AMATH 569 - Homework 4

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Problem 1

We proceed by induction. We know Holder's inequality holds when $n = 1$ and $n = 2$. Now, assume the hypothesis that for $(p_i)_{i=1}^n \subset (0, \infty]$ such that $\sum_i (1/p_i) = 1$, and for functions $f_i \in L^{p_i}(\Omega)$ we have:

$$\left\| \prod_{i=1}^n f_i \right\|_1 \leq \prod_{i=1}^n \|f_i\|_{p_i}$$

Now suppose we have matching conditions but for $(q_i)_{i=1}^{n+1} \subset (0, \infty]$. Then we have:

$$\frac{1}{p} + \frac{1}{q} := \sum_{i=1}^n \frac{1}{q_i} + \frac{1}{q_{n+1}} = 1$$

So that p and q are holder conjugates both > 0. By the inductive hypothesis we know $\prod_{i=1}^n f_i \in L^p$. We also know that $f_{n+1} \in L^q$. We can then say by Holder's inequality:

$$\begin{aligned} \left\| \prod_{i=1}^{n+1} f_i \right\|_1 &= \left\| \prod_{i=1}^n f_i \cdot f_{n+1} \right\|_1 \\ &\leq \left\| \prod_{i=1}^n f_i \right\|_p \|f_{n+1}\|_q \end{aligned}$$

Now applying the inductive hypothesis again we have:

$$\left\| \prod_{i=1}^n f_i \right\|_p \leq \prod_{i=1}^n \|f_i\|_{q_i}$$

Putting everything together, we get the desired result:

$$\left\| \prod_{i=1}^{n+1} f_i \right\|_1 \leq \prod_{i=1}^{n+1} \|f_i\|_{q_i}$$

This completes the proof. Note: The same argument works for the case $q_i = \infty$ since Hölder's inequality still holds when one exponent is infinity provided we interpret the ∞ norm as the essential supremum.

Problem 2

For now suppose that $t > 0$, and let $x = (1/t)$ so that:

$$f(t) = f\left(\frac{1}{x}\right) = \exp(-x)$$

As $t \to 0^+$ we have $x \to \infty$ so that $\exp(-x) \to 0$. Then:

$$f'(t) = -\exp(-x)\frac{dx}{dt} = \frac{1}{t^2}\exp(-x) = x^2\exp(-x)$$

And as $t \to 0^+$, we have $x \to \infty$ so that $f'(t) \to 0$ (the exponential decay kills the growth of any polynomial). By induction, we would like to show that for $j > 0$ that $f^{(j)}(t) = P_j(x)\exp(-x)$ (where $P_j(x)$ is a polynomial in x) so we can argue that as $t \to 0^+$ we have $x \to \infty$ so that $f^{(j)}(t) \to 0$. For $j = 1$, we showed the result above. Now for the sake of induction, suppose that $f^{(j)}(x) = P_j(x)\exp(-x)$, then by the product rule and chain rule:

$$\begin{aligned}
f^{(j+1)}(t) &= -P_j(x)\exp(-x)\frac{dx}{dt} + \exp(-x)P_j'(x)\frac{dx}{dt} \\
&= \frac{dx}{dt}\exp(-x)\left(P_j'(x) - P_j(x)\right) \\
&= -x^2\exp(-x)\left(P_j'(x) - P_j(x)\right) \\
&:= P_{j+1}(x)\exp(-x)
\end{aligned}$$

Where we have defined $P_{j+1}(x) = -x^2(P_j'(x) - P_j(x))$ to be the next polynomial. This proves our proposition so that as $t \to 0^+$ we have $f^{(j)}(t) \to 0$. Now when $t \leq 0$, we have $f(t) = 0$, and it is clear that $f^{(j)}(t) \to 0$ as $t \to 0^-$. Since the left and right limits agree, we have $f^{(j)}(t) \to 0$ as $t \to 0$ as desired. So for $j \geq 0$, we know:

$$f^{(j)}(t) = \begin{cases} P_j\left(\frac{1}{t}\right)\exp\left(-\frac{1}{t}\right) & t > 0 \\ 0 & t \leq 0 \end{cases}$$

When $t > 0$, $f^{(j)}(t) = P_j(x)\exp(-x)$ is continuous, and likewise when $t < 0$, $f^{(j)}(t) = 0$ Where we have just shown above that $f^{(j)}(t) \to 0$ as $t \to 0$, so $f^{(j)}(t)$ is continuous when $t = 0$. Hence for each $j \geq 0$, $f^{(j)}(t)$ exists and is continuous on all of $\mathbb{R}$, therefore f is smooth, ie. $f \in C^\infty(\mathbb{R})$.

