AMATH 569 - Homework 3

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Problem 1

At the point x_1 where u is maximized, the Laplacian satisfies $\Delta u \leq 0$, hence:

$$\Delta u = u^3 - u \leq 0 \iff u \leq 1$$

At the point x_2 where u is minimized, the Laplacian satisifes $\Delta u \geq 0$, hence:

$$\Delta u = u^3 - u \geq 0 \iff u \geq -1$$

So $-1 \leq u \leq 1$. Suppose now for the sake of contradiction that for some point $x_0 \in \Omega$ we have $u(x_0) = 1$, so that $\Delta u(x_0) = 1^3 - 1 = 0$. Therefore we have a non-negative interior maximum with $Lu(x_0) = \Delta u(x_0) = 0$ and $c = 0$, so by the strong max principle $u = 1$ (is constant in Ω), but as $u = 0$ on the boundary, this contradicts the continuity of u, therefore $u \neq 1$ anywhere inside Ω. Similarly, for the sake of contradiction, suppose that for some point $x_0 \in \Omega$ we have $u(x_0) = -1$. We again have $\Delta u(x_0) = (-1)^3 + 1 = 0$. Therefore we have a non-positive interior minimum with $Lu(x_0) = \Delta u(x_0) = 0$ and $c = 0$, so by the strong max principle $u = -1$ (is constant in Ω), but as $u = 0$ on the boundary, this contradicts the continuity of u, therefore $u \neq -1$ anywhere inside Ω. Thus on Ω, we have $-1 < u < 1$ as desired.

Problem 2

We will show that if Ω is unbounded, that we can have $Lu = Lv$ for $x \in \Omega$ and $u = v$ for $x \in \partial\Omega$ but $u \neq v$ for $x \in \overline{\Omega}$. Consider the following BVP for $L = \Delta$ and $\Omega = (0,1) \times \mathbb{R}$:

$$\begin{aligned} \Delta u &= 0, \quad \text{on } \Omega \\ u &= 0, \quad \text{on } \partial\Omega \end{aligned}$$

First off, $u = 0$ is a trivial solution to the above system. Secondly, consider $v = \exp(\pi y)\sin(\pi x)$, then:

$$\begin{aligned} \Delta v &= v_{xx} + v_{yy} \\ &= -\pi^2 \sin(\pi x) \exp(\pi y) + \pi^2 \sin(\pi x) \exp(\pi y) \\ &= 0 \end{aligned}$$

The boundary of Ω is composed of the lines defined by $x = 0$ and $x = 1$, but because of the $\sin(\pi x)$ term in v, v is surely 0 on the boundary. However $u \neq v$ clearly, so Corollary 4.4 does not hold.

Problem 3

From Theorem 4.11 in Renardy and Rodgers, if Ω is bounded and contained in the strip between two parallel planes of distance d. If $c \leq 0$, and $Lu = f$, then:

$$\max_{\overline{\Omega}} |u| \leq \max_{\partial\Omega} |u| + C \max_{\overline{\Omega}} \frac{|f|}{\lambda}$$

Where:

$$C = \exp\left((\beta + 1)d\right) - 1$$

$$\lambda(x) = \min_{v \in \mathbb{R}^n / \{0\}} \frac{v^T A(x) v}{|v|^2}$$

$$\beta = \max_{x \in \overline{\Omega}} \frac{|b(x)|}{\lambda(x)}$$

Now consider:

$$\begin{aligned} \Delta u + au &= 0, \quad \text{on } \Omega \\ u &= 0, \quad \text{on } \partial\Omega \end{aligned}$$

First we show $a > 0$. To do so, for the sake of contradiction, first suppose $a \leq 0$. Then $Lu = (\Delta u + a)u = 0 := f$ with $c = a \leq 0$. Since $u = 0$ on $\partial\Omega$, u achieves a non-negative maximum in the interior (if $u > 0$ anywhere), and u is not constant. But by the maximum principle, if $c \leq 0$, u cannot attain a non-negative maximum in the interior unless it is constant. So unless $u = 0$, this is a contradiction to the maximum principle. Therefore we must have $a > 0$. Now as $a > 0$, we can rearrange the PDE to now be $Lu = \Delta u = -au := f$, since $c = 0$ we can apply Theorem 4.11, where $A(x) = I_2$, $b(x) = [0, 0]^T$, to get:

$$\lambda(x) = \min_{v \in \mathbb{R}^2 / \{0\}} \frac{v^T I v}{|v|^2} = \min_{v \in \mathbb{R}^2 / \{0\}} 1 = 1$$

