AMATH 567

Homework 4

Nate Whybra

1) c) Put $z = Re^{i\theta}$, then $\bar{z} = Re^{-i\theta}$, so $\frac{1}{\bar{z}} = \frac{1}{R}e^{i\theta}$, so our integral become's

$$\oint_C f(z)\,dz = \int_0^{2\pi} \frac{1}{R}e^{i\theta} \cdot \underbrace{iRe^{i\theta}}_{\text{derivative}}\,d\theta$$

of $\frac{1}{\bar{z}}$ w/ respect to θ

$$= i\int_0^{2\pi} e^{2i\theta}\,d\theta$$

$$= i\left[\int_0^{2\pi} \cos(2\theta)\,d\theta + i\int_0^{2\pi} \sin(2\theta)\,d\theta\right]$$

$$= i\left[\frac{1}{2}\overset{0}{\sin(2\theta)}\Big|_0^{2\pi} + (-i)\frac{1}{2}\overset{0}{\cos(2\theta)}\Big|_0^{2\pi}\right]$$

$$= 0$$

e) By 1.2.19 ,

$$e^{\bar{z}} = \sum_{n=}^{\infty} \frac{(\bar{z})^n}{n!}$$

Put $z = Re^{i\theta}$, then $\bar{z} = Re^{-i\theta}$, so the above

$$= \sum_{n=0}^{\infty} \frac{R^n}{n!} e^{-in\theta}$$

So,

$$\oint_C \overset{e^{\bar{z}}}{\overset{\shortparallel}{f(z)}}\,dz = \int_0^{2\pi} \sum_{n=0}^{\infty} \frac{R^n}{n!} e^{-in\theta} \cdot Rie^{i\theta} \, d\theta$$

$$= i\int_0^{2\pi} \sum_{n=0}^{\infty} \frac{R^{n+1}}{n!} e^{-(n-1)i\theta} \, d\theta$$

$$= i \sum_{n=0}^{\infty} \frac{R^{n+1}}{n!} \int_0^{2\pi} e^{-(n-1)i\theta} \, d\theta$$

$$= \int_0^{2\pi} \cos((n-1)\theta) \, d\theta - i \int_0^{2\pi} \sin((n-1)\theta) \, d\theta$$

(assume $n \neq 1$) $$= \frac{1}{n-1} \sin((n-1)\theta) \Big|_0^{2\pi} + \frac{i}{n-1} \cos((n-1)\theta) \Big|_0^{2\pi}$$

$$= 0$$

When $n=1$, the integral is

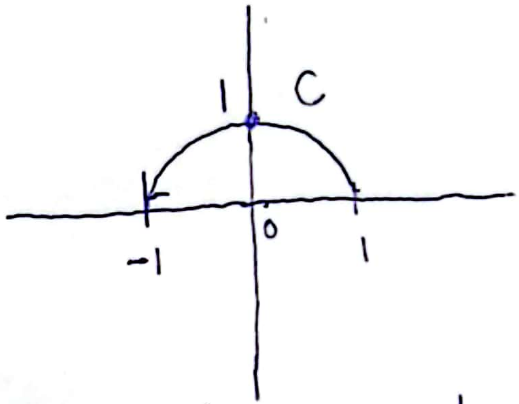
$$\int_0^{2\pi} e^{-(1-1)i\theta}\, d\theta = \int_0^{2\pi} d\theta = 2\pi.$$

So every term in the sum vanishes except at $n=1$, so our final answer is

$$= \left[2\pi i \cdot \frac{R^2}{1!} \right]$$

$$= \boxed{2\pi i} \quad \text{as } R=1$$

2) a)



Note, that

$$\int_{-1}^{1} \log(z)\,dz = -\int_{1}^{-1} \log(z)\,dz,$$

so to compute this integral we choose a half-circle contour starting at +1 and ending at -1. S. put $z = e^{i\theta}$ for $0 \leq \theta < \pi$ and consider for $\varepsilon > 0$ (note ε is approaching 0 from the right) to avoid branch

$$\int_{1}^{-1} \log(z)\,dz \overset{\circledast}{=} \int_{0}^{\pi-\varepsilon} \log(e^{i\theta})\, i e^{i\theta}\, d\theta$$

there should be $\lim_{\varepsilon \to 0}$ in front of these equalities, denote $\lim_{\varepsilon \to 0}$ by $\circledast$

$$\overset{\circledast}{=} i \int_{0}^{\pi-\varepsilon} (\log(1) + i\theta)\, e^{i\theta}\, d\theta$$

$$\overset{\circledast}{=} -\int_{0}^{\pi-\varepsilon} \theta e^{i\theta}\, d\theta$$

