AMATH 563 - Homework 1 (Theory)

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May 2025

Problem 1

First let $H_0 = \text{span}\{K(X, x) : x \in X\} \subset H$ be the pre-Hilbert space associated with H (an RKHS with kernel K). As the hint suggests, we will first prove the property holds for $f \in H_0$ and then extend the result to H by continuity. Firstly, any $f \in H_0$ takes the form, for some $\{x_1, \cdots, x_n\} \subset X$ and $\alpha \in \mathbb{R}^n$:

$$f(x) = \sum_{i=1}^{n} \alpha_i K(x, x_i)$$

Then:

$$\phi(f(x)) = \phi\left(\sum_{i=1}^{n} \alpha_i K(x, x_i)\right) = \sum_{i=1}^{n} \alpha_i \phi(K(x, x_i))$$

Now consider:

$$\langle K_\phi, f \rangle = \left\langle K_\phi, \sum_{i=1}^{n} \alpha_i K(x, x_i) \right\rangle = \sum_{i=1}^{n} \alpha_i \langle K_\phi, K(x, x_i) \rangle$$

To show $\phi(f) = \langle K_\phi, f \rangle$, all we must show is that $\phi(K(x, x_i)) = \langle K_\phi, K(x, x_i) \rangle$. However as $K_\phi \in H$, the reproducing property says $\langle K_\phi, K(x, x_i \rangle) = K_\phi(x_i) = \phi(K(x, x_i))$, as desired. This shows that the property holds on H_0. Now, since H is the completion of H_0, we know H_0 is dense in H. We also know ϕ and K_ϕ are continuous. Hence for any sequence of functions $f_i \in H_0 \to f \in H$ we have:

$$\phi(f) = \lim_{n\to\infty} \phi(f_i) = \lim_{n\to\infty} \langle K_\phi, f_i \rangle = \langle K_\phi, f \rangle$$

Which is the desired result.

Problem 2

Let $U = \text{span}\{K_{\phi_1}, \cdots, K_{\phi_m}\}$ and $V = \{f \in H : \phi_j(f) = 0 \;\text{ for } j = 1, \cdots, m\}$. From **Problem 1**, if $\phi_j(f) = 0$, for $f \in V$ we have:

$$0 = \phi_j(f) = \left\langle K_{\phi_j}, f \right\rangle$$

So clearly $f \in U^\perp$, ie $V \subset U^\perp$. Now to show $U^\perp \subset V$, we can show the equivalent contrapositive statement that if $f \notin V$, then $f \notin U^\perp$. If $f \notin V$, then for some $j \in \{1, \cdots, m\}$ we must have:

$$0 \neq \phi_j(f) = \left\langle K_{\phi_j}, f \right\rangle$$

This means f is not orthogonal to the span of the vector $K_{\phi_j} \in S$, so that $f \notin U^\perp$. Thus $U^\perp \subset V$, and we see $U = V^\perp$ as desired.

Problem 3

As θ is invertible, and $\theta_{ij} = \phi_i(K\phi_j) = \left\langle K_{\phi_i}, K_{\phi_j} \right\rangle$, we must have that the vectors $\{K_{\phi_k}\}_{k=1}^m$ are linearly independent. Now the problem takes the exact form as **Homework 2: Problem 5** where the h_j's are the K_{ϕ_j}'s, $A = \theta$ and the constraint $\phi(u) = y$ is the same as the constraints $\langle u, h_j \rangle = \left\langle u, K_{\phi_j} \right\rangle = \phi_j(u) = y_j$. Therefore our solution takes the desired form with $\alpha^* = \theta^{-1}y$:

$$u^* = \sum_{j=1}^{m} \alpha_j^* K_{\phi_j}$$

Problem 4

By the representer theorem, it is immediate that u^* takes the form:

$$u^* = \sum_{j=1}^{m} \alpha_j^* \phi_j(u) = \sum_{j=1}^{m} \alpha_j^* K_{\phi_j}$$

So:

$$\phi_i(u^*) = \langle K_{\phi_i}, u^* \rangle = \left\langle K_{\phi_i}, \sum_{j=1}^{m} \alpha_j^* K_{\phi_j} \right\rangle = \sum_{j=1}^{m} \alpha_j^* \left\langle K_{\phi_i}, K_{\phi_j} \right\rangle \implies \phi(u) = \theta \alpha^*$$

And:

$$\|u^*\|^2 = \langle u^*, u^* \rangle = \left\langle \sum_{i=1}^{m} \alpha_i^* K_{\phi_i}, \sum_{j=1}^{m} \alpha_j^* K_{\phi_j} \right\rangle = \sum_{i=1}^{m} \sum_{j=1}^{m} \alpha_i^* \alpha_j^* \left\langle K_{\phi_i}, K_{\phi_j} \right\rangle = \alpha^{*^T} \Theta \alpha^*$$

Therefore the objection function is equivalent to solving:

$$\alpha^* = \text{argmin}_{\alpha \in \mathbb{R}^m} \left(L(\theta\alpha) + \frac{\lambda}{2} \alpha^T \theta \alpha \right)$$

If we substitute $z = \theta\alpha$ so that $z^* = \theta\alpha^*$, then we see the above is equivalent to solving:

$$z^* = \text{argmin}_{z \in \mathbb{R}^m} \left(L(z) + \frac{\lambda}{2} z^T \theta^{-1} z \right)$$

This is the desired result.