3) a) we have,

$$f(z) = \frac{z}{e^z - 1} \longrightarrow \frac{1}{f(z)} = \frac{e^z - 1}{z}$$

$$= \frac{\sum_{n=0}^{\infty} \frac{z^n}{n!} - 1}{z}$$

$$= \frac{\sum_{n=1}^{\infty} \frac{z^n}{n!}}{z}$$

$$= \sum_{n=1}^{\infty} \frac{z^{n-1}}{n!}.$$

$$= 1 + \frac{z}{2!} + \sum_{n=3}^{\infty} \frac{z^{n-1}}{n!}$$

at $z = 0 \rightarrow \frac{1}{f(0)} = 1$,

so we can define $f(0) = 1$ to remove the singularity at $z = 0$.

b) Since $f(z)$ has only one singularity, and it is removable, the Taylor series radius of for $f(z)$ will have $\infty$ convergence.

c)

$$\frac{z}{e^z-1} = \sum_{n=0}^{\infty} \frac{B_n}{n!} z^n$$

$$z = (e^z-1)\sum_{n=0}^{\infty} \frac{B_n}{n!} z^n$$

$$z = \sum_{m=1}^{\infty} \frac{z^m}{m!} \sum_{n=0}^{\infty} \frac{B_n}{n!} z^n$$

$$z = \sum_{m=0}^{\infty} \frac{z^{m+1}}{(m+1)!} \sum_{n=0}^{\infty} \frac{B_n}{n!} z^n$$

Cauchy product convolution formula

$$z = \sum_{n=0}^{\infty} \overbrace{\sum_{m=1}^{n} \frac{B_{n-m}}{(m)!\,(n-m)!}}^{//\, C_n} z^n$$

The LHS is always equal to $z$ and the RHS is $\sum_{n=0}^{\infty} C_n z^n$, so $C_n = 0$ for $n \in \{0, 2, 3, 4, \ldots\}$ and $C_n = 1$ for $n=1$.

Put $n=1$, then

$$1 = \frac{B_0}{1!\,0!} = B_0$$

If $n > 1$

$$\sum_{m=1}^{n} \frac{B_{n-m}}{m!(n-m)!} = 0$$

$$\leftrightarrow \sum_{m=1}^{n} \frac{1}{n!} \binom{n}{m} B_{n-m} = 0$$

pushing $1/n!$ into the other sum

$$\leftrightarrow \sum_{m=1}^{n} \binom{n}{m} B_{n-m} = 0 \qquad (\star)$$

So $B_0 = 1$ and $B_n$ for $n > 1$ can be chosen to satisfy the above recurrence relation.

d) ~~I wrote code to evaluate the recurrence we found in (c)...~~

By repeatedly using the recurrence labeled $(\star)$, starting with $B_0 = 1$, we get

| | |
|---|---|
| $B_1 = 1$ | $B_8 = -1/30$ |
| $B_2 = -1/2$ | $B_9 = 0$ |
| $B_3 = 0$ | $B_{10} = 5/66$ |
| $B_4 = -1/30$ | $B_{11} = 0$ |
| $B_5 = 0$ | $B_{12} = -691/2730$ |
| $B_6 = 1/42$ | |
| $B_7 = 0$ | |

e) AS

$$f(z) = \sum_{n=0}^{\infty} \frac{B_n}{n!} z^n$$

For $B_{2n+1} = 0 \ \forall n \geq 1$ we need $f(z)$ to be an even function so that the odd powers of $z$ are removed from the sum, but we must treat the $z^1$ term seperately as $2n+1 \geq 3$

Notice

$$\sum_{\substack{n \neq 1 \\ n > 0}} \frac{B_n}{n!} z^n = \frac{z}{e^z} + \frac{z}{2}$$

$$= \frac{2z + z(e^z - 1)}{2(e^z - 1)}$$

$$= \frac{z + ze^z}{2e^z - 1}$$

$$= \frac{z(e^z + 1)}{2(e^z - 1)}$$

Since this function is even, by our earlier discussion $B_{2n+1} = 0 \ \forall n \geq 1.$

The RHS is an even function since plugging $z = -z$

$$\rightarrow \frac{-z(e^{-z} + 1)}{2(e^{-z} - 1)} = \frac{\cancel{e^{z}}}{\cancel{e^{-z}}} \cdot \frac{-z(e^z + 1)}{-(e^z - 1)2} = \frac{z(e^z + 1)}{2(e^z - 1)}$$

