AMATH 569 - Homework 5

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Problem 1

The first thing we show is very similar to **Lecture 21**. Since $a_{ij}, c \in L^\infty(\Omega)$ and Ω is bounded, we estimate the bilinear form:

$$B(u,v) = \sum_{i,j=1}^{n} \int_\Omega a_{ij}(x) u_{x_i} v_{x_j}\, dx + \int_\Omega c(x) uv\, dx$$

Using Hölder's inequality and boundedness of the coefficients, we have:

$$\begin{aligned}
|B(u,v)| &\leq \sum_{i,j=1}^{n} \|a_{ij}\|_{L^\infty} \int_\Omega |u_{x_i}||v_{x_j}|\, dx + \|c\|_{L^\infty} \int_\Omega |u||v|\, dx \\
&\leq \sum_{i,j=1}^{n} \|a_{ij}\|_{L^\infty} \|u_{x_i}\|_{L^2} \|v_{x_j}\|_{L^2} + \|c\|_{L^\infty} \|u\|_{L^2} \|v\|_{L^2} \\
&= C_1 + C_2 \|u\|_{L^2} \|v\|_{L^2} \quad \text{(everything is finite)} \\
&\leq C_3 \|u\|_{H_0^1(\Omega)} \|v\|_{H_0^1(\Omega)} \quad \text{(the H norms are bigger in general)},
\end{aligned}$$

Therefore $B(u,v)$ is continuous on $H_0^1(\Omega)$. Now:

$$\begin{aligned}
B(u,u) &= \sum_{i,j=1}^{n} \int_\Omega a_{ij} u_{x_i} u_{x_j}\, dx + \int_\Omega cu^2 dx \\
&\geq \theta \|\nabla u\|_{L_2}^2 + \int_\Omega cu^2 dx
\end{aligned}$$

Here $\theta > 0$ comes from the definition of uniform ellipticity in $a_{ij}'s$. Now if $c \geq -\mu$ for $\mu > 0$, then the above is:

$$\begin{aligned}
&\geq \theta \|\nabla u\|_{L_2}^2 + \int_\Omega cu^2 dx \\
&\geq \theta \|\nabla u\|_{L_2}^2 - \mu \int_\Omega u^2\ dx \\
&\geq \theta \|\nabla u\|_{L_2}^2 - \mu C_\Omega \|\nabla u\|_{L_2}^2 \quad \text{(by Poincare's inequality)} \\
&= (\theta - \mu C_\Omega) \|\nabla u\|_{L_2}^2 \\
&\geq \frac{(\theta - \mu C_\Omega)}{C_\Omega} \|u\|^2 \quad \text{(by Poincare's inequality again)}
\end{aligned}$$

To make $B = (\theta - \mu C_\Omega)/C_\Omega \geq 0$ choose $\mu \leq (\theta/C_\Omega)$. Choosing μ in this way makes $B(\cdot, \cdot)$ coercive and hence fulfills the desires of the Lax-Milgram lemma. Therefore, we are done.

Problem 2

Define the bilinear form $B(u,v) = \int_\Omega \Delta u \Delta v \; dx$ and the linear functional $F(v) = \int_\Omega fv \; dx$. Our Hilbert space $H = H_0^2(\Omega)$. We first show $B(u,v)$ is continuous:

$$|B(u,v)| = \left| \int_\Omega \Delta u \Delta v \; dx \right| \le \|\Delta u\|_{L_2} \|\Delta v\|_{L_2} \le C \|u\|_H \|v\|_H$$

Where the first inequality is Holder's inequality and the second follows by definition of the H norm. Therefore, $B(u,v)$ is continuous. Now:

$$B(u,u) = \int_\Omega (\Delta u)^2 \; dx = \|\Delta u\|_{L_2}^2$$

Now to show coerciveness, our assumptions meet the requirements for Theorem 4 in **Evans: Chapter 6.3**, so that for some $K > 0$:

$$\|u\|_H^2 \le K^2 \|\Delta u\|_{L_2}^2 \implies B(u,u) \ge (1/K^2) \|u\|_H^2$$

So $B(\cdot,\cdot)$ is coercive. Next we have:

$$|F(v)| \le \int_\Omega |f| \, |v| \; dx \le \|f\|_{L_2} \|v\|_{L_2} \le C \|v\|_H$$

Therefore F is a bounded linear functional on H. All the requirements of the Lax-Milgram lemma are met, therefore there exists a unique $u \in H$ such that for all $v \in H$:

$$B(u,v) = F(v) \iff \int_\Omega \Delta u \Delta v \; dx = \int_\Omega fv \; dx$$

This concludes the proof.

