2)

a) We have

- $\phi'(\alpha) = \frac{1}{2}\left(1 - \frac{a}{x^2}\right)\Big|_{x=\alpha=\sqrt{a}}$

$$= \frac{1}{2}\left(1 - \frac{a}{a}\right)$$

$$= 0$$

- $\phi''(\alpha) = \frac{a}{x^3}\Big|_{x=\alpha=\sqrt{a}}$

$$= \frac{a}{(\sqrt{a})^3}$$

$$= \frac{1}{\sqrt{a}} \neq 0$$

So by (4.71), $x_{n+1} = \frac{1}{2}\left(x_n + \frac{a}{x_n}\right)$ converges with order $p = 2$.

b) we have, for any $x_0 \in \mathbb{R}$

$$x_1 = \frac{a}{x_0}$$

$$x_2 = \frac{a}{x_1} = \frac{a}{\left(\frac{a}{x_0}\right)} = x_0$$

This means that $x_{2k} = x_0 \quad \forall\ k \geq 0$, and in general does not converge to $\sqrt{a}$, unless $x_0 = \sqrt{a}$.

c) we have,

$$\phi'(\alpha) = 2 + \frac{a}{x^2}\Big|_{x=\alpha=\sqrt{a}}$$

$$= 2 + \frac{a}{a}$$

$$= 3 \neq 0$$

So by (4.71), $x_{n+1} = 2x_n - \frac{a}{x_n}$ does not converge unless $x_0 = \sqrt{a}$.

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3) Let $D = [-1, 1]$, and $x, y \in D$ with $\phi(x) = \cos(x)$. From (4.87), ϕ is a contraction map on D if $\exists\ \gamma \in (0,1)$ s.t

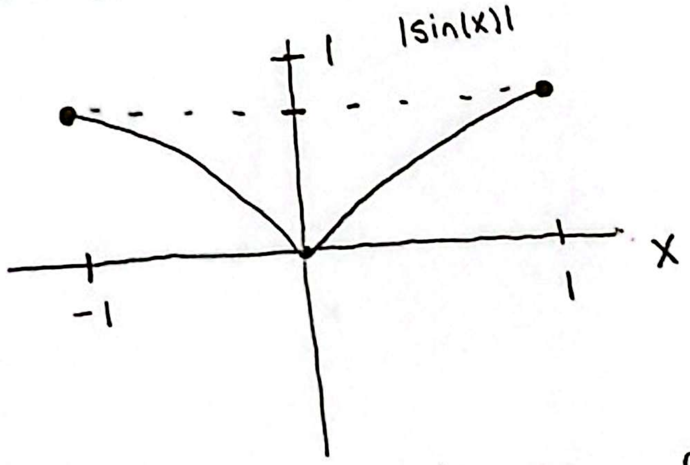
$$|\phi(x) - \phi(y)| \le \gamma\, |x-y|$$

$$\rightarrow \quad \left| \frac{\phi(x) - \phi(y)}{x-y} \right| \le \gamma$$

$$\rightarrow \quad \lim_{x \to y} \left| \frac{\phi(x) - \phi(y)}{x-y} \right| \le \gamma$$

$$\rightarrow \quad |\phi'(y)| \le \gamma$$

This means if the magnitude of the derivative of ϕ is bounded above by $\gamma \in (0,1)$ on D, then ϕ is a contraction map on D. When $\phi = \cos(x)$, $|\phi'| = |\sin(x)|$, plotting we see that,



$|\phi'| \le \overset{\gamma}{\overset{\shortparallel}{\sin(1)}} < 1$, so that on $D = [-1, 1]$, $\phi = \cos(x)$ is a contraction.

So by (Theorem 4.9.1) as D is a closed subset of $\mathbb{R}$,

$$\lim_{n \to \infty} x_n = \lim_{n \to \infty} \cos(x_{n-1}) = \alpha$$

for some $\alpha \in [-1,1]$. Noting, if you start with $x_0 \notin D$, $x_1 = \cos(x_0) \in D$, so we will always end up in D, and once we are in D we are stuck there, what is the value of α? Plugging into my calculator with $x_0 = \frac{\pi}{2}$, we get

