AMATH 569 - Homework 2

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Problem 1

Let $w(t) = \exp\left(\int_a^t v(t)\ dt\right)$, then:

$$w'(t) = \frac{d}{dt} \exp\left(\int_a^t v(t)\ dt\right)$$
$$= \exp\left(\int_a^t v(t)\ dt\right)\left(\frac{d}{dt}\int_a^t v(t)\ dt\right)$$
$$= w(t)v(t)$$

As $w(t) > 0$ for all $t \in I$, we see we can write $v(t) = w'(t)/w(t)$. Now suppose $u'(t) \leq v(t)u(t)$ for all $t \in I$, then:

$$u'(t) \leq \frac{w'(t)}{w(t)} u(t)$$

$$u'(t)w(t) \leq w'(t)u(t) \implies w(t)u'(t) - u(t)w'(t) \leq 0 \implies \frac{w(t)u'(t) - u(t)w'(t)}{w(t)^2} \leq 0$$

$$\left(\frac{u(t)}{w(t)}\right)' \leq 0 \quad \text{(Quotient Rule)}$$
$$\int_a^t \left(\frac{u(t)}{w(t)}\right)' dt \leq \int_a^t 0\ dt$$
$$\frac{u(t)}{w(t)} - \frac{u(a)}{w(a)} \leq 0$$
$$u(t) \leq \frac{u(a)w(t)}{w(a)}$$

However, $w(a) = \exp(0) = 1$, so we get the desired result:

$$u(t) \leq u(a) \exp\left(\int_a^t v(t)\ dt\right)$$

Problem 2 (a)

For notational convenience, let $x_1 = x_1(t)$, $x_2 = x_2(t)$, $\nabla f_1 = \nabla f(x_1)$, and $\nabla f_2 = \nabla f(x_2)$. Now let $u(t) = |x_1 - x_2|^2$, then:

$$u'(t) = 2(x_1' - x_2')^T(x_1 - x_2)$$

$$= 2(-\nabla f_1^T + \nabla f_2^T)(x_1 - x_2)$$

$$= 2(\nabla f_1^T - \nabla f_2^T)(x_2 - x_1)$$

Since f is $\alpha-$strongly convex, we have:

$$\begin{cases} f(x_1) \geq f(x_2) - \nabla f_1^T(x_2 - x_1) + \frac{\alpha}{2}|x_1 - x_2|^2 \\ f(x_2) \geq f(x_1) + \nabla f_2^T(x_2 - x_1) + \frac{\alpha}{2}|x_1 - x_2|^2 \end{cases}$$

$$\implies f(x_1) + f(x_2) \geq f(x_2) + f(x_1) + (\nabla f_2^T - \nabla f_1^T)(x_2 - x_1) + \alpha|x_1 - x_2|^2$$

$$0 \geq (\nabla f_2^T - \nabla f_1^T)(x_2 - x_1) + \alpha|x_1 - x_2|^2$$

$$(\nabla f_1^T - \nabla f_2^T)(x_2 - x_1) \leq -\alpha|x_1 - x_2|^2$$

$$\implies u'(t) \leq -2\alpha|x_1 - x_2|^2 = (-2\alpha)u(t)$$

So take $v(t) = -2\alpha$, so that $\int_0^t -2\alpha\ dt = -2\alpha t$. Gronwall's inequality then gives:

$$u(t) = |x_1(t) - x_2(t)|^2 \leq u(0)\exp\left(\int_0^t v(t)\ dt\right) = |x_1(0) - x_2(0)|^2 \exp(-2\alpha t)$$

Which is the desired result.