$$\beta = 0$$

$$\implies C = \exp\left(d\right) - 1$$

Since our Ω is a hexagon with side length a, a bit of high school trigonometry will tell you that the horizontal distance between the left side and the right side of the hexagon is $d = a + 2a\cos\left(\pi/6\right) = a(1 + \sqrt{3})$. Now we get the bound:

$$\max_{\overline{\Omega}} |u| \leq \max_{\partial\Omega} |u| + C \max_{\overline{\Omega}} \frac{|f|}{\lambda}$$

$$= C \max_{\overline{\Omega}} |-au| \quad (u = 0 \text{ on } \partial\Omega)$$

$$= C\lambda \max_{\overline{\Omega}} |u| \quad (\lambda > 0)$$

Therefore:

$$\lambda \geq \frac{1}{c}$$

$$= \frac{1}{\exp(a(1+\sqrt{3})) - 1}$$

Problem 4

By the strong max principle, as $Lu = \Delta u = 0$, the maximum of u must be on $\partial\Omega$. Suppose for the sake of contradiction that u is not constant. As u is maximized on $\partial\Omega$, there is an $x_0 \in \partial\Omega$ such that $u(x_0) > u(x)$ for every $x \in \Omega$. We are given that $\partial\Omega \in C^2$ and that u is smooth (hence differentiable at x_0), and we also have $c = 0$, so by Lemma 4.7 in Renardy and Rogers, it must be that $\partial u/\partial n(x_0) > 0$, however that contradicts the boundary conditition that $\partial u/\partial n = 0$ on $\partial\Omega$. Therefore, u must be some constant function, and we are done.

Problem 5

Let $D = \mathbb{R}^n \times (0, T]$, and suppose $u \in C^2(D) \cap C(\overline{D})$ is a bounded solution of the heat equation:

$$u_t = \Delta u \quad \text{on } D$$

We want to show that:

$$\sup_D u \leq \sup_{\mathbb{R}^n} u(x, 0)$$

To do this, we apply the weak maximum principle (for parabolic PDEs) on a sequence of bounded domains which approach our target domain. Let B_R be the ball of radius R centered at the origin in $\mathbb{R}^n$, and define:

$$D_R = B_R \times (0, T]$$

$$\Sigma_R = (\partial B_R \times [0, T]) \cup (B_R \times \{0\})$$

Now fix $\varepsilon > 0$, and define the function from the hint:

$$v(x, t) = u(x, t) - \varepsilon(2nt + |x|^2)$$

We compute:

$$v_t = u_t - 2n\varepsilon = \Delta u - 2n\varepsilon$$
$$\Delta v = \Delta u - \varepsilon\Delta(|x|^2) = \Delta u - 2n\varepsilon.$$

Therefore:

$$v_t = \Delta v$$

So v satisfies the heat equation on D_R. Now we have $Lv = (-\frac{\partial}{\partial t} + \Delta)v = 0$ with $c = 0$. Hence by the weak maximum principle, the maximum of v is achieved on Σ_R, so:

$$\sup_{D_R} v \leq \sup_{\Sigma_R} v$$

Since this holds for all $R > 0$, we get as $R \to \infty$ (and plugging back in for v):

$$\sup_D \left(u - \epsilon(2nt + |x|^2)\right) \leq \sup_{\mathbb{R}^n \text{ and } t=0} \left(u - \epsilon(2nt + |x|^2)\right)$$

Where we have used that $\Sigma_R \to \mathbb{R}^n \times \{0\}$ as $R \to \infty$. Finally, taking $\epsilon \to 0$ gives the desired result:

$$\sup_D u \leq \sup_{\mathbb{R}^n} u(x, 0)$$

Problem 6 (a)

As $t > 0$ and $x \in \mathbb{R}^n/\{0\}$, $\phi(x,t) > 0$, now define:

$$L = \ln(\phi(x,t)) = -\frac{|x|^2}{4t} - \frac{n}{2}\ln(4\pi t)$$

So:

$$\frac{\partial L}{\partial x_i} = -\frac{x_i}{2t}$$

$$\frac{\partial^2 L}{\partial x_i^2} = -\frac{1}{2t}$$

$$\frac{\partial L}{\partial t} = \frac{|x|^2}{4t^2} - \frac{n}{2t}$$

So that by the chain rule:

$$\phi_{x_i x_i} = \phi \cdot \left(\frac{\partial^2 L}{\partial x_i^2} + \left(\frac{\partial L}{\partial x}\right)^2\right)$$

$$= \phi \cdot \left(\frac{x_i^2}{4t^2} - \frac{1}{2t}\right)$$

And

$$\phi_t = \phi \cdot \left(\frac{\partial L}{\partial t}\right) = \phi \cdot \left(\frac{|x|^2}{4t^2} - \frac{n}{2t}\right)$$

Now:

$$\Delta\phi = \sum_{i=1}^{n} \phi_{x_i x_i}$$

$$= \phi \cdot \sum_{i=1}^{n} \left(\frac{x_i^2}{4t^2} - \frac{1}{2t}\right)$$

$$= \phi \cdot \left(\frac{|x|^2}{4t^2} - \frac{n}{2t}\right)$$

$$= \phi_t$$

As $\Delta\phi = \phi_t$, the function ϕ solves the heat equation.