$$\alpha \approx 0.739085\ldots, \in [-1,1] = D$$

4) We have,

$$j=0: \int_0^1 x\,dx = \frac{1}{2} = a_0 + a_1$$

$$j=1: \int_0^1 x^2\,dx = \frac{1}{3} = a_0 x_0 + a_1 x_1$$

$$j=2: \int_0^1 x^3\,dx = \frac{1}{4} = a_0 x_0^2 + a_1 x_1^2$$

$$j=3: \int_0^1 x^4\,dx = \frac{1}{5} = a_0 x_0^3 + a_1 x_1^3$$

Or equivalently,

$$a_0 + a_1 - \frac{1}{2} = 0$$
$$a_0 x_0 + a_1 x_1 - 1/3 = 0$$
$$a_0 x_0^2 + a_1 x_1^2 - 1/4 = 0$$
$$a_0 x_0^3 + a_1 x_1^3 - 1/5 = 0$$

So that,

$$\vec{f}\left(\begin{bmatrix} a_0 \\ a_1 \\ x_0 \\ x_1 \end{bmatrix}\right) = \begin{bmatrix} a_0 + a_1 - 1/2 \\ a_0 x_0 + a_1 x_1 - 1/3 \\ a_0 x_0^2 + a_1 x_1^2 - 1/4 \\ a_0 x_0^3 + a_1 x_1^3 - 1/5 \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \\ 0 \\ 0 \end{bmatrix}$$

The Jacobian is,

$$J_f = \begin{bmatrix} 1 & 1 & 0 & 0 \\ x_0 & x_1 & a_0 & a_1 \\ x_0^2 & x_1^2 & 2a_0x_0 & 2a_1x_1 \\ x_0^3 & x_1^3 & 3a_0x_0^2 & 3a_1x_1^2 \end{bmatrix}$$

Using NumPy, with $\begin{bmatrix} a_0 \\ a_1 \\ x_0 \\ x_1 \end{bmatrix} = \begin{bmatrix} 1/4 + \sqrt{6}/36 \\ 1/4 - \sqrt{6}/36 \\ (6+\sqrt{6})/10 \\ (6-\sqrt{6})/10 \end{bmatrix}$,

we get $\det(J_f) \approx -0.00333\ldots \neq 0$, so that J_f is nonsingular. If you put $\begin{bmatrix} a_0 \\ a_1 \\ x_0 \\ x_1 \end{bmatrix} = \begin{bmatrix} 1 \\ 1 \\ 0 \\ 0 \end{bmatrix}$, then $J_f = \begin{bmatrix} 1 & 1 & 0 & 0 \\ 0 & 0 & 1 & 1 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \end{bmatrix}$ which has zero determinant, and hence is singular.

5) we have for $i = 1, \dots, n-1$ $(\theta_0 = \alpha, \ \theta_n = \beta)$

$$\frac{1}{h^2}(\theta_{i-1} - 2\theta_i + \theta_{i+1}) + \sin(\theta_i) = 0$$

$$\downarrow$$

$$f_i = \theta_{i-1} - 2\theta_i + \theta_{i+1} + h^2 \sin(\theta_i) = 0$$

The Jacobian, J_f, has entries,

$$(J_f)_{ij} = \frac{\partial f_i}{\partial \theta_j} = \begin{cases} 1 & j = i-1 \\ -2 + h^2 \cos(\theta_i) & j = i \\ 1 & j = i+1 \\ 0 & \text{else} \end{cases}$$

To solve with Newton's method, we need to solve (iterate) for $n = 0, 1, \dots$

$$\begin{cases} J_f(x_n)\,\Delta_n = -f(x_n) \\ x_{n+1} = x_n + \Delta_n \end{cases}$$

See below for plots/ discussions.

AMATH 585 - Homework 5 - Plots, Code, and Discussions

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February 2025

Problem 1 (a)

```
# Problem 1 (a).
a, b = 4, 5
tol = 1e-6
n = int(np.ceil(np.log((b-a) / tol) / np.log(2)))

print("Problem 1(a): Bisection Method")
for i in range(n):
    x = (a + b) / 2

    if f(x) > 0:
        a = x
    else:
        b = x

    # Print interval.
    print("Iteration " + str(i) + ":", (a, b))
```

