1) For a particular $h > 0$, consider

$$I_h = \int_{ih}^{ih+\pi} \frac{e^{2ijz}}{\tan(Nz)}\, dz$$

$$= \int_{ih}^{ih+\pi} \frac{e^{2ijz}}{\tan(Nz)} \overbrace{- \frac{e^{2ijz}}{i} + \frac{e^{2ijz}}{i}}^{\text{adding } 0}\, dz$$

$$= \underbrace{\int_{ih}^{ih+\pi} \frac{e^{2ijz}}{\tan(Nz)} - \frac{e^{2ijz}}{i}\, dz}_{I_1} + \underbrace{\frac{1}{i}\int_{ih}^{ih+\pi} e^{2ijz}\, dz}_{I_2}$$

Firstly, we can easily evaluate I_2. When $j=0$,

$$I_2 = \frac{1}{i}\int_{ih}^{ih+\pi} 1\, dz = \frac{1}{i}\left[ih+\pi - ih\right] = \frac{\pi}{i} = -\pi i$$

When $j \neq 0$

$$I_2 = \frac{1}{i\,2ij}\, e^{2ijz}\Big]_{ih}^{ih+\pi} = \frac{-1}{2j}\left[e^{-2hj}\overset{1}{\cancel{e^{2ij\pi}}} - e^{-2hj}\right]$$

$$= 0$$

Now we'd like to show that as $h \to \infty$, $I_1 \to 0$.
We have

$$|I_1| = \left| \int_{ih}^{ih+\pi} e^{2ijz}\left[\cot(Nz) + i\right] dz \right| \longleftarrow \text{here I just simplified the integrand}$$

$\uparrow \le$

0

$$\leq \int_{ih}^{ih+\pi} \left|e^{2\cancel{ij}z}\right|^{1} \left| \frac{(e^{2iNz}+1)i}{e^{2iNz}-1} + i \right| |dz|$$

$$= \int_{ih}^{ih+\pi} |\cancel{i}|^{1} \left| \frac{e^{2iNz}+1}{e^{2iNz}-1} + 1 \right| |dz|$$

$$= \int_{ih}^{ih+\pi} \left| \frac{2e^{2iNz}}{e^{2iNz}-1} \right| |dz|$$

Now put $z = ih + t$ for $0 \leq t \leq \pi$, then the above is

$$= 2\int_{0}^{\pi} \left| \frac{e^{-2Nh} e^{2iNt}}{e^{-2Nh} e^{2iNt} - 1} \right| dt$$

$$\leq 2\int_{0}^{\pi} \frac{e^{-2Nh}}{e^{-2Nh}-1} \, dt$$

where we have used the reverse triangle inequality in the denominator

$$= 2\int_{0}^{\pi} \frac{1}{1-e^{2Nh}} \, dt$$

$$= \frac{2\pi}{1-e^{2Nh}}$$ which $\longrightarrow 0$ as $h \to \infty$

So as $h \to \infty$, $I_1 \to 0$ $\forall j \in (-N, N)$ and $I_2 \to -i\pi$ when $j = 0$, and $\to 0$ when $j \neq 0$. Hence

$$\lim_{h \to \infty} I_h = \begin{cases} -i\pi, & j=0 \\ 0, & \text{else} \end{cases}$$

as desired.

2) a)

Consider a semi-circular contour C_R in the UHP that encloses $z = ia$,

Consider

$$I_1 = \int_{C_R} \frac{z e^{iz}}{z^2 + a^2}\, dz \qquad a^2 > 0$$

$$= \int_{C_R} \underbrace{\frac{z e^{iz}}{(z-ia)(z+ia)}}_{f(z)}\, dz$$

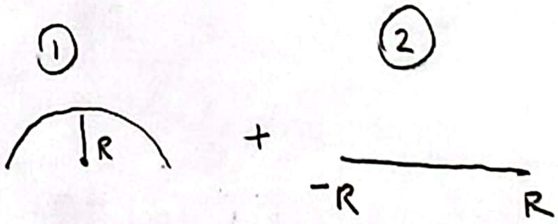
$$= 2\pi i \operatorname{Res}(g, ia)$$

$$= 2\pi i \lim_{z \to ia} \frac{z e^{iz}}{z+ia}$$

$$= 2\pi i \frac{\cancel{ia}\, e^{-a}}{2i\cancel{a}} = \pi i e^{-a}$$

We know

$I_{C_R} =$ ① + ②



we'd like to show

$$\int_{①} \frac{z e^{iz}}{z^2 + a^2}\, dz \to 0 \quad \text{as } R \to \infty$$

5 we have on ①

$$\left|\frac{z}{z^2+a^2}\right| \leq \frac{R}{R^2-a^2} \xrightarrow{R\to\infty} 0$$