Problem 2 (b)

Let x^* be the unique minimizer of the α-strongly convex function f. Then, by definition, $\nabla f(x^\star) = 0$. Define $x_1(t) = x(t)$ and $x_2(t) = x^\star$ for all $t \geq 0$. Since $x^\star$ is constant, its derivative with respect to t is 0, ie:

$$x_2'(t) = 0 = -\nabla f(x^\star)$$

So $x_2(t)$ satisfies the gradient flow equation. Applying the result from part (a), we have:

$$\left|x_1 - x_2\right|^2 \leq \left|x_1(0) - x_2(0)\right|^2 \exp(-2\alpha t)$$

$$\left|x - x^*\right|^2 \leq \left|x(0) - x^*\right|^2 \exp(-2\alpha t)$$

As desired.

Problem 3

For the sake of contradiction, and without loss of generality, suppose there exists some $x^* \in \Omega$ such that $u(x^*) > 0$. Since u is continuous, there exists $\epsilon, \delta > 0$ such that for all x in the open ball $B(x^*, \delta)$ (centered at x^* of radius δ), we have $u(x) > \epsilon$ and $B(x^*, \delta) \subset \Omega$. In simpler terms, there exists an open ball around x^* in Ω such that u is strictly positive inside the ball. Define the function:

$$v(x) = \begin{cases} \exp\left(-\frac{1}{\delta^2 - |x-x^*|^2}\right) & \text{for } |x - x^*| < \delta \\ 0 & \text{for } |x - x^*| \geq \delta \end{cases}$$

We will argue that v is smooth. The exponential function is famously smooth whenever its argument is defined, so v is smooth inside the ball. The function 0 is trivially smooth everywhere, so v is smooth outside the ball. All we must show now is that v and all of its derivatives are smooth on the boundary of the ball. To do so, for every $k \geq 0$ we can show that $D^\alpha v(x) \to 0$ as $|x - x^*| \to \delta$ where $|\alpha| = k$ so that v and all of its derivatives are continuous on the boundary of the ball. Firstly:

$$\begin{cases} \lim_{|x-x^*| \to \delta^+} v(x) = \lim_{|x-x^*| \to \delta^+} 0 = 0 \\ \lim_{|x-x^*| \to \delta^-} v(x) = \lim_{|x-x^*| \to \delta^-} \exp\left(-\frac{1}{\delta^2 - |x-x^*|^2}\right) = 0 \end{cases}$$

So $v(x)$ is continuous on the boundary of the ball (and on all of Ω). Continuing by means of the chain rule, let $r(x) = \delta^2 - |x - x^*|^2$, then $D^\alpha v(x)$ will take the form:

$$D^\alpha v(x) = \begin{cases} R_\alpha(x) \exp\left(-\frac{1}{\delta^2 - |x-x^*|^2}\right) & \text{for } |x - x^*| < \delta \\ 0 & \text{for } |x - x^*| \geq \delta \end{cases}$$

Where, $R_\alpha(x) \sim \frac{1}{r(x)^{m_\alpha}}$ for some integer $m_\alpha \geq 1$. So for some $C > 0$:

$$|D^\alpha v(x)| \leq \begin{cases} \frac{C}{r(x)^{m_\alpha}} \exp\left(-\frac{1}{r(x)}\right) & \text{for } |x - x^*| < \delta \\ 0 & \text{for } |x - x^*| \geq \delta \end{cases}$$

The exponential dominates the rational function for every α, so as $|x - x^*| \to \delta^-$ we have $D^\alpha v(x) \to 0$, and again as $|x - x^*| \to \delta^+$ we have $D^\alpha v(x) \to 0$. Hence, v and all of its derivatives exist and are continuous on the boundary of the ball, meaning v is smooth on the boundary. Since v is smooth on the ball, on the boundary of the ball, and outside the ball, v is smooth on Ω, and by definition will be 0 on $\partial\Omega$ implying that v is a valid test function. Therefore by assumption:

$$
\begin{aligned}
0 &= \int_\Omega u(x)v(x)\ dx \\
&= \int_{B(x^*,\ \delta)} u(x) \exp\left(-\frac{1}{r(x)}\right)\ dx \quad \text{(v is 0 outside of } B(x^*,\delta)) \\
&> \epsilon \int_{B(x^*,\ \delta)} \exp\left(-\frac{1}{r(x)}\right)\ dx \quad \text{(continuity of u)} \\
&> 0 \quad \text{(the exponential is strictly positive in } B(x^*,\delta))
\end{aligned}
$$

We have shown that $0 > 0$, which is a contradiction. If we had assumed $u(x^*) < 0$, we could follow the exact same argument, except that we would define $B(x^*, \delta)$ to be the open ball around x^* where u is strictly negative, and would derive a similar contradiction. Thus $u(x) = 0$ for all $x \in \Omega$.

