AMATH 561 Autumn 2024
Problem Set 5

Due: Wed 11/6 at 10am

Note: Submit electronically to Canvas.

1. Let X and $Y_0, Y_1, Y_2, ...$ be random variables on a probability space $(\Omega, \mathcal{F}, P)$ and suppose $E|X| < \infty$. Define $\mathcal{F}_n = \sigma(Y_0, Y_1, ..., Y_n)$ and $X_n = E(X|\mathcal{F}_n)$. Show that the sequence $X_0, X_1, X_2, ...$ is a martingale with respect to the filtration $(\mathcal{F}_n)_{n\geq 0}$.

2. Let $X_0, X_1, ...$ be i.i.d Bernoulli random variables with parameter p (i.e., $P(X_i = 1) = p, P(X_i = 0) = 1 - p$). Define $S_n = \sum_{i=1}^{n} X_i$ where $S_0 = 0$. Define

$$Z_n = \left(\frac{1-p}{p}\right)^{2S_n - n}, \quad n = 0, 1, 2, ...$$

Let $\mathcal{F}_n = \sigma(X_0, X_1, ..., X_n)$. Show that Z_n is a martingale with respect to this filtration.

3. Let ξ_i be a sequence of random variables such that the partial sums

$$X_n = \xi_0 + \xi_1 + ... + \xi_n, \quad n \geq 1,$$

determine a martingale. Show that the summands are mutually uncorrelated, i.e. that $E(\xi_i \xi_j) = E(\xi_i)E(\xi_j)$ for $i \neq j$.

4. Galton and Watson who invented the process that bears their names were interested in the survival of family names. Suppose each family has exactly 3 children but coin flips determine their sex. In the 1800s, only male children kept the family name so following the male offspring leads to a branching process with $p_0 = 1/8, p_1 = 3/8, p_2 = 3/8, p_3 = 1/8$. Compute the probability ρ that the family name will die out when $Z_0 = 1$. What is ρ if we assume that each family has exactly 2 children?