where the convergence is uniform as it doesn't depend on z

So with $K=1$

we can apply Jordan's lemma to say $\int_{①} f(z)\,dz \to 0$ as $R\to\infty$

we have

$$\int_{-R}^{R} f(z)\,dz = \int_{C_R} f(z)\,dz + \int_{①} f(z)\,dz$$

so as $R\to\infty$

$$\int_{-\infty}^{\infty} f(z)\,dz = \int_{C_\infty} f(z)\,dz = \pi i e^{-a}$$

The integral we are interested in

$$\int_{-\infty}^{\infty} \frac{x\sin x}{x^2+a^2}\,dx = \mathrm{Im}\left(\int_{-\infty}^{\infty} f(z)\,dz\right)$$

$$= \boxed{\pi e^{-a}}$$

h) we have

$$I = \int_0^{2\pi} \frac{1}{(5-3\sin\theta)^2}\, d\theta$$

(for now)

Consider the unit circle contour C, so that

$$\sin\theta = \frac{z - \frac{1}{z}}{2i} = \frac{z^2-1}{2iz}, \quad d\theta = \frac{dz}{iz}$$

So that

$$I = \oint_C \frac{1}{\left(5 - \frac{3(z^2-1)}{2iz}\right)^2} \frac{dz}{iz}$$

$$= \frac{1}{i} \oint_C \frac{1}{\left(\frac{10iz - 3z^2 + 3}{2iz}\right)^2} \frac{dz}{z}$$

$$= \frac{-4}{i} \oint_C \frac{z}{(-3z^2 - 10iz - 3)^2}\, dz$$

$$= -4i \oint_C \underbrace{\frac{z}{(z-3i)^2(3z-i)^2}}_{g}\, dz$$

[pole at $w_0 = 3i$ (order 2)
pole at $w_1 = \frac{i}{3}$ (order 2)]

The integrand has second order poles at w_0 and w_1

however

$$|w_0| = 3 > 1$$

$$|w_1| = \frac{1}{3} < 1$$

So only w_1 is contained in our contour, hence our integral equals

$$4 \cdot i \cdot 2\pi i \; \mathrm{Res}(g, w_1) \qquad -8\pi$$

$$= \overset{2\pi i}{4i} \cdot \lim_{z \to w_1} \frac{d}{dz} \frac{z}{(z-w_0)^2} = \lim_{z \to w_1} \frac{(z-w_0)^2 - 2z(z-w_0)}{(z-w_0)^4}$$

$$= -8\pi \lim_{z \to w_1} \frac{(z-w_0) - 2z}{(z-w_0)^3} = -8\pi \lim_{z \to w_1} \frac{-z-w_0}{(z-w_0)^3}$$

$$= 8\pi \frac{(w_0 + w_1)}{(w_1 - w_0)^3} = \frac{8\pi \left(\frac{10i}{3}\right)}{\left(-\frac{8i}{3}\right)^3} = \frac{\cancel{\tfrac{1}{3}}\, \overset{10}{\cancel{80}}\, \pi \cancel{i}}{\frac{1}{3^3} 8^3 i}$$

$$= \frac{10\pi}{64} = \boxed{\frac{5\pi}{32}}$$

3) a) we have

$$\text{Res}(f(z)\cot(\pi z), k) = \lim_{z\to k} (z-k) f(z) \cot(\pi z)$$

$$= \lim_{z\to k} \frac{(z-k) f(z)}{\tan(\pi z)} \longrightarrow \frac{0}{0}$$

$$= \lim_{z\to k} \frac{(z-k) f'(z) + f(z)}{\pi \sec^2(\pi z)} \quad \Bigg] \text{ L'Hopital, } f \text{ is analytic}$$

$$= \frac{\cancel{(k-k)}^{0} f'(k) + f(k)}{\pi \underbrace{\sec^2(\pi k)}_{=1}}$$

$$= \frac{f(k)}{\pi}$$

and this calculation is justified because $\cot(\pi z)$ has simple poles at each integer $z = k$.

