The following problems, unless specifically noted, refer to the exercises in the book *Numerical Linear Algebra*, by Lloyd N. Trefethen and David Bau, III, SIAM 1997.

Homework 6

Reading: Lectures 23-24.

Problems: Exercise 24.1.

Four additional problems:

A1. Find the LU decomposition with partial pivoting (identify the matrices P, L, and U) for the following matrix by hand:

$$\begin{bmatrix} 0 & 2 & 1 \\ 1 & 0 & 2 \\ 2 & 1 & 0 \end{bmatrix}.$$

A2. Find the LU decomposition without pivoting (identify the matrices L and U) and Cholesky decomposition (identify the matrix R) of the following matrix by hand:

$$\begin{bmatrix} 2 & -1 & -1 \\ -1 & 3 & -1 \\ -1 & -1 & 2 \end{bmatrix}.$$

A3. For $A \in \mathbb{R}^{m\times m}$, define the matrix exponential

$$e^A = I + A + \frac{A^2}{2!} + \frac{A^3}{3!} + \cdots = \sum_{k=0}^{\infty} \frac{A^k}{k!}.$$

(a) Show that the matrix-valued function $Y(t) := e^{tA}$ satisfies $Y'(t) = AY(t)$, $t > 0$, $Y(0) = I$.
(b) Suppose A is diagonalizable: $A = X\Lambda X^{-1}$. Show that $e^{tA} = Xe^{t\Lambda}X^{-1}$, where

$$e^{t\Lambda} = \begin{bmatrix} e^{t\lambda_1} & & \\ & \ddots & \\ & & e^{t\lambda_m} \end{bmatrix}.$$

A4. Prove the *Gerschgorin's theorem*: for any matrix $A \in \mathbb{C}^{m\times m}$, define the disk

$$G_i(A) = \{z \in \mathbb{C} : |z - a_{ii}| \le \sum_{j=1, j\ne i}^{m} |a_{ij}|\}, \quad i = 1, \ldots, m,$$

then $\Lambda(A) \subset \bigcup_{i=1}^m G_i(A)$, where $\Lambda(A)$ is the set of eigenvalues of A. In other words, this theorem says that every eigenvalue of A lies in at least one of the m Gerschgorin disks of A in complex plane. (Hint: Let λ be any eigenvalue of A and $\mathbf{x}$ the corresponding eigenvector, focus on the index i that $|x_i| = \|\mathbf{x}\|_\infty$.)