AMATH 584 - Homework 2

Nate Whybra

October 2024

Problem 1 (3.2):

Let $x \in \mathbb{C}^m$ be the eigenvector of A corresponding to the eigenvalue λ with the largest absolute value, so that $\rho(A) = |\lambda|$. We then have:

$$\lambda x = Ax$$

$$|\lambda| \, \|x\| = \|\lambda x\| = \|Ax\| \leq \|A\| \, \|x\|$$

By definition of eigenvector, $\|x\| \neq 0$. So the above implies (by dividing both sides by $\|x\|$):

$$|\lambda| = \rho(A) \leq \|A\|$$

Which is the desired result.

Problem 2 (3.5):

Yes the claim is true, to prove so suppose $u, v \in \mathbb{C}^m$ and $A = uv^*$ where:

$$u = \begin{bmatrix} u_1 \\ u_2 \\ \vdots \\ u_m \end{bmatrix} \quad v = \begin{bmatrix} v_1 \\ v_2 \\ \vdots \\ v_m \end{bmatrix}$$

Then the j-th column c_j of A is $v_j^* u$. Notice we can write:

$$\|A\|_F = \left(\sum_{i=1}^{m} \sum_{j=1}^{m} |a_{ij}|^2 \right)^{\frac{1}{2}} = \left(\sum_{j=1}^{m} c_j^* c_j \right)^{\frac{1}{2}} = \left(\sum_{j=1}^{m} \|c_j\|_F^2 \right)^{\frac{1}{2}}$$

Where the second equality follows by considering the sums of squared-modulus of A as the sum of the sums of squared-modulus of the columns of A. The last equality follows since $\|.\|_2 = \|.\|_F$ for vectors. Continuing by substituting in our expression for c_j:

$$\|A\|_F = \left(\sum_{j=1}^{m} \left\|v_j^* u\right\|_F^2\right)^{\frac{1}{2}} = \left(\sum_{j=1}^{m} |v_i|^2 \|u\|_F^2\right)^{\frac{1}{2}} = \left(\|u\|_F^2 \sum_{j=1}^{m} |v_i|^2\right)^{\frac{1}{2}}$$

$$= \left(\|u\|_F^2 \|v\|_F^2\right)^{\frac{1}{2}}$$

$$= \|u\|_F \|v\|_F$$

Which is the desired result.

Problem 3 (4.1):

We must compute the SVD's of 5 matrices. For (a), start with $A = \begin{bmatrix} 3 & 0 \\ 0 & -2 \end{bmatrix}$.

We want $A = U\Sigma V^*$, or equivalently $A^* = V\Sigma U^*$. This implies that $A^*A = V\Sigma U^* U\Sigma V^* = V\Sigma^2 V^*$. We have just written A^*A as an eigenvector diagonalization. By computing the eigenvalues and eigenvectors of of A^*A, we can find V and Σ, which we can then use to solve for U using the SVD equation. I will be using this approach for all 5 matrices, so I have explained it here and won't explain much in my calculations. So we have:

$$A^*A = \begin{bmatrix} 3 & 0 \\ 0 & -2 \end{bmatrix} \begin{bmatrix} 3 & 0 \\ 0 & -2 \end{bmatrix} = \begin{bmatrix} 9 & 0 \\ 0 & 4 \end{bmatrix}$$

To find the eigenvalues, we must compute the roots of the characteristic polynomial:

$$\det A - \lambda I = \det\left(\begin{bmatrix} 9-\lambda & 0 \\ 0 & 4-\lambda \end{bmatrix}\right)$$

$$p(\lambda) = (9-\lambda)(4-\lambda) = 0$$

So the eigenvalues are $\lambda \in \{9, 4\}$. Now to compute the eigenvectors, we must compute the nullspace of $A - \lambda I$ for each eigenvalue:

$$(A^*A - 9I)x = \begin{bmatrix} 0 & 0 \\ 0 & -5 \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \end{bmatrix}$$

Which yields the eigenvector $v_1 = \begin{bmatrix} 1 \\ 0 \end{bmatrix}$. Next:

$$(A^*A - 4I)x = \begin{bmatrix} 5 & 0 \\ 0 & 0 \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \end{bmatrix}$$

