AMATH 567

HW 5

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1) Suppose f and g are two entire functions with no zeroes. Then the function $h(z) = \frac{f(z)}{g(z)}$ has no singularities and is also entire. Since h is analytic on $\mathbb{C}$, it is continuous, hence bounded on any compact subset of $\mathbb{C}$ excluding the point at ∞. However we know that at $z = \infty$, $h(z) = 1$, hence it is also bounded at ∞. So h is entire and bounded (including at ∞), so by Liouville's Theorem h is constant, say $h(z) = \frac{f(z)}{g(z)} = C$ for some $C \in \mathbb{C}$, so

$$f(z) = C\, g(z) \quad \forall\, z \in \mathbb{C} \cup \{\infty\}$$

So at $z = \infty$, $C = 1$, implying $f(z) = g(z)$ $\forall\, z \in \mathbb{C} \cup \{\infty\}$.

2) (2.6.10) Cauchy's Integral formula says

$$f(z) = \frac{1}{2\pi i} \oint_C \frac{f(w)}{w - z} \, dw$$

Put $w = e^{i\theta}$, then $dw = ie^{i\theta} \, d\theta$, so the above

$$= \frac{1}{2\pi i} \int_0^{2\pi} \frac{f(w)}{w-z} \cdot i \underbrace{e^{i\theta}}_{w} \, d\theta$$

$$= \frac{1}{2\pi} \int_0^{2\pi} \frac{f(w) \cdot w}{w-z} \, d\theta$$

If z lies inside the circle, then $|z| < 1$ and $|\bar{z}| < 1$, so $\frac{1}{|\bar{z}|} > 1$, meaning $\frac{1}{\bar{z}}$ is outside the unit circle, so if $z := \frac{1}{\bar{z}}$, and $f(z)$ is analytic, we get from Cauchy's theorem that

$$0 = \frac{1}{2\pi} \int_0^{2\pi} \frac{f(w) \cdot w}{w - \frac{1}{\bar{z}}} \, d\theta$$

Now put $w = 1/\bar{w}$, then

$$f(z) = \frac{1}{2\pi} \int_0^{2\pi} \frac{f(1/\bar{w}) \, \frac{1}{\bar{w}}}{\frac{1}{\bar{w}} - z} \, d\theta$$

$$= \frac{1}{2\pi} \int_0^{2\pi} \frac{f(1/\bar{\omega})\,\frac{1}{\bar{\omega}}}{\frac{1}{\bar{\omega}} - \frac{1}{\bar{z}}}\, d\theta \quad \text{with } z := \frac{1}{\bar{z}}$$

on C, $\frac{1}{\bar{\omega}} = \omega$

$$= \frac{1}{2\pi} \int_0^{2\pi} \frac{f(1/\bar{\omega})\,\bar{z}}{\bar{z} - \bar{\omega}}\, d\theta = \frac{1}{2\pi} \int_0^{2\pi} \frac{f(\omega)\,\bar{z}}{\bar{z} - \bar{\omega}}\, d\theta$$

which should equal 0 as stated previously since $z := \frac{1}{\bar{z}}$. We have

$$f(z) = \frac{1}{2\pi} \int_0^{2\pi} \frac{f(\omega)\cdot\omega}{\omega - z}\, d\theta \pm 0$$

just adding or subtracting 0 doesn't change the sum

$$= \frac{1}{2\pi} \int_0^{2\pi} \frac{f(\omega)\,\omega}{\omega - z}\, d\theta \pm \frac{1}{2\pi} \int_0^{2\pi} \frac{f(\omega)\,\bar{z}}{\bar{z} - \bar{\omega}}\, d\theta$$

$$= \frac{1}{2\pi} \int_0^{2\pi} f(\omega) \left[\frac{\omega}{\omega - z} \pm \frac{\bar{z}}{\bar{\omega} - \bar{z}} \right] d\theta$$

where we have swapped the denominator to match the problem which we can do b/c of the $\pm$