Problem 4 (a)

The coefficients a_n for the sine series of $f(x) = x^2 - x$ on $[0,1]$ are given by (with $L = 1$):

$$a_n = 2\int_0^1 (x^2 - x)\sin(n\pi x)\ dx := 2I_1$$

We proceed by integration by parts on I_1 with:

$$\begin{cases} u = (x^2 - x) \to du = (2x-1)\ dx \\ dv = \sin(n\pi x)\ dx \to v = -\frac{1}{n\pi}\cos(n\pi x) \end{cases}$$

$$\begin{aligned} \implies I_1 &= -\frac{1}{n\pi}(x^2 - x)\cos(n\pi x)\Big|_0^1 + \frac{1}{n\pi}\int_0^1 (2x-1)\cos(n\pi x)\ dx \\ &= \frac{1}{n\pi}\int_0^1 (2x-1)\cos(n\pi x)\ dx := \frac{1}{n\pi}I_2 \end{aligned}$$

We now proceed by integration by parts on I_2 with:

$$\begin{cases} u = (2x-1) \to du = 2\ dx \\ dv = \cos(n\pi x)\ dx \to v = \frac{1}{n\pi}\sin(n\pi x) \end{cases}$$

$$\begin{aligned} \implies I_2 &= \frac{1}{n\pi}(2x-1)(\sin(n\pi x))\Big|_0^1 - \frac{2}{n\pi}\int_0^1 \sin(n\pi x)\ dx \\ &= 0 + \frac{2}{(n\pi)^2}\cos(n\pi x)\Big|_0^1 \\ &= \frac{2}{(n\pi)^2}\left((-1)^n - 1\right) \\ &= \begin{cases} 0 & \text{if n is even} \\ -\frac{4}{(n\pi)^2} & \text{if n is odd} \end{cases} \end{aligned}$$

So that finally:

$$\begin{aligned} \implies a_n = 2I_1 &= 2\left(\frac{2}{n\pi}\right)I_2 \\ &= \begin{cases} 0 & \text{if n is even} \\ -\frac{16}{(n\pi)^3} & \text{if n is odd} \end{cases} \end{aligned}$$

Problem 4 (b)

The coefficients a_n for the sine series of $f(x) = \cos(\pi x/4)$ on $[0, 1]$ are given by (with $L = 1$):

$$a_n = 2\int_0^1 \cos\left(\frac{\pi x}{4}\right)\sin(n\pi x)\ dx := 2I_1$$

We proceed by integration by parts on I_1 with:

$$\begin{cases} u = \cos\left(\frac{\pi x}{4}\right) \to du = -\frac{\pi}{4}\sin\left(\frac{\pi x}{4}\right)\ dx \\ dv = \sin(n\pi x)\ dx \to v = -\frac{1}{n\pi}\cos(n\pi x) \end{cases}$$

$$\implies I_1 = -\frac{1}{n\pi}\cos\left(\frac{\pi x}{4}\right)\cos(n\pi x)\Big|_0^1 - \frac{1}{4n}\int_0^1 \sin\left(\frac{\pi x}{4}\right)\cos(n\pi x)\ dx$$

$$= \frac{1}{n\pi}\left(1 - (-1)^n(\sqrt{2})^{-1}\right) - \frac{1}{4n}\int_0^1 \sin\left(\frac{\pi x}{4}\right)\cos(n\pi x)\ dx$$

