1)

a) Let S_K be the set of points in each step of the construction. Then

$$\mu(S_0) = \mu([0,1]\times[0,1]) = 1$$

$$\mu(S_1) = \frac{8}{9}\mu(S_0) = 8/9$$

$$\mu(S_2) = \frac{8}{9}\mu(S_1) = \frac{8^2}{9^2}\mu(S_0) = \left(\frac{8}{9}\right)^2$$

$$\vdots$$

$$\mu(S_K) = \left(\frac{8}{9}\right)^K, \quad K \geq 0$$

As $K \to \infty$, as $8/9 < 1$,

$$\mu\left(S_\infty = \begin{array}{c}\text{Sierpinski}\\ \text{carpet}\end{array}\right) = 0$$

So the fractal has measure 0.

(b) The fractal makes $m = 8$ copies (pattern is copied 8 times each with iteration) with scale factor $r = 3$ (og box shrinks by 1/3), so the similarity dimension is $d = \frac{\ln(m)}{\ln(r)} = \boxed{\frac{\ln(8)}{\ln(3)}}$

(c) Let $\varepsilon_n = (1/3)^n$ for $n \geq 0$. Then when $n=0$, we need 1 box of size $\varepsilon_0 = 1$ to cover S_0. When $n=1$ we need 8 boxes of size $\varepsilon_1 = 1/3$. When $n=2$ we need 8^2 boxes of size $\varepsilon_2 = (1/3)^2$, ..., and so forth so that $N(\varepsilon_n) = 8^n$ for $n \geq 0$. So

$$d = \lim_{n \to \infty} \frac{\ln(8^n)}{\ln(3^n)} = \lim_{n \to \infty} \frac{\cancel{n} \ln(8)}{\cancel{n} \ln(3)}$$

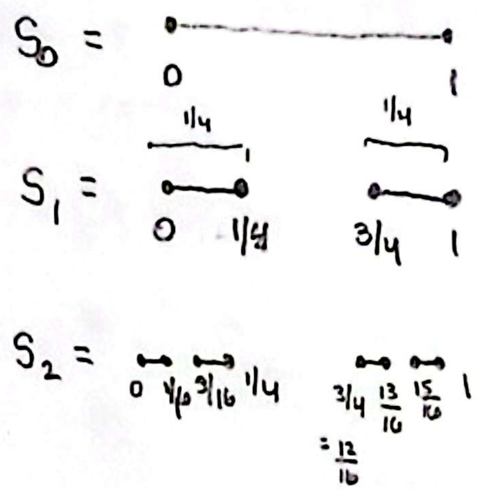
$= \boxed{\ln(8) / \ln(3)}$ just like in (b).

(d) Zooming in on the box $[0, 1/3] \times [0, 1/3]$, for $y = \frac{1}{2} \cdot \frac{1}{3} = \frac{1}{6}$, the segment $L = \{x \in [0, 1/3], y = 1/6\}$ contains precisely the Cantor set as a result of the construction, and this Cantor set is a subset of the fractal. Since the Cantor set is uncountable, it must be that the fractal is uncountable.

CamScanner

2)

a)



b) makes $m = 2$ copies, where each copy is scaled down by 4. Hence the similarity dimension is $d = \frac{\ln(m)}{\ln(r)} = \frac{\ln(2)}{\ln(4)} = \frac{\ln(2)}{\ln(2^2)} = \frac{\cancel{\ln(2)}}{2\cancel{\ln(2)}}$

$= \boxed{\frac{1}{2}}$

c) The measure at each step is (number of intervals) $\cdot$ (length of intervals)

$= 2^K \cdot \frac{1}{4^K} = \left(\frac{1}{2}\right)^K$, so as $K \to \infty$, the measure of the fractal $\to 0$ as $1/2 < 1$.

3).

(a) The set starts with $\mu(S_0) = 1$.