Using the $(+)$ sign, and combining the fractions we get

$$= \frac{1}{2\pi} \int_0^{2\pi} \frac{(\omega(\bar{\omega} - \bar{z}) + \bar{z}(\omega - z))\, f(\omega)}{|\omega - z|^2}\, d\theta$$

$$= \frac{1}{2\pi}\int_0^{2\pi} \frac{(\overset{1}{w\bar{w}} - w\bar{z} + w\bar{z} - z\bar{z})\, f(w)}{|w-z|^2}\, d\theta$$

$$= \frac{1}{2\pi}\int_0^{2\pi} f(w)\left[\frac{1-|z|^2}{|w-z|^2}\right] d\theta \quad \left(\begin{matrix} w\bar{w}=1 & \text{since} \\ w \in C & \end{matrix}\right)$$

as desired.

a) If $z = re^{i\phi}$ and $w = e^{i\theta}$, the above equals (using Euler's formula)

$$f(z) = \frac{1}{2\pi}\int_0^{2\pi} f(w)\left[\frac{1-r^2}{(\cos\theta - r\cos\phi)^2 + (\sin\theta - r\sin\phi)^2}\right] d\theta$$

Focusing on the denominator, that can be simplified to

$$\cos^2\theta - 2r\cos\theta\cos\phi + r^2\cos^2\phi + \sin^2\theta - 2r\sin\theta\sin\phi + r^2\sin^2\phi$$

$$= r^2 + 1 - 2r(\cos\theta\cos\phi + \sin\theta\sin\phi)$$

$$= 1 - 2r\cos(\phi - \theta) + r^2$$

Also,

$$f(w) = u(w) + i\,v(w)$$

$$= u(r=1, \theta=\theta) + i\,v(r=1, \theta=\theta)$$

So $\operatorname{Re}(f(w)) = u(1,\theta) = u(\theta)$

Substituting these expressions into the integral we get

this is a real number

$$Re(f(z)) = \frac{1}{2\pi}\int_0^{2\pi} Re(f(w)) \cdot \left[\frac{1-r^2}{1-2r\cos(\phi-\theta)+r^2}\right] d\theta$$

$$= \frac{1}{2\pi}\int_0^{2\pi} u(\theta)\left[\frac{1-r^2}{1-2r\cos(\phi-\theta)+r^2}\right] d\theta$$

as desired.

b) Using the minus sign

$$f(z) = \frac{1}{2\pi}\int_0^{2\pi} f(w)\left[\frac{w}{w-z} - \frac{\bar{z}}{\bar{w}-\bar{z}}\right] d\theta$$

$$w\bar{w} - wz - \bar{z}w + z\bar{z}$$

sorry for jamming this here

$$= \frac{1}{2\pi}\int_0^{2\pi} f(w)\left[\frac{w(\bar{w}-z) - \bar{z}(w-z)}{|w-z|^2}\right] d\theta$$

$$e^{i\theta}(re^{i\phi} - re^{-i\phi}) = re^{i\theta}e^{-i\phi}(2)$$

$$= \frac{1}{2\pi}\int_0^{2\pi} f(w)\left[\frac{1-w(z-\bar{z})+r^2}{1-2r\cos(\phi-\theta)+r^2}\right] d\theta$$

$$= \frac{1}{2\pi}\int_0^{2\pi} f(w)\left[\frac{1+r^2-2re^{i(\theta-\phi)}}{1-2r\cos(\phi-\theta)+r^2}\right] d\theta$$

So $\hookrightarrow$ $$Im(z) = \frac{1}{2\pi}\int_0^{2\pi} Im\left[f(w)\left(\quad \right)\right] d\theta$$

$$= \frac{1}{2\pi}\int_0^{2\pi} \frac{1}{1-2r\cos(\phi-\theta)+r^2} \operatorname{Im}\left(f(w)\left(1+r^2-2re^{i(\theta-\phi)}\right)\right) d\theta$$

$$= \frac{1}{2\pi}\int_0^{2\pi} \frac{(1+r^2)\overbrace{\operatorname{Im}(f(w))}^{v(\theta)} + \overbrace{\operatorname{Im}\left(f(w)\cdot -2re^{i(\theta-\phi)}\right)}^{(*)}}{1-2r\cos(\phi-\theta)+r^2}\, d\theta$$