$$:= \frac{1}{n\pi}\left(1 - (-1)^n(\sqrt{2})^{-1}\right) - \frac{1}{4n}I_2$$

We now proceed by integration by parts on I_2 with:

$$\begin{cases} u = \sin\left(\frac{\pi x}{4}\right) \to du = \frac{\pi}{4}\cos\left(\frac{\pi x}{4}\right)\ dx \\ dv = \cos(n\pi x)\ dx \to v = \frac{1}{n\pi}\sin(n\pi x) \end{cases}$$

$$\implies I_2 = \frac{1}{n\pi}\sin\left(\frac{\pi x}{4}\right)\sin(n\pi x)\Big|_0^1 - \frac{1}{4n}\int_0^1 \cos\left(\frac{\pi x}{4}\right)\sin(n\pi x) dx$$

$$= 0 - \frac{1}{4n}I_1$$

So we see:

$$2I_1 = 2\left(\frac{1}{n\pi}\left(1 - (-1)^n(\sqrt{2})^{-1}\right) - \frac{1}{4n}I_2\right)$$

$$= 2\left(\frac{1}{n\pi}\left(1 - (-1)^n(\sqrt{2})^{-1}\right) + \frac{I_1}{16n^2}\right)$$

$$= \frac{2 - 2(-1)^n(\sqrt{2})^{-1}}{n\pi} + \frac{I_1}{8n^2}$$

$$\implies I_1 = \frac{\frac{2-2(-1)^n(\sqrt{2})^{-1}}{n\pi}}{2 - \frac{1}{8n^2}} = \frac{\frac{1-(-1)^n(\sqrt{2})^{-1}}{n\pi}}{1 - \frac{1}{16n^2}}$$

$$= \frac{16n}{\pi} \cdot \frac{1 - (-1)^n(\sqrt{2})^{-1}}{16n^2 - 1}$$

Therefore:

$$a_n = 2I_1 = \frac{32n}{\pi(16n^2 - 1)} \left(1 - \frac{(-1)^n}{\sqrt{2}}\right)$$

Problem 5

We aim to verify that Laplace's equation in polar coordinates is given by:

$$u_{rr} + \frac{1}{r}u_r + \frac{1}{r^2}u_{\theta\theta} = 0$$

To do so, we start with:

$$u_{xx} + u_{yy} = 0$$

Expressing u as a function of r and θ, we use the chain rule to relate derivatives in Cartesian coordinates to those in polar coordinates (with $x = r\cos(\theta)$, $y = r\sin(\theta)$, $r = \sqrt{x^2 + y^2}$). The necessary derivatives are:

$$\begin{cases} r_x = \cos\theta \\ r_y = \sin\theta \\ \theta_x = -\sin(\theta)/r \\ \theta_y = \cos(\theta)/r \end{cases}$$

Using the chain rule:

$$u_x = u_r r_x + u_\theta \theta_x = u_r \cos\theta - \frac{u_\theta \sin\theta}{r}$$

$$u_y = u_r r_y + u_\theta \theta_y = u_r \sin\theta + \frac{u_\theta \cos\theta}{r}$$

Differentiating again:

$$u_{xx} = \frac{\partial}{\partial x}(u_x)$$

$$= \frac{\partial}{\partial x}\left(u_r \cos\theta - \frac{u_\theta \sin\theta}{r}\right)$$

$$= u_{rr}\cos^2\theta - 2\frac{u_{r\theta}\sin\theta\cos\theta}{r} + \frac{u_{\theta\theta}\sin^2\theta}{r^2} + \frac{u_r \sin^2\theta}{r} + \frac{u_\theta \sin\theta\cos\theta}{r^2}$$