4) b) From (3), we can write

$$g(z) = \frac{1}{\sum_{n=0}^{\infty} \frac{B_n}{n!} z^n}$$

Take $G(z)$

$$= \frac{1}{\sum_{n=0}^{12} \frac{B_n}{n!} z^n}$$

where $B_{0:12}$ are from problem 3,

To show $|G(3^{-n}) - g(3^{-n})| < 10^{-10}$ $\forall n$,

$g(z)$ can be written as $\sum_{n=1}^{\infty} \frac{z^{n-1}}{n!}$, so

$$|G(z) - g(z)| = \left| \frac{1}{\sum_{n=0}^{12} \frac{B_n}{n!} z^n} - \sum_{n=1}^{\infty} \frac{z^{n-1}}{n!} \right|$$

$$\leq \left| \frac{1}{(3^{-n})^n} - \sum_{n=1}^{\infty} \frac{(3^{-n})^{n-1}}{n!} \right|$$

$$= \left| 3^{-n^2} - \frac{e^{3^{-n}} - 1}{3^{-n}} \right| < 10^{-10}$$

since this function decreases with $n$ growing

and with $n=1$ $\left| \frac{1}{3} - \frac{e^{1/3} - 1}{3^{-1/3}} \right| < 10^{-10}$

5) a) we use the ratio test,

$$\lim_{n\to\infty} \left| \frac{z^{n+1}}{z^n} \right| = \lim_{n\to\infty} \frac{|z|^{n+1}}{|z|^n} = \lim_{n\to\infty} |z| = |z|$$

So $\sum_{n=0}^{\infty} z^n$ is analytic for $|z|<1$.

b) $$\frac{1}{1-z} = \frac{1}{1-1/2+1/2-z}$$

$$= \frac{1}{3/2-(z+1/2)}$$

$$= \frac{2}{3}\cdot\frac{1}{1-\frac{3(z+1/2)}{2}}$$

$$= \frac{2}{3}\sum_{n=0}^{\infty} \frac{3^n}{2^n}(z+1/2)^n$$

$$= \sum_{n=0}^{\infty} \frac{3^{n-1}}{2^{n-1}}(z+1/2)^n$$

we now use the ratio test

$$\lim_{n\to\infty} \left| \frac{\frac{3^n}{2^n}(z+1/2)^{n+1}}{\frac{3^{n-1}}{2^{n-1}}(z+1/2)^n} \right| = \lim_{n\to\infty} \left| \frac{3}{2}(z+1/2) \right|$$

$$= \frac{3}{2}|z+1/2|$$

This limit is $<1$ when $|z+1/2| < 2/3$, hence that is where this function $G(z)$ is analytic.

6) Let's use partial fractions

$$\frac{z^{n-1}}{(1-z^n)(1-z^{n+1})} = \frac{A(z)}{1-z^n} + \frac{B(z)}{1-z^{n+1}}$$

$$\leftrightarrow \quad z^{n-1} = A(1-z^{n+1}) + B(1-z^n)$$

Put $B = -A$, then

$$z^{n-1} = A(z^n - z^{n+1})$$

$$\frac{1}{A} = \frac{z^n - z^{n+1}}{z^{n-1}} = \frac{\cancel{z^n}}{\cancel{z^{n-1}}}(z - z^2) = z(1-z)$$

$$A = \frac{1}{z(1-z)} \qquad B = -\frac{1}{z(1-z)}$$

So we have

$$\sum_{n=1}^{\infty} \frac{z^{n-1}}{(1-z^n)(1-z^{n+1})} = \sum_{n=1}^{\infty} \frac{1}{z(1-z)(1-z^n)} - \frac{1}{z(1-z)(1-z^{n+1})}$$

$$= \lim_{M\to\infty} \sum_{n=1}^{M} \frac{1}{z(1-z)}\left[\frac{1}{(1-z^n)} - \frac{1}{(1-z^{n+1})}\right]$$

This sum telescopes, so

$$= \lim_{M\to\infty} \frac{1}{z(1-z)}\left[\frac{1}{1-z} - \frac{1}{1-z^{M+1}}\right]$$

$$= \frac{1}{z(1-z)} \lim_{M\to\infty} \frac{1}{1-z} - \frac{1}{1-z^{M+1}}$$

If $|z|<1$, the limit converges to $\frac{1}{1-z} - 1$

$$= \frac{1-(1-z)}{1-z} = \frac{z}{1-z},$$ meaning the sum

Simplifies to $\frac{1}{z(1-z)} \cdot \frac{z}{(1-z)} = \frac{1}{(1-z)^2}$

If $|z|>1$, then the limit converges to $\frac{1}{1-z}$, and the sum becomes $\frac{1}{z(1-z)} \cdot \frac{1}{(1-z)}$

$= \frac{1}{z(1-z)^2}$ which is the desired result.