We have

$$(*)\ \operatorname{Im}\left((u+iv)\cdot\left(-2r\cos(\theta-\phi) - i2r\sin(\theta-\phi)\right)\right)$$

$$= \operatorname{Im}\left(-2r\cos(\theta-\phi)u - i2ru\sin(\theta-\phi) - i2rv\cos(\theta-\phi) + 2vr\sin(\theta-\phi)\right)$$

$$= -2ru\sin(\theta-\phi) - 2rv\cos(\theta-\phi)$$

So the integral becomes

$$= \frac{1}{2\pi}\int_0^{2\pi} \frac{(1+r^2)v(\theta)}{1-2r\cos(\phi-\theta)+r^2} d\theta \,-\, \frac{1}{2\pi}\int \frac{2r\overset{v(\theta)}{v}\cos(\theta-\phi)}{1-2r\cos(\phi-\theta)+r^2} d\theta$$

$$- \frac{1}{\cancel{2}\pi}\int \frac{\cancel{2}r\sin(\theta-\phi)u(\theta)}{1-2r\cos(\phi-\theta)+r^2} d\theta$$

cos is even so $\cos(\theta-\phi) = \cos(\phi-\theta)$

$$= \frac{1}{2\pi}\int_0^{2\pi} \frac{\cancel{(1-2r\cos(\theta-\phi)+r^2)}}{\cancel{(1-2r\cos(\theta-\phi)+r^2)}} v(\theta)\, d\theta - \frac{1}{\pi}\int \frac{r\sin(\theta-\phi)u(\theta)}{1-2r\cos(\theta-\phi)+r^2} d\theta$$

sin is odd so $\sin(\theta-\phi) = -\sin(\phi-\theta)$

$$= \frac{1}{2\pi}\int_0^{2\pi} v(\theta)\, d\theta + \frac{1}{\pi}\int_0^{2\pi} \frac{r\sin(\phi-\theta)}{1-2r\cos(\phi-\theta)+r^2} u(\theta)\, d\theta$$

so

$$\operatorname{Im}(f(z)) = v(r,\phi) = C + \frac{1}{\pi}\int \frac{r\sin(\phi-\theta)}{1-2r\cos(\phi-\theta)+r^2}\,u(\theta)\,d\theta$$

where $C = \frac{1}{2\pi}\int_0^{2\pi} v(\theta)\,d\theta$ as desired,

$\qquad\qquad\qquad\qquad \underbrace{\shortparallel\ v(r=0)}$

(I explain this last step at the end of prob) here

c) We have

we computed the denom earlier

$$\frac{w+z}{w-z} = \frac{e^{i\theta}+re^{i\phi}}{e^{i\theta}-re^{i\phi}} = \frac{(\cos\theta+r\cos\phi)+i(\sin\theta+r\sin\phi)}{(\cos\theta-r\cos\phi)+i(\sin\theta-\sin\phi)}$$

$$= \frac{[(\cos\theta+r\cos\phi)+i(\sin\theta+r\sin\phi)][(\cos\theta-r\cos\phi)-i(\sin(\theta)-r\sin\phi)]}{1+r^2-2r\cos(\phi-\theta)}$$

we multiplied by the conjugate of the denom on top and bottom

Focusing on the numerator, we get 2 terms with imaginary components

$$i(\sin\theta+r\sin\phi)(\cos\theta-r\cos\phi) - i(\sin\theta-r\sin\phi)(\cos\theta+r\cos\phi)$$

$$= i\,[\cancel{\sin\theta\cos\theta} + r\sin\phi\cos\theta - r\sin\theta\cos\phi - \cancel{r^2\sin\phi\cos\phi} - \cancel{\sin\theta\cos\theta} + r\sin\phi\cos\theta - r\sin\theta\cos\phi + \cancel{r^2\sin\phi\cos\phi}]$$

$$= 2ir\,[\sin\phi\cos\theta - \sin\theta\cos\phi]$$

$$= 2ir\sin(\phi-\theta)$$

So the imaginary part

$$\operatorname{Im}\left(\frac{w+z}{w-z}\right) = \frac{2r\sin(\phi-\theta)}{1-2r\cos(\phi-\theta)+r^2}$$

as desired. Hence our result from part (b) may be expressed as

$$V(r,\phi) = \underset{\substack{\parallel \\ v(0)}}{V(r=0)} + \frac{\operatorname{Im}}{2\pi}\left[\int_0^{2\pi} u(\theta)\,\frac{w+z}{w-z}\,d\theta\right]$$

