AMATH 567
HW 7
Nate Whybra

3.5.1)

b) The function $f(z) = \frac{e^{2z} - 1}{z^2}$ fails to be analytic at $z = 0$, so we find the Laurent series for $f(z)$ centered at $z_0 = 0$,

$$f(z) = \frac{1}{z^2}\left(e^{2z} - 1\right) = \frac{1}{z^2}\left(\sum_{n=0}^{\infty} \frac{2^n}{n!} z^n - 1\right)$$

$$= \frac{1}{z^2} \sum_{n=1}^{\infty} \frac{2^n}{n!} z^n$$

$$= \sum_{n=1}^{\infty} \frac{2^n}{n!} z^{n-2}$$

Put $n - 2 = K$, then the above

$$= \sum_{K=-1}^{\infty} \frac{2^{K+2}}{(K+2)!} z^K$$

So $f(z)$ has a pole of order 1 with strength $= \frac{2^{-1+2}}{(-1+2)!} = \frac{2^1}{1!} = 2$ at $z = 0$.

(c) From example 3.5.3 in the textbook, the function $g(z) = \tan(z)$ has simple poles of strength -1 at $z = \frac{\pi}{2} + m\pi$, and the Laurent series centered at $z_0 = \frac{\pi}{2} + m\pi$ starts like
$(m \in \mathbb{Z})$

$$g(z) = \tan(z) = -\frac{1}{z-z_0} + \frac{1}{3}(z-z_0) + \text{rest of terms}$$

So

$$f(z) = e^{\tan(z)} = \sum_{n=0}^{\infty} \frac{(\tan(z))^n}{n!} = \sum_{n=0}^{\infty} \frac{1}{n!}\left(\frac{-1}{z-z_0} + \frac{1}{3}(z-z_0) + \ldots\right)^n$$

$$= \sum_{n=0}^{\infty} \frac{1}{n!}\left(-\frac{1}{z-z_0}\right)^n + \ldots$$

the $\frac{1}{n!}$ is here too

$$= \sum_{n=0}^{\infty} \frac{1}{n!}(-1)^n (z-z_0)^{-n} + \ldots$$

$$= \sum_{n=-\infty}^{0} \frac{1}{(-n)!}(-1)^{-n}(z-z_0)^n + \ldots$$

The series has infinitely many terms $(z-z_0)^n$ with n being negative, hence $f(z) = e^{\tan(z)}$ has an essential singularity at $z = \frac{\pi}{2} + m\pi$ for every $m \in \mathbb{Z}$.

(d) we have the function

$$f(z) = \frac{z^3}{z^2+z+1} = \frac{z^3}{(z-w_+)(z-w_-)}$$

where $w_\pm = -\frac{1}{2} \pm \frac{\sqrt{3}}{2} i = \pm e^{i\frac{2\pi}{3}}$. The function has a simple pole for $z = w_+$ and $z = w_-$.
By similar argument to (Example 3.5.1) in the textbook, the leading term of the Laurent series near $z = w_+$ is (the strength)

$$\frac{w_+^3}{w_+ - w_-} = \frac{e^{i2\pi}}{2e^{i2\pi/3}} = \frac{1}{2} e^{i4\pi/3}$$

and the strength of the pole at $z = w_-$ is

$$\frac{w_-^3}{w_- - w_+} = \frac{(-e^{i2\pi/3})^3}{-2e^{i2\pi/3}} = \frac{1}{2} e^{i4\pi/3}.$$

3.5.3)

a) $f(z) = \dfrac{z}{z^4+2} = \dfrac{z}{(z-w_0)(z-w_1)(z-w_2)(z-w_3)}$

where $w_j = \sqrt[4]{2}\, e^{i(\frac{\pi}{4}+\frac{\pi}{2}j)}$ for $j = 0, 1, 2, 3.$