Problem 6 (b)

Let $w = (x/\sqrt{4t})$ so that $x = \sqrt{4t}w$ and $dx = (4t)^{\frac{n}{2}} dw$. Then:

$$
\begin{aligned}
I = \int_{\mathbb{R}^n} \phi(x,t) dx &= \frac{1}{(4\pi t)^{\frac{n}{2}}} \int_{\mathbb{R}^n} \exp\left(-\frac{|x|^2}{4t}\right) dx \\
&= \frac{1}{(4\pi t)^{\frac{n}{2}}} \int_{\mathbb{R}^n} \exp\left(-\frac{|x|^2}{4t}\right) dx \\
&= \frac{(4t)^{\frac{n}{2}}}{(4\pi t)^{\frac{n}{2}}} \int_{\mathbb{R}^n} \exp\left(-|w|^2\right) dw \\
&= \frac{1}{\pi^{\frac{n}{2}}} \int_{\mathbb{R}^n} \exp\left(-|w|^2\right) dw
\end{aligned}
$$

However, from AMATH 561, we know the integral over the above Gaussian function in $\mathbb{R}^n$ is $\pi^{\frac{n}{2}}$, so that $I = 1$ as desired.

Problem 6 (c)

Let:

$$u(x,t) = \int_{\mathbb{R}^n} \phi(x-y,t)g(y)dy$$

Then using the Leibniz rule, we can pass the time and space derivatives inside/outside of the integral, and also use (a) to see:

$$\begin{aligned} u_t &= \int_{\mathbb{R}^n} \phi_t(x-y,t)g(y)dy \\ &= \int_{\mathbb{R}^n} \Delta_x \phi(x-y,t)g(y)dy \\ &= \Delta_x \left(\int_{\mathbb{R}^n} \phi(x-y,t)g(y)dy \right) \\ &= \Delta u \end{aligned}$$

We see that $u_t = \Delta u$. Now we must verify that $u(x,0) = g(x)$, to do so we compute the limit:

$$\begin{aligned} \lim_{t\to 0} u(x,t) &= \lim_{t\to 0} \int_{\mathbb{R}^n} \phi(x-y,t)g(y)dy \\ &= \int_{\mathbb{R}^n} \left(\lim_{t\to 0} \phi(x-y,t) \right) g(y)dy \end{aligned}$$

According to Wikipedia, the dirac-delta function can be represented heuristically as some function satisfying $\delta(x) = 0$ if $x \neq 0$ and $\delta(x) = \infty$ if $x = 0$, and whose integral over all of $\mathbb{R}^n$ is 1. We'd like to argue that:

$$\delta(x) = \lim_{t\to 0} \phi(x,t)$$

We've already shown that the integral of $\phi(x,t)$ with respect to x is 1 for all t, so in the limit as $t \to 0$, the integral is still 1, so that part is satisfied. Now if $x = 0$, then $\phi \sim (1/\sqrt{t})^{\frac{n}{2}}$ which blows up to ∞ as $t \to 0$. If $x \neq 0$, then the exponential dominates, and the limit is 0. Therefore the above does indeed satisfy the criteria to be the delta function. In the end we get the desired result from the rules of integrating against the dirac function.

$$u(x,0) = \lim_{t\to 0} u(x,t) = \int_{\mathbb{R}^n} \delta(x)g(y)dy = g(x)$$

Problem 6 (d)

As $\phi(x-y,t) > 0$ for all $x, y \in \mathbb{R}^n$ and $t > 0$, and $g(y) > 0$ on some set with positive volume, we must have that $u(x,t) > 0$ for all $x \in \mathbb{R}^n$ and $t > 0$. Therefore the support of u is the whole space (does not have compact support). This means that information from g propagates infinitely fast because at time $t = 0$, the information about g to u is restricted to a finite volume set (the set that g has compact support over), but the instant we have $t > 0$, parts of g are incorporated into $u(x,t)$ over the whole space.