1) Let $V(x,y) = -x^2 - 4y^2$, then

$$-\nabla V = \begin{bmatrix} -(-2x) \\ -(-8y) \end{bmatrix} = \begin{bmatrix} 2x \\ 8y \end{bmatrix} = \begin{bmatrix} \dot{x} \\ \dot{y} \end{bmatrix}$$

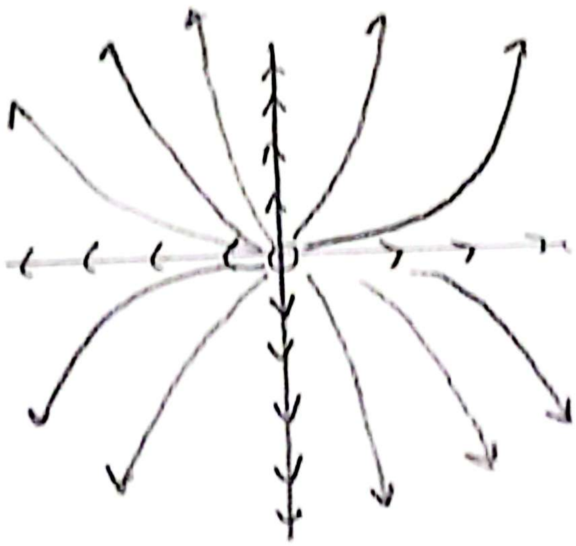
So the system is a gradient system.

Notice,

$$\dot{\vec{x}} = \overbrace{\begin{pmatrix} 2 & 0 \\ 0 & 8 \end{pmatrix}}^{A} \vec{x}$$

with $\Delta = 16 > 0$, $\tau = 10$, $\tau^2 - 4\Delta = 36 > 0$

So the f.p at $\vec{x} = \vec{0}$ is an unstable node.



Note that $\vec{x} = \begin{bmatrix} 1 \\ 0 \end{bmatrix}$ and $\vec{x} = \begin{bmatrix} 0 \\ 1 \end{bmatrix}$ are eigenvectors of A.

2) For $V(x,y) = ax^2 + by^2$ to be a Liaponov function for our system, we need

i) $V(\vec{x}) > 0$, $\forall \vec{x} \neq \vec{x}^*$ and $V(x^*) = 0$

ii) $\dot{V} < 0$, $\forall x \neq \vec{x}^*$

we first find the f.p's

$\dot{x} := 0 = y - x^3$, $\dot{y} := 0 = -x - y^3$

$\rightarrow y = x^3$ $\qquad x = -y^3$

So, $y = (-y^3)^3 = -y^9 \rightarrow y(1+y^8) = 0$

$\rightarrow y = 0$ [ignore complex values]

$\rightarrow x = 0$

So $\vec{x}^* = \vec{0}$ is the only f.p. Now (i) is satisfied if both $a, b > 0$. For (ii),

$$\dot{V} = 2ax \cdot \dot{x} + 2by \cdot \dot{y}$$

$$= 2ax(y - x^3) - 2by \cdot (x + y^3)$$

$$= 2axy - 2ax^4 - 2bxy - 2by^4$$

$$= 2[\underbrace{(a-b)xy}_{A} - \underbrace{a(x^4 + y^4)}_{B}]$$

The above can only be negative if A is, as $B > 0$ $\forall \vec{x} \neq \vec{0}$. To make A not matter, all we need is to set $a = b$. So put $a = b = 1$, then $V(x, y) = x^2 + y^2$ is a Liaponov function for our system.

3) Fix $y \in \mathbb{R}$. Then as

$x \to \infty$, $\dot{x} \to 3$ and $\dot{y} \to 0$ (1)

$x \to -\infty$, $\dot{x} \to \infty$ and $\dot{y} \to \infty$ (2)

This means that regardless on the initial conditions, eventually (from (1)) any trajectory is either moving strictly to the right, or (from (2)) strictly vertically. This implies all trajectories move towards infinity and cannot be confined to any trapping region. Hence our system cannot have any closed orbits.

