1) a) The method can be re-written like

$$0 \cdot y_K - 1 y_{K+1} + 1 \cdot y_{K+2} = h\left(b_0 f_K + b_1 f_{K+1} + b_2 f_{K+2}\right)$$

So that with $m = 2$, $\begin{bmatrix} a_0 = 0 \\ a_1 = -1 \\ a_2 = 1 \end{bmatrix}$. As the method is third order, we must satisfy the theorem for $p = 3$, meaning $j \in \{1, 2, 3\}$, leading to the following system of equations

$$\begin{cases} a_1 + 2a_2 = b_0 + b_1 + b_2, & j = 1 \\ a_1 + 4a_2 = 2b_1 + 4b_2, & j = 2 \\ a_1 + 8a_2 = 3b_1 + 12b_2, & j = 3 \end{cases}$$

$$\downarrow$$

$$\begin{cases} 1 = b_0 + b_1 + b_2 & (1) \\ 3 = 2b_1 + 4b_2 & (2) \\ 7 = 3b_1 + 12b_2 & (3) \end{cases}$$

(3) - (2)

$\rightarrow \quad 4 = b_1 + 8b_2 \rightarrow b_1 = 4 - 8b_2 \rightarrow b_1 = 4 - \frac{40}{12} = \frac{8}{12}$

(3) $\rightarrow \quad 3 = 8 - 12b_2 \rightarrow b_2 = 5/12$

(1) $\rightarrow \quad b_0 + 8/12 + 5/12 = 1 \rightarrow b_0 = -1/12$

So, $\boxed{b_0 = -1/12, \quad b_1 = 2/3, \quad b_2 = 5/12}$

(b) Using Newton's form, our polynomial takes the form

$$p(s) = a + b(s - t_{K+1}) + c(s - t_{K+1})(s - t_{K+2})$$

Where we can find a, b, c with the divided difference table

$$\begin{array}{llll}
t_{K+1} & f_{K+1} & & \\[2ex]
t_{K+2} & f_{K+2} & \dfrac{f_{K+2} - f_{K+1}}{t_{K+2} - t_{K+1}} = \dfrac{f_{K+2} - f_{K+1}}{h} & \\[3ex]
t_K & f_K & \dfrac{f_K - f_{K+2}}{t_K - t_{K+2}} = \dfrac{f_{K+2} - f_K}{2h} & \dfrac{\dfrac{f_{K+2} - f_K}{2h} - \dfrac{f_{K+2} - f_{K+1}}{h}}{-h}
\end{array}$$

So that $a = f_{K+1}$, $b = \dfrac{f_{K+2} - f_{K+1}}{h}$, $c = \dfrac{f_{K+2} - 2f_{K+1} + f_K}{2h^2}$

we then have

$$y_{K+2} = y_{K+1} + \int_{t_{K+1}}^{t_{K+2}} a + b(s - t_{K+1}) + c(s - t_{K+1})(s - t_{K+2}) \, ds$$

$$= y_{K+1} + a(t_{K+2} - t_{K+1}) + \frac{b}{2}(s - t_{K+1})^2 \Big|_{t_{K+1}}^{t_{K+2}}$$

$$+ c\left(\frac{s^3}{3} - \frac{s^2}{2}(t_{K+1} + t_{K+2}) + t_{K+1} t_{K+2} s\right) \Big|_{t_{K+1}}^{t_{K+2}}$$

$$= y_{k+1} + ah + \frac{b}{2}\left[(\overbrace{t_{k+2} - t_{k+1}}^{h})^2\right] + c\left[\frac{1}{3}(t_{k+2}^3 - t_{k+1}^3)\right.$$

$$\left. - \frac{1}{2}(t_{k+1} + t_{k+2})(t_{k+2}^2 - t_{k+1}^2) + t_{k+1}t_{k+2}(\overbrace{t_{k+2} - t_{k+1}}^{h})\right]$$

$$= y_{k+1} + ah + \frac{b}{2}h^2 + c\left[\frac{1}{3}(\overbrace{t_{k+2} - t_{k+1}}^{h})(t_{k+2}^2 + t_{k+2}t_{k+1} + t_{k+1}^2)\right.$$