So again, by similar argument to Example 3.5.1, there is a simple pole for every root $z = w_j$. Denote the strengths of each pole as s_j, then

$$s_0 = \frac{w_0}{(w_0-w_1)(w_0-w_2)(w_0-w_3)}$$

Note that with $j \neq k$

$$w_j - w_k = 2^{1/4} e^{i\pi/4}\left(e^{\frac{\pi}{2}ij} - e^{\frac{\pi}{2}ik}\right)$$
$$= 2^{1/4} e^{i\pi/4}\left(i^j - i^k\right)$$

So,
$$s_0 = \frac{2^{1/4} e^{i\pi/4}}{2^{3/4} e^{i3\pi/4}(1-i)(1+1)(1+i)}$$
$$= \frac{2^{1/4} e^{i\pi/4}}{2^2 \cdot 2^{3/4} e^{i3\pi/4}} = 2^{-5/2} e^{-i\pi/2} = -i\, 2^{-5/2}$$
$$= \frac{-i}{2^{5/2}}$$

$$S_1 = \frac{\omega_1}{(\omega_1-\omega_0)(\omega_1-\omega_2)(\omega_1-\omega_3)}$$

$$= \frac{2^{1/4}\cancel{e^{i\frac{3\pi}{4}}}}{2^{3/4}\cancel{e^{i\frac{3\pi}{4}}}(i-1)(i+1)(i+i)}$$

$$= \frac{1}{2^{1/2}(-2)(2i)} = \frac{i}{2^{1/2}\cdot 2^2} = \frac{i}{2^{5/2}}$$

$$S_2 = \frac{\omega_2}{(\omega_2-\omega_0)(\omega_2-\omega_1)(\omega_2-\omega_3)}$$

$$= \frac{2^{1/4} e^{i5\pi/4}}{2^{3/4} e^{i\frac{3\pi}{4}}(-1-1)(-1-i)(-1+i)}$$

$$= \frac{e^{i\pi/2}}{2^{1/2}\cdot 2\cdot(i-1)(i+1)}$$

$$= \frac{i}{2^{3/2}\cdot(-2)} = \frac{-i}{2^{5/2}}$$

$$S_3 = \frac{\omega_3}{(\omega_3-\omega_0)(\omega_3-\omega_1)(\omega_3-\omega_2)}$$

$$= \frac{2^{1/4} e^{i7\pi/4}}{2^{3/4} e^{i3\pi/4}(-i-1)(-i-i)(-i+1)}$$

$$= \frac{-1}{2^{1/2}(1-i)(1+i)2i} = \frac{i}{2^{1/2}\cdot 2^1\cdot 2^1} = \frac{i}{2^{5/2}}$$

c) For the function $f(z) = \frac{z}{\sin^2(z)}$, there are singularities whenever $z = m\pi$ for $m \in \mathbb{Z}$. So that we can write $\sin^2(z) = \frac{1-\cos(2z)}{2}$

$$f(z) = \frac{2z}{1-\cos(2z)} = \frac{2z}{\sum_{n=1}^{\infty} \frac{(-1)^{n+1}}{(2n)!} \cdot 2^{2n} z^{2n}} \quad ①$$

$$= \frac{\cancel{z} \cdot 2}{\cancel{z} \sum_{n=1}^{\infty} \frac{(-1)^{n+1}}{(2n)!} 2^{2n} z^{2n-1}}$$

$$= \frac{2}{z(2 - 4z^2 + \dots)}$$

So $f(z)$ has a pole of order 1 at $z=0$ with strength $\frac{2}{2} = 1$. Now if we plug $z = z_0 + m\pi$ into ①, $(m \neq 0)$

$$f(z) = \frac{2z_0}{(z_0+m\pi)^2[2-4(z_0+m\pi)^2+\dots]} + \frac{2m\pi}{(z_0+m\pi)^2[2-4(z_0+m\pi)^2+\dots]}$$

if $z = m\pi$, ie $z_0 = 0$, we see the above becomes

$$= \frac{2\pi m}{z^2[2-4z^2+\dots]}$$

So there is a pole of order 2 at $z = m\pi$ with strength $= \frac{2\pi m}{2} = \pi m \quad \forall\, m \neq 0$.