Problem 3

Choose $v = u$, then by assumption:

$$\int_\Omega |\nabla u|^2 \, dx = \lambda \int_\Omega u^2 dx$$

By the Poincare inequality, we have:

$$\int_\Omega u^2 \ dx \leq C_\Omega \int_\Omega |\nabla u|^2 \ dx = \lambda C_\Omega \int_\Omega u^2 \ dx$$

So with $X = \int_\Omega u^2 dx$ we see:

$$X \leq \lambda C_\Omega X$$

As $X \geq 0$, the above implies that $(1/C_\Omega) \leq \lambda$ as desired.

Problem 4

Let $u, v \in \mathcal{S}(\mathbb{R}^n)$ and let α be a multi-index. Then firstly,

$$\begin{aligned}
\mathcal{F}\left(u+v\right)\left(\xi\right) &= (2\pi)^{-n/2} \int_{\mathbb{R}^n} e^{-i\xi\cdot x}(u(x)+v(x))\,dx \\
&= (2\pi)^{-n/2} \left(\int_{\mathbb{R}^n} e^{-i\xi\cdot x} u(x)\,dx + \int_{\mathbb{R}^n} e^{-i\xi\cdot x} v(x)\,dx \right) \\
&= \mathcal{F}\left(u\right)\left(\xi\right) + \mathcal{F}\left(v\right)\left(\xi\right)
\end{aligned}$$

So $\mathcal{F}\left(u+v\right) = \mathcal{F}\left(u\right) + \mathcal{F}\left(v\right)$. Next, integrating by parts:

$$\begin{aligned}
\mathcal{F}\left(D^\alpha u\right)\left(\xi\right) &= (2\pi)^{-n/2} \int_{\mathbb{R}^n} e^{-i\xi\cdot x} D^\alpha u(x)\,dx \\
&= (2\pi)^{-n/2}(-1)^{|\alpha|} \int_{\mathbb{R}^n} D^\alpha \left(e^{-i\xi\cdot x}\right) u(x)\,dx \\
&= (2\pi)^{-n/2}(-1)^{|\alpha|} \int_{\mathbb{R}^n} (-i\xi)^\alpha e^{-i\xi\cdot x} u(x)\,dx \\
&= (i\xi)^\alpha \mathcal{F}\left(u\right)\left(\xi\right)
\end{aligned}$$

So $\mathcal{F}\left(D^\alpha u\right) = (i\xi)^\alpha \mathcal{F}\left(u\right)$. Now, differentiating under the integral sign:

$$\begin{aligned}
D^\alpha \mathcal{F}\left(u\right)\left(\xi\right) &= D^\alpha \left[(2\pi)^{-n/2} \int_{\mathbb{R}^n} e^{-i\xi\cdot x} u(x)\,dx \right] \\
&= (2\pi)^{-n/2} \int_{\mathbb{R}^n} D^\alpha \left(e^{-i\xi\cdot x}\right) u(x)\,dx \\
&= (2\pi)^{-n/2} \int_{\mathbb{R}^n} (-ix)^\alpha e^{-i\xi\cdot x} u(x)\,dx \\
&= \mathcal{F}\left((-ix)^\alpha u\right)\left(\xi\right)
\end{aligned}$$

So $D^\alpha \mathcal{F}\left(u\right) = \mathcal{F}\left((-ix)^\alpha u\right)$. Now for the product identity, we see:

$$\begin{aligned}
\mathcal{F}\left(uv\right)\left(\xi\right) &= (2\pi)^{-n/2} \int_{\mathbb{R}^n} e^{-i\xi\cdot x} u(x)v(x)\,dx \\
&= (2\pi)^{-n/2} \int_{\mathbb{R}^n} u(x) \left[(2\pi)^{-n/2} \int_{\mathbb{R}^n} e^{-i(\xi-\eta)\cdot x} \mathcal{F}\left(v\right)\left(\eta\right)\,d\eta \right] dx \\
&= (2\pi)^{-n} \int_{\mathbb{R}^n} \mathcal{F}\left(v\right)\left(\eta\right) \left[\int_{\mathbb{R}^n} e^{-i(\xi-\eta)\cdot x} u(x)\,dx \right] d\eta \\
&= (2\pi)^{-n} \int_{\mathbb{R}^n} \mathcal{F}\left(v\right)\left(\eta\right) \mathcal{F}\left(u\right)\left(\xi-\eta\right)\,d\eta = (2\pi)^{-n/2} \left(\mathcal{F}\left(u\right) * \mathcal{F}\left(v\right)\right)\left(\xi\right)
\end{aligned}$$