$$\overset{\circledast}{=} -\left[\int_{0}^{\pi-\varepsilon} \theta\cos(\theta)\,d\theta + i\int_{0}^{\pi-\varepsilon} \theta\sin(\theta)\,d\theta\right]$$

which after integration by parts becomes

$$\overset{\circledast}{=} -\left[\cos(\theta) + \theta\sin\theta \Big|_{0}^{\pi-\varepsilon} + i(\sin\theta - \theta\cos\theta)\Big|_{0}^{\pi-\varepsilon}\right]$$

$$\overset{\circledast}{=} -\left[\cos(\pi-\varepsilon) + (\pi-\varepsilon)\sin(\pi-\varepsilon) - 1 + i\left[\sin(\pi-\varepsilon) - (\pi-\varepsilon)\cos(\pi-\varepsilon)\right]\right]$$

$$= \left(1 - \cos(\pi - \varepsilon) - (\pi-\varepsilon)\sin(\pi-\varepsilon)\right) + i\left((\pi-\varepsilon)\cos(\pi-\varepsilon) - \sin(\pi-\varepsilon)\right)$$

The reason we introduced ε is to avoid the branch cut of the logarithm on the negative real axis. So as $\varepsilon \to 0$, our contour C approaches the branch cut, so as $\varepsilon \to 0$ the above reduces to

$$\Rightarrow \left(1 - \cos(\pi) - \pi\sin(\pi)\right) + i\left(\pi\cos(\pi) - \sin(\pi)\right)$$

$$= \quad 2 + i(-\pi) = 2 - i\pi$$

we have computed $+\int_{1}^{-1} \log(z)\,dz$, so we need to negate our solution above, hence

$$\int_{-1}^{1} \log(z)\,dz = \boxed{-2 + i\pi}$$

b) For this part, we use the same contour as in part (a) noting that

$$\int_{-1}^{1} z^{1/2}\, dz = -\int_{1}^{-1} z^{1/2}\, dz$$

So if $z = Re^{i\theta} \rightarrow z^{1/2} = R^{1/2} e^{i\theta/2}$, so (where $\varepsilon \to 0$ from the right to avoid the branch)

$$\int_{1}^{-1} z^{1/2}\, dz = \lim_{\varepsilon \to 0} \int_{0}^{\pi-\varepsilon} R^{1/2} e^{i\theta/2} \cdot Ri e^{i\theta}\, d\theta \checkmark$$

$$= \lim_{\varepsilon \to 0} i \int_{0}^{\pi-\varepsilon} e^{i 3\theta/2}\, d\theta \quad (\text{as } R = 1)$$

$$= \lim_{\varepsilon \to 0} i \left[\int_{0}^{\pi-\varepsilon} \cos(3\theta/2)\, d\theta + i \int_{0}^{\pi-\varepsilon} \sin(3\theta/2)\, d\theta \right]$$

$$= \lim_{\varepsilon \to 0} i \left[\frac{2}{3} \sin\left(\frac{3\theta}{2}\right) \Big|_{0}^{\pi-\varepsilon} - i \left[\frac{2}{3} \cos(3\theta/2)\right]_{0}^{\pi-\varepsilon} \right]$$

$$= \lim_{\varepsilon \to 0} i \left[\frac{2}{3} \sin\left(\frac{3}{2}(\pi-\varepsilon)\right) - \frac{2i}{3}\left[\cos\left(\frac{3(\pi-\varepsilon)}{2}\right) - 1\right]\right]$$

$$= \frac{2}{3}\left[\cancel{\cos\left(\frac{3\pi}{2}\right)}^{0} - 1\right] + \frac{2}{3} i \sin\left(\frac{3\pi}{2}\right)$$

$$= -\frac{2}{3} - \frac{2}{3} i,$$ so taking the negative we have

$$\int_{-1}^{1} z^{1/2}\, dz = \boxed{\frac{2}{3} + \frac{2}{3} i}$$

3) Firstly, to show

$$|f(z)| \leq \frac{1}{R^2 - a^2} \quad R > a$$

we must show

$$\frac{1}{|f(z)|} \geq R^2 - a^2$$

we have

$$\frac{1}{|f(z)|} = |z^2 + a^2| = |(z - ia)(z + ia)|$$

$$= |(z - ia)||(z + ia)|$$

By the triangle inequality, $|z - ia| \geq ||z| - |ia||$ (1) $\geq |R - a| = R - a$ (as $R > a$). For $|z + ia|$, we need to do more work