Then we remove $\frac{2}{7} \rightarrow \mu(S_1) = 1 - 2/7$,

then $2 \cdot \left(\frac{2}{7}\right)^2 \rightarrow \mu(S_2) = 1 - 2/7 - 2 \cdot (2/7)^2$,

then $4 \cdot \left(\frac{2}{7}\right)^3 \rightarrow \mu(S_3) = 1 - 2/7 - 2 \cdot (2/7)^2 - 4(2/7)^3$

$- \dots$

$$\rightarrow \mu(S_n) = 1 - \sum_{K=1}^{n} 2^{K-1} (2/7)^K$$

So as $n \rightarrow \infty$

the fractal

$$\mu(S) = 1 - \sum_{K=1}^{\infty} \frac{2^{2K-1}}{7^K}$$

$$= 1 - \frac{1}{2} \sum_{K=1}^{\infty} \left(\frac{4}{7}\right)^K$$

$$= 1 - \frac{1}{2}\left(\frac{1}{1 - 4/7} - 1\right)$$

$$= 1 - \frac{1}{2}\left(\frac{4}{3}\right)$$

$$= 1 - \frac{2}{3} = \boxed{1/3}$$

(b) The fractal is not self similar, the <u>scale factor changes</u> in each step.

4)

(a) we have $x_1 = \begin{cases} r x_0 & 0 \le x_0 \le 1/2 \\ r(1-x_0) & 1/2 < x_0 \le 1 \end{cases}$

So if $x_0 \in [0, 1/2]$, x_0 escapes if $r x_0 > 1 \leftrightarrow x_0 > \frac{1}{r}$, so if $x_0 \in (1/r, 1/2]$.
If $x_0 \in [1/2, 1]$, then $r(1-x_0) > 1 \leftrightarrow \frac{1}{r} + x_0 < 1 \leftrightarrow x_0 < 1 - 1/r$, so if $x_0 \in [1/2, 1-1/r)$.
So the set of x_0 that escapes in 1 iteration is $\{ x_0 \in (1/r, 1-1/r) \}$.

(b) For 2 iterations, choose $x_0 \in [0, 1/r] \cup [1-1/r, 1]$.

If $x_0 \in [0, 1/r]$, then $x_1 = r x_0$. If $x_0 \le \frac{1}{2r}$, then $x_1 \le 1/2$ so that $x_2 = r x_1 = r^2 x_0 > 1 \leftrightarrow \underbrace{x_0 > 1/r^2}$. If $x_0 > 1/2r$, then $x_1 > 1/2$ so that $x_2 = r(1-x_1) = r(1 - r x_0)$
$= r - r^2 x_0 > 1$
$\leftrightarrow -r^2 x_0 > 1 - r \rightarrow \underbrace{x_0 < \frac{r-1}{r^2}}$.

If $x_0 \in [1/2, 1-1/r]$, then $x_1 = r(1-x_0)$. If $x_0 \geq 1 - \frac{1}{2r}$, then $x_1 \leq 1/2$ so that

$$x_2 = rx_1 = r^2(1-x_0)$$

$$\rightarrow \quad r^2(1-x_0) > 1$$

$$\Leftrightarrow \quad 1-x_0 > 1/r^2$$

$$\Leftrightarrow \quad x_0 < 1 - 1/r^2$$

If $x_0 < 1 - 1/2r$, then $x_1 \geq 1/2$ so that

$$x_2 = r(1-x_1) = r(1-r(1-x_0))$$

$$= r - r^2(1-x_0)$$

$$= r - r^2 + r^2 x_0$$

$$\rightarrow \quad r - r^2 + r^2 x_0 \geq 1$$

$$\Leftrightarrow \quad r^2 x_0 > 1 - r + r^2$$

$$\Leftrightarrow \quad x_0 > \frac{1-r+r^2}{r^2} = 1 - \frac{1}{r} + \frac{1}{r^2}$$

So x_0 escapes after 2 iterations when

$$x_0 \in \left(\frac{1}{r^2}, \frac{r-1}{r^2}\right) \cup \left(1 - \frac{1}{r} + \frac{1}{r^2}, 1 - \frac{1}{r^2}\right)$$

c) For x_0 to never escape, when $r = 3$, following the pattern from (a) and (b), we see in each iteration the set of points that don't escape are contained in the traditional Cantor set by removing middle thirds in each iteration. So when r is general, the set of points that never escape is the Cantor set generated by removing the middle interval of scale $1/r$.

(d) Put $\varepsilon_n = (1/r)^n$. Then the number of boxes needed to cover the set in each iteration is 2^n, ie $N(\varepsilon_n) = 2^n$, so the box dimension is

$$\lim_{n\to\infty} \frac{\ln(N(\varepsilon_n))}{\ln(1/\varepsilon_n)} = \lim_{n\to\infty} \frac{\ln(2^n)}{\ln(r^n)}$$

$$= \lim_{n\to\infty} \frac{\cancel{n}\ln(2)}{\cancel{n}\ln(r)} = \boxed{\frac{\ln(2)}{\ln(r)}}$$