Now suppose we want a function $g : \mathbb{R} \to \mathbb{R}$ such that $g \in C^\infty(\mathbb{R})$, $g(t) > 0$ on $(-1, 1)$ and $\operatorname{supp} g = [-1, 1]$, we can define:

$$g(t) = f(1-t^2) = \begin{cases} \exp\left(-\frac{1}{1-t^2}\right) & 1-t^2 < 0 \\ 0 & 1-t^2 \geq 0 \end{cases} = \begin{cases} \exp\left(-\frac{1}{1-t^2}\right) & -1 < t < 1 \\ 0 & \text{else} \end{cases}$$

Then by our earlier argument, $g(t)$ satisfies the above desired properties. Now suppose we want to construct a function $\psi \in C_c^\infty(\mathbb{R}^n)$ satisfying:

$$\begin{cases} (1) \ \int_{\mathbb{R}^n} \psi(x)\ dx = 1 \\ (2) \ \psi \geq 0 \text{ on } \mathbb{R}^n \\ (3) \ \psi(x) = 0 \text{ for } \ |x| > 1 \end{cases}$$

Then we can take:

$$\psi(x) = \begin{cases} C \exp\left(-\frac{1}{1-|x|^2}\right) & \text{for } \ |x| < 1 \\ 0 & \text{for } \ |x| \geq 1 \end{cases}$$

Where:

$$C = \frac{1}{\int_{B_1(0)} \exp\left(-\frac{1}{1-|x|^2}\right)\ dx}$$

Which makes sense as the integral is surely positive. To justify the smoothness of ψ, we realize:

$$\psi(x) = Cf(h(x)) = Cf(1-|x|^2)$$

As the the function $h(x) = 1 - |x|^2 = 1 - x_1^2 - \cdots - x_n^2 \in C^\infty(\mathbb{R}^n)$, and $f \in C^\infty(\mathbb{R})$, ψ is the composition of smooth functions which is scaled by a constant, and is hence smooth on $\mathbb{R}^n$. To not violate the smoothness of ψ, ψ must vanish to 0 as $|x| \to 1$, hence the compact support of ψ must be the unit ball $B_1(0) \subset \mathbb{R}^n$, so that $\psi \in C_c^\infty(\mathbb{R}^n)$. This shows (2) and (3). The integral condition (1) is trivially satisfied as:

$$\int_{\mathbb{R}^n} \psi(x)\ \mathrm{dx} = C \int_{B_1(0)} \exp\left(-\frac{1}{1-|x|^2}\right)\ dx = C \cdot \frac{1}{C} = 1$$

Therefore ψ satisfies all the desired properties, and we are done.

Problem 3

Let $u = (x/\epsilon)$. Then $\epsilon^n du = dx$, and we have:

$$\begin{aligned}\int_{\mathbb{R}^n} \psi_\epsilon(x)\ dx &= \frac{\epsilon^n}{\epsilon^n} \int_{\mathbb{R}^n} \psi(u)\ du \\ &= \int_{\mathbb{R}^n} \psi(u)\ du \\ &= 1\end{aligned}$$