$$|z + ia| = |x + (y + a)i|$$

$$= \sqrt{x^2 + (y+a)^2}$$

$$= \sqrt{x^2 + y^2 + a^2 + 2ya}$$

$$\geq \sqrt{y^2 + 2ya + a^2} \quad \forall\, y \text{ on } C$$

So pick $y = R$, then

$$= \sqrt{R^2 + 2Ra + a^2}$$

$$= \sqrt{(R+a)^2}$$

$$= R + a \quad (2)$$

Putting ① and ② together we se

$$\frac{1}{|f(z)|} \geq |z-ia||z+ia| \geq (R-a)(R+a)$$
$$= R^2 - a^2$$

Hence,

$$|f(z)| \leq \frac{1}{R^2-a^2} \quad \text{as desired}$$

Next we know the following bound

$$\left| \int_C f(z)\, dz \right| \leq \sup_{z\in C} |f(z)| \cdot L$$

where L is the arc length of C,
Since our contour is half a circle of radius R, the arc length is half the circumfrence, hence $L = \frac{2\pi R}{2} = \pi R$,

So finally

$$\left| \int_C f(z)\, dz \right| \leq \sup_{z\in C} |f(z)| \cdot L \leq \frac{1}{R^2-a^2} \cdot \pi R$$

$$= \frac{\pi R}{R^2-a^2}$$

which is the desired result,

4) Firstly,

$$|z^3+1| = |z^3-(-1)| \geq ||z^3|-|-1||$$
$$= |R^3-1|$$
$$= R^3-1 \quad \text{as } R>1$$

Also, the arclength of C, $L = \theta r = \frac{\pi}{3}R$.

So $$\left|\int_C \frac{1}{z^3+1}\,dz\right| \leq \int_C \left|\frac{1}{z^3+1}\right| dz \leq \sup_{z\in C}\left|\frac{1}{z^3+1}\right| \cdot L$$

$$\leq \frac{1}{R^3-1}\cdot\frac{\pi}{3}R = \frac{\pi}{3}\left(\frac{R}{R^3-1}\right)$$

Now, using the above

$$0 \leq \left|\int_{C_R} \frac{1}{z^3+1}\,dz\right| \leq \frac{\pi}{3}\left(\frac{R}{R^3-1}\right)$$

$$\lim_{R\to 0} 0 \leq \lim_{R\to 0}\left|\int_{C_R} \frac{1}{z^3+1}\,dz\right| \leq \lim_{R\to\infty}\frac{\pi}{3}\left(\frac{R}{R^3-1}\right)$$

$$0 \leq \lim_{R\to\infty}\left|\int_{CR} \frac{1}{z^3+1}\,dz\right| \leq 0$$

So by the squeeze theorem, the modulus of the integral $\to 0$ as $R\to\infty$, so the integral itself must also approch 0 as $R\to\infty$, so we are done.

5) b) It is common knowledge that both $f_1(z) = e^z$ and $f_2(z) = z^2$ are both analytic on $\mathbb{C}$. On page 38 of the textbook it says the composition of analytic functions is also analytic, so $f(z) = e^{z^2}$ is also analytic, so $f(z) = e^{z^2}$ is also analytic on the whole plane $\mathbb{C}$. So by Cauchy's Theorem, we get for free that (since the unt circle is a closed contour in $\mathbb{C}$)

$$\oint_C e^{z^2} dz = 0$$

e) we have

$$\oint_C \frac{1}{2z^2+1} dz = \frac{1}{2} \oint \frac{1}{z^2+1/2} dz$$

$$\left. = \frac{1}{2} \oint \frac{1}{(z-i/\sqrt{2})(z+i/\sqrt{2})} dz \right] \quad ①$$

Let's compute the partial fraction decomposition of the integrand

$$\frac{1}{(z-i/\sqrt{2})(z+i/\sqrt{2})} = \frac{A}{(z-i/\sqrt{2})} + \frac{B}{(z+i/\sqrt{2})}$$