Which yields the eigenvector $v_2 = \begin{bmatrix} 0 \\ 1 \end{bmatrix}$. So we have:

$$\Sigma^2 = \begin{bmatrix} 9 & 0 \\ 0 & 4 \end{bmatrix} \to \Sigma = \begin{bmatrix} 3 & 0 \\ 0 & 2 \end{bmatrix}$$

$$V = \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix}$$

So since $AV = U\Sigma$, we know $U = AV\Sigma^{-1}$. Hence:

$$U = \begin{bmatrix} 3 & 0 \\ 0 & -2 \end{bmatrix} \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix} \begin{bmatrix} \frac{1}{3} & 0 \\ 0 & \frac{1}{2} \end{bmatrix} = \begin{bmatrix} 1 & 0 \\ 0 & -1 \end{bmatrix}$$

Thus:

$$A = U\Sigma V^* = \begin{bmatrix} 1 & 0 \\ 0 & -1 \end{bmatrix} \begin{bmatrix} 3 & 0 \\ 0 & 2 \end{bmatrix} \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix}$$

Now for part (b), redefine $A = \begin{bmatrix} 2 & 0 \\ 0 & 3 \end{bmatrix}$. Then:

$$A^*A = \begin{bmatrix} 2 & 0 \\ 0 & 3 \end{bmatrix} \begin{bmatrix} 2 & 0 \\ 0 & 3 \end{bmatrix} = \begin{bmatrix} 4 & 0 \\ 0 & 9 \end{bmatrix}$$

To find the eigenvalues, we must compute the roots of the characteristic polynomial:

$$\det A - \lambda I = \det \left(\begin{bmatrix} 4 - \lambda & 0 \\ 0 & 9 - \lambda \end{bmatrix} \right)$$

$$p(\lambda) = (4 - \lambda)(9 - \lambda) = 0$$

So the eigenvalues are $\lambda \in \{4, 9\}$. Now to compute the eigenvectors, we must compute the nullspace of $A - \lambda I$ for each eigenvalue:

$$(A^*A - 4I)x = \begin{bmatrix} 0 & 0 \\ 0 & 5 \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \end{bmatrix}$$

Which yields the eigenvector $v_1 = \begin{bmatrix} 1 \\ 0 \end{bmatrix}$. Next:

$$(A^*A - 9I)x = \begin{bmatrix} -5 & 0 \\ 0 & 0 \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \end{bmatrix}$$

Which yields the eigenvector $v_2 = \begin{bmatrix} 0 \\ 1 \end{bmatrix}$. By reordering the eigenvectors, we can make it so that the singular values are in decreasing order. So we have:

$$V = \begin{bmatrix} 0 & 1 \\ 1 & 0 \end{bmatrix}$$

$$\Sigma^2 = \begin{bmatrix} 9 & 0 \\ 0 & 4 \end{bmatrix} \to \Sigma = \begin{bmatrix} 3 & 0 \\ 0 & 2 \end{bmatrix}$$

So since $AV = U\Sigma$, we know $U = AV\Sigma^{-1}$. Hence:

$$U = \begin{bmatrix} 2 & 0 \\ 0 & 3 \end{bmatrix} \begin{bmatrix} 0 & 1 \\ 1 & 0 \end{bmatrix} \begin{bmatrix} \frac{1}{3} & 0 \\ 0 & \frac{1}{2} \end{bmatrix} = \begin{bmatrix} 0 & 1 \\ 1 & 0 \end{bmatrix}$$

Thus:

$$A = U\Sigma V^* = \begin{bmatrix} 0 & 1 \\ 1 & 0 \end{bmatrix} \begin{bmatrix} 3 & 0 \\ 0 & 2 \end{bmatrix} \begin{bmatrix} 0 & 1 \\ 1 & 0 \end{bmatrix}$$

Now for part (d), redefine $A = \begin{bmatrix} 1 & 1 \\ 0 & 0 \end{bmatrix}$. Then:

$$A^*A = \begin{bmatrix} 1 & 0 \\ 1 & 0 \end{bmatrix} \begin{bmatrix} 1 & 1 \\ 0 & 0 \end{bmatrix} = \begin{bmatrix} 1 & 1 \\ 1 & 1 \end{bmatrix}$$

To find the eigenvalues, we must compute the roots of the characteristic polynomial:

$$\det A^*A - \lambda I = \det \left(\begin{bmatrix} 1-\lambda & 1 \\ 1 & 1-\lambda \end{bmatrix} \right)$$

$$p(\lambda) = (1-\lambda)(1-\lambda) - 1 = \lambda^2 - 2\lambda = \lambda(\lambda - 2) = 0$$

