AMATH 569 Homework 1

Due: Monday, April 7 at Midnight.

This is a warm-up homework. Please submit your solutions to Gradescope. Typesetting your assignments using Latex is highly recommended but not required. If handing in written solutions please make sure they are readable. Show your work and describe all logical steps. If you are using other results or inequalities make sure to mention them clearly.

(1) (Taylor's theorem) Suppose $f : \mathbb{R} \to \mathbb{R}$ is smooth. Prove that for any integer $k \geq 0$, it holds that

$$f(x) = \sum_{i=0}^{k} \frac{f^{(i)}(0)}{i!} x^i + \frac{f^{(k+1)}(\xi)}{(k+1)!} x^{k+1},$$

for some point $\xi \in [0, x]$.

(2) (Hölder's inequality)

(a) Prove Young's inequality for products: Suppose $a, b \geq 0$ and $p, q > 1$ satisfy $1/p + 1/q = 1$. Then it holds that

$$ab \leq \frac{a^p}{p} + \frac{b^q}{q},$$

with equality holding if and only if (iff) $a^p = b^q$. *Hint: disintegrate the functions on the right hand side and think about their graphs.*

(b) Use this inequality to prove Hölder's inequality: For two smooth functions $f, g : [0, 1] \to \mathbb{R}$ and numbers $p, q \in (0, +\infty)$ satisfying $1/p + 1/q = 1$ [1] we have the inequality

$$\int_0^1 |f(x)g(x)| \mathrm{d}x \leq \left(\int_0^1 |f(x)|^p \mathrm{d}x \right)^{1/p} \left(\int_0^1 |g(x)|^q \mathrm{d}x \right)^{1/q}.$$

(3) (from Renardy and Rogers) Consider the boundary-value problem

$$\begin{cases} u_{xx}(x) - \lambda u(x) = 0, & x \in (0, 1), \\ u(0) = u_x(1) = 0. \end{cases}$$

with a parameter $\lambda \geq 0$. Show that

(a) Whenever $\lambda = \lambda_n = ((2n + 1)^2 \pi^2)/4$ for $n = 1, 2, 3, \ldots$, then the boundary-value problem has a family of solutions of the form

$$u_n(x) = A \sin\left(\frac{(2n + 1)\pi}{2} \right),$$

with a free parameter $A \in (-\infty, +\infty)$.

(b) For all other values of λ the only solution of the boundary-value problem is the trivial solution $u(x) = 0$.

[1] we often say p, q are "Hölder conjugates" when this relationship holds.

(4) Use separation of variables to solve the following problem:

$$\begin{cases} u_{xx} + u_{yy} = 0, \quad (x, y) \in (0,1) \times (0,1), \\ u(0, y) = u(1, y) = 0, \quad u(x, 0) = u(x, 1) = \sin(2\pi x). \end{cases}$$