AMATH 563 - Homework 1 (Theory)

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Problem 1

We will show that $\|x\|_p$ fails the parallelogram identity except when $p = 2$. Consider $x = [1, 1, 0, \cdots, 0]^T \in \mathbb{R}^n$ and $y = [1, -1, 0, \cdots, 0]^T \in \mathbb{R}^n$. Then:

$$\|x\|_p = (1^p + 1^p)^{\frac{1}{p}} = 2^{\frac{1}{p}}$$

$$\|y\|_p = (1^p + 1^p)^{\frac{1}{p}} = 2^{\frac{1}{p}}$$

$$\|x + y\|_p = \left\| [2, 0, \cdots, 0]^T \right\|_p = (2^p)^{\frac{1}{p}} = 2$$

$$\|x - y\|_p = \left\| [0, 2, \cdots, 0]^T \right\|_p = (2^p)^{\frac{1}{p}} = 2$$

For the parallelogram identity to be true, we'd need:

$$\|x + y\|_p^2 + \|x - y\|_p^2 = 2(\|x\|_p^2 + \|y\|_p^2)$$

$$4 + 4 = 2(2^{\frac{2}{p}} + 2^{\frac{2}{p}})$$

$$8 = 4 \cdot 2^{\frac{2}{p}}$$

$$2 = 2^{\frac{2}{p}}$$

$$p = 2$$

Since the parallelogram identity only holds for $p = 2$, $\mathbb{R}^n$ equipped with the l^p norm is only a Hilbert space when $p = 2$.

Problem 2 (a) and (b)

To show **(a)**, we work with the contrapositive statement. Suppose that x and y are both non-zero and linearly dependent, that is there exists some $s \in F$ such that $y = sx$. Then:

$$|\langle x, y\rangle| = |\langle x, sx\rangle| = |s| \, \|x\|^2 > 0$$

Where the last inequality follows as $\|x\| = 0 \iff x = 0$. Since $\langle x, y\rangle \neq 0$, x and y are not orthogonal. We have shown that if x and y are linearly dependent, then they are not orthogonal, therefore by contrapositive, if x and y are orthogonal, then they must be linearly independent. To show **(b)**, we proceed by induction. We have already shown the case when $n = 2$ in part (a). Now suppose we have $n \geq 3$ mutually orthogonal vectors $(x_i)_{i=1}^n$, and for some finite sequence $(a_i)_{i=1}^n \subset F$ we have:

$$a_1 v_1 + \cdots + a_n v_n = 0$$

Then for any $i \in [1 \to n]$:

$$0 = \langle v_i, 0\rangle = \langle v_i, a_1 v_1 + \cdots + a_n v_n\rangle = \sum_{j=1}^{n} \langle v_i, a_j v_j\rangle$$

$$= \sum_{j=1}^{n} \overline{a_j} \langle v_i, v_j\rangle = \overline{a_i} \, \|v_i\|^2$$

But as $v_i \neq 0$, it must be that $a_i = 0$. This holds for any i, therefore all the a_i's must be 0, meaning the x_i's are linearly independent.

Problem 3

Consider:

$$\|x_n - x\|^2 = \langle x_n - x, x_n - x \rangle$$

$$= \langle x_n - x, x_n \rangle - \langle x_n - x, x \rangle$$

$$= \langle x_n, x_n \rangle - \langle x, x_n \rangle - \langle x_n, x \rangle + \langle x, x \rangle$$

$$= \|x_n\|^2 - 2 \langle x_n, x \rangle + \|x\|^2$$

As $n \to \infty$, by assumption we have $\|x_n\| \to \|x\|$ and $\langle x_n, x \rangle \to \langle x, x \rangle$, hence:

$$\lim_{n\to\infty} \|x_n - x\|^2 = 2 \|x\|^2 - 2 \langle x, x \rangle = 2(\|x\|^2 - \|x\|^2) = 0$$

$$\implies \lim_{n\to\infty} \|x_n - x\| - 0$$

Which is the desired result.