So $\mathcal{F}(uv) = (2\pi)^{-n/2}\mathcal{F}(u) * \mathcal{F}(v)$. Finally, for the convolution identity:

$$\begin{aligned}
\mathcal{F}(u * v)(\xi) &= (2\pi)^{-n/2} \int_{\mathbb{R}^n} e^{-i\xi\cdot x} \int_{\mathbb{R}^n} u(y)v(x-y)\,dy\,dx \\
&= (2\pi)^{-n/2} \int_{\mathbb{R}^n} u(y) \left[\int_{\mathbb{R}^n} v(x-y)e^{-i\xi\cdot x} dx\right] dy \\
&= (2\pi)^{-n/2} \int_{\mathbb{R}^n} u(y)e^{-i\xi\cdot y} \left[\int_{\mathbb{R}^n} v(z)e^{-i\xi\cdot z} dz\right] dy \\
&= (2\pi)^{-n/2}\mathcal{F}(u)(\xi)\mathcal{F}(v)(\xi) = (2\pi)^{n/2}\mathcal{F}(u)\cdot\mathcal{F}(v)
\end{aligned}$$

So $\mathcal{F}(u * v) = (2\pi)^{n/2}\mathcal{F}(u)\cdot\mathcal{F}(v)$, as desired.

Problem 5

From example **5.68** in Renardy and Rogers we have:

$$(\mathcal{F}(1), \phi) = (2\pi)^{n/2}\phi(0)$$

Meaning the FT of 1 is $(2\pi)^{n/2}\delta$.

$$u = \sum_{|\alpha|\leq m} a_\alpha x^\alpha$$

Now for any multi-index α such that $|\alpha| \leq m$, we have from **Problem 4** with $f = 1$:

$$\mathcal{F}\left((-ix)^\alpha 1\right) = D^\alpha (2\pi)^{n/2}\delta = (2\pi)^{n/2} D^\alpha \delta$$

So if $u = \sum_{|\alpha|\leq m} a_\alpha x^\alpha$, then:

$$\mathcal{F}(u) = \sum_{|\alpha|\leq m} a_\alpha \mathcal{F}(x^\alpha) = \sum_{|\alpha|\leq m} \frac{a_\alpha}{(-i)^\alpha} \mathcal{F}\left((-ix)^\alpha\right) = \sum_{|\alpha|\leq m} \frac{a_\alpha}{(-i)^\alpha} (2\pi)^{n/2} D^\alpha \delta$$

$$:= \sum_{|\alpha|\leq m} c_\alpha D^\alpha \delta$$

As desired!

Problem 6

For $u(x) : \mathbb{R}^n \to \mathbb{R} = \exp(-\|x\|^2) = \exp(-x \cdot x)$, we have:

$$\begin{aligned}
\hat{u}(v) &= (2\pi)^{-n/2} \int_{\mathbb{R}^n} \exp(-iv \cdot x) \exp(-x \cdot x)\ dx \\
&= (2\pi)^{-n/2} \prod_{i=1}^{n} \int_{\mathbb{R}} \exp\left(-iv_i x_i - x_i^2\right)\ dx_i \\
&= (2\pi)^{-n/2} \prod_{i=1}^{n} \int_{\mathbb{R}} \exp\left(-\left(x_i^2 + iv_i x_i - \frac{v_i^2}{4}\right) - \frac{v_i^2}{4}\right)\ dx_i \\
&= (2\pi)^{-n/2} \prod_{i=1}^{n} \exp\left(-\frac{v_i^2}{4}\right) \prod_{i=1}^{n} \int_{\mathbb{R}} \exp\left(-\left(x_i + i\frac{v_i}{2}\right)^2\right)\ dx \\
&= (2\pi)^{-n/2} \exp\left(\frac{-\|v\|^2}{4}\right) \prod_{i=1}^{n} \int_{\mathbb{R}} \exp(-u^2)\ du \\
&= (2\pi)^{-n/2} \exp\left(-\frac{\|v\|^2}{4}\right) \prod_{i=1}^{n} \sqrt{\pi} \\
&= 2^{-n/2} \exp\left(-\frac{\|v\|^2}{4}\right)
\end{aligned}$$

