6.1) To show $I-2P$ is unitary, we must show that either $(I-2P)^*(I-2P) = I$ or $(I-2P)(I-2P)^* = I$. We have

$$(I-2P)(I-2P)^* = I^2 - 2PI - I2P^* + 4PP^*$$

$$= (I-2P)(I^*-2P^*)$$

$$= I^2 - 2P - 2P^* + 4PP^*$$

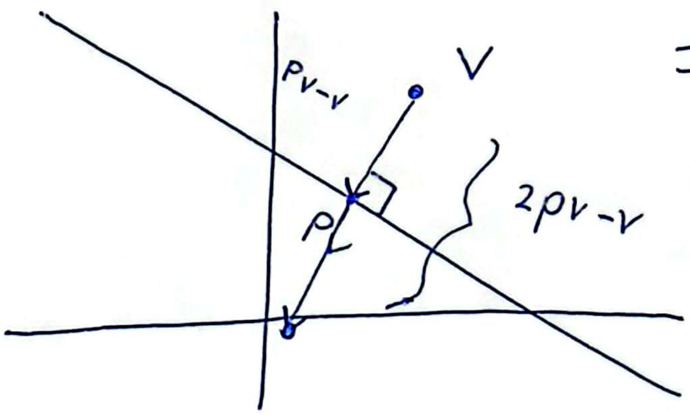
since P is an orthogonal projector, we know $P = P^*$, so

$$= I^2 - 4P + 4P^2$$

$$= I - 4P + 4P \quad (\text{since } P^2 = P)$$

$$= I$$

Geometric Interpretation



The operator $I-2P$ ~~map~~ reflects V over the range of P. Applying $I-2P$ a second time will take us back to range(P) where we started, ie $(I-2P)^2 = I$.

6.2) E is an orthogonal projector. To show this, we must show that $P^2 = P$. ($E^2 = E$) and also $E^* = E$. Fix $x \in \mathbb{C}^m$, then

$$E^2 x = E(Ex) =$$

$$E\left[\frac{x + Fx}{2}\right] = \frac{1}{2}\left[Ex + E(Fx)\right]$$

$$= \frac{1}{2}\left[\frac{x + Fx}{2}\right] + \frac{1}{2}\left[\frac{Fx + F^2 x}{2}\right]$$

Note that since F is the flip operator, $F^2 = I$. (2 flips is equivalent to not flipping).

So the above

$$= \frac{1}{4}\left[x + Fx + Fx + x\right]$$

$$= \frac{1}{4}\left[2x + 2Fx\right]$$

$$= \frac{x + Fx}{2} = Ex$$

So $E^2 = E$, and E is ~~an orthogonal~~ a projector.

To see what the entries of E are, we can just see where E maps the vectors e_j for $1 \le j \le m$,

$$E e_j = \frac{1}{2}\left[e_j + Fe_j\right]$$

at the entry level, this formula becomes

$$E_{ij} = \frac{1}{2}(e_{ij} + e_{m-i+1,j})$$

we must

I almost forgot to show that $E = E^*$, to do so we can use the above formula

$$(E_{ij})^* = \left[\frac{1}{2}(e_{ij} + e_{m-i+1,j})\right]^*$$

$$= \frac{1}{2}\left[e_{ij}^* + e_{m-i+1,j}^*\right]$$

$$= \frac{1}{2}\left[e_{ji} + e_{j,m-i+1}\right]$$

where this formula is describing that the columns of E are the rows of E^*

but since the entries above are for the transpose, then the above

$$= \frac{1}{2}\left[e_{i'j'} + e_{m-i'+1,j'}\right]$$

where i',j' represents the indexing for the rows and columns of E^*, however this is the same as $E_{i'j'} = E_{ij}$, so $E = E^*$.

