AMATH 569 Homework 5

Due: Monday, June 2 at Midnight.

Please submit your solutions to Gradescope. Typesetting your assignments using Latex is highly recommended but not required. If handing in written solutions please make sure they are readable. Show your work and describe all logical steps. If you are using other results or inequalities make sure to mention them clearly. You may also acknowledge collaborations with your classmates.

Note: The use of AI models such as ChatGPT is strictly prohibited and will result in a report to UW Academic Misconduct without warning.

1. (Evans) Consider a uniformly elliptic differential operator

$$\mathcal{L} = -\sum_{i,j=1}^{n} (a_{ij} u_{x_i})_{x_j} + cu$$

with coefficients $a_{i,j}, c \in L_\infty(\Omega)$ for a bounded open set Ω. Show that there exists a constant $\mu > 0$ such that the corresponding bilinear form $B(\cdot,\cdot)$ satisfies the conditions of Lax-Milgram for $c(x) \geq -\mu$ for all $x \in \Omega$.

2. (Evans) Consider the biharmonic equation

$$\begin{aligned} \Delta^2 u = f \quad &\text{in } \Omega, \\ u = \partial_n u \quad &= 0 \quad \text{on } \partial\Omega, \end{aligned}$$

where $\partial_n u$ is the normal derivative of u on the boundary. We say that u is a weak solution of the biharmonic equation if

$$\int_\Omega \Delta u \Delta v = \int_\Omega f v$$

for all $v \in H_0^2(\Omega)$. Prove that there exists a unique weak solution for $f \in L^2(\Omega)$.

3. We say that a function $u \in H_0^1(\Omega)$ is an eigenfunction of the Dirichlet Laplacian if

$$\int_\Omega \nabla u \cdot \nabla v = \lambda \int_\Omega uv \qquad \forall v \in H_0^1(\Omega).$$

with eigenvalue $\lambda > 0$. Assuming such an eigenpair exists, show that $1/C_\Omega \leq \lambda$ where C_Ω is the Poincaré constant for Ω.

4. Prove the following Fourier identities for $u, v \in \mathcal{S}(\mathbb{R}^n)$ and a multi-index α:

- $\mathcal{F}[D^\alpha u] = (i\xi)^\alpha \mathcal{F}[u]$
- $D^\alpha \mathcal{F}[u] = \mathcal{F}[(-ix)^\alpha u]$
- $\mathcal{F}[u+v] = \mathcal{F}[u] + \mathcal{F}[v]$
- $\mathcal{F}[uv] = (2\pi)^{-n/2} \mathcal{F}[u] * \mathcal{F}[v]$
- $\mathcal{F}[u * v] = (2\pi)^{n/2} \mathcal{F}[u] \cdot \mathcal{F}[v]$

5. Show that Fourier transforms of polynomials are linear combinations of the derivatives of the Dirac delta distribution. That is, if $u = \sum_{|\alpha|\le m} a_\alpha x^\alpha$ is a polynomial of degree m, then $\mathcal{F}[u] = \sum_{|\alpha|\le m} c_\alpha D^\alpha \delta(\xi)$, where $\delta(\xi)$ is the Dirac delta distribution and c_α are constants that depend on a_α.

6. (R&R) Compute the Fourier transforms of the following functions:

- $u(x) = \exp(-\|x\|^2), \quad x \in \mathbb{R}^n$
- $u(x) = 1/(1 + |x|^2), \quad x \in \mathbb{R}$
- $u(x) = \frac{\sin(x)}{1+x^2}, \quad x \in \mathbb{R}$

7. (R&R) Use Fourier transforms to compute the fundamental solution of the Laplace equation in $\mathbb{R}^n$, i.e., $\Delta G = \delta$.