$$\left. - \frac{1}{2}(t_{k+1} + t_{k+2})^2 h + t_{k+1}t_{k+2}h\right]$$

$$= y_{k+1} + ah + \frac{b}{2}h^2 + ch\left[\frac{1}{3}t_{k+2}^2 + \frac{1}{3}t_{k+2}t_{k+1} + \frac{1}{3}t_{k+1}^2\right.$$

$$\left. - \frac{1}{2}t_{k+2}^2 - \cancel{t_{k+1}t_{k+2}} - \frac{1}{2}t_{k+1}^2 + \cancel{t_{k+1}t_{k+2}}\right]$$

$$= y_{k+1} + ah + \frac{b}{2}h^2 + ch\left[-\frac{1}{6}t_{k+2}^2 + \frac{1}{3}t_{k+2}t_{k+1} - \frac{1}{6}t_{k+1}^2\right]$$

$$= y_{k+1} + ah + \frac{b}{2}h^2 - \frac{ch}{6}\left[\overbrace{(t_{k+2} - t_{k+1})^2}^{h^2}\right]$$

$$= y_{k+1} + ah + \frac{b}{2}h^2 - \frac{c}{6}h^3$$

Now substituting for a, b, c, we have

$$= y_{k+1} + hf_{k+1} + \frac{f_{k+2} - f_{k+1}}{2\cancel{h}}h^{\cancel{2}} - \frac{f_{k+2} - 2f_{k+1} + f_k}{12\cancel{h^2}}h^{\cancel{3}}$$

$$= y_{k+1} + h\left[\frac{12f_{k+1} + 6f_{k+2} - 6f_{k+1} - f_{k+2} + 2f_{k+1} - f_k}{12}\right]$$

$$= y_{k+1} + h\left[-\frac{1}{12} f_k + \frac{2}{3} f_{k+1} + \frac{5}{12} f_{k+2}\right]$$

So just as in (a), $\boxed{b_0 = -1/12\ ,\ b_1 = 2/3\ ,\ b_2 = 5/12}$

2) (a) By re-indexing, we can write the method as

$$y_{k+2} - y_k = h\left[f_{k+2} - 3f_{k+1} + 4f_k\right]$$

So that $[a_0 = -1,\ a_1 = 0,\ a_2 = 1]$ and $[b_0 = 4,\ b_1 = -3,\ b_2 = 1]$

The characteristic polynomial is

$$\lambda^2 - 1 := 0$$

$$\leftrightarrow \quad \lambda = \pm 1$$

So as each λ is distinct and $|\lambda| \leq 1$, the method is <u>zero stable</u>. Now we check the order of convergence,

$$a_0 + a_1 + a_2 = -1 + 0 + 1 = 0 \quad \checkmark$$

$$\overset{a_0}{0 \cdot (-1)} + \overset{a_1}{1(0)} + \overset{a_2}{2(1)} = 2 = \overset{4}{b_0} + \overset{-3}{b_1} + \overset{1}{b_2} = 2 \qquad p = 1$$

$$0^2 \cdot (-1) + 1^2 \cdot 0 + 2^2 \cdot 1 = 4 \neq 2(-3+2) = -2 \qquad p = 2$$

So by the Dahlquist Equivalence Theorem (DET), this method is <u>convergent</u> with <u>LTE $O(h)$</u>.

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(b) Again re-indexing, we have

$$y_{k+2} - 2y_{k+1} + y_k = h\left(f_{k+2} - f_{k+1}\right)$$

So that $[a_0 = 1, a_1 = -2, a_2 = 1]$ and $[b_0 = 0, b_1 = -1, b_2 = 1]$

The characteristic polynomial is

$$\lambda^2 - 2\lambda + 1 := 0$$

$$\leftrightarrow \quad (\lambda - 1)^2 = 0 \quad \leftrightarrow \quad \lambda = 1, \text{ w/ mult} = 2$$

So both roots have magnitude 1, and fails the root condition, so that this method is <u>not convergent</u>, by the (DET). We now find the LTE, we have

$$1 + (-2) + 1 = 0 \quad \checkmark$$

$$0 \cdot 1 + 1 \cdot (-2) + 2 \cdot 1 = 0 = 0 + (-1) + 1 \quad \checkmark \qquad p = 1$$

$$0^2 \cdot 1 + 1^2 \cdot (-2) + 2^2 \cdot 1 = 2 = 2(-1 + 2) = 2 \quad \checkmark \qquad p = 2$$

$$0^3 \cdot 1 + 1^3 \cdot (-2) + 2^3 \cdot 1 = 6 \neq 3(-1 + 4) = 9 \quad \times \qquad p = 3$$

So the <u>LTE is $O(h^2)$</u>.