Another way to see $E = E^*$ is to consider

$$E = \frac{1}{2}[I + F] \rightarrow E^* = \frac{1}{2}[I^* + F^*]$$

But $F^2 = I$, also and $F^* = F$, so the above equals $\rightarrow$ $E^* = \frac{1}{2}[I + F] = E$

6.4) From the chapter if A^*A is nonsingular then we can write the orthogonal projector onto the range of A as

$$P = A(A^*A)^{-1}A^*$$

So for <u>a)</u>

$$A = \begin{bmatrix} 1 & 0 \\ 0 & 1 \\ 1 & 0 \end{bmatrix} \qquad A^*A = \begin{bmatrix} 1 & 0 & 1 \\ 0 & 1 & 0 \end{bmatrix}\begin{bmatrix} 1 & 0 \\ 0 & 1 \\ 1 & 0 \end{bmatrix} = \begin{bmatrix} 2 & 0 \\ 0 & 1 \end{bmatrix}$$

$$(A^*A)^{-1} = \begin{bmatrix} 1/2 & 0 \\ 0 & 1 \end{bmatrix}$$

So

$$P = \begin{bmatrix} 1 & 0 \\ 0 & 1 \\ 0 & 0 \end{bmatrix}\begin{bmatrix} 1/2 & 0 \\ 0 & 1 \end{bmatrix}\begin{bmatrix} 1 & 0 & 1 \\ 0 & 1 & 0 \end{bmatrix} = \begin{bmatrix} 1/2 & 0 & 1/2 \\ 0 & 1 & 0 \\ 1/2 & 0 & 1/2 \end{bmatrix}$$

Now let's see where P maps $\begin{bmatrix} 1 \\ 2 \\ 3 \end{bmatrix}$

$$P\begin{bmatrix} 1 \\ 2 \\ 3 \end{bmatrix} = \begin{bmatrix} 1/2 & 0 & 1/2 \\ 0 & 1 & 0 \\ 1/2 & 0 & 1/2 \end{bmatrix}\begin{bmatrix} 1 \\ 2 \\ 3 \end{bmatrix} = \begin{bmatrix} 2 \\ 2 \\ 2 \end{bmatrix}$$

For <u>b)</u>

$$B = \begin{bmatrix} 1 & 2 \\ 0 & 1 \\ 1 & 0 \end{bmatrix} \qquad B^*B = \begin{bmatrix} 1 & 2 & 1 \\ 0 & 1 & 0 \end{bmatrix}\begin{bmatrix} 1 & 2 \\ 0 & 1 \\ 1 & 0 \end{bmatrix} = \begin{bmatrix} 2 & 2 \\ 2 & 5 \end{bmatrix}$$

$$(B^*B)^{-1} = \begin{bmatrix} 5/6 & -1/3 \\ -1/3 & 1/3 \end{bmatrix}$$

↑ just use 2D matrix inverse formula

So,

$$P = \begin{bmatrix} 1 & 2 \\ 0 & 1 \\ 1 & 0 \end{bmatrix} \begin{bmatrix} 5/6 & -1/3 \\ -1/3 & 1/3 \end{bmatrix} \begin{bmatrix} 1 & 2 & 1 \\ 0 & 1 & 0 \end{bmatrix}$$

$$= \begin{bmatrix} 5/6 & 1/3 & 1/6 \\ 1/3 & 1/3 & -1/3 \\ 1/6 & -1/3 & 5/6 \end{bmatrix}$$

Now let's see where P maps $\begin{bmatrix} 1 \\ 2 \\ 3 \end{bmatrix}$

$$P\begin{bmatrix} 1 \\ 2 \\ 3 \end{bmatrix} = \begin{bmatrix} 5/6 & 1/3 & 1/6 \\ 1/3 & 1/3 & -1/3 \\ 1/6 & -1/3 & 5/6 \end{bmatrix} \begin{bmatrix} 1 \\ 2 \\ 3 \end{bmatrix}$$

$$= \begin{bmatrix} 2 \\ 0 \\ 2 \end{bmatrix}$$

7.1)

a) It is easier to first find the reduced QR factorization and then extend to the full QR factorization.
I will use graham-schmidt (to compute $\hat{Q}$) and direct computation to compute $\hat{R}$, We have
$\uparrow (\hat{R})$

$$A = \begin{bmatrix} 1 & 0 \\ 0 & 1 \\ 1 & 0 \end{bmatrix}$$

Take $q_1 = a_1 / \|a_1\|_2 = \begin{bmatrix} 1/\sqrt{2} \\ 0 \\ 1/\sqrt{2} \end{bmatrix}$