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b) we have

$$|\cot(\pi z)| = \left| \frac{e^{2\pi i z}+1}{e^{2\pi i z}-1} \right| = \left| \frac{e^{2\pi i x}\, e^{-2\pi y}+1}{e^{2\pi i x}\, e^{-2\pi y}-1} \right| \quad (\star)$$

when $x = (N+\frac{1}{2})$ we have

$$(\star) = \left| \frac{\overset{1}{\cancel{e^{2\pi i N}}}\, \overset{-1}{\cancel{e^{\pi i}}}\, e^{-2\pi y}+1}{\underset{1}{\cancel{e^{2\pi i N}}}\, \underset{-1}{\cancel{e^{\pi i}}}\, e^{-2\pi y}-1} \right| = \left| \frac{1-e^{-2\pi y}}{-e^{-2\pi y}-1} \right|$$

$$= \left| \frac{1-e^{-2\pi y}}{1+e^{-2\pi y}} \right| \le \frac{1}{1+e^{-2\pi y}} \le 1 \le 2$$

This same bound happens when $x = -(N+\frac{1}{2})$ by almost the exact same algebra. (there will be an extra -1, but it gets taken care of by the $|\cdot|$). Now notice when $y = (N+\frac{1}{2})$

$$(\star) = \left| \frac{e^{2\pi i x}\, e^{-2\pi(N+\frac{1}{2})}+1}{e^{2\pi i x}\, e^{-2\pi(N+1/2)}-1} \right|$$

$$\le \frac{1+e^{-2\pi(N+\frac{1}{2})}}{1-e^{-2\pi(N+\frac{1}{2})}} = \frac{1+u}{1-u}$$

In general $\frac{1+u}{1-u} \le 2$ when $1+u \le 2-2u$

$\rightarrow u \le 1/3$

So we see if $e^{-2\pi(N+1/2)} \le 1/3$

we have

$$e^{-2\pi(N+\frac{1}{2})} = \frac{1}{e^{2\pi(N+1/2)}} \leq \frac{1}{3^{2\pi(N+1/2)}} \leq \frac{1}{3}$$

as $2\pi(N+1/2) > 1$ when $N \geq 1$, so $|\cot(\pi z)| \leq 2$ here as well. when $y = -(N+1/2)$ we would just multiply the numerator / denominator of (*) by $e^{-2\pi y}$ and do the same algebra again. Hence, $|\cot(\pi z)| \leq 2$ on Γ_n $\forall\, n \geq 1$.

c) we have

$$\left| \oint_{\Gamma_n} \frac{p(z)}{q(z)} \cot(\pi z)\, dz \right|$$

$$\leq \oint_{\Gamma_n} \left| \frac{p(z)}{q(z)} \right| \left| \cot(\pi z) \right| |dz| = (*)$$

we have $|\cot(\pi z)| \leq 2$ on Γ_n

If

$$\left| \frac{p(z)}{q(z)} \right| = \frac{|p_0 + p_1 z + \dots + p_{K-2} z^{K-2}|}{|q_0 + q_1 z + \dots + q_K z^K|}$$

$$\leq \frac{|p_0| + |p_1||z| + \dots + |p_{K-2}||z|^{K-2}}{\left| |q_K||z|^K - |q_0 + \dots + q z^{K-1}| \right|}$$

$$\leq \frac{M_1 |z|^{K-2}}{\left| |q_K||z|^K - \sum_{j=0}^{K-1} |q_K| |z|^j \right|}$$

(As $|z| > 1$, $|z|^a < |z|^b$ if $a < b$, take $M_1 = \max_{0 \leq j \leq K-2} |p_j|$)

$$\leq \frac{M_1 |z|^{K-2}}{\left| M_2 |z|^K - M_2 \sum_{j=0}^{K-1} |z|^j \right|}$$

(with $M_2 = \min_{0 \leq j \leq K} |q_j|$)

$$= \frac{M_1 |z|^{K-2}}{M_2 \left| |z|^K - \frac{|z|^K - 1}{|z| - 1} \right|}$$

✓ summing geometric series

$$\leq \frac{M |z|^{K-2}}{\left| |z|^{K+1} - 2|z|^K + 1 \right|}$$

$$\leq \frac{M\, 2^{K-2} N^{K-2}}{| N^{K+1} - 2N^K + 1 |}$$ where have $|z| > N$ and $|z| < 2N$

$$= \frac{C\, N^{K-2}}{| N^{K+1} - 2N^K + 1 |}$$ where $C = M 2^{K-2}$

As $\oint_{\Gamma_N} |dz| = 8(N + 1/2)$, we have

$$(*) \quad \leq \frac{2 \cdot 8(N + 1/2) \cdot C\, N^{K-2}}{| N^{K+1} - 2N^K + 1 |}$$

where taking $N \to \infty$ here gives us 0 as the degree of N is larger in the denominator ($K-1$ vs. $K+1$) and these are all real numbers, as desired.