Which shows the first condition as desired. To argue the third condition, with the same substituion, we can write:

$$\psi_\epsilon(x) = \frac{1}{\epsilon^n}\psi(u)$$

As $\psi(x) = 0$ for $|x| > 1$, we have $\psi(u) \geq 0$ for $|u| = |(x/\epsilon)| > 1 \implies |x| > \epsilon$, as desired. For the second condition, as both ϵ and ψ are non-negative, it is easy to see $\psi_\epsilon \geq 0$. All that is left to discuss is whether $\psi_\epsilon \in C^\infty(\mathbb{R}^n)$. As ψ_ϵ is just a scaled and dilated version of ψ, it is clearly still smooth. We can also realize that if $\psi(x) = 0$, then $\psi_\epsilon(u) = (1/\epsilon^n)\psi(u/\epsilon) = 0$ when $x = u/\epsilon$. Therefore if K is the support of ψ, we can define the compact support of ψ_ϵ to be $K_\epsilon = \{u \in \mathbb{R}^n : u = x/\epsilon \quad \forall x \in K\}$, so that $\psi_\epsilon \in C_c^\infty(\mathbb{R}^n)$. Therefore, ψ_ϵ has all the desired properties and we are done.

Problem 4

Define ψ_ϵ as in **Problem 3**. Now for all $\epsilon > 0$, define the set $A_\epsilon = \{y \in \mathbb{R}^n : \text{dist}(y, A) < \epsilon\}$, and then we let:

$$\chi_\epsilon(x) = (1_{A_\epsilon} * \psi_\epsilon) = (\psi_\epsilon * 1_{A_\epsilon})$$

Expanding we have:

$$\begin{aligned}\chi_\epsilon(x) &= \int_{\mathbb{R}^n} \psi_\epsilon(x-y) 1_{A_\epsilon}(y)\ dy \\ &= \int_{A_\epsilon} \psi_\epsilon(x-y)\ dy \quad \textbf{(1)}\end{aligned}$$

$\psi_\epsilon(x-y)$ has support on $B_\epsilon(x)$. But if $x \in A$, then $B_\epsilon(x) \subset A_\epsilon$ so that the above:

$$\begin{aligned}&= \int_{B_\epsilon(x)} \psi_\epsilon(x-y)\ dy \\ &= 1\end{aligned}$$

If $\text{dist}(x, A) \geq \epsilon$, then $B_\epsilon(x) \cap A_\epsilon = \emptyset$, so that **(1)** $= 0$, and $\chi_\epsilon(x) = 0$. If $0 < \text{dist}(x, A) < \epsilon$, then **(1)** implies:

$$\begin{aligned}\chi_\epsilon(x) &= \int_{A_\epsilon \cap B_\epsilon(x)} \psi_\epsilon(x-y)\ dy \\ &\leq \int_{B_\epsilon(x)} \psi_\epsilon(x-y)\ dy \\ &= 1\end{aligned}$$

Since $\psi_\epsilon \geq 0$, so is χ_ϵ, hence if $0 < \text{dist}(x, A) < \epsilon$, we have $0 \leq \chi_\epsilon \leq 1$. Also from **Lecture 15.1**, we have (we note that the functions meet the requirements for this later):

$$\begin{aligned}|D^\alpha \chi_\epsilon(x)| &= |(D^\alpha \psi_\epsilon * 1_{A_\epsilon})| \\ &= \left| \int_{A_\epsilon} D^\alpha \psi_\epsilon(x-y)\ dy \right| \\ &\leq \int_{B_\epsilon(x)} |D^\alpha \psi_\epsilon(x-y)|\ dy\end{aligned}$$

Now as $\psi_\epsilon(x) = \epsilon^{-n}\psi(x/\epsilon)$, we can take ψ to be the same as from **Problem 2**. The integrand above looks like:

$$\begin{aligned}
|D^\alpha \psi_\epsilon(x-y)| &= \left| \frac{\partial^{|\alpha|}\psi_\epsilon(x-y)}{\partial x_1^{\alpha_1}\cdots\partial x_n^{\alpha_n}} \right| \\
&= \frac{1}{\epsilon^{n+|\alpha|}} \left| \frac{\partial^{|\alpha|}\psi\left(\frac{x-y}{\epsilon}\right)}{\partial x_1^{\alpha_1}\cdots\partial x_n^{\alpha_n}} \right| \\
&\leq \frac{C}{\epsilon^{n+|\alpha|}}
\end{aligned}$$

