2)

a) We have

- $\phi'(\alpha) = \frac{1}{2}\left(1 - \frac{a}{x^2}\right)\Big|_{x=\alpha=\sqrt{a}}$

$$= \frac{1}{2}\left(1 - \frac{a}{a}\right)$$

$$= 0$$

- $\phi''(\alpha) = \frac{a}{x^3}\Big|_{x=\alpha=\sqrt{a}}$

$$= \frac{a}{(\sqrt{a})^3}$$

$$= \frac{1}{\sqrt{a}} \neq 0$$

So by (4.71), $x_{n+1} = \frac{1}{2}\left(x_n + \frac{a}{x_n}\right)$ converges with order $p = 2$.

b) we have, for any $x_0 \in \mathbb{R}$

$$x_1 = \frac{a}{x_0}$$

$$x_2 = \frac{a}{x_1} = \frac{a}{\left(\frac{a}{x_0}\right)} = x_0$$

This means that $x_{2k} = x_0 \quad \forall\, k \geq 0$, and in general does not converge to $\sqrt{a}$, unless $x_0 = \sqrt{a}$.

c) we have,

$$\phi'(\alpha) = 2 + \frac{a}{x^2}\Big|_{x=\alpha=\sqrt{a}}$$

$$= 2 + \frac{a}{a}$$

$$= 3 \neq 0$$

So by (4.71), $x_{n+1} = 2x_n - \frac{a}{x_n}$ does not converge unless $x_0 = \sqrt{a}$.

3) Let $D = [-1,1]$, and $x, y \in D$ with $\phi(x) = \cos(x)$. From (4.87), ϕ is a contraction map on D if $\exists\, \gamma \in (0,1)$ s.t

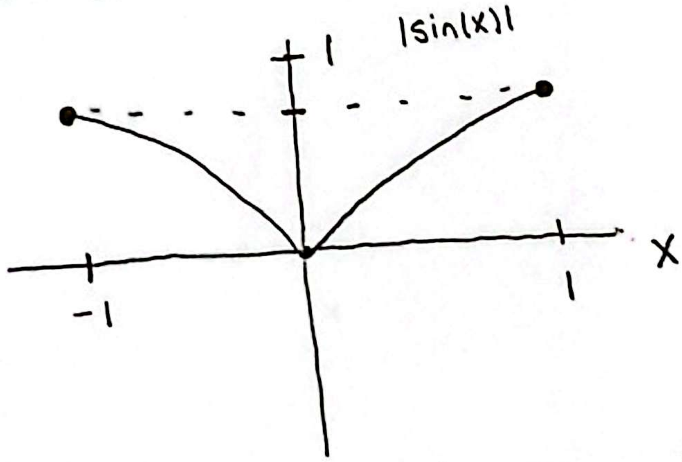
$$|\phi(x) - \phi(y)| \leq \gamma\, |x-y|$$

$$\rightarrow \quad \left| \frac{\phi(x) - \phi(y)}{x-y} \right| \leq \gamma$$

$$\rightarrow \quad \lim_{x \to y} \left| \frac{\phi(x) - \phi(y)}{x-y} \right| \leq \gamma$$

$$\rightarrow \quad |\phi'(y)| \leq \gamma$$

This means if the magnitude of the derivative of ϕ is bounded above by $\gamma \in (0,1)$, on D, then ϕ is a contraction map on D. When $\phi = \cos(x)$, $|\phi'| = |\sin(x)|$, plotting we see that,



$$|\phi'| \leq \underset{\gamma}{\underset{\parallel}{\sin(1)}} < 1,$$ so that on $D = [-1,1]$, $\phi = \cos(x)$ is a contraction.

