AMATH 561
Homework 3
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1) Let $\Omega = \{a, b, c\}$, $F = 2^{\Omega}$, and P be any probabity meausure on Ω. Consider the random variable

$$X(\omega) = \begin{cases} r_0 & \text{if } \omega = a \\ r_1 & \text{if } \omega = b \\ r_2 & \text{if } \omega = c \end{cases} \quad \left\{ \begin{array}{l} \text{where} \\ \text{each } r_i \in \mathbb{R} \\ \text{and are} \\ \text{distinct} \end{array} \right.$$

Then $\sigma(X) = \{X^{-1}(A) : A \in B(\mathbb{R})\}$

$= \{C \subseteq \Omega : X(C) = A \ \forall A \in B(\mathbb{R})\}$

Any interval $I_1 \in B(R)$ containing $\{r_0, r_1, r_2\}$ will have its pre-image under X be the whole space Ω. Any interval only containing $\{r_i, r_j\}$ $i \neq j$ will have pre-image $\{a, b\}$, $\{b, c\}$, or $\{a, c\}$, and any interval only containing $\{r_i\}$ will have pre-image $\{a\}$, $\{b\}$, or $\{c\}$. Also any interval that contains none of the r_i's will have $\emptyset$ as the pre-image. All of these sets together form 2^{Ω}, so since this is the largest σ-algebra we can have on Ω, $\sigma(X) = 2^{\Omega}$. Now consider the following function

(Note we didn't have to use intervals here, the same argument holds for Borel sets in general)

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$$f(x) = \begin{cases} q_1 & \text{if } x = r_0 \quad (\omega = a) \\ q_2 & \text{if } x \neq r_0 \quad (\omega \in \{b, c\}) \end{cases} \quad \text{for } q_1 \neq q_2 \text{ in } \mathbb{R}$$

we must now find $\sigma(f(x))$. Notice that the range of $f(x)$ is $\{q_1, q_2\}$. So ~~—~~ any Borel set A containing both q_1 and q_2 will have its pre-image be Ω under $f(x)$. If A contains only one of the q_i's, the pre-image will be $\{a\}$ if $i=1$ and $\{b,c\}$ if $i=2$. If A contains neither q_1 or q_2, the pre-image will be ϕ. This means

$$\sigma(f(x)) = \{\phi, \{a\}, \{b,c\}, \Omega\}$$

which is easily seen to be a σ-algebra. Also notice that

$$\sigma(f(x)) \subset \sigma(x)$$

↑ strict subset

So we are done.

Now suppose we have a function

$$g(x) = 1 \quad (g(x(\omega)) = 1 \quad \forall \omega \in \Omega)$$

Since the pre-image of this function is Ω for any Borel set A that is non-empty, and is the empty set if $A = \phi$,

$$\sigma(g(x)) = \{ \phi, \Omega \}$$

2) Let $\Omega = \{a_1, a_2, a_3, a_4, a_5, a_6, a_7, a_8\}$,

$F = 2^{\Omega}$, and $P(\{a_i\}) = 1/8$. Let

$$A = \{a_1, a_2, a_3, a_4\}$$

$$B = \{a_1, a_2, a_5, a_6\}$$

$$C = \{a_1, a_3, a_7, a_8\}$$

Then

$$A \cap B = \{a_1, a_2\}$$

$$A \cap C = \{a_1, a_3\}$$

$$B \cap C = \{a_1\}$$

$$A \cap B \cap C = \{a_1\}$$

So $P(A) = P(B) = P(C) = 1/2$, $P(A \cap B) = P(A \cap C) = 1/4$, and $P(B \cap C) = P(A \cap B \cap C) = 1/8$.

Notice $P(A)P(B) = \frac{1}{2} \cdot \frac{1}{2} = \frac{1}{4} = P(A \cap B)$ ✓

$P(A)P(C) = 1/4 = P(A \cap C)$ ✓

$P(A)P(B)P(C) = (1/2)^3 = 1/8 = P(A \cap B \cap C)$ ✓

But

$$P(B)P(C) = 1/4 \neq 1/8 = P(B \cap C)$$

as desired.