$$1 = A(z+i/\sqrt{2}) + B(z-i/\sqrt{2})$$

$$z = i/\sqrt{2} \rightarrow A = \frac{1}{\sqrt{2}i}$$

$$z = -i/\sqrt{2} \rightarrow B = -\frac{1}{\sqrt{2}i}$$

So

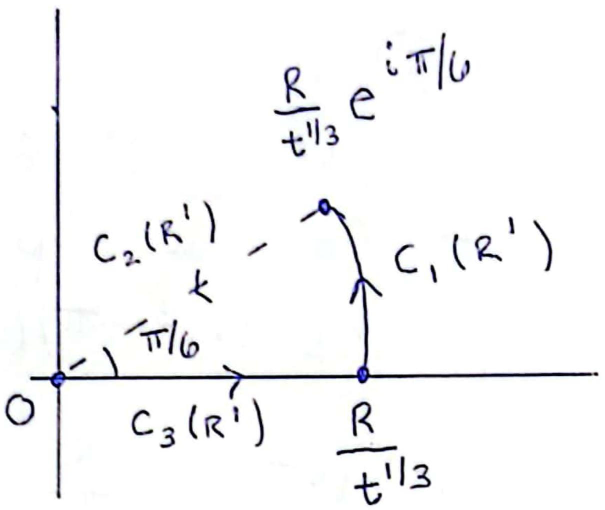
$$① = \frac{1}{2}\left[\frac{1}{\sqrt{2}i}\oint_C \frac{1}{z-i/\sqrt{2}}dz - \frac{1}{\sqrt{2}i}\oint_C \frac{1}{z+i/\sqrt{2}}dz\right]$$

By example 2.5.2, since both of the roots $z_1 = i/\sqrt{2}$, $z_2 = -i/\sqrt{2}$ lie inside the unit circle C (our closed contour), both integrals evaluate to $2\pi i$, so

$$① \ \frac{1}{2}\left[\frac{1}{\sqrt{2}i}\cdot 2\pi i - \frac{1}{\sqrt{2}i}\cdot 2\pi i\right]$$

$$= \boxed{0}$$

6) Consider the following contour



which is the circular sector with boundary points R', 0, $R' e^{i\pi/6}$ where $R' = \frac{R}{t^{1/3}}$ for $t > 0$. Following the ideas of exercise 2.5.5, first denote $C(R/t^{1/3}) = \sum_{i=1}^{3} C_i(R/t^{1/3})$, which is the combination of the 3 segments shown in the diagram that creates the full sector. We are interested in the contour C_3,

$$\int_{C_3(R/t^{1/3})} e^{iz^3 t}\, dz \overset{①}{=} \underbrace{\oint_{C(R')} e^{iz^3 t}\, dz}_{\text{whole loop}} - \underbrace{\int_{C_1(R')} e^{iz^3 t}\, dz}_{\text{arc } ②}$$

$R/t^{1/3} = R'$

$$- \underbrace{\int_{C_2(R')} e^{iz^3 t}\, dz}_{\text{diagonal line } ③}$$

we are interested in what happens as $R \to \infty$. Firstly since $e^{iz^3 t}$ is analytic everywhere, the loop integral (1) vanishes for every $R > 0$. Secondly, we want to show that as $R \to \infty$, (2) $\to 0$. To do so, put

$z = \frac{R}{t^{1/3}} e^{i\theta}$ for $0 \le \theta \le \pi/6$. Then

$z^3 = \frac{R^3}{t} e^{i3\theta}$ and $dz = \frac{iR}{t^{1/3}} e^{i\theta}$ so

$$(2) = \int_{C_1(R')} e^{iz^3 t}\, dz = \int_0^{\pi/6} e^{i\left(\frac{R^3}{t} e^{i3\theta}\right)\cdot t} \cdot \frac{iR}{t^{1/3}} e^{i\theta}\, d\theta$$

$$= \frac{iR}{t^{1/3}} \int_0^{\pi/6} e^{i(R^3 e^{i3\theta})} e^{i\theta}\, d\theta$$

$$= \frac{iR}{t^{1/3}} \int_0^{\pi/6} [e^{i[R^3 \cos(3\theta) + iR^3 \sin(3\theta)]} e^{i\theta}\, d\theta$$

$$= \frac{iR}{t^{1/3}} \int_0^{\pi/6} e^{-R^3 \sin(3\theta)} e^{i(R^3 \cos(3\theta) + \theta)}\, d\theta$$

So,

$$|\textcircled{2}| = \left| \frac{iR}{t^{1/3}} \int_0^{\pi/6} e^{-R^3 \sin(3\theta)} e^{i(R^3\cos(3\theta)+\theta)} \, d\theta \right|$$

$$\leq \frac{R}{t^{1/3}} \int_0^{\pi/6} \Big| \overbrace{e^{-R^3 \sin(3\theta)}}^{\text{this is } \geq 0} \Big| \cdot \Big| \underbrace{e^{i(R^3\cos 3\theta + \theta)}}_{1} \Big| \, d\theta$$

$$= \frac{R}{t^{1/3}} \int_0^{\pi/6} e^{-R^3 \sin(3\theta)} \, d\theta$$

Now as $0 \leq \theta \leq \pi/6$, $0 \leq 3\theta \leq \pi/2$, so we can use the inequality mentioned in the problem

$$\sin(3\theta) \geq \frac{2(3\theta)}{\pi} = \frac{6\theta}{\pi}$$

$$\leftrightarrow \quad -\sin(3\theta) \leq -\frac{6\theta}{\pi}$$

So the above is

$$\leq \frac{R}{t^{1/3}} \int_0^{\pi/6} e^{-\frac{R^3 6\theta}{\pi}} \, d\theta = \frac{-\pi R}{6R^3 t^{1/3}} \cdot \left[e^{-\frac{R^3 6\theta}{\pi}} \right]_0^{\pi/6}$$

$$= \frac{-\pi}{6R^2 t^{1/3}} \left[e^{-R^3} - 1 \right] = \frac{\pi}{6R^2 t^{1/3}} \left[1 - e^{-R^3} \right]$$