(c) Again re-indexing, we have

$$y_{k+2} - y_{k+1} - y_k = h\left[f_{k+2} - f_{k+1}\right]$$

So that $[a_0 = -1, a_1 = -1, a_2 = 1]$ and $[b_0 = 0, b_1 = -1, b_2 = 1]$.

However, notice that $a_0 + a_1 + a_2 = -1 \neq 0$. So this method cannot have LTE $O(h^p)$ for any $p \geq 1$, and immediately fails the (DET) so that this method is <u>not convergent</u>.

3) a) I used the forward Euler method because I figured it would be good to try. To back up my choice, I compared the numerical derivative of the computed solution with the real dy/dt and $d\alpha/dt$ with the L2 norm. The approximation seems very accurate except at the "cusps" of the oscillations as indicated by the figures on the following pages. I needed to have h be small in order to converge to this solution, if it was too large, the solutions blow up.

b) The Jacobian is $\begin{bmatrix} 1+h(x_{k+1}^2-1) & -h \\ \varepsilon h & 1 \end{bmatrix}$.

which is found by the system

$$0 = x_{k+1} - x_k - h\left(-\frac{x_{k+1}^3}{3} + x_{k+1} + \alpha_{k+1}\right)$$

$$0 = \alpha_{k+1} - \alpha_k + h\varepsilon x_{k+1} .$$

I was able to use larger step values and still get a good solution.

4) we send the method through the test equation

$y' = \lambda y$,

$$\rightarrow \quad y_{k+1} = y_k + h\lambda \left(\frac{y_k + y_{k+1}}{2} \right)$$

$$\rightarrow \quad y_{k+1} - \frac{h\lambda}{2} y_{k+1} = y_k + \frac{h\lambda}{2} y_k$$

$$\rightarrow \quad y_{k+1} \left(1 - \frac{h\lambda}{2} \right) = y_k \left(1 + \frac{h\lambda}{2} \right)$$

$$\rightarrow \quad y_{k+1} = \frac{1 + \frac{h\lambda}{2}}{1 - \frac{h\lambda}{2}} y_k$$

So if $z = h\lambda$, the region of absolute stability is defined by

$$\left| \frac{1 + \frac{z}{2}}{1 - \frac{z}{2}} \right| \leq 1$$

$$\leftrightarrow \quad \frac{\left| 1 + \frac{1}{2}x + \frac{1}{2}iy \right|}{\left| 1 - \frac{1}{2}x - \frac{1}{2}iy \right|} \leq 1$$

$$\frac{\sqrt{(1 + \frac{1}{2}x)^2 + (\frac{1}{2}y)^2}}{\sqrt{(1 - \frac{1}{2}x)^2 + (-\frac{1}{2}y)^2}} \leq 1$$

$$\leftrightarrow \quad (1+\tfrac{1}{2}x)^2 + \cancel{\tfrac{1}{4}y^2} \le (1-\tfrac{1}{2}x)^2 + \cancel{\tfrac{1}{4}y^2}$$

$$\leftrightarrow \quad |1+\tfrac{1}{2}x| \le |1-\tfrac{1}{2}x|$$

which is true when $x \le 0$. As $x = Re(z)$, the region of absolute stability is the entire left half plane, hence the method is A-stable.

5) We have

$$|y_{k+1} - w^{j+1}| = \left| \frac{h\lambda}{2} (y_{k+1} - w^j) \right|$$

$$= \left| \frac{h\lambda}{2} \right| |y_{k+1} - w^j|$$

This implies convergence to y_{k+1} immediately if this a contraction, ie if

$$\left| \frac{h\lambda}{2} \right| < 1 \iff \boxed{|h\lambda| < 2.}$$