This function does not violate analytic continuation because at $z=1$, both functions are undefined, ie this function (these 2 functions) are not the same on the boundary (unit circle); so it's okay that the functions are different inside and outside the boundary (the unit circle).

7) Consider the sequence of functions

$$\hat{f}_n(z) = \int_0 e^{izx} f(x)\, dx \,. \quad (1)$$

Since $e^{izx}$ and $f(x)$ are continuous, the integrand is continuous, and if $\hat{f}(x)$ exists $\lim_{n\to\infty} \hat{f}_n(z) = \hat{f}(z)$. Let's show $\hat{f}(z)$ is bounded

$$|\hat{f}(z)| = \left| \int_0^\infty e^{izx} f(x)\, dx \right|$$

$$\leq \int_0^\infty |e^{izx}|\,|f(x)|\, dx$$

$$\leq M \int_0^\infty |e^{ix(a+ib)}|\, dx$$

$$= M \int_0^\infty |e^{ixa} e^{-xb}|\, dx$$

$$= M \int_0^\infty e^{-xb}\, dx \quad (\text{as } x \text{ and } b > 0)$$

$$= M \left[ -\frac{1}{b} e^{-xb} \right]_0^\infty$$

$$= -\frac{M}{b} \left[ \lim_{x\to\infty} \frac{1}{e^{xb}} - 1 \right] = \frac{M}{b}$$

So the integral exists.

The integral (1) is the same as

$$\hat{f}_n(x) = \int_0^\infty g_n(z,x)\,dx \quad ; \quad g_n(z,x) = \begin{cases} g(z,x) & x \in [0,n] \\ 0 & \text{else} \end{cases}$$

where $g(z,x) = f(z)e^{izx}$. $g_n \to g$ uniformly since the bound doesn't depend on $z$ (the limit is independent of $z$), so from previous homework $\hat{f}_n \to f$ uniformly. If we can show $\hat{f}_n$ is analytic $\forall n$, then we can conclude our limit function $f$ is analytic as well. $g_n(z,x)$ is analytic since it is bounded and continuous, it's loop integral will vanish because of Cauchy's theorem, and morera's theorem then tells us $g_n(z,x)$ is analytic. Since $g_n(z,x)$ is analytic, we can use Theorem 3.11 from Bernard's notes, to say $\hat{f}_n(z)$ is analytic, Since $g_n(z,x)$ is continuous for all $x \in (0,n)$, which by our earlier discussion. Concludes the argument that $\hat{f}(z)$ is analytic.

8) From previous homework, both functions have a branch cut on $[-1,1]$. There was a theorem presented in lecture saying that if on the domain both functions are defined, if they are equal for a sequence of points in the domain, and this sequence of points has an accumulation point inside the domain, then both functions are equal on the whole domain. So consider the sequence of points $z_n = 2 + \frac{1}{n}$, then

$$\sqrt{z_n - 1}\,\sqrt{z_n + 1} = \sqrt{1+\frac{1}{n}} \cdot \sqrt{3+\frac{1}{n}}$$

$$= \frac{\sqrt{1+\frac{1}{n}}}{\sqrt{1+\frac{1}{n}}} \cdot \sqrt{3+1/n}\,\sqrt{1+\frac{1}{n}}$$

$$= \left(1+\frac{1}{n}\right)\sqrt{\frac{3+1/n}{1+1/n}}$$

$$= \sqrt{z_n - 1} \cdot \sqrt{\frac{z_n+1}{z_n-1}}$$

where the manipulations above are justified be-cause of real numbers. $z_n \to 2$ as $n \to \infty$, and $2 \notin [-1,1]$. So we get $\sqrt{z-1}\,\sqrt{z+1} = (z-1)\sqrt{\frac{z+1}{z-1}}$ where they are defined, Now put $z = \frac{1}{t}$, as $t \to 0$, $z \to \infty$, so