$\leq \dfrac{\pi}{6R^2 t/3}$. So as $R \to \infty$,

$$\lim_{R\to\infty} 0 \leq \lim_{R\to\infty} |(2)| \leq \lim_{R\to\infty} \frac{\pi}{6R^2 t^{1/3}} = 0$$

which implies $(2) \to 0$ as $R \to \infty$ as desired.

Now let's take a look at (3). We can put $z = \frac{R}{t^{1/3}} e^{i\pi/6}$ so that

$z^3 = \frac{R^3}{t} e^{i\pi/2} = \frac{R^3}{t} \cdot i$ and $dz = \frac{dr}{t^{1/3}} \cdot e^{i\pi/6}$

This is technically a parametrization for $-C_2(R')$, so after computing the next integral, we need to remember to negate the result, so

$$(3) = -\int_0^R e^{i(r^3/t)\cdot i \cdot t} \cdot \frac{1}{t^{1/3}} e^{i\pi/6}\, dr$$

$$= -\frac{e^{i\pi/6}}{t^{1/3}} \int_0^R e^{-r^3}\, dr.$$

So as $R \to \infty$, $(3) \to -\dfrac{e^{i\pi/6}}{t^{1/3}} \displaystyle\int_0^\infty e^{-r^3}\, dr$

So finally,

$$\int_0^\infty e^{iz^3 t}\, dz = \lim_{R\to 0} \int_{C_3(R')} e^{iz^3 t}\, dz$$

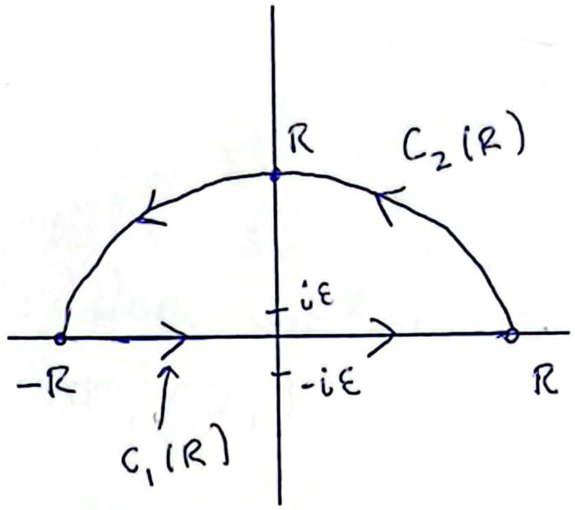
$$= \lim_{R\to\infty} \cancel{(1)}^{\,0} - \cancel{(2)}^{\,0} - (3)$$

$$= -\left(-\frac{e^{i\pi/6}}{t^{1/3}} \int_0^\infty e^{-r^3}\, dr \right)$$

$$= \boxed{\frac{e^{i\pi/6}}{t^{1/3}} \int_0^\infty e^{-r^3}\, dr}$$

which completes our computation.

7) consider the following contour



which is a half circle in the upper-half plane. You can see the whole contour $C(R) = C_1(R) + C_2(R)$ where $C_1(R)$ is the horizontal line from $-R$ to R and $C_2(R)$ is the arc the circle. We are interested in the contour $C_1(R)$.

Notice

$$\int_{C_1(R)} \frac{\varepsilon}{z^2+\varepsilon^2}\,dz = \underbrace{\int_{C(R)} \frac{\varepsilon}{z^2+\varepsilon^2}\,dz}_{1} - \underbrace{\int_{C_2(R)} \frac{\varepsilon}{z^2+\varepsilon^2}\,dz}_{2}$$

we are interested in what happens as $R \to \infty$. Firstly, notice that

$$\frac{\varepsilon}{z^2+\varepsilon^2} = \varepsilon \frac{1}{(z-i\varepsilon)(z+i\varepsilon)} = \frac{\varepsilon}{2i\varepsilon}\left[\frac{1}{z-i\varepsilon} - \frac{1}{z+i\varepsilon}\right]$$

So

$$① = \frac{1}{2i}\int_{C(R)} \frac{1}{z-i\varepsilon} - \frac{1}{z+i\varepsilon}\, dz$$