Similarly:

$$u_{yy} = \frac{\partial}{\partial y}(u_y)$$

$$= \frac{\partial}{\partial y}\left(u_r \sin\theta + \frac{u_\theta \cos\theta}{r}\right)$$

$$= u_{rr}\sin^2\theta + 2\frac{u_{r\theta}\sin\theta\cos\theta}{r} + \frac{u_{\theta\theta}\cos^2\theta}{r^2} + \frac{u_r \cos^2\theta}{r} - \frac{u_\theta \sin\theta\cos\theta}{r^2}$$

Therefore:

$$0 = u_{xx} + u_{yy} = u_{rr}(\cos^2\theta + \sin^2\theta) + \frac{u_{\theta\theta}}{r^2}(\sin^2\theta + \cos^2\theta) + \frac{u_r}{r}(\sin^2\theta + \cos^2\theta)$$

$$= u_{rr} + \frac{1}{r}u_r + \frac{1}{r^2}u_{\theta\theta}$$

Thus, as desired:

$$u_{rr} + \frac{1}{r}u_r + \frac{1}{r^2}u_{\theta\theta} = 0$$

We now solve the above equation using separation of variables. Assume a solution of the following form:

$$u(r,\theta) = R(r)\Theta(\theta)$$

Substituting into Laplace's equation:

$$R''\Theta + \frac{1}{r}R'\Theta + \frac{1}{r^2}R\Theta'' = 0$$

Dividing both sides by $R\Theta$ (when $u \neq 0$):

$$\frac{R''}{R} + \frac{1}{r}\frac{R'}{R} + \frac{1}{r^2}\frac{\Theta''}{\Theta} = 0$$

$$\implies r^2\frac{R''}{R} + r\frac{R'}{R} + \frac{\Theta''}{\Theta} = 0$$

Rewriting:

$$\frac{\Theta''}{\Theta} = -\left(r^2\frac{R''}{R} + r\frac{R'}{R}\right) = -\lambda$$

Where $-\lambda$ is a constant as the LHS only depends on θ and the RHS only depends on r. This yields two ordinary differential equations:

$$\begin{cases} \Theta'' + \lambda\Theta = 0, \\ r^2R'' + rR' - \lambda R = 0. \end{cases}$$

The first equation has solutions:

$$\begin{cases} \Theta(\theta) = A\cos(\sqrt{\lambda}\theta) + B\sin(\sqrt{\lambda}\theta), & \lambda > 0, \\ \Theta(\theta) = A + B\theta, & \lambda = 0, \\ \Theta(\theta) = Ae^{\sqrt{-\lambda}\theta} + Be^{-\sqrt{-\lambda}\theta}, & \lambda < 0. \end{cases}$$

The second equation is an Euler equation. So assume the ansatz $R(r) = Cr^n$, then:

$$Cn(n-1)r^n + Cnr^n - \lambda Cr^n = 0$$

$$R(r)(n^2 - \lambda) = 0$$

So either $R(r) = 0 \implies u(r,\theta) = 0$ or $n = \pm\sqrt{\lambda}$, so that $R(r) = Cr^{\sqrt{\lambda}} + Dr^{-\sqrt{\lambda}} = Cr^n + Dr^{-n}$, and $\lambda = n^2 > 0$. Therefore for each $n \geq 0$, there is a solution:

$$u_n(r,\theta) = (A_n \cos(n\theta) + B_n \sin(n\theta))(C_n r^n + D_n r^{-n})$$

Finally the general solution would take the form:

$$u(r,\theta) = \sum_{n\geq 0} u_n(r,\theta)$$

Problem 6

Let $u(x,t) = \int_0^t w(x,t,\tau)\ d\tau$. We will show that u satisfies the nonhomogeneous heat equation and its boundary conditions. From the well-posedness of the heat-equation, this u will be the unique solution. We have:

$$u_t = w(x,t,t) + \int_0^t w_t(x,t,\tau)\ d\tau \quad \text{(Leibniz rule)}$$

$$= f(x,t) + \int_0^t w_{xx}(x,t,\tau)\ d\tau = f(x,t) + \frac{d^2}{dx^2}\int_0^t w(x,t,\tau)\ d\tau$$

