1)

a) The fixed points x satisfy

$$x = x - \frac{g(x)}{g'(x)}$$

$$0 = -\frac{g(x)}{g'(x)}$$

$$\leftrightarrow \quad g(x) = 0 \quad \text{and} \quad g'(x) \neq 0$$

So by definition the simple roots of g are fixed points of the Newton map.

b) If $f(x) = x - \frac{g(x)}{g'(x)}$, then

$$f'(x) = 1 - \frac{g'(x) \cdot g'(x) - g(x)\, g''(x)}{(g'(x))^2}$$ quotient rule

$$= \cancel{1} - \cancel{1} - \frac{g(x)\, g''(x)}{(g'(x))^2}$$

$$= -\frac{g(x)\, g''(x)}{(g'(x))^2}$$

So if x is a fixed point, from (a), $f'(x) = 0$, so that x is superstable.

2)

(a) The fixed points x must satisfy

$$x = 3x - x^3 = f(x)$$

$$\rightarrow \quad 2x - x^3 = 0$$

$$\rightarrow \quad x(2-x^2) = 0$$

So $x \in \{0, -\sqrt{2}, \sqrt{2}\}$.

As $f'(x) = 3 - 3x^2$, the multipliers are

- $|f'(0)| = 3 > 1 \rightarrow x = 0$ is <u>unstabe</u>
- $|f'(\sqrt{2})| = |3 - 3\cdot 2| = |-3| = 3 > 1 \rightarrow x = \sqrt{2}$ is <u>unstable</u>
- $|f'(-\sqrt{2})| = |3 - 3\cdot 2| = |-3| = 3 > 1 \rightarrow x = -\sqrt{2}$ is <u>unstable</u>

b) we have $f'(x) = 3 - 3x^2 := 0$

$$\rightarrow \quad 3(1-x^2) = 0$$

$$x = \pm 1$$

By the intermediate value theorem when $x \in [-2,2]$, the extrema can only be at $x = -2, -1, 1, 2$. we have $f(-2) = 2$, $f(-1) = -2$, $f(1) = 2$, $f(2) = -2$, so $|f(x)| \leq 2$ when $x \in [-2,2]$. when $|x| > 2$, then $x^2 > 4 \rightarrow x^2 - 3 > 1$, so

$$|f(x)| = |x(3-x^2)| = |x||x^2-3| > |x|\cdot 1 = |x| \quad (>2)$$

as desired.

When $x_0 = 1.9$, this means the orbit $x_0, x_1, x_2, \ldots$ stays in the box $|x| \leq 2$.
When $x_0 = 2.1$, the orbit stays outside of the box, which is consistent with the diagrams.

(c) we have $f(2) = 6-8 = -2$ and $f(-2) = -6+8 = 2$.
So with $p = 2$, $q = -2$, we have

$$\begin{cases} f(p) = q \\ f(q) = p \end{cases}$$

hence $(2,-2)$ repeating is a 2-cycle. I would call this 2-cycle a <u>center</u>.

3) a) The fixed points x of the ODE must satisfy, $\underline{f(x) = 0}$. The fixed points of the Euler map must satisfy,

$$x = x + h \cdot f(x)$$

$$\leftrightarrow \quad 0 = h\, f(x)$$

$$\leftrightarrow \quad \underline{f(x) = 0}$$

So the fixed points correspond.

b) For ODE: $f'(x^*) < 0 \rightarrow$ stable
$f'(x^*) > 0 \rightarrow$ unstable

For FE map: $|1 + h f'(x^*)| < 1$

$$\leftrightarrow \quad -1 < 1 + h f'(x^*) < 1$$

$$\leftrightarrow \quad -2 < h\, f'(x^*) < 0$$

$$\leftrightarrow \quad f'(x^*) \in \left(-\tfrac{2}{h}, 0\right) \rightarrow \text{stable}$$

and

$$f'(x^*) \in \left(-\infty, -\tfrac{2}{h}\right) \cup (0, \infty) \rightarrow \text{unstable}$$

So the stability of the fixed points match when $x \in \left(-\frac{2}{h}, 0\right) \cup (0, \infty)$ and don't match otherwise.

c) From (b), the FE map is stable whenever $f'(x^*) \in (-\frac{2}{h}, 0)$, ie when

$$-\frac{2}{h} < f'(x^*) < 0$$

$$\Leftrightarrow \quad \frac{2}{h} > -f'(x^*)$$

$$\Leftrightarrow \quad h < -\frac{2}{f'(x^*)}$$

So if we pre-compute $f'(x^*)$ and it's negative, choose h like the above.

(d) Oscillations can occur when $|1+hf'(x^*)| = 1$ (the derivative of the map)

where $1+hf'(x^*) = g'(x^*)$, with $g(x) = x + hf(x)$.

For this to occur, we'd need

$$hf'(x^*) = -2 \quad \text{or} \quad \cancel{f'(x^*) = 0}$$

$$f'(x^*) = -\frac{2}{h} \qquad \text{(but we can ignore this one)}$$

It is common to see this condition when $f'(x^*) \approx -\frac{2}{h}$, ie if h is large and $f'(x^*)$ is near 0_- or if h is small and $f'(x^*)$ ~~is~~ has a large magnitude. So one should choose $h << \frac{-2}{f'(x^*)}$.

e) The FE map for this ODE is

$$x_{n+1} = x_n(1+hK) = g(x_n)$$

For there to be a 2-cycle, we need $x_{n+2} = x_n$, ie

$$x_{n+2} = x_{n+1}(1+hK) = x_n(1+hK)^2 := x_n$$

$$\Leftrightarrow \quad x_n\left((1+hK)^2 - 1\right) = 0$$

$$\Leftrightarrow \quad x_n\left(2hK + (hK)^2\right) = 0$$

$$\Leftrightarrow \quad \underbrace{x_n \neq 0}_{\text{don't care}} \quad \text{or} \quad hK(2+hK) = 0$$

$$\Leftrightarrow \quad \underbrace{hK \neq 0}_{\text{degenerate}} \quad \text{or} \quad hK = -2$$

To be neutrally stable, we need $|g'(x^*)| = |1+hK| = 1$, which is true when $hK = -2$ like above. In this scenario, we have

$$x_{n+1} = x_n(1-2) = -x_n$$

So that the orbit <u>alternates signs</u>.

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