1) a)

$$\delta(c) = \int_{-1}^{1} |f(t) - c| \, dt$$

$$= \int_{0}^{1} |1 - c| \, dt \quad + \int_{-1}^{0} |-1 - c| \, dt$$

$$= |1-c| t \Big]_0^1 + |1+c| t \Big]_0^1$$

$$= |1-c| + |1+c|$$

$$= \begin{cases} ① \; 1 - c + 1 + c = 2 & \text{if } -1 \le c \le 1 \\ ② \; c - 1 + 1 + c = 2c & \text{if } c > 1 \\ ③ \; -c - 1 + 1 - c = -2c & \text{if } c > -1 \end{cases}$$

As ②, ③ > 2, $\delta(c)$ is minimized for any value of c between $\underline{-1}$ and $\underline{1}$.

b) $\delta(c) = \sum_{i=1}^{n} | f(t_i) - c| \ dt$

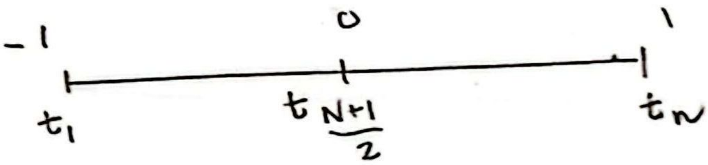
As $t_i = -1 + \frac{2(i-1)}{N-1} := 0$

$\rightarrow \quad 1 - N + 2(i-1) := 0$

$\rightarrow \quad i - 1 = \frac{N-1}{2}$

$\rightarrow \quad i = \frac{N-1}{2} + \frac{2}{2} = \frac{N+1}{2}$

So when $i = \frac{N+1}{2}$, $t_i = 0$



$i = \frac{N+1}{2}$

$\downarrow$

So

$$\delta(c) = \sum_{i < \frac{N+1}{2}} |-1-c| + \sum_{i > \frac{N+1}{2}} |1-c| + |-c|$$

$$= \frac{N-1}{2}\left(|1+c| + |1-c| \right) + |c|$$

$$= \begin{cases} \frac{N-1}{2}(2) + |c| & \text{if } -1 \le c \le 1 \\ \frac{N-1}{2}(2c) + |c| & \text{if } c > 1 \\ \frac{N-1}{2}(-2c) + |c| & \text{if } c > -1 \end{cases}$$

$$= \begin{cases} N-1+|c| & \text{if } -1 \le c \le -1 \\ Nc & \text{if } c > 1 \\ -Nc & \text{if } c < -1 \end{cases}$$

which is minimized when $N-1+|c|$ is minimized, implying $\delta(c)$ is minimized when $c=0$.

c)

$$\delta(c) = \left(\int_{-1}^{1} |f(t)-c|^2 \, dt \right)^{1/2}$$

$$\rightarrow \delta^2(c) = \int_{-1}^{1} |f(t)-c|^2 \, dt$$

$$= \int_{-1}^{0} (1+c)^2 \, dt + \int_{0}^{1} (1-c)^2 \, dt$$

$$= (1+c)^2 + (1-c)^2$$

$$= 1 + \cancel{2c} + c^2 + 1 - \cancel{2c} + c^2$$

$$= 2(c^2+1)$$

which is minimized when $c=0$ with $\delta^2(0) = 2 \rightarrow \delta(0) = \sqrt{2}$.

d) $$\delta(c) = \sqrt{\sum_{i=1}^{N} |f(t_i) - c|^2}$$

$$\rightarrow \delta^2(c) = \sum_{i=1}^{n} |f(t_i) - c|^2$$

$$= \sum_{i < \frac{N+1}{2}} |1+c|^2 + \sum_{i > \frac{N+1}{2}} |1-c|^2 + \underbrace{c^2}_{i=\frac{N+1}{2}}$$

$$= \frac{N-1}{2}\left([1+c]^2 + [1-c]^2\right) + c^2$$

$$= N-1\,(c^2+1) + c^2$$

which is minimized when $c=0$, making

$\delta(c) = \sqrt{(N-1)}$.

e)

$$\delta(c) = \max(|1+c|, |c|, |1-c|)$$

$$= \begin{cases} 1+c & \text{if } c \geq 0 \\ 1-c & \text{if } c < 0 \end{cases}$$

So $\delta(c)$ is minimized when $c = 0$, with $\delta(c) = 1$.

f) Again, here because N is odd

$$\delta(c) = \max(|1+c|, |c|, |1-c|)$$

so this has the same solution as part (e).