So by (Theorem 4.9.1) as D is a closed subset of $\mathbb{R}$,

$$\lim_{n \to \infty} x_n = \lim_{n \to \infty} \cos(x_{n-1}) = \alpha$$

for some $\alpha \in [-1, 1]$. Noting, if you start with $x_0 \notin D$, $x_1 = \cos(x_0) \in D$, so we will always end up in D, and once we are in D we are stuck there, what is the value of α? Plugging into my calculator with $x_0 = \frac{\pi}{2}$, we get

$$\alpha \approx 0.739085\ldots, \in [-1, 1] = D$$

4) We have,

$$j=0: \int_0^1 x\,dx = \frac{1}{2} = a_0 + a_1$$

$$j=1: \int_0^1 x^2\,dx = \frac{1}{3} = a_0 x_0 + a_1 x_1$$

$$j=2: \int_0^1 x^3\,dx = \frac{1}{4} = a_0 x_0^2 + a_1 x_1^2$$

$$j=3: \int_0^1 x^4\,dx = \frac{1}{5} = a_0 x_0^3 + a_1 x_1^3$$

Or equivalently,

$$a_0 + a_1 - \frac{1}{2} = 0$$

$$a_0 x_0 + a_1 x_1 - 1/3 = 0$$

$$a_0 x_0^2 + a_1 x_1^2 - 1/4 = 0$$

$$a_0 x_0^3 + a_1 x_1^3 - 1/5 = 0$$

So that,

$$\vec{f}\left(\begin{bmatrix} a_0 \\ a_1 \\ x_0 \\ x_1 \end{bmatrix}\right) = \begin{bmatrix} a_0 + a_1 - 1/2 \\ a_0 x_0 + a_1 x_1 - 1/3 \\ a_0 x_0^2 + a_1 x_1^2 - 1/4 \\ a_0 x_0^3 + a_1 x_1^3 - 1/5 \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \\ 0 \\ 0 \end{bmatrix}$$

The Jacobian is,

$$J_f = \begin{bmatrix} 1 & 1 & 0 & 0 \\ x_0 & x_1 & a_0 & a_1 \\ x_0^2 & x_1^2 & 2a_0x_0 & 2a_1x_1 \\ x_0^3 & x_1^3 & 3a_0x_0^2 & 3a_1x_1^2 \end{bmatrix}$$

Using NumPy, with $\begin{bmatrix} a_0 \\ a_1 \\ x_0 \\ x_1 \end{bmatrix} = \begin{bmatrix} 1/4 + \sqrt{6}/36 \\ 1/4 - \sqrt{6}/36 \\ (6+\sqrt{6})/10 \\ (6-\sqrt{6})/10 \end{bmatrix}$,

we get $\det(J_f) \approx -0.00333\ldots \neq 0$, so that J_f is nonsingular. If you put $\begin{bmatrix} a_0 \\ a_1 \\ x_0 \\ x_1 \end{bmatrix} = \begin{bmatrix} 1 \\ 1 \\ 0 \\ 0 \end{bmatrix}$, then $J_f = \begin{bmatrix} 1 & 1 & 0 & 0 \\ 0 & 0 & 1 & 1 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \end{bmatrix}$ which has zero determinant, and hence is singular.

5) we have for $i = 1, \ldots, n-1$ $(\theta_0 = \alpha, \theta_n = \beta)$

$$\frac{1}{h^2}(\theta_{i-1} - 2\theta_i + \theta_{i+1}) + \sin(\theta_i) = 0$$

$$f_i = \theta_{i-1} - 2\theta_i + \theta_{i+1} + h^2 \sin(\theta_i) = 0$$

The Jacobian, J_f, has entries,

$$(J_f)_{ij} = \frac{\partial f_i}{\partial \theta_j} = \begin{cases} 1 & j = i-1 \\ -2 + h^2\cos(\theta_i) & j = i \\ 1 & j = i+1 \\ 0 & \text{else} \end{cases}$$

To solve with Newton's method, we need to solve (iterate) for $n = 0, 1, \ldots$

$$\begin{cases} J_f(x_n)\Delta_n = -f(x_n) \\ x_{n+1} = x_n + \Delta_n \end{cases}$$

See below for plots/discussions.