In the third to last line we made the obvious u substitution, and in the second to last line used that the 1D Gaussian integral evaluates to $\sqrt{\pi}$. Next, for $u(x) : \mathbb{R} \to \mathbb{R} = 1/(1 + |x|^2) = 1/(1 + x^2)$ we have:

$$\begin{aligned}
\hat{u}(v) &= \int_{\mathbb{R}} \frac{\exp(-ivx)}{1 + x^2}\ dx \\
&:= \int_{\mathbb{R}} g(x)
\end{aligned}$$

The above integrand has poles at $z = \pm i$. As $1/(1 + x^2) < 1/x^2$, by Jordan's lemma, the above integral is equivalent to a contour integral in the upper or lower half plane when $v > 0$ and when $v < 0$ respectively (with opposite contour directions). So when $v > 0$, by the Residue theorem:

$$\hat{u}(v) = 2\pi i \mathrm{Res}(g, i) = 2\pi i \frac{\exp(-v)}{2i} = \pi \exp(-v)$$

Finally when $v < 0$, we have:

$$\hat{u}(v) = -2\pi i \mathrm{Res}(g, -i) = -2\pi i \frac{\exp(v)}{-2i} = \pi \exp(v)$$

When $v = 0$:

$$\hat{u}(v) = \int_{\mathbb{R}} \frac{1}{1+x^2}\, dx = \pi$$

Therefore:

$$\hat{u}(v) = \begin{cases} \pi \exp(v) & v < 0 \\ \pi & v = 0 \\ \pi \exp(-v) & v > 0 \end{cases} \implies \exp(-|v|)$$

Now for $u(x) : \mathbb{R} \to \mathbb{R} = \sin(x)/(1+x^2)$, we have:

$$\begin{aligned} \hat{u}(v) &= \int_{\mathbb{R}} \frac{\exp(-ivx)\sin(x)}{1+x^2}\, dx \\ &:= \int_{\mathbb{R}} g(x)\, dx \end{aligned}$$

The above integrand has poles at $z = \pm i$. As $|\sin(x)/(1+x^2)| < 1/x^2$, by Jordan's lemma, the above integral is equivalent to a contour integral in the upper or lower half plane when $v > 0$ and when $v < 0$ respectively (with opposite contour directions). So when $v > 0$, by the Residue theorem:

$$\hat{u}(v) = 2\pi i \mathrm{Res}(g, i) = 2\pi i \frac{\exp(-v)}{2i} \sin(i) = \pi \sin(i) \exp(-v)$$

When $v < 0$:

$$\hat{u}(v) = -2\pi i \mathrm{Res}(g, -i) = -2\pi i \frac{\exp(v)}{-2i} \sin(-i) = -\pi \sin(i) \exp(v)$$

When $v = 0$, the integrand $\sin(x)/(1+x^2)$ is an odd function over a symmetric domain, so the integral is 0. Hence:

$$\hat{u}(v) = \begin{cases} -\pi \exp(v) \sin(i) & v < 0 \\ 0 & v = 0 \\ \pi \exp(-v) \sin(i) & v > 0 \end{cases} \implies \mathrm{sign}(v) \exp(-|v|) \sin(i)$$

Problem 7

Since Bamdad said it was okay, we only consider $n \geq 3$. First we note that $G \in S'(\mathbb{R}^n)$ by assumption. By the properties proved in **Problem 4** we see, if $\Delta G = \delta$ then:

$$\begin{aligned}
\mathcal{F}\left(\Delta G\right)(v) &= \mathcal{F}\left(\delta\right)(v) \\
\sum_{i=1}^{n} \mathcal{F}\left(\sum_{i=1}^{n} \partial_{x_i}^2 G\right)(v) &= (2\pi)^{-n/2} \\
\hat{G}(v)\sum_{i=1}^{n}(iv_i)^2 &= (2\pi)^{-n/2}
\end{aligned}$$

The above readily implies if $v \neq 0$:

$$\hat{G}(v) = \frac{(2\pi)^{-n/2}}{-\sum_{i=1}^{n} v_i^2} = -\frac{(2\pi)^{-n/2}}{\|v\|^2}$$

So to recover G we can apply the inverse transform:

$$G(x) = \mathcal{F}^{-1}\left(\hat{G}(v)\right) = -(2\pi)^{-n}\int_{\mathbb{R}^n} \frac{\exp(iv \cdot x)}{\|v\|^2}\, dv := -(2\pi)^{-n} I$$