$$= f(x,t) + u_{xx}$$

$$\implies u_t - u_{xx} = f(x,t)$$

So the PDE (first condition) is satisfied. Next:

$$u(x,0) = \int_0^0 w(x,t,\tau)\ d\tau = 0$$

$$u(0,t) = \int_0^t w(0,t,\tau)\ d\tau = \int_0^t 0\ dt = 0$$

$$u(1,t) = \int_0^t w(1,t,\tau)\ d\tau = \int_0^t 0\ dt = 0$$

Hence, all of the boundary conditions are also satisfied. Therefore, $u(x,t) = \int_0^t w(x,t,\tau)\ d\tau$ as desired.

Problem 7 (a)

Firstly, a strong solution to the wave equation satisfies $u_{tt} = u_{xx}$. Now consider the weak solution integral:

$$\begin{aligned} I &= \int_{-\infty}^{\infty}\int_{-\infty}^{\infty} u\left(\phi_{tt} - \phi_{xx}\right)\ dxdt \\ &= \int_{-\infty}^{\infty}\int_{-\infty}^{\infty} u\phi_{tt}\ dtdx - \int_{-\infty}^{\infty}\int_{-\infty}^{\infty} u\phi_{xx}\ dxdt \\ &:= I_1 - I_2 \end{aligned}$$

Where in the second line we split the integral, and then changed the order of integration for the integral on the left. Integration by parts on I_1 gives:

$$\begin{aligned} I_1 &= \int_{-\infty}^{\infty}\left(u\phi_t|_{-\infty}^{\infty} - \int_{-\infty}^{\infty} u_t\phi_t\ dt\right) dx \\ &= -\int_{-\infty}^{\infty}\int_{-\infty}^{\infty} u_t\phi_t\ dtdx \end{aligned}$$

Where we have used that $\phi_t \to 0$ as $t \to \pm\infty$ since $\phi \in C_0^2(\mathbb{R}^2)$. Applying integration by parts again, we see:

$$\begin{aligned} &= -\int_{-\infty}^{\infty}\left(u_t\phi|_{-\infty}^{\infty} - \int_{-\infty}^{\infty} u_{tt}\phi\ dt\right) dx \\ &= \int_{-\infty}^{\infty}\int_{-\infty}^{\infty} u_{tt}\phi\ dtdx \end{aligned}$$

Where we have again used that $\phi \in C_0^2(\mathbb{R}^2)$ to say $\phi \to 0$ as $t \to \pm\infty$. By a symmetric argument that is almost identically the same as the one above, we can see:

$$I_2 = \int_{-\infty}^{\infty}\int_{-\infty}^{\infty} u_{xx}\phi\ dxdt = \int_{-\infty}^{\infty}\int_{-\infty}^{\infty} u_{xx}\phi\ dtdx \quad \text{(switching the order)}$$

Therefore:

$$I = I_1 - I_2 = \int_{-\infty}^{\infty}\int_{-\infty}^{\infty} (u_{tt} - u_{xx})\phi\ dtdx = 0$$

Where we have used $u_{xx} = u_{tt}$ to conclude the above. Thus, as $I = 0$, the strong solution to the wave equation is also a weak solution.

Problem 7 (b)

Firstly:

$$H(x-t) = \begin{cases} 0, & x < t \\ 1, & x \geq t \end{cases}$$

So the weak solution integral is:

$$I = \int_{-\infty}^{\infty}\int_{-\infty}^{\infty} H(x-t)(\phi_{tt} - \phi_{xx})dtdx$$

$$= \int_{-\infty}^{\infty}\int_{-\infty}^{\infty} H(x-t)\phi_{tt}\ dtdx - \int_{-\infty}^{\infty}\int_{-\infty}^{\infty} H(x-t)\phi_{xx}\ dxdt$$

$$\int_{-\infty}^{\infty}\int_{-\infty}^{x} \phi_{tt}\ dtdx - \int_{-\infty}^{\infty}\int_{t}^{\infty} \phi_{xx}\ dxdt$$

$$= \int_{-\infty}^{\infty} \phi_t|_{-\infty}^{x}\ dx - \int_{-\infty}^{\infty} \phi_x|_{t}^{\infty}\ dt$$

$$= \int_{-\infty}^{\infty} \phi_t(x,x)\ dx + \int_{-\infty}^{\infty} \phi_x(t,t)\ dt$$

