AMATH 563 - Homework 1 (Theory)

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April 2025

Problem 1

On the interval $[a,b]$, let $x_n(t) = 0$ if $t \in [a, \frac{a+b}{2}]$, $x_n(t) = 1$ if $t \in [\frac{a+b}{2} + \frac{1}{n}, b]$, and $x_n(t) = n(x - \frac{a+b}{2})$ in between (where eventually n is large enough to guarantee that the intervals will be contained in $[a,b]$). These are piecewise-linear functions, so they are clearly in $C([a,b])$. We will show this sequence is Cauchy in $C([a,b])$ with the $L^2([a,b])$ norm, but does not converge to a continuous function on $[a,b]$. Consider:

$$|x_n(t) - x_m(t)| = \begin{cases} 0 & \text{if } a \leq t \leq \frac{a+b}{2} \\ 0 & \text{if } \frac{a+b}{2} + \frac{1}{\min(n,m)} \leq t \leq b \\ |n-m|\,(x - \frac{a+b}{2}) & \text{if } \frac{a+b}{2} \leq t \leq \frac{a+b}{2} + \frac{1}{\max(n,m)} \end{cases}$$

Hence:

$$\begin{aligned} \|x_n(t) - x_m(t)\|^2 &= (n-m)^2 \int_{\frac{a+b}{2}}^{\frac{a+b}{2} + \frac{1}{\max(n,m)}} \left(t - \frac{a+b}{2}\right)^2 dt \\ &= (n-m)^2 \int_0^{\frac{1}{\max(n,m)}} u^2 \, du \\ &= \frac{(n-m)^2}{3\max(n,m)^3} \\ &\leq \frac{\max(n,m)^2}{3\max(n,m)^3} = \frac{1}{3\max(n,m)} \end{aligned}$$

By choosing $m, n > \frac{1}{3\epsilon^2}$ (for $\epsilon > 0$) we see the above is bounded by ϵ^2, so that:

$$\|x_n(t) - x_m(t)\| < \epsilon$$

Therefore, the sequence of functions is Cauchy in $C[a,b]$ with our chosen norm. However, now consider $x_n(t)$ as $n \to \infty$. In the limit, we should get the function:

$$x(t) = \begin{cases} 0 & \text{if } a \leq t < \frac{a+b}{2} \\ 1 & \text{if } \frac{a+b}{2} < t \leq b \end{cases}$$

Which is not continuous on $[a,b]$. Thus, our Cauchy sequence does not converge to a function in $C([a,b])$, so $C([a,b])$ equipped with the $L^2([a,b])$ norm is not a Banach space.

Problem 2

A normed space is a vector space with a norm. So to show that $(X, \|\cdot\|)$ is a normed space with $X = X_1 \times X_2$ and $\|x\| = \max\left(\|x_1\|_1, \|x_2\|_2\right)$, all we must do is show that X is a vector space and that $\|\cdot\|$ is a valid norm on X. To show X is a vector space, consider two vectors $u, v \in X$. Then $u = (u_1, u_2)$ and $v = (v_1, v_2)$ for $u_1, v_1 \in X_1$ and $u_2, v_2 \in X_2$. We must show X is closed under addition and scalar multiplication. So take $a, b \in F$, then:

$$au + bv = a(u_1, u_2) + b(v_1, v_2) = (au_1 + bv_1, au_2 + bv_2) = (w_1, w_2)$$