<u>Note:</u> By the first equation in the problem

$$f(z) = \frac{1}{2\pi}\int_0^{2\pi} \frac{f(w)\cdot w}{w-z}\,d\theta \quad , \text{ so if } z=0$$

$$f(0) = \frac{1}{2\pi}\int_0^{2\pi} \frac{f(w)\cdot \cancel{w}}{\cancel{w}}\,d\theta$$

$$= \frac{1}{2\pi}\int_0^{2\pi} u(1,\theta) + i v(1,\theta)\,d\theta$$

So,

$$v(0) = \operatorname{Im}(f(0)) = \frac{1}{2\pi}\int_0^{2\pi} v(\theta)\,d\theta$$

as we have previously claimed.

3) our goal is to use Morera's theorem, so we want to pick $f(z_0)$ so that on Ω f is continuous, and we also want to show $\oint_C f(z)\,dz = 0$ for every simple closed contour C lying in Ω. We can start with the latter. Since f is analytic in any closed contour in Ω not containing z_0, Cauchy's Theorem implies $\oint_C f(z)\,dz = 0$ in this case. If C encloses z_0, we can use Cauchy's Theorem again to deform the contour to $C_\varepsilon(z_0)$, a circle of radius $\varepsilon > 0$ centered at z_0, and it suffices to show that as $\varepsilon \to 0$, $\oint_{C_\varepsilon(z_0)} f(z)\,dz \to 0$. We have

$$\left| \oint_{C_\varepsilon(z_0)} f(z)\,dz \right| \le \oint_{C_\varepsilon(z_0)} |f(z)| \cdot |dz|$$

$$\le \oint_{C_\varepsilon(z_0)} M|z-z_0|^{-\gamma}\,|dz|$$

We can say $|z-z_0| = \varepsilon$ and $\oint_{C_\varepsilon(z_0)} |dz| \le 2\pi\varepsilon$ (the arclength of the contour)

So the above is

$$\leq \varepsilon^{-\gamma} M \oint_{C_\varepsilon(z_0)} |dz|$$

$$\leq \varepsilon^{-\gamma} M \cdot 2\pi\varepsilon = \varepsilon^{1-\gamma} \cdot 2\pi M$$

and as $\varepsilon \to 0$

$$0 \leq \lim_{\varepsilon \to 0} \left| \oint_{C_\varepsilon(z_0)} f(z)\, dz \right| \leq \lim_{\varepsilon \to 0} \varepsilon^{1-\gamma} \cdot 2\pi M$$

$$= 0 \text{ since } \gamma > 1$$

So we get that $\oint_C f(z)\, dz = 0$ for every closed contour in C. We now need to make f continuous. Now consider the following diagram

Any closed contour C in Ω enclosing z_0 can be deformed into something depicted in the diagram. Namely

$$C = \lim_{\varepsilon \to 0} C_\varepsilon(z_0) + L_1(\varepsilon) + L_2(\varepsilon) + C_0(\varepsilon)$$

In the limit, $\int_{L_1(\varepsilon)} f(z)\, dz = -\int_{L_2(\varepsilon)} f(z)\, dz$

So the contributions from the lines cancel, so the integral over $C'(\varepsilon)$ is 0 by Cauchy's theorem as z_0 is not enclosed in $C(\varepsilon)$ and f is analytic here

So in C', cauchy's integral theorem applies to a point z away from z_0, and

define $$f(z_0) = \frac{1}{2\pi i} \oint_{C'(\varepsilon)} \frac{f(w)}{w - z_0}\, dw$$