So the eigenvalues are $\lambda \in \{2, 0\}$ (but 0 is degenerate). Now to compute the eigenvectors, we must compute the nullspace of $A - \lambda I$ for each eigenvalue:

$$(A^*A - 2I)x = \begin{bmatrix} -1 & 1 \\ 1 & -1 \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \end{bmatrix}$$

Which yields the eigenvector $v_1 = \begin{bmatrix} \frac{1}{\sqrt{2}} \\ \frac{1}{\sqrt{2}} \end{bmatrix}$. Next:

$$(A^*A - 0I)x = \begin{bmatrix} 1 & 1 \\ 1 & 1 \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \end{bmatrix}$$

Which yields the eigenvector $v_2 = \begin{bmatrix} \frac{1}{\sqrt{2}} \\ -\frac{1}{\sqrt{2}} \end{bmatrix}$. So we have:

$$V = \begin{bmatrix} \frac{1}{\sqrt{2}} & \frac{1}{\sqrt{2}} \\ \frac{1}{\sqrt{2}} & -\frac{1}{\sqrt{2}} \end{bmatrix}$$

$$\Sigma^2 = \begin{bmatrix} 2 & 0 \\ 0 & 0 \end{bmatrix} \rightarrow \Sigma = \begin{bmatrix} \sqrt{2} & 0 \\ 0 & 0 \end{bmatrix}$$

So since $AV = U\Sigma$, but Σ is not invertible in this case, we must use the pseudo inverse Σ^+. We need $AV\Sigma^+ = U$. So:

$$U = \begin{bmatrix} 1 & 1 \\ 0 & 0 \end{bmatrix} \begin{bmatrix} \frac{1}{\sqrt{2}} & \frac{1}{\sqrt{2}} \\ \frac{1}{\sqrt{2}} & -\frac{1}{\sqrt{2}} \end{bmatrix} \begin{bmatrix} \frac{1}{\sqrt{2}} & 0 \\ 0 & 0 \end{bmatrix} = \begin{bmatrix} \frac{1}{\sqrt{2}} & 0 \\ 0 & 0 \end{bmatrix}$$

However, we need U to be orthogonal, we must normalize the the matrix and also fill the 0 column with something orthogonal to the first column. Doing this we get:

$$U = \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix}$$

$$A = U\Sigma V^* = \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix} \begin{bmatrix} \sqrt{2} & 0 \\ 0 & 0 \end{bmatrix} \begin{bmatrix} \frac{1}{\sqrt{2}} & \frac{1}{\sqrt{2}} \\ \frac{1}{\sqrt{2}} & -\frac{1}{\sqrt{2}} \end{bmatrix}$$

Now for part (e), redefine $A = \begin{bmatrix} 1 & 1 \\ 1 & 1 \end{bmatrix}$. Then:

$$A^*A = \begin{bmatrix} 1 & 1 \\ 1 & 1 \end{bmatrix} \begin{bmatrix} 1 & 1 \\ 1 & 1 \end{bmatrix} = \begin{bmatrix} 2 & 2 \\ 2 & 2 \end{bmatrix}$$

To find the eigenvalues, we must compute the roots of the characteristic polynomial:

$$\det A^*A - \lambda I = \det \left(\begin{bmatrix} 2-\lambda & 2 \\ 2 & 2-\lambda \end{bmatrix} \right)$$

$$p(\lambda) = (2-\lambda)(2-\lambda) - 4 = \lambda^2 - 4\lambda = \lambda(\lambda - 4) = 0$$

So the eigenvalues are $\lambda \in \{4, 0\}$ (with 0 being a degenerate case). Now to compute the eigenvectors, we must compute the nullspace of $A - \lambda I$ for each eigenvalue:

$$(A^*A - 4I)x = \begin{bmatrix} -2 & 2 \\ 2 & -2 \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \end{bmatrix}$$

Which yields the eigenvector $v_1 = \begin{bmatrix} \frac{1}{\sqrt{2}} \\ \frac{1}{\sqrt{2}} \end{bmatrix}$. Next:

$$(A^*A - 0I)x = \begin{bmatrix} 2 & 2 \\ 2 & 2 \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \end{bmatrix}$$