Figure 1: Implementation

```
Problem 1(a): Bisection Method
Iteration 0: (4.5, 5)
Iteration 1: (4.75, 5)
Iteration 2: (4.875, 5)
Iteration 3: (4.9375, 5)
Iteration 4: (4.9375, 4.96875)
Iteration 5: (4.953125, 4.96875)
Iteration 6: (4.9609375, 4.96875)
Iteration 7: (4.96484375, 4.96875)
Iteration 8: (4.96484375, 4.966796875)
Iteration 9: (4.96484375, 4.9658203125)
Iteration 10: (4.96484375, 4.96533203125)
Iteration 11: (4.965087890625, 4.96533203125)
Iteration 12: (4.965087890625, 4.9652099609375)
Iteration 13: (4.965087890625, 4.96514892578125)
Iteration 14: (4.965087890625, 4.965118408203125)
Iteration 15: (4.9651031494140625, 4.965118408203125)
Iteration 16: (4.965110778808594, 4.965118408203125)
Iteration 17: (4.965110778808594, 4.965114593505859)
Iteration 18: (4.965112686157227, 4.965114593505859)
Iteration 19: (4.965113639831543, 4.965114593505859)
```

Figure 2: Smallest known interval containing root after each iteration, accurate to 6 decimal places. Here iteration 0 is not the starting point, but rather the first step in the iteration.

Without running the code further, from (4.33), with $tol = 10^{-12}$ and $b - a = 5 - 4 = 1$, we'd need $n = \lceil \frac{\log 10^{12}}{\log 2} \rceil = 40$ iterations total to achieve the tolerance. To get to 6 decimal place accuracy ($tol = 10^{-6}$), we had to go for $n = \lceil \frac{\log 10^{6}}{\log 2} \rceil = 20$ iterations.

Problem 1 (b)

```python
# Problem 1 (b).
error = np.inf
tol = 1e-8
x = 5
k = 0

print("Problem 1(b): Newton's Method")
while error > tol:
    # Get function value.
    fofx = f(x)
    print("Iteration " + str(k) + ":" + " (x = " + str(x) + ", f(x) = " + str(fofx) + ")")

    # Update.
    x = x - (fofx / df(x))
    error = np.abs(fofx)
    k = k + 1
```

Figure 3: Implementation

```
Problem 1(b): Newton's Method
Iteration 0: (x = 5, f(x) = -5.0)
Iteration 1: (x = 4.966310265004573, f(x) = -0.16564277761091706)
Iteration 2: (x = 4.965115686301458, f(x) = -0.00020120180609861 65)
Iteration 3: (x = 4.96511423174643, f(x) = -2.978897128969038e-10)
```

Figure 4: Here iteration 0 is the starting point.

From iteration $1 \to 2$ we improved by 3 decimal places. From $2 \to 3$ we improved 6 decimal places. If this trend continued, we'd improve 9 decimal places from $3 \to 4$, so I predict it would take 1 more step to achieve 16 decimal point accuracy.

Problem 1 (c)

```python
# Problem 1 (c).
error = np.inf
tol = 1e-8
k = 0
x_prev = 4
x_curr = 5

print("Problem 1(c): Secant Method")
while error > tol:
    # Print before.
    print("Iteration " + str(k) + ":" + " (x = " + str(x_prev) + ", f(x) = " + str(f(x_prev)) + ")")

    # Assign temp values.
    f_curr = f(x_curr)
    f_prev = f(x_prev)

    # Update.
    x = x_curr - f_curr * (x_curr - x_prev) / (f_curr - f_prev)
    error = np.abs(f(x))
    k = k + 1
    x_prev = x_curr
    x_curr = x

# Print after.
print("Iteration " + str(k) + ":" + " (x = " + str(x) + ", f(x) = " + str(f(x)) + ")")
```

Figure 5: Implementation

```
Problem 1(c): Secant Method
Iteration 0: (x = 4, f(x) = 49.598150033144236)
Iteration 1: (x = 5, f(x) = -5.0)
Iteration 2: (x = 4.908421805556329, f(x) = 7.402024407938184)
Iteration 3: (x = 4.963079336311798, f(x) = 0.2808942198028612)
Iteration 4: (x = 4.965235312126352, f(x) = -0.016750499040933065)
Iteration 5: (x = 4.965114231713327, f(x) = 4.280998666672531e-09)
```

Figure 6: Here iteration 0 is the starting point.

From iteration $2 \to 3$ we improved by 1 decimal place. From $3 \to 4$ we improved by 1 decimal place. From $4 \to 5$ we improved by 7 decimal places. The rate of convergence increases dramatically here, if this trend were to continue I suspect it would take 1 or 2 more iterations to get to 16 decimal place accuracy.