As X_1 is a vector space and $u_1, v_1 \in X_1$, we must have $w_1 \in X_1$. Also, as X_2 is a vector space and $u_2, v_2 \in X_2$, we must have $w_2 \in X_2$. Therefore, $au + bv = (w_1, w_2) \in X_1 \times X_2 = X$, and X is a vector space. Now define $\|x\| = \max\left(\|x_1\|_1, \|x_2\|_2\right)$ for $x = (x_1, x_2) \in X$. We must show that $\|\cdot\|$ satisfies the properties of a norm. To show positive definiteness, if $\|x\| = 0$, then $\max\left(\|x_1\|_1, \|x_2\|_2\right) = 0$. As both $\|\cdot\|_1$ and $\|\cdot\|_2$ are norms, they are both non-negative. So if the maximum is 0, they must be 0, hence it must be that $(x_1, x_2) = (0, 0)$ which is the 0 vector in X. To show absolute homogeneity, let $s \in F$, then:

$$\|sx\| = \|(sx_1, sx_2)\| = \max\left(\|sx_1\|_1, \|sx_2\|_2\right) = |s| \max\left(\|x_1\|_1, \|x_2\|_2\right) = |s| \, \|x\|$$

Where the second to last equality follows from the absolute homogeneity of $\|\cdot\|_1$ and $\|\cdot\|_2$. Finally, to show the triangle inequality, let $x = (x_1, x_2), y = (y_1, y_2) \in X$, then:

$$\|x + y\| = \|(x_1, x_2) + (y_1, y_2)\|$$

$$= \|(x_1 + y_1, x_2 + y_2)\|$$

$$= \max\left(\|x_1 + y_1\|_1, \|x_2 + y_2\|_2\right) := M$$

There are two cases here. If $M = \|x_1 + y_1\|_1$ then:

$$\|x + y\| = \|x_1 + y_1\|_1 \leq \|x_1\|_1 + \|y_1\|_1$$

$$\leq \max\left(\|x_1\|_1, \|x_2\|_2\right) + \max\left(\|y_1\|_1, \|y_2\|_2\right) = \|x\| + \|y\|$$

On the other hand, if $M = \|x_2 + y_2\|_2$, then:

$$\|x+y\| = \|x_2+y_2\|_2 \leq \|x_2\|_2 + \|y_2\|_2$$

$$\leq \max\left(\|x_1\|_1, \|x_2\|_2\right) + \max\left(\|y_1\|_1, \|y_2\|_2\right) = \|x\| + \|y\|$$

So in either case, the triangle inequality is satisfied. Thus, $\|\cdot\|$ is a norm on X and $(X, \|\cdot\|)$ is a normed space.

Problem 3

Let $T : X \to Y$ and $S : Y \to Z$ be linear operators, and also define $H(x) = ST(x) = S(T(x)) : X \to Z$ to be the composition of S and T. We want to show that H is also also a linear operator, so take $x, y \in X$ and $a, b \in F$, then:

$$H(ax + by) = S(T(ax + by)) = S(aT(x) + bT(y))$$

$$= aS(T(x)) + bS(T(y))$$

$$= aH(x) + bH(y)$$

Hence, the composition H is a linear operator.

Problem 4

Suppose we have the functional, with $x \in C([a,b])$:

$$f(x) = \max_{t\in[a,b]} x(t)$$

This functional cannot be linear in general. For instance take $x(t) = \cos^2(t)$ and $y(t) = \sin^2(t)$ on $[a,b] = [0,2\pi]$ (both functions are continuous on $[0,2\pi]$). Then:

$$f(x+y) = \max_{t\in[a,b]} (\cos^2(t) + \sin^2(t)) = \max_{t\in[a,b]}(1) = 1$$

$$f(x) + f(y) = \max_{t\in[a,b]}(\cos^2(t)) + \max_{t\in[a,b]}(\sin^2(t)) = 1 + 1 = 2$$

So $f(x+y) \neq f(x) + f(y)$, and f cannot be linear as stated earlier. The functional should be bounded. Consider $f(x(t))$ when $\|x(t)\| = \sup_{t\in[a,b]} |x(t)| = 1$. Now:

$$\|f\| = \sup_{\|x\|=1} f(x) = \sup_{\|x\|=1} \max_{t\in[a,b]} x(t)$$

$$= \sup_{\|x\|=1} \sup_{t\in[a,b]} x(t) \leq \sup_{\|x\|=1} \sup_{t\in[a,b]} |x(t)| = 1$$

Where the third equality follows from the Extreme Value Theorem. Hence, the norm of the functional is bounded.