Pg 119, Problem 5

a) If $e^x \approx 1 + cx$, we want to find c such that $e^x - 1 \approx cx$. From equation (2.18), with $\pi_1(x) = x$ and $f(x) = e^x - 1$, we have

$$(\pi_1, \pi_1) c = (\pi_1, f)$$

$$c \int_0^1 x^2 \, dx = \int_0^1 x(e^x - 1) \, dx$$

$$\rightarrow \quad c \frac{x^3}{3} \Big|_0^1 = x(e^x - x) \Big|_0^1 - \int_0^1 e^x - x \, dx$$

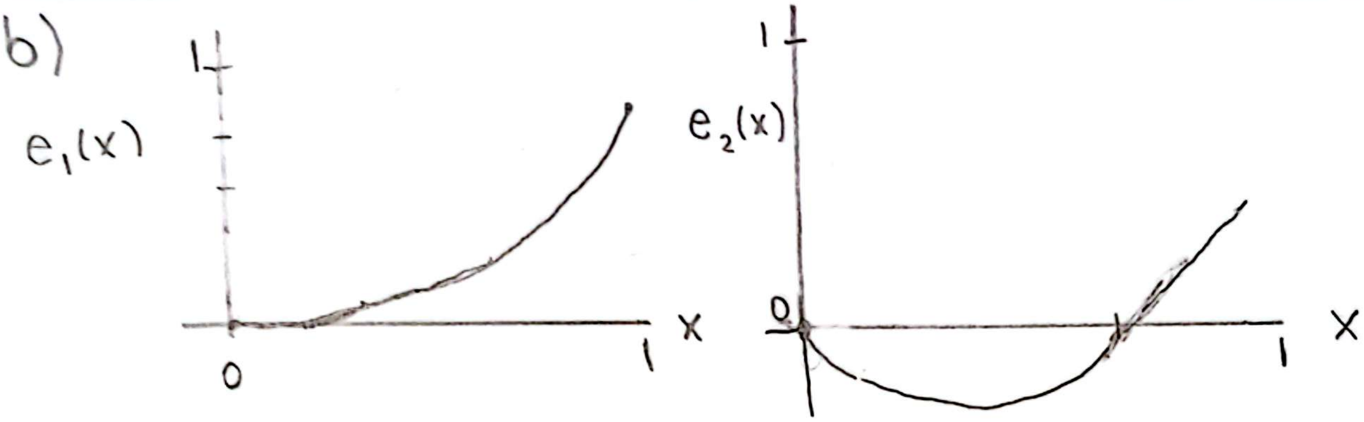
$$\rightarrow \quad \frac{1}{3} c = e - 1 - \left(e^x - \frac{x^2}{2}\right) \Big|_0^1$$

$$\rightarrow \quad \frac{1}{3} c = \cancel{e} - \cancel{1} - \left[\cancel{e} - \frac{1}{2} - \cancel{1}\right]$$

$$\rightarrow \quad \frac{1}{3} c = \frac{1}{2}$$

$$\rightarrow \quad \boxed{c = \frac{3}{2}}$$

b)



$$\max_{0 \le x \le 1} |e_1(x)| = e^1 - (1+1) = e^1 - 2 \approx 0.718$$

$$\max_{0 \le x \le 1} |e_2(x)| = e^1 - \left(1 + \frac{3}{2}\right) \approx 0.218$$

c) Now we want $e^x - 1 \approx c_1 x + c_2 x^2$. So put $\pi_1(x) = x$, $\pi_2(x) = x^2$, and $f(x) = e^x - 1$.