Problem 3

```python
# Problem 3.
c = 0.5
x = c * np.pi
n = 100

for i in range(n):
    x = np.cos(x)

    if i == n-1:
        print(x)
```

```
0.7390851332151607
```

Figure 7: Hitting cosine over and over for Problem 3.

Problem 4

```python
# Problem 4.
a0 = 1/4 + np.sqrt(6) / 36
a1 = 1/4 - np.sqrt(6) / 36
x0 = (6 + np.sqrt(6)) / 10
x1 = (6 - np.sqrt(6)) / 10

J = np.array([[1, 1, 0, 0], [x0, x1, a0, a1], [x0 ** 2, x1 ** 2, 2*a0*x0, 2*a1*x1], [x0 ** 3, x1 ** 3, 3 * a0 * (x0 ** 2), 3 * a1 * (x1 ** 2)]])
det_J = np.linalg.det(J)
print("Jacobian determinant:", det_J)
```

```
Jacobian determinant: -0.0033333333333333319
```

Figure 8: Computing the determinant of the Jacobian for Problem 4.

Problem 5

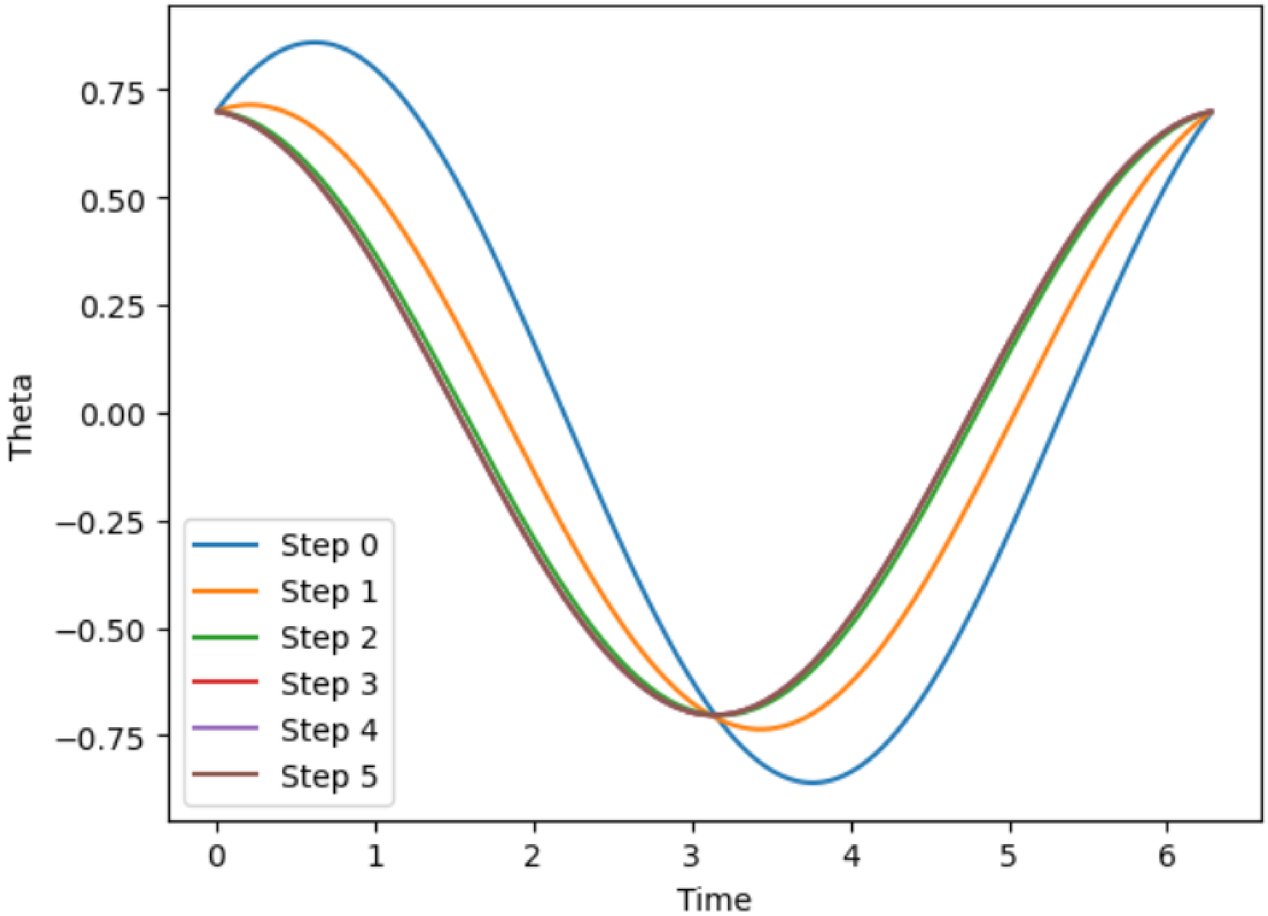

Figure 9: Solution 1. Iteration with $\theta^0 = 0.7\cos(t) + 0.5\sin(t)$ and $n = 1000$.