Then

$$|f(z) - f(z_0)| = \frac{1}{2\pi} \left| \oint_{C'(\varepsilon)} \frac{f(w)}{w-z}\, dz - \oint_{C'(\varepsilon)} \frac{f(w)}{w - z_0}\, dz \right|$$

$$= \frac{1}{2\pi} \left| \oint_{C'(\varepsilon)} f(w) \left[\frac{z - z_0}{(w-z)(w-z_0)} \right] dz \right|$$

$$\leq \frac{1}{2\pi} \left| \oint_{C'(\varepsilon)} |f(w)| \frac{|z - z_0|}{|w-z||w-z_0|} |dz| \right|$$

$$\leq \frac{1}{2\pi} \oint_{C'(\varepsilon)} \frac{M |z - z_0|^{1-\gamma}}{|w-z||w-z_0|} |dz|$$

The distances $|w-z|$ and $|w-z_0|$ are strictly positive, and for any $\varepsilon > 0$ can be bounded below by some constant $M_1(\varepsilon)$ (their product)

so $$\frac{1}{|w-z||w-z_0|} \leq \frac{1}{M_1(\varepsilon)},$$ so the above is

$$\leq \frac{M}{2\pi M_1(\varepsilon)} \oint_{C'(\varepsilon)} |z - z_0|^{1-\gamma} |dz|$$

Take $|z - z_0| = 2\varepsilon$, then (so the bubbles in the diagram never overlap)

$$\leq \frac{M\varepsilon^{1-\gamma}\, 2^{1-\gamma}}{2\pi M_1(\varepsilon)} \oint_{c'(\varepsilon)} |dz|$$

$$= \frac{2^{1-\gamma} M \varepsilon^{1-\gamma}}{2\pi M_1(\varepsilon)} \cdot L(\varepsilon)$$ ✓ arclength of $c'(\varepsilon)$

as $\varepsilon \to 0$ $L(\varepsilon) \to L$ (the arc length of c).

$\frac{1}{M_1(\varepsilon)}$ can be bounded above by a constant M_1 that does not depend on ε, so as $\gamma > 1$ as $\varepsilon \to 0$ we can make $f(z)$ and $f(z_0)$ arbitrarily close to each other, meaning we can define $f(z_0) = \frac{1}{2\pi i} \oint_c \frac{f(z)}{z - z_0} dz$ and our function is continuous with this definition.

Our function is continuous in Ω and $\oint_{\text{Loop}} f(z) \cancel{dz} = 0$ for every closed loop in Ω, so Morera's theorem implies f is analytic on all of Ω with this choice of $f(z_0)$.

4) we have

$$\left| \int_{\Gamma} f(z)\,dz - \int_{\Gamma_n} f(z)\,dz \right|$$

$$= \left| \int_a^b f(z(t))\,z'(t)\,dt - \int_a^b f(z_n(t))\,z_n'(t)\,dt \right|$$

$$= \left| \int_a^b f(z(t))\,z'(t) - \underbrace{f(z_n(t))\,z_n'(t)}_{\text{"}} \,dt \right|$$

$$= \left| \int_a^b f(z(t))\,z'(t) \overbrace{- f(z(t))\,z_n'(t) + f(z(t))\,z_n'(t)} - f(z_n(t))\,z_n'(t)\,dt \right|$$

(where we have left out the t's)

$$= \left| \int_a^b f(z)\,[z' - z_n'] + z_n'\,[f(z) - f(z_n)]\,dt \right|$$

(triangle inequality)

$$\leq \int_a^b |f(z)| \cdot |z' - z_n'| + |z_n'| \cdot |f(z) - f(z_n)|\,dt$$

As $z_n'(t)$ is bounded, say $|z_n'| \leq M_1$.
As f is continuous on $\overline{\Omega}$ (a compact set) it is also bounded, say $|f(z)| \leq M_2$, then the above is

① $\leq M_1(b-a)\sup_{t\in[a,b]}|z'-z_n'| + M_2(b-a)\sup_{t\in[a,b]}|f(z)-f(z_n)|$