By examples 2.5.2 and 2.5.3 in the textbook since $z = i\varepsilon$ is inside the half circle $C(R)$ and $z = -i\varepsilon$ is outside $C(R)$, the above equals

$$2\pi i \cdot \frac{1}{2i} \cdot 1 = \pi \quad \text{for any } R > 0$$

Now let's take care of ②. Problem 3 (2.4.7) exactly applies here, so we have

$$0 \le |②| = \left| \int_{C_2(R)} \frac{\varepsilon}{z^2+\varepsilon^2}\, dz \right| \le \frac{\varepsilon \pi R}{R^2-\varepsilon^2}$$

$$\lim_{R\to 0} 0 \le \lim_{R\to\infty} \left| \int_{C_2(R)} \frac{\varepsilon}{z^2+\varepsilon^2}\, dz \right| \le \lim_{R\to\infty} \frac{\varepsilon \pi R}{R^2-\varepsilon^2} = 0$$

So as $R \to \infty$, ② $\to 0$

Putting this all together

$$\int_{-\infty}^{\infty} \frac{\varepsilon}{z^2+\varepsilon^2}\, dz = \lim_{R\to\infty} \int_{C_1(R)} \frac{\varepsilon}{z^2+\varepsilon^2}\, dz$$

$$= \lim_{R\to\infty} (1) - \cancel{(2)}^{\to 0}$$

$$= \boxed{\pi}$$

Confirming with real integration

$$\int_{-\infty}^{\infty} \frac{\varepsilon}{x^2+\varepsilon^2}\, dx = \int_{-\infty}^{\infty} \frac{\varepsilon/\varepsilon^2}{x^2/\varepsilon^2+1}\, dx$$

$$= \frac{1}{\varepsilon}\int_{-\infty}^{\infty} \frac{1}{(x/\varepsilon)^2+1}\, dx \qquad \text{put } u = x/\varepsilon \quad du = \frac{1}{\varepsilon}dx \to dx = \varepsilon du$$

$$= \frac{\cancel{\varepsilon}^{1}}{\cancel{\varepsilon}} \int_{-\infty}^{\infty} \frac{1}{u^2+1}\, du$$

$$= \operatorname{atan}(u) \Big|_{-\infty}^{\infty} = \pi/2 - (-\pi/2) = \boxed{\pi}$$

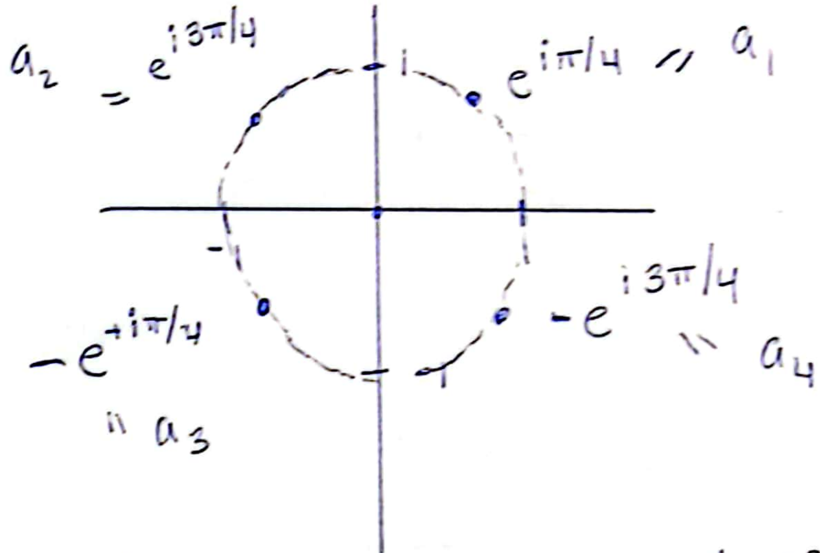
So both computations give the same result.

8) Let's start by examining the function $f(z) = \frac{1}{1+z^4}$, we have

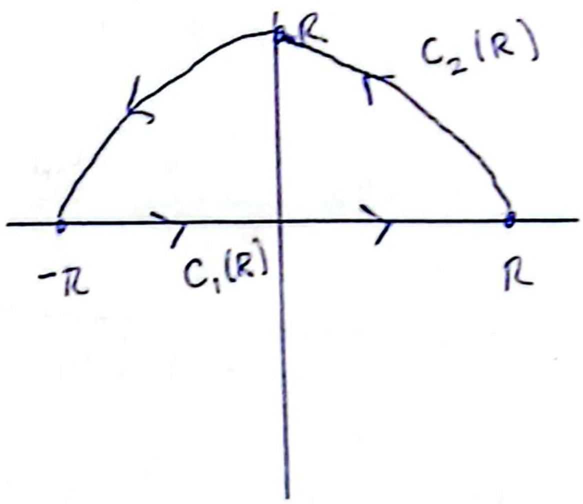
$$f(z) = \frac{1}{z^4+1} = \frac{1}{(z^2-i)(z^2+i)} = \frac{1}{(z-\sqrt{i})(z+\sqrt{i})(z-\sqrt{-i})(z+\sqrt{-i})}$$