Then $q_2 = \dfrac{a_2 - r_{12} q_1}{r_{22}}$

$$= \frac{\begin{bmatrix} 0 \\ 1 \\ 0 \end{bmatrix} - (q_1^* a_2) q_1}{r_{22}} = \frac{\begin{bmatrix} 0 \\ 1 \\ 0 \end{bmatrix} - \overbrace{\begin{bmatrix} 1/\sqrt{2} & 0 & 1/\sqrt{2} \end{bmatrix} \begin{bmatrix} 0 \\ 1 \\ 0 \end{bmatrix}}^{0 \,=\, r_{12}} \begin{bmatrix} 1/\sqrt{2} \\ 0 \\ 1/\sqrt{2} \end{bmatrix}}{r_{22}}$$

$$= \begin{bmatrix} 0 \\ 1 \\ 0 \end{bmatrix} \Big/ \underbrace{\left\| \begin{bmatrix} 0 \\ 1 \\ 0 \end{bmatrix} \right\|_2}_{=1} = \begin{bmatrix} 0 \\ 1 \\ 0 \end{bmatrix}$$

So $\hat{Q} = \begin{bmatrix} 1/\sqrt{2} & 0 \\ 0 & 1 \\ 1/\sqrt{2} & 0 \end{bmatrix}$. We can compute $\hat{R}$

as $\begin{bmatrix} r_{11} & r_{12} \\ r_{21} & r_{22} \end{bmatrix} = \begin{bmatrix} \cancel{q} \; \|a_1\|^2 & 0 \\ \underset{=0}{q_2^* a_1} & \|a_2\|^2 \end{bmatrix} = \begin{bmatrix} \sqrt{2} & 0 \\ 0 & 1 \end{bmatrix}$

To extend to the full QR, we take a unit vector orthogonal to both q_1 and q_2, say $q_3 = \begin{bmatrix} 0 \\ 0 \\ 1 \end{bmatrix}$ (very clearly orthogonal to q_1 and q_2), and say

$$\underset{3\times 3}{Q = \begin{bmatrix} 1/\sqrt{2} & 0 & 0 \\ 0 & 1 & 0 \\ 1/\sqrt{2} & 0 & 1 \end{bmatrix}}, \quad \underset{3\times 2}{R = \begin{bmatrix} \sqrt{2} & 0 \\ 0 & 1 \\ 0 & 0 \end{bmatrix}}$$

b) We have $B = \begin{bmatrix} 1 & 2 \\ 0 & 1 \\ 1 & 0 \end{bmatrix}$. Take $q_1 = \dfrac{a_1}{\underbrace{\|a_1\|_2}_{=r_{11}}} = \begin{bmatrix} 1/\sqrt{2} \\ 0 \\ 1/\sqrt{2} \end{bmatrix}$.

$r_{12} = \sqrt{2}$

$$\text{Then } r_{22}q_2 = a_2 - (\overbrace{q_1^* a_2}^{r_{12}})q_1 = \begin{bmatrix} 2 \\ 1 \\ 0 \end{bmatrix} - \overbrace{\begin{bmatrix} 1/\sqrt{2} & 0 & 1/\sqrt{2} \end{bmatrix}\begin{bmatrix} 2 \\ 1 \\ 0 \end{bmatrix}}^{=\sqrt{2}}\begin{bmatrix} 1/\sqrt{2} \\ 0 \\ 1/\sqrt{2} \end{bmatrix}$$

$$= \begin{bmatrix} 2 \\ 1 \\ 0 \end{bmatrix} - \sqrt{2}\begin{bmatrix} 1/\sqrt{2} \\ 0 \\ 1/\sqrt{2} \end{bmatrix} = \begin{bmatrix} 1 \\ 1 \\ -1 \end{bmatrix} \; (=1)$$

So $r_{22} = \sqrt{1^2 + 1^2 + (-1)^2} = \sqrt{3}$, so $q_2 = \begin{bmatrix} 1/\sqrt{3} \\ 1/\sqrt{3} \\ -1/\sqrt{3} \end{bmatrix}$

So $\hat{Q} = \begin{bmatrix} 1/\sqrt{2} & 1/\sqrt{3} \\ 0 & 1/\sqrt{3} \\ 1/\sqrt{2} & -1/\sqrt{3} \end{bmatrix}$ and $\hat{R} = \begin{bmatrix} r_{11} & r_{12} \\ r_{21} & r_{22} \end{bmatrix}$