So we have,

$$\sum_{j=1}^{2} (\pi_i, \pi_j) c_j = (\pi_i, f), \quad i = 1, 2$$

$$i = 1 \rightarrow \quad (\pi_1, \pi_1) c_1 + (\pi_1, \pi_2) c_2 = (\pi_1, f)$$

$$c_1 \int_0^1 x^2 \, dx + c_2 \int_0^1 x^3 \, dx = \int_0^1 x (e^x - 1) \, dx$$

$$\rightarrow \quad \frac{1}{3} c_1 + \frac{1}{4} c_2 = \frac{1}{2}$$

$i = 2$,

$$c_1 (\pi_2, \pi_1) + c_2 (\pi_2, \pi_2) = (\pi_2, f)$$

$$c_1 \int_0^1 x^3 \, dx + c_2 \int_0^1 x^4 \, dx = \int_0^1 x^2 (e^x - 1) \, dx$$

$$\rightarrow \quad \frac{1}{4} c_1 + \frac{1}{5} c_2 = x^2 (e^x - x) \Big|_0^1 - x \left(e^x - \frac{x^2}{2} \right) \Big|_0^1 + \int_0^1 e^x - \frac{x^2}{2} \, dx$$

$$\rightarrow \quad \frac{1}{4} c_1 + \frac{1}{5} c_2 = (e - 1) - \left(e - \frac{1}{2} \right) + \left(e^x - \frac{x^3}{6} \right) \Big|_0^1$$

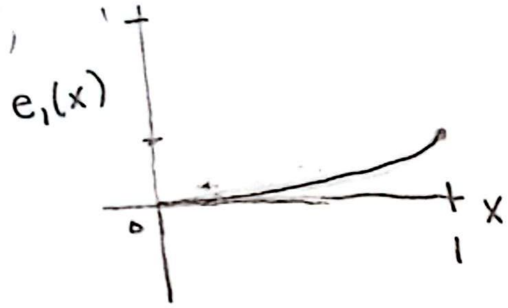
$$\rightarrow \quad \frac{1}{4} c_1 + \frac{1}{5} c_2 = e - 7/3$$

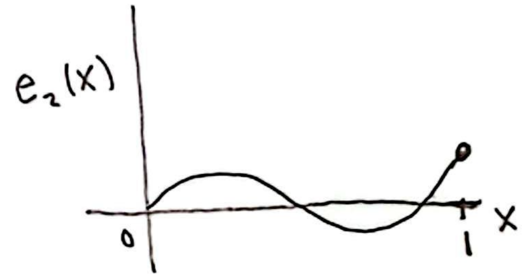
By solving the system,

$$\begin{pmatrix} \frac{1}{3} & \frac{1}{4} \\ 1/4 & 1/5 \end{pmatrix} \begin{pmatrix} c_1 \\ c_2 \end{pmatrix} = \begin{pmatrix} 1/2 \\ e - 7/3 \end{pmatrix}$$

we get $c_1 \approx 0.903$ and $c_2 \approx 0.796$

c)