As $z_n' \xrightarrow{n\to\infty} z'$ uniformly, we can choose, $\forall \varepsilon>0$, an N_1 such that $\forall\ n \geq N_1$,

$$|z'-z_n'| \leq \varepsilon \quad \forall t\in[a,b]$$

As f is continuous on a compact set $\overline{\Omega}$, f is uniformly continuous on $\overline{\Omega}$.
So $\forall \varepsilon>0$, $\exists\ \delta>0$ such that

$$|z-z_n| \leq \delta \rightarrow |f(z)-f(z_n)| \leq \varepsilon$$

As $z_n \xrightarrow{n\to\infty} z$ uniformly, we can choose $\forall\ \delta>0$ an N_2 such that

$$|z-z_n| \leq \delta \quad \forall t\in[a,b]$$

So for all $n \geq N = \max(N_1, N_2)$,

① $\leq M_1(b-a)\varepsilon + M_2(b-a)\varepsilon$

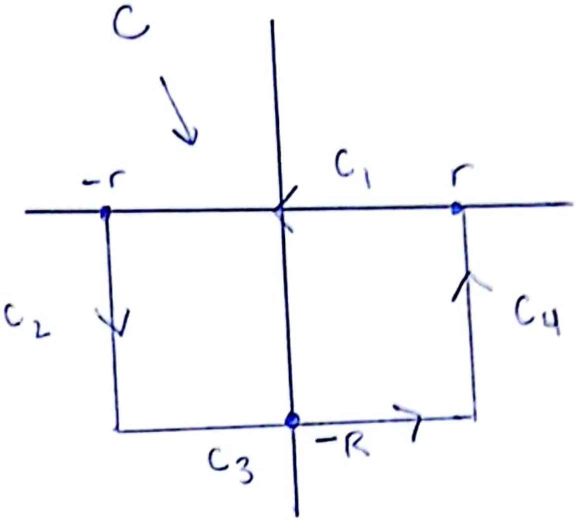
$= (b-a)(M_1+M_2)\varepsilon$

$= C\varepsilon$ for $C=(b-a)(M_1+M_2)$

which establishes that

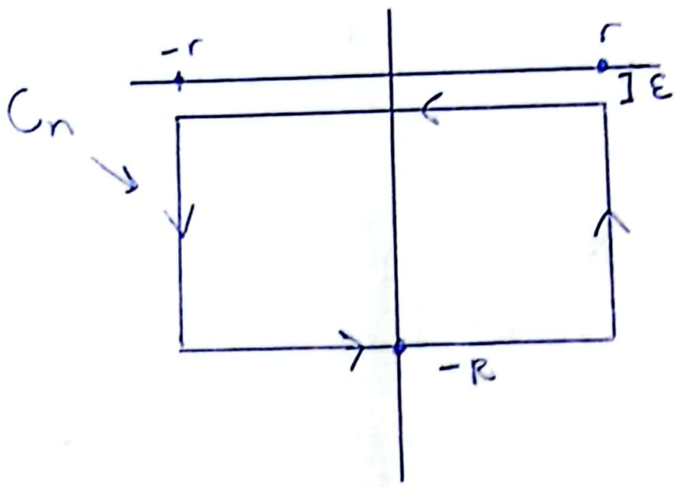
$$\int_{\Gamma_n} f(z)\,dz \longrightarrow \int_{\Gamma} f(z)\,dz$$

5) Consider the contour C defined in the problem for $r, R > 0$



$$z(t) = \begin{cases} r - 2r t & 0 \le t \le 1 \\ i(0 - R(t-1)) & 1 < t \le 2 \\ -r + 2r(t-3) & 2 < t \le 3 \\ i(-R + R(t-4)) & 3 < t \le 4 \end{cases}$$

Now $\forall\, 0 < \varepsilon < \frac{R}{2}$, consider the contour C_ε



$$z_\varepsilon(t) = \begin{cases} (r - 2rt) - i\varepsilon & 0 \le t \le 1 \\ i(-\varepsilon - R(t-1)) & 1 \le t \le 2 \\ -r + 2r(t-3) & 2 \le t \le 3 \\ i(-R + (R-\varepsilon)(t-4)) & 3 < t \le 4 \end{cases}$$

replace epsilon with $1/n$

Define $z_n(t) = z_{\varepsilon = 1/n}(t)$. Then as $n \to \infty$, $z_n(t) \to z(t)$ uniformly and $z'_n(t) \to z'(t)$ uniformly. The function $f(z)\sqrt{z-1}\sqrt{z+1}$ has a branch cut on the line $[-1, 1]$, but is continuous on C_n. By Lemma 1 (problem 4) we have

$$\oint_{C_n} f(z)\sqrt{z-1}\sqrt{z+1}\, dz \xrightarrow{n\to\infty} \oint_{C} f(z)\sqrt{z-1}\sqrt{z+1}\, dz$$