Which yields the eigenvector $v_2 = \begin{bmatrix} -\frac{1}{\sqrt{2}} \\ \frac{1}{\sqrt{2}} \end{bmatrix}$. So we have:

$$V = \begin{bmatrix} \frac{1}{\sqrt{2}} & -\frac{1}{\sqrt{2}} \\ \frac{1}{\sqrt{2}} & \frac{1}{\sqrt{2}} \end{bmatrix}$$

$$\Sigma^2 = \begin{bmatrix} 4 & 0 \\ 0 & 0 \end{bmatrix} \to \Sigma = \begin{bmatrix} 2 & 0 \\ 0 & 0 \end{bmatrix}$$

So since $AV = U\Sigma$, but Σ is not invertible in this case, we must use the pseudo inverse Σ^+. We need $AV\Sigma^+ = U$. So:

$$U = \begin{bmatrix} 1 & 1 \\ 1 & 1 \end{bmatrix} \begin{bmatrix} \frac{1}{\sqrt{2}} & -\frac{1}{\sqrt{2}} \\ \frac{1}{\sqrt{2}} & \frac{1}{\sqrt{2}} \end{bmatrix} \begin{bmatrix} \frac{1}{2} & 0 \\ 0 & 0 \end{bmatrix} = \begin{bmatrix} \frac{1}{2} & 0 \\ \frac{1}{2} & 0 \end{bmatrix}$$

However, we need U to be orthogonal, we must normalize the the matrix and also fill the 0 column with something orthogonal to the first column. Doing this we get:

$$U = \begin{bmatrix} \frac{1}{\sqrt{2}} & \frac{1}{\sqrt{2}} \\ \frac{1}{\sqrt{2}} & -\frac{1}{\sqrt{2}} \end{bmatrix}$$

$$A = U\Sigma V^* = \begin{bmatrix} \frac{1}{\sqrt{2}} & \frac{1}{\sqrt{2}} \\ \frac{1}{\sqrt{2}} & -\frac{1}{\sqrt{2}} \end{bmatrix} \begin{bmatrix} 2 & 0 \\ 0 & 0 \end{bmatrix} \begin{bmatrix} \frac{1}{\sqrt{2}} & -\frac{1}{\sqrt{2}} \\ \frac{1}{\sqrt{2}} & \frac{1}{\sqrt{2}} \end{bmatrix}$$

Problem 4 (5.4):

Suppose $A \in \mathbb{C}^{mxm}$ such that $A = U\Sigma V^*$ and we have:

$$B = \begin{bmatrix} 0 & A^* \\ A & 0 \end{bmatrix}$$

Where B is a $2m$ x $2m$ Hermitian matrix. Notice how the formula for the SVD of A gives us the following two equations (where the second equation follows from taking the adjoint on both sides of the first equation):

$$AV = U\Sigma$$

$$A^*U = V\Sigma$$

We will play with the matrix B to try and use the above two relationships. First consider the $2m$ x m matrix:

$$X = \begin{bmatrix} V \\ U \end{bmatrix}$$

We have:

$$BX = \begin{bmatrix} 0 & A^* \\ A & 0 \end{bmatrix} \begin{bmatrix} V \\ U \end{bmatrix} = \begin{bmatrix} A^*U \\ AV \end{bmatrix} = \begin{bmatrix} V\Sigma \\ U\Sigma \end{bmatrix}$$

Next consider another $2m$ x m matrix:

$$Y = \begin{bmatrix} -V \\ U \end{bmatrix}$$

So that we have:

$$BY = \begin{bmatrix} 0 & A^* \\ A & 0 \end{bmatrix} \begin{bmatrix} -V \\ U \end{bmatrix} = \begin{bmatrix} A^*U \\ -AV \end{bmatrix} = \begin{bmatrix} V\Sigma \\ -U\Sigma \end{bmatrix}$$

Finally consider the matrix Z:

$$Z = \begin{bmatrix} V & -V \\ U & U \end{bmatrix}$$

So we have:

$$BZ = \begin{bmatrix} 0 & A^* \\ A & 0 \end{bmatrix} \begin{bmatrix} V & -V \\ U & U \end{bmatrix} = \begin{bmatrix} V\Sigma & V\Sigma \\ U\Sigma & -U\Sigma \end{bmatrix} = \begin{bmatrix} V & V \\ U & -U \end{bmatrix} \begin{bmatrix} \Sigma & 0 \\ 0 & \Sigma \end{bmatrix}$$

