AMATH 567 FALL 2024
HOMEWORK 7 — DUE NOVEMBER 11 ON GRADESCOPE BY 1:30PM

All solutions must include significant justification to receive full credit. If you handwrite your assignment you must either do so digitally or if it is written on paper you must *scan* your work. A standard photo is not sufficient.

If you work with others on the homework, you must name your collaborators.

1: From A&F: 3.5.1 b, c, d (Only consider singularities in the finite complex plane)

2: From A&F: 3.5.3 a, c, d

3: Introducing the Gamma function: Do A&F: 3.6.6. This is the same Gamma function you may have seen defined as

$$\Gamma(z) = \int_0^\infty t^{z-1}e^{-t}dt$$

This better known representation is only valid for $\mathrm{Re}(z) > 0$. The representation given here is valid in all of $\mathbb{C}$. It takes a bit of work to show that our representation is an analytic continuation of the integral representation (this requires the Dominated Convergence Theorem), but it is quite doable. Not now though.

4: Consider a sequence of numbers $(a_n)_{n\geq 0}$ such that $|a_n| < 1$ and

$$\sum_{n=0}^{\infty}(1 - |a_n|) < \infty.$$

Define a Blaschke factor

$$B(a, z) = \begin{cases} \frac{|a|}{a}\frac{a-z}{1-\bar{a}z} & a \neq 0, \\ z & a = 0. \end{cases}$$

- Show that

$$H(z) = \prod_{n=0}^{\infty} B(a_n, z),$$

defines an analytic function in the open unit disk $|z| < 1$.
- Show that $H(z)$ has zeros at $z = a_n$ for every n. It might seem that this construction of an analytic function with an infinite number of zeros in a bounded region implies that $H(z) = 0$ for all z. Why is this not the case?

5: We define the Weierstrass $\wp$-function as

$$\wp(z) = \frac{1}{z^2} + \sum_{j,k=-\infty}^{\infty}\left(\frac{1}{(z - j\omega_1 - k\omega_2)^2} - \frac{1}{(j\omega_1 + k\omega_2)^2}\right),$$

where $(j,k)=(0,0)$ is excluded from the double sum. Also, you may assume that ω_1 is a positive real number, and that ω_2 is on the positive imaginary axis. All considerations below are meant for the entire complex plane, except the poles of $\wp(z)$.

(a) Show that $\wp(z+M\omega_1+N\omega_2)=\wp(z)$, for any two integers M, N. In other words, $\wp(z)$ is a doubly-periodic function: it has two independent periods in the complex plane. Doubly periodic functions are called elliptic functions.

(b) Establish that $\wp(z)$ is an even function: $\wp(-z)=\wp(z)$.

(c) Find Laurent expansions for $\wp(z)$ and $\wp'(z)$ in a neighborhood of the origin in the form

$$\wp(z)=\frac{1}{z^2}+\alpha_0+\alpha_2 z^2+\alpha_4 z^4+\dots$$

and

$$\wp'(z)=-\frac{2}{z^3}+\beta_1 z+\beta_3 z^3+\dots$$

Give expressions for the coefficients introduced above.

(d) Show that $\varphi(z)$ satisfies the differential equation

$$(\wp')^2=a\wp^3+b\wp^2+c\wp+d,$$

for suitable choices of a, b, c, d. Find these constants. You may need to invoke Liouville's theorem to obtain this final result. It turns out that the function $\varphi(z)$ is determined by the coefficients c and d, implying that it is possible to recover ω_1 and ω_2 from the knowledge of c and d.