d) we have

$$f(z) = \frac{e^z - 1 - z}{z^4} = \frac{\sum_{n=2}^{\infty} \frac{z^n}{n!}}{z^4}$$

$$= \sum_{n=2}^{\infty} \frac{z^{n-4}}{n!}$$

$$= \sum_{n=-2}^{\infty} \frac{z^n}{(n+4)!}$$

Since f has singularities at $z=0$, we see f has a pole of order 2 at $z=0$ with strength $= \frac{1}{(+2)!} = \frac{1}{2}$

3.6.6)

a) Taking the principle branch of the logarithm, recall that $\frac{d}{dz}\ln(f(z)) = \frac{f'(z)}{f(z)}$, so

$$\frac{d}{dz}\ln\left(\frac{1}{f(z)}\right) = \frac{d}{dz}\ln\left(f(z)^{-1}\right) = \frac{d}{dz} - \ln(f(z)) = \frac{-f'(z)}{f(z)}$$

So if,

$$\frac{1}{\Gamma(z)} = z e^{\gamma z} \prod_{n=1}^{\infty} \left(1 + \frac{z}{n}\right) e^{-z/n}$$

$$-\log(\Gamma(z)) = \log(z) + \gamma z + \sum_{n=1}^{\infty} \log\left(1+\frac{z}{n}\right) - \frac{z}{n}$$

$$\xrightarrow{\frac{d}{dz}} \quad \frac{-\Gamma'(z)}{\Gamma(z)} = \frac{1}{z} + \gamma + \sum_{n=1}^{\infty} \frac{\frac{1}{n}}{1+z/n} - \frac{1}{n}$$

$$= \frac{1}{z} + \gamma + \sum_{n=1}^{\infty} \frac{1}{z+n} - \frac{1}{n}$$

$$\longrightarrow \quad \frac{\Gamma'(z)}{\Gamma(z)} = \frac{-1}{z} - \gamma - \sum_{n=1}^{\infty} \frac{1}{z+n} - \frac{1}{n}$$

as desired.

b) we have

$$\frac{\Gamma'(z+1)}{\Gamma(z+1)} - \frac{\Gamma'(z)}{\Gamma(z)} - \frac{1}{z}$$

which we'd like to show $= 0$. So to that effect, we can show $\frac{\Gamma'(z+1)}{\Gamma(z+1)} = \frac{1}{z} + \frac{\Gamma'(z)}{\Gamma(z)}$.
From (a), we see

$$\frac{\Gamma'(z+1)}{\Gamma(z+1)} = -\frac{1}{z+1} - \gamma - \sum_{n=1}^{\infty} \frac{1}{z+1+n} - \frac{1}{n}$$

and

$$\frac{1}{z} + \frac{\Gamma'(z)}{\Gamma(z)} = \frac{1}{z} - \frac{1}{z} - \gamma - \sum_{n=1}^{\infty} \frac{1}{z+n} - \frac{1}{n}$$

$$= -\gamma - \sum_{n=1}^{\infty} \frac{1}{z+n} - \frac{1}{n}$$

$$= -\frac{1}{z+1} - \gamma - \left(\sum_{n=2}^{\infty} \frac{1}{z+n} - \frac{1}{n}\right) + 1$$

$$= -\frac{1}{z+1} - \gamma - \left(\sum_{n=1}^{\infty} \frac{1}{z+n+1} - \frac{1}{n+1}\right) + 1$$

$$= -\frac{1}{z+1} - \gamma - \left[\left(\frac{1}{z+2} - \frac{1}{2}\right) + \left(\frac{1}{z+3} - \frac{1}{3}\right) + \dots + - 1\right]$$

$$= -\frac{1}{z+1} - \gamma - \left[\left(\frac{1}{z+2} - 1\right) + \frac{1}{2} + \left(\frac{1}{z+3} - \frac{1}{2}\right) + \frac{1}{6} + \dots - 1\right]$$

$$= \frac{-1}{z+1} - \gamma + 1 - \sum_{n=1}^{\infty} \frac{1}{z+1+n} - \frac{1}{n} + \left(\frac{1}{n} - \frac{1}{n+1}\right)$$