$r_{21} = q_2^* a_1 = 0$

$$= \begin{bmatrix} \sqrt{2} & \sqrt{2} \\ 0 & \sqrt{3} \end{bmatrix}$$

Problem continues on next page

To get our 3rd orthogonal vector q_3, we solve a system of equations

$$q_1^* q_3 = 0 \longrightarrow \begin{bmatrix} 1/\sqrt{2} & 0 & 1/\sqrt{2} \end{bmatrix} \begin{bmatrix} x \\ y \\ z \end{bmatrix} = 0$$

$$q_2^* q_3 = 0 \longrightarrow \begin{bmatrix} 1/\sqrt{3} & 1/\sqrt{3} & -1/\sqrt{3} \end{bmatrix} \begin{bmatrix} x \\ y \\ z \end{bmatrix} = 0$$

So,

$$\frac{x}{\sqrt{2}} + \frac{z}{\sqrt{2}} = 0 \longrightarrow x = -z$$

plugging in

$$\frac{x}{\sqrt{3}} + \frac{y}{\sqrt{3}} + \frac{z}{\sqrt{3}} = 0 \longrightarrow -2z + y = 0 \rightarrow y = 2z$$

put $z = 1$, then $x = -1$, and $y = 2$

So the vector $q_3 = \begin{bmatrix} -1 \\ 2 \\ 1 \end{bmatrix} \Big/ \left\| \begin{bmatrix} -1 \\ 2 \\ 1 \end{bmatrix} \right\|_2 = \begin{bmatrix} -1/\sqrt{6} \\ 2/\sqrt{6} \\ 1/\sqrt{6} \end{bmatrix}$

is orthogonal and unit length. So the full QR decomposition is

$$Q = \begin{bmatrix} 1/\sqrt{2} & 1/\sqrt{3} & -1/\sqrt{6} \\ 0 & 1/\sqrt{3} & 2/\sqrt{6} \\ 1/\sqrt{2} & -1/\sqrt{3} & 1/\sqrt{6} \end{bmatrix} \qquad R = \begin{bmatrix} \sqrt{2} & \sqrt{2} \\ 0 & \sqrt{3} \\ 0 & 0 \end{bmatrix}$$

Problem A1:

cqr.m | mqr.m | experiment.m | qr_calcs.m | +

```matlab
% Compute the QR factorization using algorithm 7.1 (Classical QR Decomposition).
function [Q, R] = cqr(A)
    [m, n] = size(A);

    % Allocate memory for Q and R.
    Q = zeros(m, n);
    R = zeros(n, n);

    % Initialize the first column of Q by taking the first column vector of A and normalizing.
    Q(:, 1) = A(:, 1) / norm(A(:, 1), 2);

    % Compute the entries of Q and R.
    for j = 1:n
        v_j = A(:, j);

        for i = 1:j-1
            R(i, j) = Q(:, i)' * A(:, j);
            v_j = v_j - R(i, j) * Q(:, i);
        end

        R(j, j) = norm(v_j, 2);
        Q(:, j) = v_j / R(j, j);
    end
end
```

Editor - C:\Users\Nwhybra\Desktop\UW AMATH Masters\AMATH 584\HW\HW3\mqr.m

cqr.m | mqr.m | experiment.m | qr_calcs.m | +

```matlab
% Compute the QR factorization using algorithm 8.1 (Modified QR Decomposition).
function [Q, R] = mqr(A)
    [m, n] = size(A);

    % Allocate memory for Q and R.
    Q = zeros(m, n);
    R = zeros(n, n);

    % This is basically the same as the first for loop setting v_i = a_i.
    V = A;

    % Compute the entries of Q and R.
    for i = 1:n
        R(i, i) = norm(V(:, i), 2);
        Q(:, i) = V(:, i) / R(i, i);

        for j = i+1:n
            R(i, j) = Q(:, i)' * V(:, j);
            V(:, j) = V(:, j) - R(i, j) * Q(:, i);
        end
    end
end
```

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cqr.m | mqr.m | experiment.m | qr_calcs.m

```
[U, X] = qr(randn(80));
[V, X] = qr(randn(80));
S = diag(2 .^ (-1:-1:-80));
A = U*S*V';

[QC, RC] = cqr(A);
[QM, RM] = mqr(A);

r_vals_c = log(diag(RC));
r_vals_m = log(diag(RM));
j = (1:80)';

figure
scatter(j, r_vals_c);
hold on;
scatter(j, r_vals_m);
hold off;

xlabel('j');
ylabel('log(R_{jj})');
title('log(R_{jj}) vs. j');
legend('Classical QR (GS)', 'Modified QR (GS)');
```