However, for every n, Cauchy's theorem says

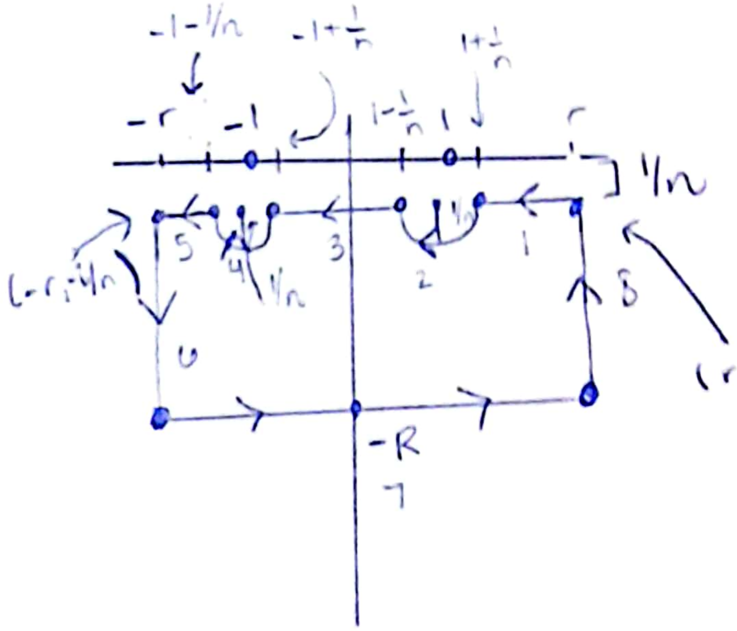
$$\oint_{C_n} f(z)\sqrt{z-1}\,\sqrt{z+1}\,dz = 0$$

as $f(z)\sqrt{z-1}\,\sqrt{z+1}$ is analytic on (and in) each C_n.

So $$\lim_{n\to\infty}\oint_{C_n} f(z)\sqrt{z-1}\sqrt{z+1}\,dz = \lim_{n\to\infty} 0 = 0$$

$$= \oint_{C} f(z)\sqrt{z-1}\sqrt{z+1}\,dz$$ as desired.

For the next part consider the following contour C_n when $\frac{1}{n} < R/2$



$$z_n(t) = \begin{cases} r-(1+\frac{1}{n})t-\frac{i}{n} & 0 \le t \le 1 \\ (1-\frac{1}{n})+\frac{1}{n}e^{-i\pi(t-1)}-i/n & 1 \le t \le 2 \\ (1-\frac{1}{n})-(2-\frac{2}{n})(t-2)-i/n & 2 \le t \le 3 \\ (-1+\frac{1}{n})+\frac{1}{n}e^{-i\pi(t-3)}-i/n & 3 \le t \le 4 \\ (-1-\frac{1}{n})-(-\frac{1}{n}-1-r)(t-4)-\frac{i}{n} & \text{for } 4 \le t \le 5 \\ i(-\frac{1}{n}-R(t-5)) & 5 \le t \le 6 \\ -r+2r(t-6) & 6 \le t \le 7 \\ i(-R+(R-\frac{1}{n})(t-7)) & 7 \le t \le 8 \end{cases}$$

The function $\frac{f(z)}{\sqrt{z-1}\sqrt{z+1}}$ has a branch cut on the line $[-1,1]$ and is not defined at $z=1$ or $z=-1$. By defining C_n the way we have, $z_n(t) \xrightarrow{n\to\infty} z(t)$ uniformly and $z'_n(t) \xrightarrow{n\to\infty} z'(t)$ uniformly (this is very easy to see, please don't hurt me hahaha) and the function $\frac{f(z)}{\sqrt{z-1}\sqrt{z+1}}$ is analytic and defined on C_n $\forall n$ satisfying $\frac{1}{n} < R/2$. So Lemma 1 (problem 4) gives us

$$\oint_{C_n} \frac{f(z)}{\sqrt{z-1}\sqrt{z+1}}\, dz \xrightarrow{n\to\infty} \oint_{C} \frac{f(z)}{\sqrt{z-1}\sqrt{z+1}}\, dz$$

But Cauchy's Theorem says the LHS is 0 for every $n > \frac{2}{R}$, so we get that

$$\oint \frac{f(z)}{\sqrt{z-1}\sqrt{z+1}}\, dz = 0$$ as desired.