(*)
$$= \frac{-1}{z+1} - \gamma + 1 - \left(\sum_{n=1}^{\infty} \frac{1}{z+1+n} - \frac{1}{n}\right) - \sum_{n=1}^{\infty}\left(\frac{1}{n} - \frac{1}{n+1}\right)$$

where the splitting of the above sum is allowed since $\sum_{n=1}^{\infty} \frac{1}{n} - \frac{1}{n+1} = \sum_{n=1}^{\infty} \frac{1}{n(n+1)} < \sum_{n=1}^{\infty} \frac{1}{n^2} = \frac{\pi^2}{6} < \infty$

But notice

$$\sum_{n=1}^{\infty} \frac{1}{n} - \frac{1}{n+1} = 1 - \frac{1}{2} + \frac{1}{2} - \frac{1}{3} + \frac{1}{3} + \dots$$

$$= \lim_{N\to\infty} \sum_{n=1}^{N} \frac{1}{n} - \frac{1}{n+1}$$

telescopes

$$= \lim_{N\to\infty} 1 - \frac{1}{N+1}$$

$$= 1$$

So (*)

$$= -\frac{1}{z+1} - \gamma - \sum_{n=1}^{\infty} \frac{1}{z+1+n} - \frac{1}{n} = \frac{\Gamma'(z+1)}{\Gamma(z)}$$ as

desired, establishing

$$\frac{\Gamma'(z+1)}{\Gamma(z+1)} - \frac{\Gamma'(z)}{\Gamma(z)} - \frac{1}{z} = 0$$

By integrating both sides of the equation (w/ respect to z) we see

$$\ln(\Gamma(z+1)) - \ln(\Gamma(z)) - \ln(z) = \underset{\text{(some constant)}}{C}$$

$$\ln\left(\frac{\Gamma(z+1)}{z\,\Gamma(z)}\right) = C$$

$$\frac{\Gamma(z+1)}{z\,\Gamma(z)} = e^{C}$$

$$\Gamma(z+1) = \underbrace{e^{C}}_{=e^{c}} \cdot z \cdot \Gamma(z) = c' z \cdot \Gamma(z)$$

also as desired,

c) $$\lim_{z \to 0} z \cdot \Gamma(z) = \lim_{z \to 0} \frac{z}{\frac{1}{\Gamma(z)}}$$

$$= \lim_{z \to 0} \frac{\cancel{z}}{\cancel{z}\, e^{\gamma z} \prod_{n=1}^{\infty} \left(1 + \frac{z}{n}\right) e^{-z/n}}$$

$$= \frac{1}{e^{\gamma(0)} \prod_{n=1}^{\infty} 1} = \frac{1}{1} = 1$$

So $$\lim_{z \to 0} \Gamma(z+1) = \lim_{z \to 0} c' z \cdot \Gamma(z)$$

$$\longrightarrow \quad \Gamma(1) = c' \cdot 1 = c'$$

as desired.

d) If we want $\Gamma(1)=1$, that is equivalent to $\frac{1}{\Gamma(1)} = 1$, so when $z=1$

$$1 := \frac{1}{\Gamma(1)} = e^{\gamma} \prod_{n=1}^{\infty} \left(1+\frac{1}{n}\right) e^{-1/n}$$

$$\rightarrow \quad e^{-\gamma} = \prod_{n=1}^{\infty} \left(1+\frac{1}{n}\right) e^{-1/n} \quad \text{(as desired).}$$

e) we have,

$$\prod_{n=1}^{\infty} \left(1+\frac{1}{n}\right) e^{-1/n} = \prod_{n=1}^{\infty} \frac{n+1}{n} e^{-1/n}$$

