(1-z)(1+z)}{\sqrt{z-1}\sqrt{z+1}}$$

where the limit is fine since we are away from the branch cut on [-1,1]

$$= \frac{(1-z_0)(1+z_0)}{\sqrt{z_0-1}\sqrt{z_0+1}} = \frac{1-z_0^2}{\sqrt{z_0-1}\sqrt{z_0+1}}$$

so we get

$$(*) = 2\left[-z_0 - \frac{(1-z_0^2)}{\sqrt{z_0-1}\sqrt{z_0+1}}\right]$$

$$= -2\left[z_0 + \frac{(1-z_0^2)}{\sqrt{z_0-1}\sqrt{z_0+1}}\right]$$ and we are done.