Nate Whybra

AMATH 561

Homework 5

1) To show $(X_n)_{n\geq 0}$ is a martingale with respect to the filtration $(F_n)_{n\geq 0}$ we need to show

(i) $E[|X_n|] < \infty \quad \forall n \geq 0$

(ii) $X_n \in F_n \quad \forall n \geq 0$

(iii) $E[X_{n+1} | F_n] = X_n \quad \forall n \geq 0$

For (i), using Jensen's inequality with the absolute value function (who's convexity is easily seen by a 1 line application of the triangle inequality), we can say, as

$X_n = E[X|F_n]$

$$E[|X_n|] \leq E[E[|X||F_n]]$$
$$= E[|X|] < \infty$$

For (ii), $E[X|F_n] \in F_n$ by definition of conditional expectation.

For (iii),

$$E[X_{n+1} | F_n] = E[E[X|F_{n+1}] | F_n]$$

Since $F_n \subseteq F_{n+1}$

$$= E[X|F_n] = X_n$$

which concludes that $(X_n)_{n\geq 0}$ is a martingale.

2) To show Z_n is a martingale with respect to the filtration defined by $F_n = \sigma(X_0, X_1, \dots, X_n)$, we must show

(i) $E[|Z_n|] < \infty \quad \forall n \geq 0$

(ii) $Z_n \in F_n \quad \forall n \geq 0$

(iii) $E[Z_{n+1} | F_n] = Z_n \quad \forall n \geq 0$

For (i), as $Z_n = \left(\frac{1-p}{p}\right)^{2S_n - n}$ and $q = \frac{1-p}{p} > 0$, $Z_n \geq 0 \;\; \forall n \geq 0$, hence $|Z_n| = Z_n$.

So,

$$E[|Z_n|] = E[q^{2S_n - n}]$$

$$= q^{-n} E[(q^2)^{S_n}]$$

$$= q^{-n} G_{S_N}(q^2)$$

since S_n is a sum of iid random variables

$$= q^{-n} (G_X(q^2))^n$$

$$= q^{-n} \left[\sum_{K \geq 0} p_K q^{2K}\right]^n$$

$$= q^{-n} \left[(1-p) q^0 + p q^2\right]^n$$

$$= q^{-n} \left[(1-p) + p \cdot \frac{(1-p)^2}{p^2}\right]^n$$

$$= q^{-n}\left[1-p+\frac{1}{p}-2+p\right]^n$$

$$= q^{-n}\left[-1+\frac{1}{p}\right]^n$$

$$= q^{-n}\left[\frac{1-p}{p}\right]^n = q^{-n}q^n = 1 < \infty$$

$$\forall n \geq 0$$

For (ii),

$$Z_n = q^{-n}\, q^{2S_n}$$

$$= q^{-n}\, q^{2(X_1+\dots+X_n)}$$

$$= q^{-n}\prod_{i=1}^{N} q^{2X_i}$$

since $F_n = \sigma(X_0, X_1, \dots, X_n)$, and the product of F_n measurable functions is F_n measurable, because $q^{2X_i} \in F_n \ \forall\, 0 \leq i \leq n$, we can say $Z_n \in F_n$.

For (iii),

$$E[Z_{n+1} \mid F_n] = E\left[q^{2S_{N+1}-(n+1)} \mid F_n\right]$$

$$= E\left[q^{2S_n} q^{2X_{n+1}} q^{-(n+1)} \mid F_n\right]$$

$$= q^{2S_n - n}\, E\left[q^{-1} q^{2X_{n+1}} \mid F_n\right]$$

$$= Z_n\, E\left[q \cdot q^{2X_{n+1}}\right]$$

since $X_{n+1} \notin F_n$

$$= q^{-1} G_{X_{n+1}}(q^2) \cdot Z_n$$

$$= Z_n \, q^{-1} \left((1-p) \cdot q^0 + p \cdot q^2 \right)$$

from earlier calculation

$$= Z_n \, q^{-1} q$$

$$= Z_n$$

as desired. So Z_n is a martingale under this filtration.

3) Put $\varepsilon_i = X_i - X_{i-1}$, then notice

$$E[\varepsilon_i \mid F_{i-1}] = E[X_i - X_{i-1} \mid F_{i-1}]$$

since $X_i \in F_i$ and X_n is a martingale

$$= E[X_i \mid F_{i-1}] - E[X_{i-1} \mid F_{i-1}]$$

$$= X_{i-1} - X_{i-1}$$

$$= 0$$

$(i \neq j)$

We want to show $E[\varepsilon_i \varepsilon_j] = E[\varepsilon_i] E[\varepsilon_j]$ or equivalently $E[\varepsilon_i \varepsilon_j - \varepsilon_i \cdot E[\varepsilon_j]] = 0$.

The above can be written as $E[\varepsilon_i (\varepsilon_j - E[\varepsilon_j])]$.

Now consider (and WLOG take $j < i$) (can swap i w/ j)

$$E[\varepsilon_i (\varepsilon_j - E[\varepsilon_j]) \mid F_{i-1}]$$

$$= (\varepsilon_j - E[\varepsilon_j]) \cdot \underbrace{E[\varepsilon_i \mid F_{i-1}]}_{= 0}$$

(since $j < i$, $\varepsilon_j - E[\varepsilon_j]$ is F_{i-1} measurabe)

$$= 0$$

Taking the expectation now on both sides yields

$$E[\varepsilon_i (\varepsilon_j - E[\varepsilon_j])] = 0$$

which by earlier discussion yields the desired result.

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4) This is equivalent to computing the extinction probability of the branching process, so if $\phi(s) = \sum_{K \geq 0} p_K s^K$, we need to solve

$$p = \phi(p)$$

$$p = \frac{1}{8} + \frac{3}{8}p + \frac{3}{8}p^2 + \frac{1}{8}p^3$$

$$\leftrightarrow \quad p^3 + 3p^2 - 5p + 1 = 0$$

$$\leftrightarrow \quad (p-1)(p^2+4p-1) = 0$$

$$p = 1 \qquad\qquad p = -2 \pm \sqrt{5}$$

However $p=1$ is non-sensical, and $p = -2-\sqrt{5} < 0$ is also not possible, hence $p = \sqrt{5} - 2 > 0$.

If each family has exactly 2 children, then the above formula becomes

$$p = \phi(p)$$

$$p = \frac{1}{4} + \frac{1}{2}p + \frac{1}{4}p^2 \quad \left.\right\} \begin{array}{l} p_0 = 1/4 \\ p_1 = 1/2 \\ p_2 = 1/4 \end{array}$$

$$\leftrightarrow \quad p^2 - 2p + 1 = 0$$

$$\leftrightarrow \quad (p-1)^2 = 0$$

So $p = 1$ is the only solution, hence the family name will die with probability 1.