$$= \lim_{N\to\infty} \prod_{n=1}^{N} \frac{n+1}{n} e^{-1/n} \quad \Big\downarrow \text{ grouping terms}$$

$$= \lim_{N\to\infty} \left(\frac{2}{1}\,\frac{3}{2}\,\frac{4}{3} \cdots \frac{N+1}{N}\right)\left(e^{-1/1}\, e^{-1/2} \cdots e^{-1/N}\right)$$

$$= \lim_{N\to\infty} \left(\frac{2}{1}\,\frac{3}{2}\,\frac{4}{3} \cdots \frac{N+1}{N}\right) e^{-S(N)} \quad \text{(as desired)}$$

$$= \lim_{N\to\infty} (N+1)\, e^{-S(N)} \quad \text{(as also desired)}$$

(by cancelling fractions)

So,

$$e^{-\gamma} = \prod_{n=1}^{\infty} \left(1+\frac{1}{n}\right)e^{-1/n} = \lim_{n\to\infty} (n+1)\, e^{-S(n)}$$

$$-\gamma = \log\left(\lim_{n\to\infty} (n+1)\, e^{-S(n)}\right)$$

$$= \lim_{n\to\infty} \log\left([n+1]\, e^{-S(n)}\right) \quad \left[\text{as the input is } > 0 \text{ and } \log \text{ is continuous}\right]$$

$$= \lim_{n\to\infty} \log(n+1) - S(n)$$

$$\to \gamma = \lim_{n\to\infty} S(n) - \log(n+1)$$

$$= \lim_{n\to\infty} \left[\sum_{k=1}^{n} \frac{1}{k} - \log(n+1)\right]$$

as desired.

4) without loss of generality, assume $a_n \neq 0$ for any n, then (WLOG since finitely many z's with $|z|<1$ won't bother the convergence of the product).

$$H(z) = \prod_{n=0}^{\infty} B(a_n, z)$$

$$= \prod_{n=0}^{\infty} 1 + (-1 + B(a_n, z))$$

$$= \prod_{n=0}^{\infty} 1 + \Big(\overbrace{\frac{|a_n|}{a_n} \frac{a_n - z}{1 - \bar{a}_n z} - 1}^{\circledast} \Big)$$

we'd like to use Weirstrass M-Test, so we'd like to bound $\circledast$ by some constant M_n such that $\sum_{n=0}^{\infty} M_n < \infty$. So

$$|\circledast| = \left| \frac{|a_n| a_n - |a_n| z - a_n + |a_n|^2 z}{a_n (1 - \bar{a}_n z)} \right|$$

find common denominator

$$= \left| \frac{(|a_n| - 1)(a_n - |a_n| z)}{a_n (1 - \bar{a}_n z)} \right|$$

divide the right term by a_n

$$= \left| \frac{(|a_n| - 1)(1 - \bar{a}_n z)}{1 - \bar{a}_n z} \right|$$

$$\leq \frac{|1 - |a_n|| \, \cancel{|1 - \bar{a}_n z|}}{\cancel{|1 - \bar{a}_n z|}}$$

$$= 1 - |a_n|$$

Since we are given that $\sum_{n=0} 1-|a_n| < \infty$,
we choose our M_n's to be $1-|a_n|$. Each $B(a_n, z)$ is analytic everywhere except when $z = \frac{1}{\bar{a}_n}$, so by the M-test, we get that $H(z)$ is analytic $\forall z$ in the unit disk other than at the points just mentioned.

To see the zeroe's of H, this occurs when

$$B(a_n, z) = 0$$

$$\frac{|a_n|}{a_n} \frac{a_n - z}{1 - \bar{a}_n z} = 0$$

$$\leftrightarrow \quad z = a_n .$$

And if $a = 0$, then $z = 0$ also is a zero.
so H has countably many zeroes inside the unit disk.