Where in the second line we used the definition of ψ_ϵ. Since $\psi \in C_c^\infty(\mathbb{R}^n)$, we know ψ and all of it's derivatives are bounded, so we can upper bound the second line by a positive constant C. Going back, we now have:

$$\begin{aligned}
|D^\alpha \chi_\epsilon(x)| &\leq \int_{B_\epsilon(x)} |D^\alpha \psi_\epsilon(x-y)| \; dy \\
&\leq \int_{B_\epsilon(x)} \frac{C}{\epsilon^{n+|\alpha|}} \; dy \\
&= \mathrm{Vol}(B_\epsilon(x)) \cdot \frac{C}{\epsilon^{n+|\alpha|}} \\
&= C' \cdot \frac{\epsilon^n}{\epsilon^{n+|\alpha|}} \\
&= C'\epsilon^{-|\alpha|}
\end{aligned}$$

Where we have used that the volume of $B_\epsilon(x)$ is proportional to ϵ^n and combined the other constant part with C into $C' > 0$. One final note, $\chi_\epsilon \in C^\infty(\mathbb{R})$, as it is the convolution of a smooth function with compact support (ψ_ϵ) with an L^∞ function (1_{A_ϵ}) (this also justifies our use of **15.1**). This completes this problem.

Problem 5

Using **Definition 5.36** from Renardy and Rogers, we have:

$$(H', \phi) = -(H, \phi') = -\int_0^\infty \phi'(x)\ dx = \phi(0) = (\delta, \phi)$$

Where in the second to last equality, we use that the test function ϕ decays to 0 towards ∞. This completes the calculation.

Problem 6

The hint was very nice, but the "only if" direction is actually completely proved in **Lemma 5.16** by Renardy and Rogers. Therefore, it remains to show the forward direction. Suppose that for each compact set $K \subset \Omega$, there exists $m, C_{k,m} \geq 0$ such that:

$$|l(\phi)| \leq C_{K,m} \sum_{|\alpha| \leq m} \sup_{x \in K} |D^\alpha \phi(x)|$$

To show sequential continuity, take $\phi_n \to \phi$ in $D(K)$. Then:

$$\begin{aligned} |l(\phi) - l(\phi_n)| &= |l(\phi - \phi_n)| \\ &\leq C_{K,m} \sum_{|\alpha| \leq m} \sup_{x \in K} |D^\alpha (\phi(x) - \phi_n(x))| \end{aligned}$$

Where the first equality follows as l is linear. As $n \to \infty$ we hve $\phi_n \to \phi$ so that the RHS above approaches 0 and:

$$\lim_{n \to \infty} l(\phi_n) = l(\phi)$$

We are justified in passing the limit since ϕ, ϕ_n are smooth functions with compact support (ie $\phi_n \to \phi$ uniformly on K). This proves the sequential continuity of l, and we are done.

Problem 7

We can write:

$$u(x,t) = (G(\cdot,t) * u_0) + \int_0^t (G(\cdot,t-s) * f(\cdot,s))\,ds$$

We interpret G_t and LG in the sense of distributions, so all operations are justified via distributional derivatives and convolution with test functions. So differentiating with respect to t, with Leibniz rule and the FTC we see:

$$\begin{aligned}
u_t &= (G_t(\cdot,t) * u_0) + (G_t(\cdot,0) * f(\cdot,t)) + \int_0^t (G_t(\cdot,t-s) * f(y,s))\,ds \\
&= (LG(\cdot,t) * u_0) + \int_0^t (LG(\cdot,t-s) * f(y,s))\,ds + (\delta * f(\cdot,t)) \\
&= L\left((G(\cdot,t) * u_0) + \int_0^t (G(\cdot,t-s) * f(\cdot,s))\,ds\right) + f(x,t) \\
&= Lu + f
\end{aligned}$$

This shows the first condition for the PDE is satisfied by u. We now check the boundary condition:

$$\begin{aligned}
u(x,0) &= (G(\cdot,0) * u_0) + \int_0^0 (G(\cdot,-s) * f(\cdot,s))\,ds \\
&= (\delta * u_0) + 0 \\
&= u_0
\end{aligned}$$

Therefore, the boundary condition is also satisfied and $u(x,t)$ solves the PDE.