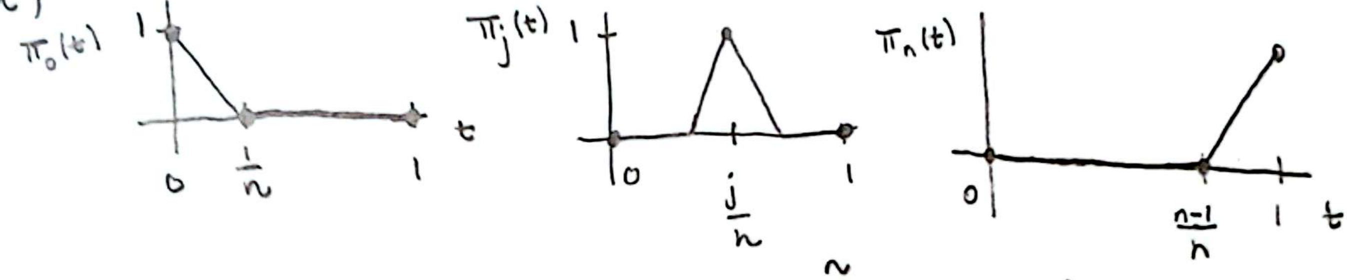


$$\max_{0 \le x \le 1} |e_1(x)| = e^1 - 1 - 1 - \frac{1}{2} \approx 0.218$$

$$\max_{0 \le x \le 1} |e_2(x)| = e^1 - 1 - 0.903\ldots - 0.796\ldots \approx 0.019$$

Pg 121, Problem 13

a) $1 \le j \le n-1$



A linear combination $\pi(t) = \sum_{j=0}^{n} c_j \pi_j(t)$ is a continuous piecewise function of linear functions over each interval Δ_n.

b) $$\pi_j(K/n) = \begin{cases} 0 & \text{if } j \neq K \\ 1 & \text{if } j = K \end{cases} = \delta_{jK}$$

c) Suppose that $\sum_{j=0}^{n} c_j \pi_j(t) = 0$ on $[0,1]$.

Then from (b), with $0 \le K \le n$,

$$\sum_{j=0}^{n} c_j \pi_j(K/n) = \sum_{j=0}^{n} c_j \delta_{jK} = c_K = 0$$

which implies $c_j = 0 \;\forall\; 0 \le j \le n$, implying that the $\{\pi_j\}$'s are independent. This same exact argument holds for the subdivision points, so yes the system is independent there too.

d) we have to compute

$$A_{ij} = \int_0^1 \pi_i(t)\pi_j(t)\, dt$$

Firstly, $\pi_0(t) = \begin{cases} 1-nt, & 0 \le t \le \frac{1}{n} \\ 0, & t > \frac{1}{n} \end{cases}$

$1 \le j \le n-1$

$$\pi_j(t) = \begin{cases} nt-j+1, & \frac{j-1}{n} \le t \le \frac{j}{n} \\ -nt+j+1, & j/n \le t \le \frac{j+1}{n} \\ 0, & \text{else} \end{cases}$$

$$\pi_n(t) = \begin{cases} nt-n+1, & \frac{n-1}{n} \le t \le 1 \\ 0 & \end{cases}$$

If $|i-j| > 1$, then $\pi_i \pi_j = 0 \ \forall t$, so here $A_{ij} = 0$. If $j = i+1$, we have

$$A_{i,i+1} = \int_{i/n}^{i+1/n} (-nt+i+1)(nt-i)\, dt$$

which after plugging into Sympy, reduces to $\frac{1}{6n}$. (whoops, I realized the u-sub is pretty straightforward later)

If $j = i-1$, the integral is identical to $A_{i,i+1}$, so again we'd get $A_{i,i-1} = \frac{1}{6n}$.

If $i = j$, we get (if $i \neq 0$ and $i \neq n$)

$$A_{ii} = \int_{\frac{i-1}{n}}^{i/n} (nt - i + 1)^2 \, dt + \int_{i/n}^{\frac{i+1}{n}} (-nt + i + 1)^2 \, dt$$

$u = nt - i + 1$

$du = n\,dt \rightarrow \frac{du}{n} = dt$

$v = -nt + i + 1$

$dv = -n\,dt \rightarrow -\frac{dv}{n} = dt$

$$= \frac{1}{n}\int_0^1 u^2 \, du - \frac{1}{n}\int_1^0 v^2 \, dv$$

$$= \frac{1}{n}\left[\frac{u^3}{3} \Big|^1 + \frac{v^3}{3} \Big|_0^1 \right]$$

$$= \frac{1}{n}\left[\frac{1}{3} + \frac{1}{3} \right]$$

$$= \frac{2}{3n}$$

If $i=0$ we have

$$A_{00} = \int_0^{1/n} (1-nt)^2 \, dt \rightarrow \begin{array}{l} u = 1-nt \\ du = -n\,dt \end{array}$$

$$= -\frac{1}{n} \int_1^0 u^2 \, du$$

$$= \frac{1}{n} \int_0^1 u^2 \, du = \frac{1}{3n}$$

Because of symmetry with A_{00}, $A_{N,N} = \frac{1}{3N}$ also.