$H(z)$ need not be 0 $\forall |z| \leq 1$ because our function at $a=1$ is 1 for all z. =
Since $\sum_{n=0}^{\infty} 1 - |a_n| < \infty$, we get for free that $\lim_{n\to\infty} 1-|a_n| = 0 \rightarrow \lim_{n\to\infty} |a_n| = 1$. Since our function has 0's at every a_n but not at the limit point $a=1$, we have that the zeros of H are not dense inside the unit disk.

5) a) we can write with $w = (w_1, w_2)$

$$p(z, w) = \frac{1}{z^2} + \sum_{\substack{i=(j,k)\\ i \neq (0,0)}} \frac{1}{(z - w \cdot i)^2} - \frac{1}{(w \cdot i)^2}$$

Put $z' = z + Mw_1 + Nw_2 = z + T \cdot w \quad (T = (M, N))$,

then

$$p(z', w) = \frac{1}{(z + T \cdot w)^2} + \sum_{\substack{i=(j,k)\\ i \neq (0,0)}} \frac{1}{(z + T \cdot w - i \cdot w)^2} - \frac{1}{(w \cdot i)^2}$$

$$= \frac{1}{(z + T \cdot w)^2} + \sum_{\substack{i=(j,k)\\ i \neq (0,0)}} \frac{1}{(z + (T - i) \cdot w)^2} - \frac{1}{(w \cdot i)^2}$$

Put $l = (T - i)$, then the above becomes

$$= \frac{1}{(z + T \cdot w)^2} + \sum_{\substack{l=(j,k)\\ l \neq T}} \frac{1}{(z + l \cdot w)^2} - \frac{1}{(w \cdot (T - l))^2}$$

swapping 1 term from inside and the one outside

$$= \frac{1}{z^2} + \sum_{\substack{l=(j,k)\\ l \neq 0}} \frac{1}{(z + l \cdot w)^2} - \frac{1}{(w \cdot (T - l))^2}$$

replacing $T - l$ with i by the sub definition and realizing l and i now have the same index range

$$= \frac{1}{z^2} + \sum_{\substack{i=(j,k)\\ i \neq 0}} \frac{1}{(z + i \cdot w)^2} - \frac{1}{(w \cdot i)^2}$$

$= p(z, w)$ (as desired).

So p is doubly periodic.

b)

$$p(-z,w) = \frac{1}{(-z)^2} + \sum_{\substack{i=(j,k) \\ i \neq (0,0)}} \frac{1}{(-z-w\cdot i)^2} - \frac{1}{(w\cdot i)^2}$$

Replacing i with $-i$ won't change the sum, so

$$= \frac{1}{z^2} + \sum_{\substack{i=(j,k) \\ i \neq (0,0)}} \frac{1}{(-z+w\cdot i)^2} - \frac{1}{(w\cdot i)^2}$$

this won't change b/c of square

$$= \frac{1}{z^2} + \sum_{\substack{i=(j,k) \\ i \neq (0,0)}} \frac{1}{[(-1)(z-w\cdot i)]^2} - \frac{1}{(w\cdot i)^2}$$

$$= \frac{1}{z^2} + \sum_{\substack{i=(j,k) \\ i \neq (0,0)}} \frac{1}{(z-w\cdot i)^2} - \frac{1}{(w\cdot i)^2}$$

$$= p(z,w)$$

so p is an even function in z.

c) $$p(z,w) = \frac{1}{z^2} + \sum_{\substack{i=(j,k)\\ i\neq(0,0)}} \frac{1}{(z-w\cdot i)^2} - \frac{1}{(w\cdot i)^2}$$

$$= \frac{1}{z^2} + \sum_{\substack{i=(j,k)\\ i\neq(0,0)}} \frac{1}{(w\cdot i)^2}\left[\frac{1}{(\frac{z}{w\cdot i}-1)^2} - 1\right]$$