$$= \frac{1}{(z-e^{i\pi/4})(z+e^{i\pi/4})(z-e^{i3\pi/4})(z+e^{i3\pi/4})}$$

Let's see where the roots of the denominator are in the plane



we can repeat the same procedure as in Problem 7, with the following contour



$C(R) = C_1(R) + C_2(R)$

we want to focus our attention on $C_1(R)$ as $R \to \infty$

$$\int_{C_1(R)} f(z)\,dz = \underbrace{\int_{C(R)} f(z)\,dz}_{①} - \underbrace{\int_{C_2(R)} f(z)\,dz}_{②}$$

To compute ① we must compute the partial fraction for $f(z)$ (where the a_i's are the roots)

$$f(z) = \frac{1}{(z-a_1)(z-a_2)(z-a_3)(z-a_4)}$$

$$= \frac{A}{z-a_1} + \frac{B}{z-a_2} + \frac{C}{z-a_3} + \frac{D}{z-a_4}$$

$$\to 1 = A(z-a_2)(z-a_3)(z-a_4) + B(z-a_1)(z-a_3)(z-a_4)$$
$$+ C(z-a_1)(z-a_2)(z-a_4) + D(z-a_1)(z-a_2)(z-a_3)$$

$$z = a_2 \to B = \frac{1}{(a_2-a_1)(a_2-a_3)(a_2-a_4)}$$

$$z = a_1 \to A = \frac{1}{(a_1-a_2)(a_1-a_3)(a_1-a_4)}$$

$$z = a_3 \to C = \frac{1}{(a_3-a_1)(a_3-a_2)(a_3-a_4)}$$

$$z = a_4 \to D = \frac{1}{(a_4-a_1)(a_4-a_2)(a_4-a_3)}$$

Again referencing examples 2.5.2 and 2.5.3 the contributions from the "C and D" fractions will integrate to be 0, so we only need to find A and B.

$$\frac{1}{A} = (e^{i\pi/4} - e^{i3\pi/4})(2e^{i\pi/4})(e^{i\pi/4} + e^{i3\pi/4})$$

$$= 2e^{i\pi/4} \cdot e^{i\pi/4}(1-i) \cdot e^{i\pi/4}(1+i)$$

$$= 4e^{i3\pi/4}$$

$$\boxed{A = \frac{1}{4}e^{-i3\pi/4}}$$

$$\frac{1}{B} = (e^{i3\pi/4} - e^{i\pi/4})(e^{i3\pi/4} + e^{i\pi/4})(2e^{i3\pi/4})$$

$$= e^{i\pi/4}(i-1)\, e^{i\pi/4}(i+1) \cdot 2e^{i3\pi/4}$$

$$= -4e^{5\pi/4}$$

$$\boxed{B = -\frac{1}{4}e^{-5\pi/4}}$$

So,

$$① = \frac{1}{4}e^{-\frac{i3\pi}{4}} \int_{C(R)} \frac{1}{(z - e^{i\pi/4})} dz - \frac{1}{4}e^{-5\pi/4} \int_{C(R)} \frac{1}{(z - e^{i3\pi/4})} dz$$

$$+ C \int_{C(R)} \cancel{\frac{1}{(z + e^{\frac{i\pi}{4}})}}^{\,0} dz + D \int_{C(R)} \cancel{\frac{1}{(z + e^{i3\pi/4})}}^{\,0} dz$$

$$= \frac{1}{4} e^{\frac{-i3\pi}{4}} \cdot 2\pi i - \frac{1}{4} e^{\frac{-5\pi}{4}} \cdot 2\pi i \quad \star$$

$$= \frac{\pi i}{2} \left[e^{5\pi/4\, i} - e^{3\pi/4\, i} \right]$$

$$= \frac{\pi}{2} i \left[-\frac{1}{\sqrt{2}} - \frac{1}{\sqrt{2}} i - \left(-\frac{1}{\sqrt{2}}\right) + i \cdot \frac{1}{\sqrt{2}} \right]$$

$$= \frac{\pi}{2} i \left[\cancel{-\frac{1}{\sqrt{2}}} - \frac{1}{\sqrt{2}} i + \cancel{\frac{1}{\sqrt{2}}} - \frac{i}{\sqrt{2}} \right]$$

$$= \frac{\pi}{2} i \left[\frac{-2i}{\sqrt{2}} \right]$$

$$= \boxed{\pi / \sqrt{2}} \longleftarrow$$

i circled this but we aren't done

where at (*) we have used examples 2.5.2 and 2.5.3 to say the integrals should compute to $2\pi i$, since the corresponding roots lie inside $C(R)$. Now for (2), notice that

$$|z^4+1| = |z^4-(-1)|$$
$$\geq \big||z^4|-1\big|$$
$$= R^4-1 \quad \text{for } R>1$$