Problem 4

If $x \perp y$, then by the Pythagorean theorem:

$$\|x + \alpha y\|^2 = \|x\|^2 + |\alpha|^2 \|y\|^2 \geq \|x\|^2$$

Which proves the first direction. For the second direction we proceed by means of the contrapositive. Suppose x and y are not orthogonal so that $\langle x, y \rangle \neq 0$. Now consider the function:

$$f(\alpha) = \|x + \alpha y\|^2 = \langle x + \alpha y, x + \alpha y \rangle$$

$$= \|x\|^2 + 2\alpha \langle x, y \rangle + \alpha^2 \|y\|^2 \quad (\text{as } \alpha \in \mathbb{R})$$

This is a convex parabola in α, and is minimized when its derivative is 0, ie:

$$\frac{df}{d\alpha} = 2 \langle x, y \rangle + 2\alpha \|y\|^2 := 0$$

$$\implies \alpha_{min} = -\frac{\langle x, y \rangle}{\|y\|^2}$$

Therefore, by setting $\alpha := \alpha_{min}$:

$$\|x + \alpha y\|^2 = \|x\|^2 - \frac{2 \langle x, y \rangle^2}{\|y\|^2} + \frac{\langle x, y \rangle^2}{\|y\|^2} = \|x\|^2 - \frac{\langle x, y \rangle^2}{\|y\|^2} < \|x\|^2$$

$$\implies \|x + \alpha y\| < \|x\|$$

Where the last line follows by assumption as, $\langle x, y \rangle^2 / \|y\|^2 > 0$. Thus, if $\|x + \alpha y\| \geq \|x\|$, it must be that $x \perp y$, and the second direction is shown.

Problem 5

Let $A = \{u \in H : \langle u, h_j \rangle = y_j\}$ (the set of vectors in H which satisfy the constraints), and let $S = span\{h_1, \cdots, h_n\}$. The whole space $H = S \oplus S^\perp$ meaning any element $u \in H$ can be expressed as $u = u_1 + u_2$ where $u_1 \in S$ and $u_2 \in S^\perp$. So:

$$\langle u, h_j \rangle = \langle u_1 + u_2, h_j \rangle = \langle u_1, h_j \rangle + \langle u_2, h_j \rangle$$

$$= \langle u_1, h_j \rangle \quad (\text{as } u_2 \in S^\perp \implies \langle u_2, h_j \rangle = 0)$$

So $u \in A \iff u_1 \in A$, and by the Pythagorean theorem:

$$\|u\|^2 = \|u_1 + u_2\|^2 = \|u_1\|^2 + \|u_2\|^2$$

Therefore for any $u \in A$, $\|u_1\| \leq \|u\|$, and we can restrict our attention to vectors in S. Since u is in the span of the the h_j's, for some vector of scalars $c \in F^n$, we have:

$$u = \sum_{i=1}^{n} c_i h_i := c \cdot [h_1, \cdots, h_n] \quad (\text{defining notation})$$

Plugging in the constraints:

$$y_j = \langle u, h_j \rangle = \left\langle \sum_{i=1}^{n} c_i h_i, h_j \right\rangle$$

$$= \sum_{i=1}^{n} c_i \langle h_i, h_j \rangle$$

$$= \sum_{i=1}^{n} A_{ij} c_i \quad (\text{defining } A_{ij} := \langle h_i, h_j \rangle)$$

$$\implies y = Ac$$

As shown in class, since the h_j's are linearly independent, the matrix A is positive definite and symmetric, and therefore invertible, meaning the c above is the unique solution to the above system of equations, and hence (after solving) u is the unique vector in H that actually satisfies the constraints (is in A). As u is the only vector in A, it is the unique minimizer to $\|u\|$ such that $u \in A$. Therefore as desired, the minimizer u^* exists, is unique, and is given by:

$$u^* = (A^{-1}y) \cdot [h_1, \cdots, h_n]$$