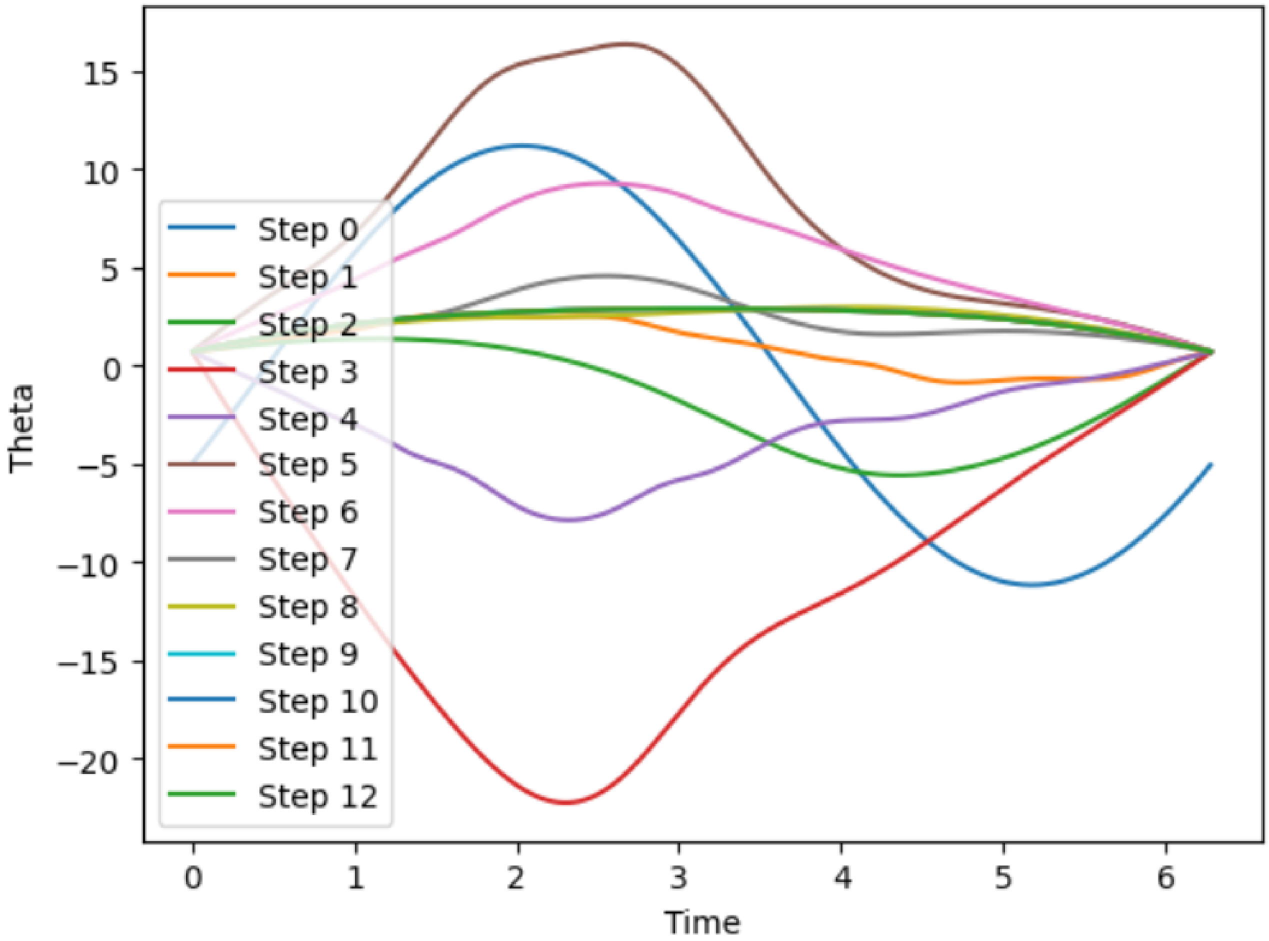

Figure 10: Solution 2. Iteration with $\theta^0 = -5\cos(t) + 10\sin(t)$ and $n = 1000$.