The sets $\{A, B, C\}$ are not independent because they are not mutually independent as we have demonstrated above,

3) Suppose that it <u>is</u> possible for there to exist a countable collection of events $A_1, A_2, \ldots \in F$ which are independent such that $P(A_i) = 1/2$ for all $i \geq 1$. Since $P(A_i) > 0$, each A_i must be non-empty. We will now show that no pair of sets A_i and A_j can be disjoint. Suppose for the sake of contradiction that the A_i's are pairwise disjoint. Then

$$P\left(\bigcup_{i \geq 1} A_i\right) \underset{\substack{\uparrow \\ \text{countable additivity}}}{=} \sum_{i \geq 1} P(A_i) = \sum_{i \geq 1} \frac{1}{2} = \infty$$

which is a contradiction because

$$P\left(\bigcup_{i \geq 1} A_i\right) \leq P(\Omega) = 1$$

Since the A_i's can't be disjoint, there exists some $\omega \in \Omega$ such that $\omega \in \bigcap_{i \geq 1} A_i$, so

$$P(\omega) \leq P\left(\bigcap_{i \geq 1} A_i\right) \underset{\substack{\uparrow \\ \text{from independence}}}{=} \prod_{i \geq 1} P(A_i) = \prod_{i \geq 1} \frac{1}{2}$$

$$= \lim_{n \to \infty} \left(\frac{1}{2}\right)^n$$

$= 0$ (since $1/2 < 1$) $\longrightarrow$ implying $P(\omega) = 0 \ \forall \omega \in \Omega$

This leads to a contradiction since

$$1 = P(\Omega) = P\left(\bigcup_{\omega \in \Omega} \{\omega\}\right) \overset{\text{countable additivity}}{=} \sum_{\omega \in \Omega} P(\{\omega\})$$

$$= \sum_{\omega \in \Omega} 0 = 0 \longrightarrow \text{but } 1 \neq 0.$$

So our initial claim is false and we are done.

4a) Put $Z = XY$. To find the distribution function $F_Z(c)$, we must compute $P(Z \leq c) = P(XY \leq c)$.

$$P(XY \leq c) = E\left[\mathbb{1}_{\{xy \leq c\}}\right]$$

$$= E\left[\mathbb{1}_{\{x \leq c/y\}}\right]$$

since X, Y are independent

$$= \int_{\mathbb{R}} \mathbb{1}_{\{x \leq c/y\}}\, \mu(dx)\, \nu(dy)$$

$$= \int_{\mathbb{R}} \underline{P(X \leq c/y)\, \mu(dx)}\, \nu(dy)$$

$$= \int_{\mathbb{R}} F(c/y)\, dG(y)$$

b) To find the density function of $Z = X \cdot Y$, we can compute the derivative of the distribution function from (a). We will do some manipulations on $F_Z(c)$, and then take $\frac{d}{dc}$ at the end. Firstly (and we can do this because X, Y are continuous)

$$F_Z(c) = \int_{\mathbb{R}} F^{(X)}(c/y)\, dG(y)$$

since F has a density function

$$= \int_{\mathbb{R}} \int_{-\infty}^{c/y} f(u)\, du\, dG(y)$$

Put $u = w/y$, then $du = 1/y\, dw$ and the above

$$= \int_{\mathbb{R}} \int_{-\infty}^{c} f(w/y)/y\, dw\, dG(y)$$

$$= \int_{-\infty}^{c} \int_{\mathbb{R}} f(w/y)/y\, dG(y)\, dw \quad \text{(by Fubini's Theorem)}$$

So the density function is

$$h(c) = \frac{d}{dc} \int_{-\infty}^{c} \int_{\mathbb{R}} f(w/y)/y\, \underbrace{dG(y)}_{\text{since } Y \text{ has a density}}\, dw \quad \text{(by Fundamental Theorem of Calculus)}$$

$$= \int_{\mathbb{R}} \left[f(c/y)/y\right] \cdot g(y)\, dy$$

However the above integral can be written

$$\int_{(0,\infty)} \frac{f(c/y)\,g(y)}{y}\,dy + \int_{(-\infty,0)} \overbrace{\frac{f(c/y)\,g(y)}{y}}^{i(y)}\,dy + \int_{\{0\}} \frac{f(c/y)\,g(y)}{y}\,dy$$

denote the integrand as $i(y)$

$$= \int_{-\infty}^{0} i(y)\,dy + \int_{0}^{\infty} i(y)\,dy + \underbrace{\cancelto{0}{P(Y=0)} \cdot \frac{f(\infty)\,g(0)}{0}}_{\text{Since } Y \text{ is a continuous r.v, } P(Y=0)=0}$$

So finally our density is

$$h(c) = \int_{-\infty}^{0} \frac{f(c/y)\,g(y)}{y}\,dy + \int_{0}^{\infty} \frac{f(c/y)\,g(y)}{y}\,dy$$

we've made the assumption that X Y > 0

4) c. From problem 4b we can write the density function of $X \cdot Y$ as

$$h(c) = \int_0^\infty \lambda e^{-\lambda(c/y)} \cdot \lambda e^{-\lambda y} \, dy$$

$$= \lambda^2 \int_0^\infty e^{-\lambda(y + c/y)} \, dy$$

where we have ignored the $\int_{-\infty}^{0}$ term since the exponential density function is 0 when the input is negative.