We are almost done. We want the matrix with the Σ's to be the diagonal matrix of eigenvalues, but since B is hermitian, we know it needs to have distinct eigenvalues. We can do the following:

$$\begin{bmatrix} 0 & A^* \\ A & 0 \end{bmatrix} \begin{bmatrix} V & -V \\ U & U \end{bmatrix} = \begin{bmatrix} V & V \\ U & -U \end{bmatrix} \begin{bmatrix} \Sigma & 0 \\ 0 & \Sigma \end{bmatrix}$$
$$\begin{bmatrix} 0 & A^* \\ A & 0 \end{bmatrix} \begin{bmatrix} V & -V \\ U & U \end{bmatrix} = \begin{bmatrix} V & V \\ U & -U \end{bmatrix} \begin{bmatrix} I & 0 \\ 0 & -I \end{bmatrix} \begin{bmatrix} I & 0 \\ 0 & -I \end{bmatrix} \begin{bmatrix} \Sigma & 0 \\ 0 & \Sigma \end{bmatrix}$$

Where I is the m x m identity matrix. This is allowed because:

$$\begin{bmatrix} I & 0 \\ 0 & -I \end{bmatrix}^2 = I_{2m}$$

Now notice the following two things:

$$\begin{bmatrix} V & V \\ U & -U \end{bmatrix} \begin{bmatrix} I & 0 \\ 0 & -I \end{bmatrix} = \begin{bmatrix} V & V \\ U & -U \end{bmatrix}$$
$$\begin{bmatrix} I & 0 \\ 0 & -I \end{bmatrix} \begin{bmatrix} \Sigma & 0 \\ 0 & \Sigma \end{bmatrix} = \begin{bmatrix} \Sigma & 0 \\ 0 & -\Sigma \end{bmatrix}$$

So, substituting into above:

$$\begin{bmatrix} 0 & A^* \\ A & 0 \end{bmatrix} \begin{bmatrix} V & -V \\ U & U \end{bmatrix} = \begin{bmatrix} V & -V \\ U & U \end{bmatrix} \begin{bmatrix} \Sigma & 0 \\ 0 & -\Sigma \end{bmatrix}$$

So:

$$B = \begin{bmatrix} 0 & A^* \\ A & 0 \end{bmatrix} = \begin{bmatrix} V & -V \\ U & U \end{bmatrix} \begin{bmatrix} \Sigma & 0 \\ 0 & -\Sigma \end{bmatrix} \begin{bmatrix} V & -V \\ U & U \end{bmatrix}^{-1} = \Delta D \Delta^{-1}$$

Hence, the matrix B has the above diagonalization.

Problem 5 (A1):

Let $x \in \mathbb{C}^m$ be such that $\|x\|_\infty = 1$. Then:

$$\begin{aligned} \|Ax\|_\infty &= \max_{1\leq i\leq m} \sum_{j=1}^{n} x_j a_{ij} \leq \max_{1\leq i\leq m} \sum_{j=1}^{n} |x_j a_{ij}| \\ &= \max_{1\leq i\leq m} \sum_{j=1}^{n} |x_j|\,|a_{ij}| \leq \max_{1\leq i\leq m} \sum_{j=1}^{n} |a_{ij}| \end{aligned}$$

Where we have used that $|x_j| \leq 1$, which is true since $\|x\|_\infty = 1$. Now take a vector $v \in \mathbb{C}^m$ defined in the following way. Find the row a_i in A with the largest row sum, ie:

$$i = \arg\max_k \left(\max_{1\leq k\leq m} \sum_{j=1}^{n} |a_{kj}| \right)$$

Define v such that $v_j = 1$ when $a_{ij} > 0$ and $v_j = -1$ when $a_{ij} < 0$, then $v_j a_{ij} = |a_{ij}|$. Note that $\|v\|_\infty = 1$. This way we have:

$$\|Av\|_\infty = \left\| \sum_{j=1}^{n} v_j a_{ij} \right\|_\infty = \max_{1\leq i\leq m} \sum_{j=1}^{n} v_j a_{ij} = \max_{1\leq i\leq m} \sum_{j=1}^{n} |a_{ij}|$$

Denote the very last term in the above expression X. We have shown the matrix ∞-norm is bounded above by X and have also found a unit vector v with respect to the ∞-norm that attains the value X when used to compute $\|Av\|_\infty$. Thus $\|A\|_\infty = X$ and we are done.

Problem 6 (A2):

Please see the attached .m MATLAB file: whybra_hw2_script.m