d) Problem (c) implies that the

$$\mathrm{Res}\Big(\underbrace{\frac{p(z)}{q(z)} \cot(\pi z)}_{g(z)}, \infty \Big) = 0$$

But we know g has infinitely many simple poles from $\cot(\pi z)$ for $k \in \mathbb{Z}$, and some finite number j of poles coming from the zeroes of $q(z)$, we know the sum of the sum of all residues in the finite z-plane w/ the residue at ∞ sum to 0, but as $\mathrm{Res}(g, \infty) = 0$, we have the sum of the residues in the finite plane sum to 0, hence

$$0 = \sum_{z \text{ a pole}} \mathrm{Res}(g, z)$$

roots of $q(z)$

$$0 = \sum_{k \in \mathbb{Z}} \mathrm{Res}(g, k) + \sum_{j} \mathrm{Res}(g, z_j)$$

but from (a), $\mathrm{Res}(g, k) = \frac{p(k)}{q(k)} \frac{1}{\pi}$

So we get

$$\frac{1}{\pi} \sum_{k=-\infty}^{\infty} \frac{p(k)}{q(k)} = - \sum_{j} \text{Res}\left(\frac{p(z)}{q(z)} \cot(\pi z), z_j \right)$$

$$\sum_{k=-\infty}^{\infty} \frac{p(k)}{q(k)} = -\pi \sum_{j} \text{Res}\left(\frac{p(z)}{q(z)} \cot(\pi z), z_j \right)$$

as desired.

4) a) From (3)

$$\sum_{K=-\infty}^{\infty} \frac{1}{K^2+1} = \sum_{K=-\infty}^{\infty} \overbrace{\frac{1}{(K-i)(K+i)}}^{f(K)}$$

Note, none of these functions have roots at the integers, and the poly degrees are correct (2 larger in denom)

$$= -\pi \left[\operatorname{Res}(f, i) + \operatorname{Res}(f, -i) \right]$$

$$= -\pi \left[\frac{1}{2i} + \frac{1}{-2i} \right] = 0$$

b)

$$\sum_{K=-\infty}^{\infty} \frac{1}{K^4+1} = \sum_{K=-\infty}^{\infty} \frac{1}{(K^2-i)(K^2+i)}$$

f

$$= \sum_{K=-\infty}^{\infty} \frac{1}{(K-e^{\frac{i\pi}{4}})(K+e^{i\pi/4})(K+e^{3i\pi/4})(K-e^{i\pi 3/4})}$$

$$= -\pi \left[\operatorname{Res}(f, e^{i\pi/4}) + \operatorname{Res}(f, -e^{i\pi/4}) + \operatorname{Res}(f, e^{i3\pi/4}) + \operatorname{Res}(f, -e^{i3\pi/4}) \right]$$

$$= -\pi \left[\cancel{\frac{1}{4e^{i\frac{3\pi}{4}}}} - \cancel{\frac{1}{4e^{i3\pi/4}}} + \cancel{\frac{1}{4e^{i9\pi/4}}} - \frac{1}{4e^{i9\pi/4}} \right]$$

$$= 0$$

c) $$\sum_{k=-\infty}^{\infty} \frac{1}{k^2-\frac{1}{4}} = \sum_{k=-\infty}^{\infty} \overbrace{\frac{1}{(k-\frac{1}{2})(k+1/2)}}^{f}$$

$$= -\pi\left[\operatorname{Res}(f, 1/2) + \operatorname{Res}(f, -1/2)\right]$$

$$= -\pi\left[1 + -1\right]$$

$$= 0$$

d) $$\sum_{k=-\infty}^{\infty} \frac{1}{16k^4-1} = \sum_{k=-\infty}^{\infty} \overbrace{\frac{1}{(2k-1)(2k+1)(2k+i)(2k-i)}}^{f}$$

$$= -\pi\Big[\operatorname{Res}(f, 1/2) + \operatorname{Res}(f, -1/2) + \operatorname{Res}(f, -i/2) + \operatorname{Res}(f, i/2)\Big]$$

$$= -\pi\Big[\frac{1}{2}\cdot\frac{1}{1+i}\,\frac{1}{1-i} + \frac{1}{-2}\,\frac{1}{i-1}\,\frac{1}{-(i+1)} + \frac{1}{-(1+i)}\,\frac{1}{1-i}\,\frac{1}{-2i} + \frac{1}{i-1}\,\frac{1}{i+1}\,\frac{1}{2i}\Big]$$

$$= -\pi\left[\frac{1}{4} - \frac{1}{4} + \frac{1}{4i} - \frac{1}{4i}\right] = 0$$