<u>Important Note</u>: The contours chosen in this problem were chosen to be contained in the region that contains C, (and Σ).

(a) (3.1.1)

b) If $\alpha \le |z| \le \beta$ and $\alpha > 1$ then

$$\left|\frac{1}{z^n}\right| = \frac{1}{|z|^n} < \left(\frac{1}{\alpha}\right)^n$$

Since $1/\alpha < 1$ and does not depend on z, as $n \to \infty$, $\frac{1}{z^n} \to 0$ uniformly.

If $\alpha < 1$ and $\beta < 1$ then

$$\frac{1}{|z|^n} > \left(\frac{1}{\beta}\right)^n$$

Since $\frac{1}{\beta} > 1$ and does not depend on z, you might say as $n \to \infty$ the limit converges to ∞ uniformly (if this is allowed treating ∞ like any other point)

If $\alpha = \beta = 1$ then

$$\frac{1}{z^n} = \frac{1}{R^n e^{i\theta n}} = \frac{1}{e^{i\theta n}}$$

which only has a limit if θ is a integer multiple of 2π. So if $\alpha < 1$ and $\beta > 1$ the limit will also not converge uniformly.

d) We have

$$|1+n^2z^2| \geq ||1|-|n^2z^2||$$
$$= |1-n^2R^2|$$

So $$\left|\frac{1}{1+(nz)^2}\right| \leq \frac{1}{|1-n^2R^2|}$$

This bound is independent of z, so as $n \to \infty$ $\frac{1}{1+(nz)^2} \to 0$ uniformly for any $\alpha, \beta > 0$

7) (3.1.2)

a) • If $\alpha = 0$, z could be zero, meaning (is always if $|z| = 0$)

$$\frac{1}{z^n} = \frac{1}{0} = \infty \quad (\text{take the limit as } R \to \infty)$$

So if $\alpha = 0$, $\frac{1}{z^n}$ won't converge uniformly as $n \to \infty$ if $0 = \alpha \le |z| \le \beta$, since the limit depends on z,

• If $\alpha \neq 0$, $\frac{1}{1+(nz)^2}$ will be 1 when $z = 0$, $\forall n$.

This will make the convergence no longer uniform since when $z = 0$ the limit is 1 and when $z \neq 0$, the limit is 0.

b) • If $\alpha > 0$ and $\beta = \infty$, the same answers from 3.1.1 still hold, meaning if $\alpha > 1$ the limit converges uniformly to 0 but if $\alpha \le 1$ the limit depends on z and hence does not converge uniformly.

• If $\alpha > 0$ and $\beta = \infty$, nothing changes from 3.1.1, the limit still converges to 0 if $|z| \to \infty$, and the problem point $z = 0$ is no longer in our domain.

8) (3.1.3) We have

$$\lim_{n\to\infty} \int_0^1 n z^{n-1}\, dz$$

$$= \lim_{n\to\infty} \frac{\cancel{n} z^n}{\cancel{n}}\Big|_0^1 = \lim_{n\to\infty} 1 = 1$$

and now consider for $|z|<1$, say $|z| = \alpha < 1$,

then

$$|n z^{n-1}| = n|z|^{n-1} = n\alpha^{n-1}$$

and as $\alpha < 1$, as $n\to\infty$ the above becomes arbitrarily small, hence $\lim_{n\to\infty} n z^{n-1} = 0$ for $|z|<1$, so

$$\int_0^1 \lim_{n\to\infty} n z^{n-1}\, dz = \int_0^1 0\, dz = 0$$

This is not a counterexample because on $[0,1]$, the convergence is not uniform, since if $|z|=1$,

$$|n z^{n-1}| = n|z|^{n-1} = n$$

which $\longrightarrow \infty$ as $n\to\infty$.