Where in the second to last line above, we used that $\phi \in C_0^2(\mathbb{R}^2)$ to assert that the derivatives of ϕ vanish at $\pm\infty$. For some variable v, we can combine the integrals above:

$$= \int_{-\infty}^{\infty} \phi_t(v,v) + \phi_x(v,v)\ dv$$

$$= \int_{-\infty}^{\infty} \frac{\partial}{\partial v}\phi(v,v)\ dv$$

$$= \phi(v,v)|_{-\infty}^{\infty}$$

$$= 0$$

Therefore $H(x-t)$ is a weak solution. Now for $H(x+t)$ we have:

$$H(x+t) = \begin{cases} 0, & x < -t \\ 1, & x \geq -t \end{cases}$$

So the weak solution integral is:

$$I = \int_{-\infty}^{\infty}\int_{-\infty}^{\infty} H(x+t)(\phi_{tt} - \phi_{xx}) dtdx$$

$$= \int_{-\infty}^{\infty}\int_{-\infty}^{\infty} H(x+t)\phi_{tt}\ dtdx - \int_{-\infty}^{\infty}\int_{-\infty}^{\infty} H(x+t)\phi_{xx}\ dxdt$$

$$= \int_{-\infty}^{\infty}\int_{-x}^{\infty} \phi_{tt}\ dtdx - \int_{-\infty}^{\infty}\int_{-t}^{\infty} \phi_{xx}\ dxdt$$

$$= \int_{-\infty}^{\infty} \phi_t|_{-x}^{\infty}\ dx - \int_{-\infty}^{\infty} \phi_x|_{-t}^{\infty}\ dt$$

$$= \int_{-\infty}^{\infty} -\phi_t(x,-x)\ dx + \int_{-\infty}^{\infty} \phi_x(-t,t)\ dt$$

$$= \int_{-\infty}^{\infty} \phi_t(-x,x)\ dx + \int_{-\infty}^{\infty} \phi_x(-t,t)\ dt \quad (x \to -x)$$

$$= \int_{-\infty}^{\infty} \phi_t(-v,v) + \phi_x(-v,v)\ dv$$

$$:= I_1 + I_2$$

Now define $F(v) = \phi(-v,v)$ so that by the chain rule:

$$F'(v) = -\phi_x(-v,v) + \phi_t(-v,v) \implies \phi_t(-v,v) + \phi_x(-v,v) = F'(v) + 2\phi_x(-v,v) \quad \textbf{(1)}$$

Then the above:

$$= \int_{-\infty}^{\infty} F'(v) + 2\phi_x(-v,v)\ dv$$

$$= (F(v) = \phi(v,-v))|_{x\to\infty}^{x\to\infty} + 2\int_{-\infty}^{\infty} \phi_x(v,-v)\ dv$$

$$= 0 + 2\int_{-\infty}^{\infty} \phi_x(v,-v)\ dv \quad (\text{as } \phi \in C_0^2(\mathbb{R}^2))$$

$$= 2I_2$$

So we get the equation:

$$I = 2I_2 \implies I_1 = I_2$$

By making the substitution $u = -v$, we see:

$$I_2 = \int_{-\infty}^{\infty} \phi_x(-v, v)\ dv = -\int_{-\infty}^{\infty} \phi_x(u, -u)\ du = -I_2 \quad \text{(from symmetry)}$$

$$\implies I_2 = 0 \implies I_1 = 0 \implies I = 0$$

Thus $H(x+t)$ is a weak solution.