So,

$$A_{ij} = \begin{cases} \frac{1}{6N} & \text{if } |i-j| = 1 \\ \frac{1}{3N} & \text{if } i=j=0 \text{ or } i=j=N \\ \frac{2}{3N} & \text{if } i=j \text{ and } i \neq 0, \neq N \\ 0 & \text{else} \end{cases}$$

4) a) Put $\pi_1(x) = 1$, then by G.S,

$$\pi_2 = x - \frac{(x, \pi_1)}{(\pi_1, \pi_1)} \pi_1$$

$$= x - \frac{\int_0^1 x \, dx}{\int_0^1 1 \, dx} \cdot 1$$

$$= x - \frac{1}{2}$$

$$\pi_3 = x^2 - \frac{(x^2, \pi_1)}{(\pi_1, \pi_1)} \pi_1 - \frac{(x^2, \pi_2)}{(\pi_2, \pi_2)} \pi_2$$

$$= x^2 - \frac{\int_0^1 x^2 \, dx}{\int_0^1 1 \, dx} \cdot 1 - \frac{\int_0^1 x^3 - \frac{1}{2}x^2 \, dx}{\int_0^1 (x - 1/2)^2 \, dx} \left(x - \frac{1}{2}\right)$$

$$= x^2 - \frac{1}{3} - \cancel{\frac{\frac{1}{12}}{\frac{1}{12}}} (x - 1/2)$$

$$= x^2 - x - \frac{1}{6}$$

So $\{1, x - \frac{1}{2}, x^2 - x - \frac{1}{6}\}$ is an orthogonal basis. Now we want

$$t^3 \approx c_1 + c_2\left(t - \tfrac{1}{2}\right) + c_3\left(t^2 - t - \tfrac{1}{6}\right) = p_2(t)$$

minimizing $\int_0^1 (t^3 - p_2(t))^2$, is the same as minimizing the L_2 norm, so we can use the normal equations

$$\sum_{j=1}^{3} (\pi_i, \pi_j) c_j = (\pi_i, f) \qquad i = 1, 2, 3$$

$$i=1 \rightarrow (\pi_1, \pi_1) c_1 + \overset{0}{\cancel{(\pi_1, \pi_2)}} c_2 + \overset{0}{\cancel{(\pi_1, \pi_3)}} c_3 = (\pi_1, t^3)$$

$$i=2 \rightarrow \overset{0}{\cancel{(\pi_2, \pi_1)}} c_1 + (\pi_2, \pi_2) c_2 + \overset{0}{\cancel{(\pi_2, \pi_3)}} c_3 = (\pi_2, t^3)$$

$$i=3 \rightarrow \overset{0}{\cancel{(\pi_3, \pi_1)}} c_1 + \overset{0}{\cancel{(\pi_3, \pi_2)}} c_2 + (\pi_3, \pi_3) c_3 = (\pi_3, t^3)$$

After computing the integrals

$$c_1 = \frac{1}{4}$$

$$\frac{1}{12} c_2 = 3/40 \rightarrow c_2 = \frac{36}{40} = \frac{9}{10}$$

$$\frac{7}{60} c_3 = \frac{-3}{40} \rightarrow c_3 = \frac{-120}{7 \cdot 40} = -\frac{3}{7}$$

So, the minimizer in this basis is

$$p_2(t) = \frac{1}{4} + \frac{9}{10}\left(x - \frac{1}{2}\right) - \frac{3}{7}\left(x^2 - x - \frac{1}{6}\right)$$

If you used the monomial basis, you would still get the same polynomial (as it ultimately depends on the span of $\{1, x, x^2\}$), but the computation would be a lot more tedious if you used the monomial basis b/c you'd have to solve $Ax = b$ instead of diag $x = b$.