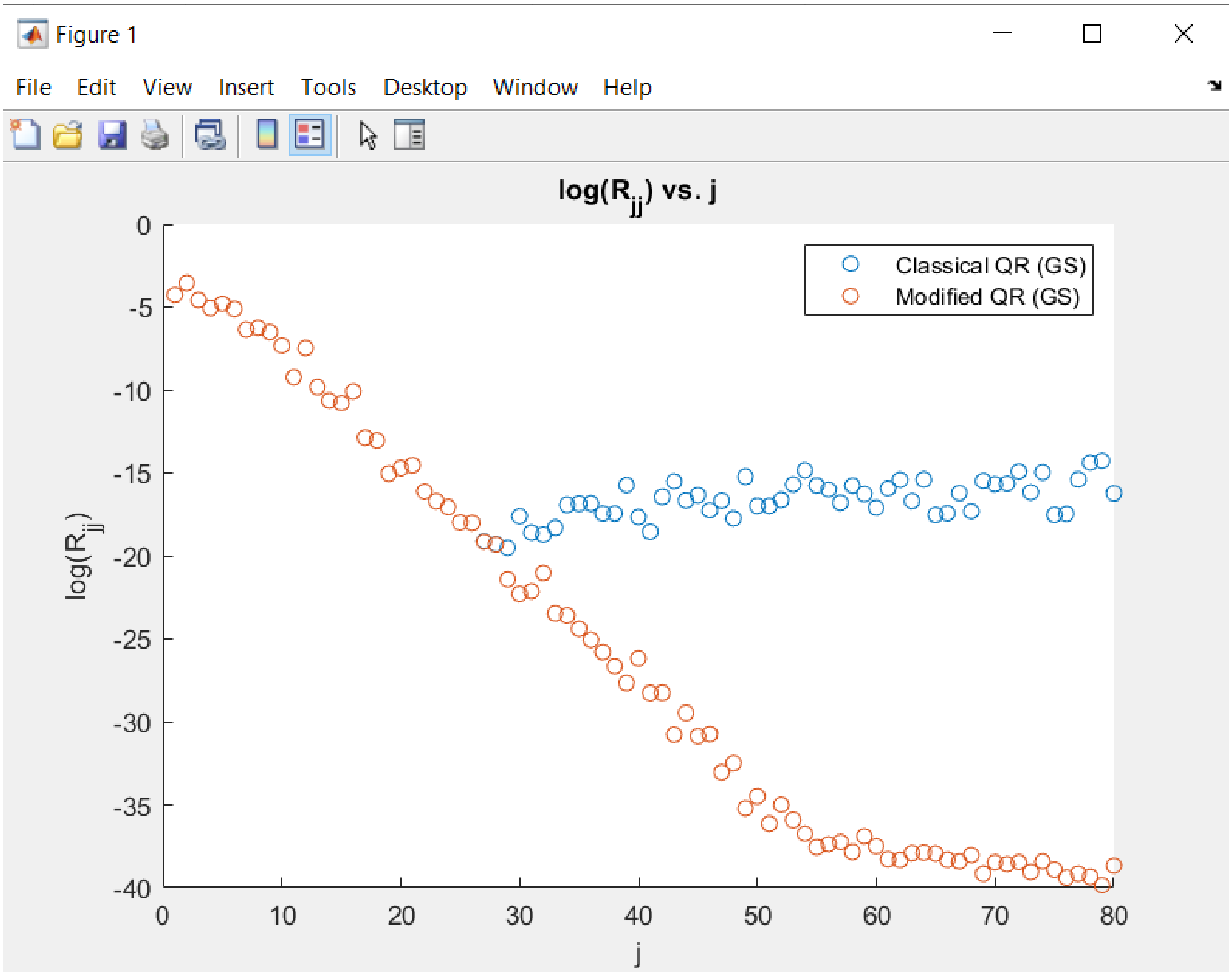
Figure 1
File Edit View Insert Tools Desktop Window Help
log(R_jj) vs. j
Classical QR (GS)
Modified QR (GS)
log(R_jj)
j
0
-5
-10
-15
-20
-25
-30
-35
-40
0
10
20
30
40
50
60
70
80