because the above 2 functions are equal

$$\sqrt{z-1}\,\sqrt{z+1} = \sqrt{(z-1)(z+1)}$$

$$= \sqrt{z^2 - 1}$$

$$= \sqrt{\frac{1}{t^2} - 1}$$

$$= \frac{1}{t}\sqrt{1-t} \quad (*)$$

The binomial exapansion then gives

$$(*) = \frac{1}{t}\left(1 - \frac{t^2}{2} + \frac{t^4}{8} + O(t^5)\right) \quad (\text{as } t \to 0)$$

$$= z\left(1 - \frac{1}{z^2 \cdot 2} + \frac{1}{z^4 \cdot 8} + O\left(\frac{1}{z^5}\right)\right) \quad (\text{as } z \to \infty)$$

$$= z - \frac{1}{2}\frac{1}{z^2} + \frac{1}{8} \cdot \frac{1}{z^4} + O\left(\frac{1}{z^5}\right)$$

$$= z + 0 - \frac{1}{2} z^{-2} + O(z^{-3}) \text{ as } (\text{as } z \to \infty)$$

so $b_0 = 0$, $b_2 = -1/2$, and $b_1 = 0$.

1) (3.3.2)

a) If $|z| < a$, then $\frac{|z|^2}{a^2} < \frac{a^2}{a^2} = 1$, so

$$\frac{z}{a^2 - z^2} = \frac{z}{a^2} \cdot \frac{1}{1 - \frac{z^2}{a^2}}$$

$$= \frac{z}{a^2} \sum_{n=0}^{\infty} \left(\frac{z^2}{a^2}\right)^n$$

$$= \sum_{n=0}^{\infty} \frac{z^{2n+1}}{a^{2n+2}}$$

b) If $|z| > a$, then $\frac{1}{|z|} < \frac{1}{a}$ and $\frac{a^2}{|z|^2} < \frac{a^2}{a^2} = 1$,

so,

$$\frac{z}{a^2 - z^2} = -\frac{z}{z^2} \frac{1}{1 - \frac{a^2}{z^2}}$$

$$= -\frac{1}{z} \sum_{n=0}^{\infty} \left(\frac{a^2}{z^2}\right)^n$$

$$= - \sum_{n=0}^{\infty} \frac{a^{2n}}{z^{2n+1}}$$

2) (3.3.5) we can use Theorem 3.12 to say

$$J_n(t) = \frac{1}{2\pi i} \oint_C \frac{e^{t/2\,(z - 1/z)}}{z^{n+1}}\, dz$$

our singularity is at $z = 0$, as $\frac{1}{z}$ is not defined there. So for convenience let's take our annulus to be $1/2 \le |z| \le 3/2$, and choose our contour $C$ to be the unit circle. Put $z = e^{i\theta}$. Then $z - \frac{1}{z} = e^{i\theta} - e^{-i\theta}$ $= 2i\sin(\theta)$, and $dz = ie^{i\theta}\,d\theta$. we can also take $-\pi \le \theta < \pi$, then our integral becomes

$$\frac{1}{2\pi i} \int_{-\pi}^{\pi} \frac{e^{t/2 \cdot 2i\sin\theta}}{e^{i(n+1)\theta}} \cdot i\, e^{i\theta}\, d\theta$$

$$= \frac{i}{2\pi i} \int_{-\pi}^{\pi} e^{t/\cancel{2}(\cancel{2} i \sin\theta) - n\theta i}\, d\theta$$

$$= \frac{1}{2\pi} \int_{-\pi}^{\pi} e^{-i(n\theta - t\sin\theta)}\, d\theta \quad \checkmark$$

(continues on next page)

$$= \frac{1}{2\pi}\int_{-\pi}^{\pi} \cos(n\theta - t\sin\theta)\,d\theta - \frac{1}{2\pi} i \int_{-\pi}^{\pi} \sin(n\theta - t\sin\theta)\,d\theta$$

(and over $-a \le \theta \le a$)

Since $\sin(x)$ is an odd function, the second integral is 0, and since $\cos(x)$ is an even function, the above can be written as

$$= \frac{1}{2\pi}\int_{0}^{\pi} 2\cos(n\theta - t\sin\theta)\,d\theta$$

$$= \frac{1}{\pi}\int_{0}^{\pi} \cos(n\theta - t\sin\theta)\,d\theta \qquad \checkmark$$

as desired.