

AMATH 561 Autumn 2024

Problem Set 6

Due: Wed 11/13 at 10am

Note: Submit electronically to Canvas.

1. Let $X \sim \text{Binomial}(n, U)$, where $U \sim \text{Uniform}((0, 1))$. What is the probability generating function $G_X(s)$ of X ? What is $P(X = k)$ for $k \in \{0, 1, 2, \dots, n\}$?

2. Consider a branching process with immigration

$$Z_0 = 1, \quad Z_{n+1} = \sum_{i=1}^{Z_n} \xi_i^{n+1} + Y_{n+1},$$

where the (ξ_i^{n+1}) are iid with common distribution ξ , the (Y_n) are iid with common distribution Y and the (ξ_i^{n+1}) and (Y_{n+1}) are independent. What is $G_{Z_{n+1}}(s)$ in terms of $G_{Z_n}(s)$, $G_\xi(s)$ and $G_Y(s)$? Write $G_{Z_2}(s)$ explicitly in terms of $G_\xi(s)$ and $G_Y(s)$.

3. (a) Let X be exponentially distributed with parameter λ . Show by elementary integration (not complex integration) that $E(e^{itX}) = \lambda/(\lambda - it)$.
(b) Find the characteristic function of the density function $f(x) = \frac{1}{2}e^{-|x|}$ for $x \in \mathbb{R}$.

4. A coin is tossed repeatedly, with heads turning up with probability p on each toss. Let N be the minimum number of tosses required to obtain k heads. Show that, as $p \rightarrow 0$, the distribution function of $2Np$ converges to that of a gamma distribution. Note that, if $X \sim \Gamma(\lambda, r)$ then

$$f_X(x) = \frac{1}{\Gamma(r)} \lambda^r x^{r-1} e^{-\lambda x} 1_{x \geq 0}.$$