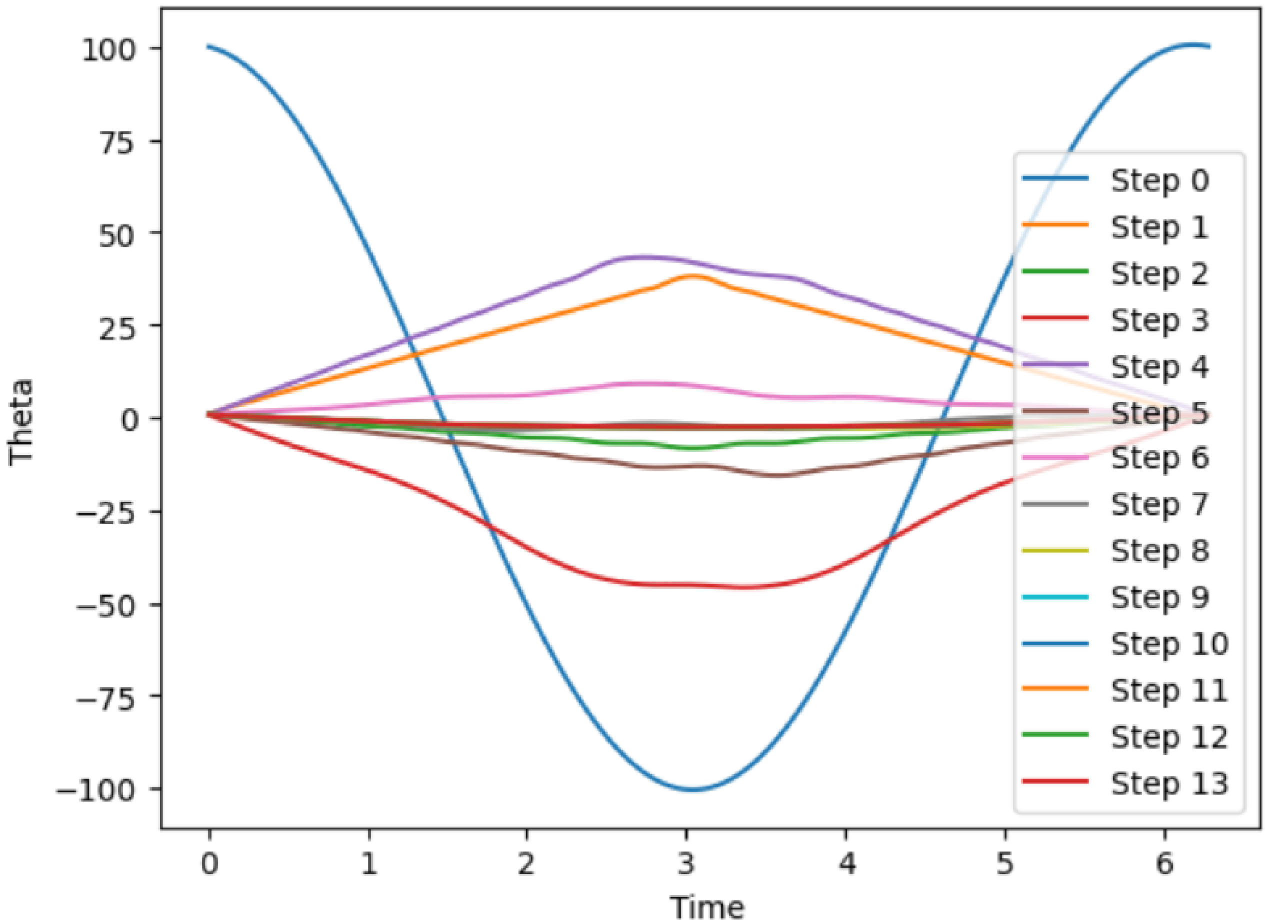

Figure 11: Solution 3. Iteration with $\theta^0 = 100\cos(t) - 10\sin(t)$ and $n = 1000$.