Taylor series for $\frac{1}{(z-1)^2}$

$$= \frac{1}{z^2} + \sum_{\substack{i=(j,k)\\ i\neq(0,0)}} \frac{1}{(w\cdot i)^2}\left[\sum_{n=1}^{\infty} n\frac{z^{n-1}}{(w\cdot i)^{n-1}} - 1\right]$$

$$= \frac{1}{z^2} + \sum_{\substack{i=(j,k)\\ i\neq(0,0)}} \sum_{n=2}^{\infty} n\frac{z^{n-1}}{(w\cdot i)^{n+1}}$$

$$= \frac{1}{z^2} + \sum_{n=2}^{\infty} \sum_{\substack{i=(j,k)\\ i\neq(0,0)}} \frac{n z^{n-1}}{(w\cdot i)^{n+1}}$$

since p is even

$$= \frac{1}{z^2} + \sum_{n=1}^{\infty} \sum_{\substack{i=(j,k)\\ i\neq(0,0)}} \frac{(2n+1)z^{2n}}{(w\cdot i)^{2n+2}}$$

So our laurent series has coefficients

$$a_{2n} = \sum_{\substack{i=(j,k)\\ i\neq(0,0)}} \frac{(2n+1)}{(w\cdot i)^{2n+2}}$$

Let's find a_0, a_2, and a_4. $a_0 = 0$.

$$a_2 = \sum_{\substack{i=(j,k)\\ i\neq(0,0)}} \frac{3}{(w\cdot i)^4}, \quad a_4 = \sum_{\substack{i=(j,k)\\ i\neq(0,0)}} \frac{5}{(w\cdot i)^6}.$$

So that

$$p(z) = \frac{1}{z^2} + a_2 z^2 + a_4 z^4 + \dots$$

$$p'(z) = -\frac{2}{z^3} + \overbrace{2a_2}^{b_1} z + \overbrace{4a_4}^{b_2} z^3 + \dots$$

d) we have,

$$(p')^2 = \left(-\frac{2}{z^3} + a_2 z \cdot 2 \; a_4 z^3 \cdot 4 + \dots\right)^2$$

$$= \frac{4}{z^6} - \frac{8a_2}{z^2} - 16a_4 + f_1(z)$$

where $f_1(z)$ has only positive powers of z.

Also,

$$p^3 = \left(\frac{1}{z^3} + a_2 z^2 + a_4 z^4 + \dots\right)^3$$

$$= \frac{1}{z^6} + \frac{3a_2}{z^2} + 3a_4 + f_2(z)$$

where $f_2(z)$ also has only positive powers of z,

we can then see

$$(p')^2 - 4p^3 = \frac{-20a_2}{z^2} - 28a_4 + f_1(z) - 4f_2(z)$$

since $p(z) = \frac{1}{z^2} + a_2 z^2 + \dots$, $-20a_2 \cdot p(z)$ only has $\frac{1}{z^2}$ as a (non positive) negative power term, and we can write

$$(p')^2 - 4p^3 + 20a_2 p + 28a_4 = f_1(z) - 4f_2(z) + f_3(z)$$

where again, $f_3(z)$ has only positive values of z. By construction, the above function has no poles, it is a polynomial with only positive powers of z. Since p is elliptic, so is the above formula, meaning it is doubly periodic. It is (analytic and hence) continuous and bounded on each lattice parallelogram $j\omega_1 + K\omega_2$ (since polynomials are entire), and since the function (periodic) repeats itself on each paralleogram, it must be analytic and bounded on all of $\mathbb{C}$. Liouville's theorem then implies the above is constant, ie,

$$(p')^2 - 4p^3 + 20a_2 p + 28a_4 = C \quad \text{for some } C \in \mathbb{C}$$

But plugging 0 into the RHS at the top of this page trivially returns 0 (only positive powers of z).

Hence,

$$(p')^2 = 4p^3 - 20a_2 p - 28a_4$$

So $a = 4$, $b = 0$, $c = -20a_2$, $d = -28a_4$.