So
$$\left|\oint_{C_2(R)} \frac{1}{z^4+1}\,dz\right| \leq \oint_{C_2(R)} \left|\frac{1}{z^4+1}\right| dz \leq \sup_{z\in C_2(R)} \left|\frac{1}{z^4+1}\right| \cdot \underbrace{L}_{\text{arc length of } C}$$
$$\leq \frac{\pi R}{R^4-1}$$

So as $R\to\infty$

$$\lim_{R\to\infty} 0 \leq \lim_{R\to\infty} |(2)| \leq \lim_{R\to\infty} \frac{\pi R}{R^4-1} = 0$$

$\Rightarrow$ which implies $(2)\to 0$ as $R\to\infty$.

wrapping this up, as $R\to\infty$

$$\int_{-\infty}^{\infty} \overset{\frac{1}{z^4+1}}{\overbrace{f(z)}}\,dz = \lim_{R\to\infty}\int_{C_1(R)} f(z)\,dz$$
$$= \lim_{R\to\infty} (1) - \cancel{(2)}^{\,0}$$
$$= \boxed{\pi/\sqrt{2}}$$

and we are done.

9) a) Since $f(z) = \sin(z)$ is analytic everywhere in $\mathbb{C}$, we can use Cauchy's Integral formula with $z=0$

$$\oint \frac{\sin(\zeta)}{\zeta}\, d\zeta = 2\pi i\, f(z=0)$$

$$= 2\pi i \sin(0)$$

$$= \boxed{0}$$

e) we have

$$f(z) = e^{z^2}\left(\frac{1}{z^2} - \frac{1}{z^3}\right) = \frac{e^{z^2}}{z^2} - \frac{e^{z^2}}{z^3}$$

$$= \frac{\sum_{n=0}^{\infty} \frac{z^{2n}}{n!}}{z^2} - \frac{\sum_{n=0}^{\infty} \frac{z^{2n}}{n!}}{z^3}$$

$$= \sum_{n=0}^{\infty} \frac{z^{2n-2}}{n!} - \sum_{n=0}^{\infty} \frac{z^{2n-3}}{n!}$$

$$= \left(\frac{1}{z^2} + 1 + \underbrace{\sum_{n=2}^{\infty} \frac{z^{2n-2}}{n!}}_{1}\right) - \left(\frac{1}{z^3} + \frac{1}{z^1} + \underbrace{\sum_{n=2}^{\infty} \frac{z^{2n-3}}{n!}}_{2}\right)$$

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The terms (1) and (2) are analytic everywhere in $\mathbb{C}$ because they are polynomials. So if we loop integrate them, they will vanish. So simplifying things, we can write

$$\oint_C f(z)\,dz = \oint_C e^{z^2}\left(\frac{1}{z^2} - \frac{1}{z^3}\right)dz$$

$$= \oint_C \frac{1}{z^2} - \frac{1}{z^3} - \frac{1}{z}\,dz$$

$$= \oint_C \frac{1}{z^2}\,dz - \oint_C \frac{1}{z^3}\,dz - \oint_C \frac{1}{z}\,dz$$

Since the function $g(z) = 1$ is analytic everywhere, Cauchy's integral formula tells us

$$\oint_C \frac{1}{z}\,dz = 2\pi i\, g(0) = 2\pi i$$

$\longrightarrow$ Theorem 2.6.2

tells us

$$+\oint_C \frac{1}{z^2}\,dz = \frac{2\pi i}{1!}\underbrace{g'(0)}_{=0 \text{ (since } g(z)=1 \text{ is constant)}} = 0$$

And Theorem 2.6.2 gives us again

$$\oint_C \frac{1}{z^3}\,dz = \frac{2\pi i}{2!}\underbrace{g''(0)}_{=0}$$

$$= 0$$

So finally,

$$\oint_C f(z)\,dz = \oint_C e^{z^2}\left(\frac{1}{z^2} - \frac{1}{z^3}\right) dz$$

$$= 0 - 0 - 2\pi i$$

$$= \boxed{-2\pi i}$$