Since the integral is radial, let's restrict our attention to computing $I(r)$, the integral on the ball of radius r, so that as $r \to \infty$ we have $I(r) \to I$. We can rotate the coordinates so that $x = (r, 0, \cdots, 0)$ and $\|x\| = r$, and also make the typical substitution of spherical coordinates $\rho = \|v\|$ so that $dv = \rho^{n-1} d\rho dS_{n-1}$. Here dS_{n-1} is the differential element on $\partial B(0, \rho)$. In this manner, we have $v \cdot x = r\rho\cos(\theta)$ with θ being the angle between x and v. From this, we can further write $dv = \rho^{n-1}\sin^{n-2}(\theta) d\rho d\theta d\Omega$ where $d\Omega$ represents the differential element of the leftover $n-2$ dimensional solid angle independent of θ. Doing this, we have:

$$\begin{aligned}
I(r) &= \int_{\Omega}\int_0^r\int_0^{\pi} \frac{\exp(ir\rho\cos(\theta))}{\rho^2}\rho^{n-1}\sin^{n-2}(\theta)\ d\theta\ d\rho\ d\Omega \\
&= \int_{\Omega}\int_0^r \rho^{n-3}\int_0^{\pi} \exp(ir\rho\cos(\theta))\sin^{n-2}(\theta)\ d\theta\ d\rho\ d\Omega \\
&= \omega_{n-2}\int_0^r \rho^{n-3}\int_0^{\pi} \exp(ir\rho\cos(\theta))\sin^{n-2}(\theta)\ d\theta\ d\rho
\end{aligned}$$

Where above we used that the integral over Ω is the surface area of the unit sphere in $\mathbb{R}^{n-2}$, call it ω_{n-2}. By making the substitution $u = \cos(\theta)$ from

Pythagoras theorem, we can write $\sin(\theta) = (1-u^2)^{1/2}$ and $du = -\sin(\theta)\, d\theta$ so that:

$$\begin{aligned}
I(r) &= \omega_{n-2} \int_0^r \rho^{n-3} \int_{-1}^1 \exp(ir\rho u)(1-u^2)^{\frac{n-3}{2}}\; du\; d\rho \\
&= \omega_{n-2} \int_0^r \rho^{n-3} \left(\int_{-1}^1 \cos(r\rho u)(1-u^2)^{\frac{n-3}{2}}\; du + \int_{-1}^1 \sin(r\rho u)(1-u^2)^{\frac{n-3}{2}}\; du \right) d\rho
\end{aligned}$$

The cosine integral is even over a symmetric domain, so we can write it as twice the same integral from 0 to 1. The sine integral is odd over a symmetric domain, so it evaluates to 0.

$$I(r) = 2\omega_{n-2} \int_0^r \rho^{n-3} \int_0^1 \cos(r\rho u)(1-u^2)^{\alpha-\frac{1}{2}}\; du\; d\rho$$

From *https://functions.wolfram.com/Bessel-TypeFunctions/BesselJ/07/01/01/0001/*, we have the following identity where J_α are Bessel functions of the first kind. Setting $\alpha = \frac{n-2}{2}$ we see:

$$J_\alpha(z) = J_{\frac{n-2}{2}}(z) = \frac{2^{\frac{n-3}{2}}}{\sqrt{\pi}\Gamma(\frac{n-1}{2})} z^{\frac{n-2}{2}} \int_0^1 \cos(zu)(1-u^2)^{\frac{n-3}{2}}\; du$$

So by setting $z = r\rho$ and making the substitution for the cosine integral, we get:

$$I(r) = 2^{\frac{1-n}{2}} \omega_{n-2} \frac{\sqrt{\pi}\; \Gamma\left(\frac{n-1}{2}\right)}{r^{\frac{n-2}{2}}} \int_0^r \rho^{\frac{n-4}{2}} J_{\frac{n-2}{2}}(r\rho)\; d\rho$$

By making the substitution $u = r\rho$, we can further simplify $I(r)$ to:

$$I(r) = 2^{\frac{1-n}{2}}\, \omega_{n-2} \cdot \frac{\sqrt{\pi}\,\Gamma\left(\frac{n-1}{2}\right)}{r^{n-2}} \int_0^{r^2} u^{\frac{n-4}{2}} J_{\frac{n-2}{2}}(u)\, du$$

Taking $r \to \infty$, we see the fundamental solution G takes the form:

$$G(x) = \frac{C_n}{r^{n-2}} = \frac{C_n}{\|x\|^{n-2}}$$

Where C_n is a constant depending on n.