Problem 5

To show that $\|\varphi\| = \sup_{\|x\|=1} |\varphi(x)|$ is a norm on X^*, we mush show that it satisfies the properties of a norm. To show non-negativity, let $\|\varphi\| = 0$, then:

$$\sup_{\|x\|=1} |\varphi(x)| = 0$$

Since $|\varphi(x)| \geq 0$ for all $x \in X$, we must have that $\varphi(x) = 0$ for the above to be true. Now to show absolute homogeneity, take $\varphi \in X^*$ and $s \in F$, then:

$$\|s\varphi\| = \sup_{\|x\|=1} |s\varphi(x)| = |s| \sup_{\|x\|=1} |\varphi(x)| = |s| \, \|\varphi\|$$

Finally, to show the triangle inequality, consider $f, g \in X^*$, then:

$$\|f + g\| = \sup_{\|x\|=1} |f(x) + g(x)|$$

$$\leq \sup_{\|x\|=1} (|f(x)| + |g(x)|) \quad \text{(Triange Inequality)}$$

$$\leq \sup_{\|x\|=1} |f(x)| + \sup_{\|x\|=1} |g(x)|$$

$$= \|f\| + \|g\|$$

So all norm properties are satisfied, hence $\|\varphi\|$ defines a norm for $\varphi \in X^*$.

Problem 6

Let $x, y \in X$ and define $z = x - \frac{\langle x,y\rangle}{\langle y,y\rangle} y$ (assuming $y \neq 0$, if $y = 0$ then the Schwarz inequality is true trivially). Then:

$$\langle z, y\rangle = \left\langle x - \frac{\langle x,y\rangle}{\langle y,y\rangle} y, y \right\rangle$$

$$= \langle x, y\rangle - \left\langle \frac{\langle x,y\rangle}{\langle y,y\rangle} y, y \right\rangle = \langle x, y\rangle - \frac{\langle x,y\rangle}{\langle y,y\rangle} \langle y, y\rangle = 0$$

Therefore, z and y are orthogonal. We can now apply the Pythagorean theorem to x:

$$\|x\|^2 = \|z\|^2 + \left\| \frac{\langle x,y\rangle}{\langle y,y\rangle} y \right\|^2$$

$$= \|z\|^2 + \left| \frac{\langle x,y\rangle}{\langle y,y\rangle} \right|^2 \|y\|^2$$

$$= \|z\|^2 + \frac{|\langle x,y\rangle|^2}{\|y\|^4} \|y\|^2$$

$$= \|z\|^2 + \frac{|\langle x,y\rangle|^2}{\|y\|^2}$$

$$\geq \frac{|\langle x,y\rangle|^2}{\|y\|^2}$$

$$\implies |\langle x,y\rangle|^2 \leq \|x\|^2 \|y\|^2$$

$$\implies |\langle x,y\rangle| \leq \|x\| \, \|y\|$$

This is the Schwarz inequality as desired. In the first inequality above, equality happens if $\|z\| = 0 \implies z = 0$. If $z = 0$, then $x = \frac{\langle x,y\rangle}{\langle y,y\rangle} y := sy$, meaning y is a scalar multiple of x (x and y are dependent). For the converse, suppose x and y are dependent, ie $y = sx$ for some $s \in F$. Then:

$$|\langle x,y\rangle| = |\langle x, sx\rangle| = |s| \, \|x\|^2$$

$$\|x\| \, \|y\| = \|x\| \, \|sx\| = |s| \, \|x\|^2$$

$$\implies |\langle x,y\rangle| = \|x\| \, \|y\|$$

Hence, equality holds iff x and y are dependent. *(Note almost the exact same proof is given on Wikipedia's page on Inner products).*