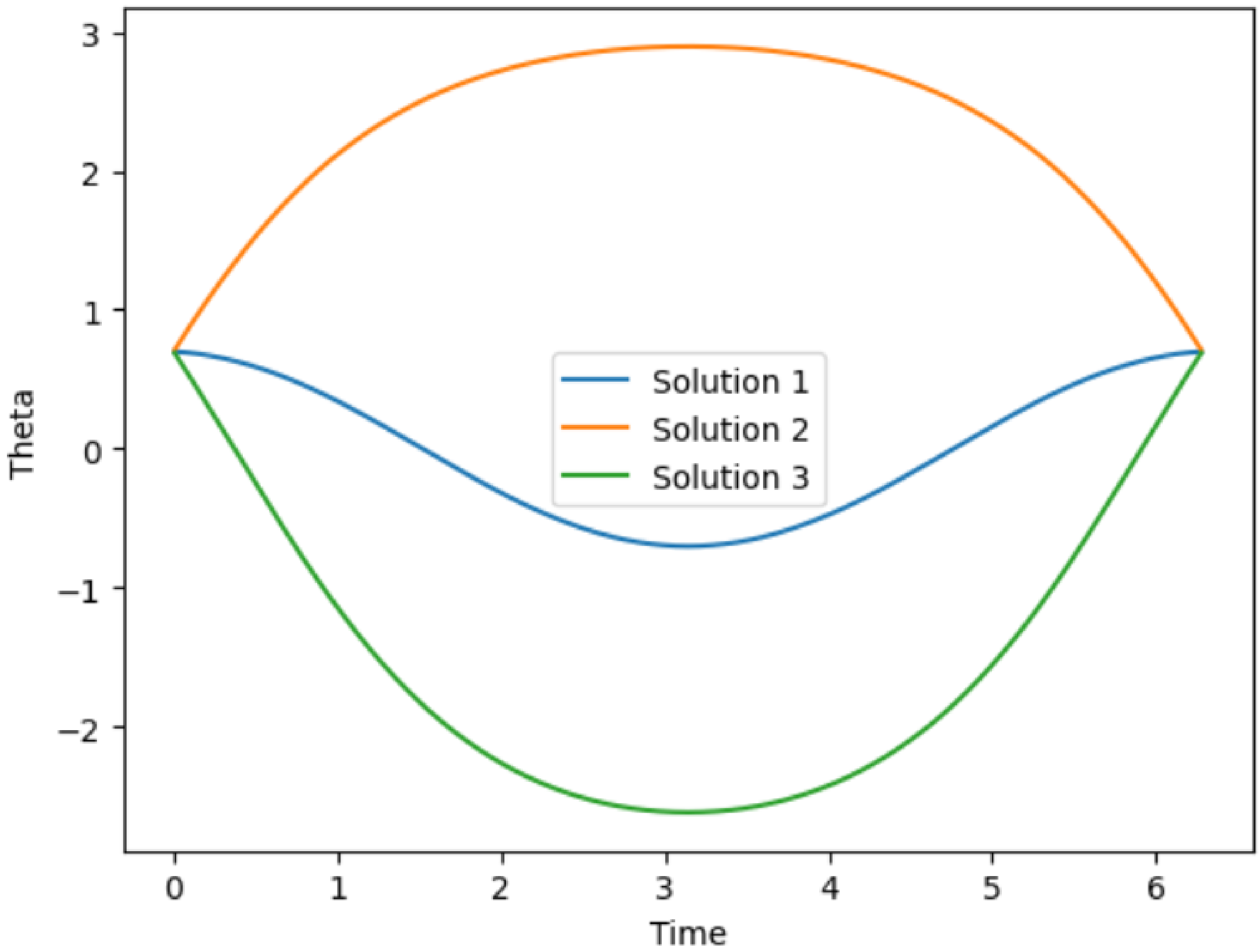

Figure 12: All solutions plotted.

By changing the initial conditions, I was able to find 3 solutions. If $\theta = 0$ corresponds to the pendulum being in the center position, Solution 1 would correspond to 1 single oscillation of a pendulum (back and forth over the center point) whereas Solutions 2 and 3 correspond to the pendulum swinging up through and back down to its starting position without passing through the center point. You can also notice that if the initial guess had a larger magnitude (appropriate norm), then it generally took more steps to converge. I used $\sum_i |\Delta_{n,i}| \leq 10^{-6}$ as my stopping criterion (when the sum of absolute changes was small enough).