AMATH 585

Homework 2

Nate Whybra

1) The interpolation polynomial is

$$\ell_i(x) = \prod_{\substack{j=k-2 \\ j \neq i}}^{k} \frac{x - x_j}{x_i - x_j}, \quad i = k-2, k-1, k$$

So the quadratic is, where

$$p(x) = \sum_{i=k-2}^{k} f(x_i)\,\ell_i(x)$$

$$= f(x_{k-2})\left[\frac{x - x_{k-1}}{x_{k-2} - x_{k-1}} \cdot \frac{x - x_k}{x_{k-2} - x_k}\right] + f(x_{k-1})\left[\frac{x - x_{k-2}}{x_{k-1} - x_{k-2}} \cdot \frac{x - x_k}{x_{k-1} - x_k}\right]$$

$$+ f(x_k)\left[\frac{x - x_{k-2}}{x_k - x_{k-2}} \cdot \frac{x - x_{k-1}}{x_k - x_{k-1}}\right]$$

Now suppose $f(x) = x^3 - 2$ w/ $x_0 = 0$, $x_1 = 1$, $x_2 = 2$,
Then, as $f(0) = -2$, $f(1) = -1$, and $f(2) = 6$,

$$p(x) = -2\left[\frac{(x-1)(x-2)}{-1 \cdot -2}\right] - \left[\frac{x(x-2)}{1 \cdot -1}\right] + 6\left[\frac{x(x-1)}{2 \cdot 1}\right]$$

$$= -(x-1)(x-2) + x(x-2) + 3x(x-1)$$

To find x_3, we solve $p(x) = 0$

$$\rightarrow \quad x(x-2) + 3x(x-1) = (x-1)(x-2)$$

$$\rightarrow \quad x(x - 2 + 3x - 3) = x^2 - 3x + 2$$

$\rightarrow \quad 4x^2 - 5x = x^2 - 3x + 2$

$\rightarrow \quad 3x^2 - 2x - 2 = 0$

$\rightarrow \quad x = \frac{2}{6} \pm \frac{1}{6}\sqrt{4 + 4 \cdot 3 \cdot 2}$

$= \frac{1}{6}\left[2 \pm \sqrt{28}\right]$

$= \frac{1}{6}\left[2 \pm 2\sqrt{7}\right]$

$= \frac{1}{3}\left[1 \pm \sqrt{7}\right] \approx 4.74, \; -2.14$

we take the value closest to x_2, so

$x_3 = \frac{1}{3}\left[1 + \sqrt{7}\right]$.

2) we want

$$\ell(x) = c_1 x + c_2$$

$$\ell(-1) = -c_1 + c_2 = a$$

$$\ell(1) = c_1 + c_2 = b$$

$$\leftrightarrow \begin{bmatrix} -1 & 1 \\ 1 & 1 \end{bmatrix} \begin{bmatrix} c_1 \\ c_2 \end{bmatrix} = \begin{bmatrix} a \\ b \end{bmatrix}$$

$$\leftrightarrow c_1 = \frac{b-a}{2}, \quad c_2 = \frac{a+b}{2}$$

Hence, $\ell(x) = \frac{b-a}{2} \cdot x + \frac{a+b}{2}$

$$= \frac{1}{2}\left[(b-a)x + (a+b)\right]$$

If we have $x_j = \cos\left(\frac{j\pi}{n}\right)$, $j = 0, 1, \dots, n$ on $[-1,1]$, and instead want our interpolation points on $[a,b]$, we can just apply our function $\ell(x)$ on the x_j's

$$x_j' = \ell(x_j) = \frac{1}{2}\left[(b-a)\cos\left(\frac{j\pi}{n}\right) + (a+b)\right]$$

These (x_j')'s are the desired solution.

5)

(a) Firstly,

$$\lambda_1(x) = \sum_{i=0}^{1} |\ell_i(x)|$$

$$= \left|\frac{x - x_1}{x_0 - x_1}\right| + \left|\frac{x - x_0}{x_1 - x_0}\right|$$

$$= \left|\frac{x - \frac{3}{2}}{1 - \frac{3}{2}}\right| + \left|\frac{x - 1}{\frac{3}{2} - 1}\right|$$

$$= 2\left|x - \tfrac{3}{2}\right| + 2|x-1| = 2\left(\left|x - \tfrac{3}{2}\right| + |x-1|\right)$$

Secondly,

$$\Lambda_1 = \max_{x \in [1,2]} \overbrace{2\left(\left|x - \tfrac{3}{2}\right| + |x-1|\right)}^{\lambda_1(x)} \quad (1)$$

As $x \geq 1$, we have $|x-1| = x-1$ and $\left|x - \frac{3}{2}\right| = x - \frac{3}{2}$ when $x \geq 3/2$ and $\left|x - \frac{3}{2}\right| = \frac{3}{2} - x$ when $x < \frac{3}{2}$, so

$$(1) = \max_{x \in [1,2]} \begin{cases} 2\left(x-1 + \frac{3}{2} - x\right) = 1, & \text{if } 1 \leq x \leq 3/2 \\ 2\left(x-1 + x - \frac{3}{2}\right) = 4x - 5, & \text{if } x > 3/2 \end{cases}$$

$4x-5$ is monotonically increasing on $[1,2]$, so it achieves its maximum when $x=2$, $\rightarrow 4(2)-5=3$. 3 is always larger than 1, so it must be that $\boxed{\Lambda_1 = 3}$

b) Let $y_1 = f(1)$ and $y_2 = f(\frac{3}{2})$, then we can write (with $x_1 = 1$ and $x_2 = 3/2$)

$$p_1(x) = \frac{y_2 - y_1}{x_2 - x_1}(x - x_1) + y_1$$

$$= 2(y_2 - y_1)(x-1) + y_1$$

Now, we'd like a continuous function f on $[1,2]$ such that

$$\max_{x \in [1,2]} |p_1(x)| = 3$$

and

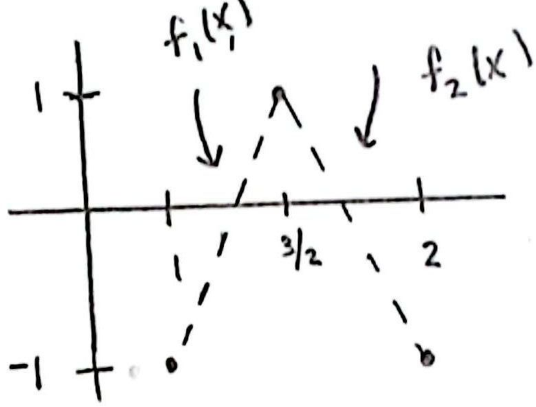
$$\max_{x \in [1,2]} |f(x)| = 1$$

Let's simplify the problem a bit, choose $y_1 = -1$ and $y_2 = 1$, so that

$$p_1(x) = 4x - 5$$

which has $\max_{[a,b]} |f(x)| = 3$ at $x = 2$.

So let's choose $f(x)$ to be a linear piecewise function



$$f(x) = \begin{cases} f_1(x), & x \in [1, 3/2] \\ f_2(x), & x \in [3/2, 1] \end{cases}$$

$$= \begin{cases} 4x - 5, & x \in [1, 3/2] \\ -4x + 7, & x \in [3/2, 1] \end{cases}$$

Clearly f is continuous and it meets the required specifications.