4)

(a) Our Jacobian is

$$J(x,y) = \begin{bmatrix} 1-3x^2-5y^2 & -1-10xy \\ 1-2xy & 1-x^2-3y^2 \end{bmatrix}$$

At $x=y=0$

$$= \begin{bmatrix} 1 & -1 \\ 1 & 1 \end{bmatrix}$$

So $\Delta = 2$, $\tau = 2$, $\tau^2 - 4\Delta = 4-8 = -4 < 0$.

Hence the f.p at $\vec{0}$ is an unstable spiral,

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(b)

$$r\dot{r} = x\dot{x} + y\dot{y}$$

$$= x(x - y - x^3 - 5xy^2) + y(x + y - yx^2 - y^3)$$

$$= x^2 - \cancel{xy} - x^4 - 5x^2y^2 + \cancel{xy} + y^2 - y^2x^2 - y^4$$

$$= x^2 + y^2 - 6x^2y^2 - x^4 - y^4$$

$$= r^2 - 6r^4 \sin^2\theta \cos^2\theta - r^4\cos^4\theta - r^4 \sin^4\theta$$

$$= r^2 - r^4\left[6\overbrace{\sin^2\theta}^{S^2}\,\overbrace{\cos^2\theta}^{C^2} + \cos^2\theta(1 - \sin^2\theta) + \sin^2\theta(1 - \cos^2\theta)\right]$$

$$= r^2 - r^4\left[6S^2C^2 + \underbrace{C^2} - C^2S^2 + \underbrace{S^2} - S^2C^2\right]$$

$$= r^2 - r^4\left[1 + 4S^2C^2\right]$$

$$= r^2 - r^4 - r^4(4\sin^2\theta\cos^2\theta)$$

$$= r^2 - r^4 - r^4\left[2\sin\theta\cos\theta\right]^2$$

$$= r^2 - r^4 - r^4\sin(2\theta)$$

and if $r \neq 0$

$$\dot{r} = r - r^3 - r^3\sin(2\theta)$$

$$\dot{\theta} = \frac{1}{r^2}\left( x\dot{y} - y\dot{x} \right)$$

$$= \frac{1}{r^2}\left( x^2 + \cancel{yx} - \cancel{yx^3} - xy^3 - \left(\cancel{xy} - y^2 - \cancel{yx^3} - 5xy^3\right)\right)$$

$$= \frac{1}{r^2}\left( x^2 + y^2 + 4xy^3 \right)$$

$$= \frac{1}{r^2}\left( r^2 + 4r^4 \cos\theta \sin^3\theta \right)$$

$$= 1 + 4r^2 \cos\theta \sin^3\theta$$

$$= 1 + 2r^2 \sin(2\theta) \cdot \sin^2\theta$$

(c) To find $r_1$, we want $\dot{r} > 0 \ \forall \theta$.

So $r - r^3 - r^3 \sin^2(2\theta) > 0$

$\leftrightarrow \quad r - r^3 > r^3 \sin^2(2\theta) \geq 0$

$r - r^3 < r^3$

$\leftrightarrow \quad 2r^3 - r > 0$

$\leftrightarrow \quad 2r^2 - 1 > 0$

$r^2 > 1/2$

$\rightarrow \quad r > 1/\sqrt{2}$

So say $r_{max} = \dfrac{1.001}{\sqrt{2}}$

(c) we want $\dot{r} > 0$, $\forall\, \theta$, so

$$r - r^3 - r^3 \sin^2(2\theta) > 0$$

$$r - r^3 > r^3 \sin^2(2\theta) \geq 0$$

small

$$1 - r^2 > 0, \quad r^2 < 1, \quad \boxed{r_2 = 1 - \varepsilon}$$

(d) we want $\dot{r} < 0$, $\forall\, \theta$, so

$$r - r^3 - r^3 \sin^2(2\theta) < 0$$

$$r - r^3 < r^3 \sin^2(2\theta) \leq r^3$$

small

$$2r^3 - r > 0$$

$$2r^2 - 1 > 0 \rightarrow r^2 > 1/2 \rightarrow \boxed{r_1 = \sqrt{1/2} + \varepsilon}$$

e) $\dot{r} := 0 \Leftrightarrow r - r^3\left(1 + \sin^2(2\theta)\right) = 0$

$$\Leftrightarrow 1 - r^2\left(1 + \sin^2(2\theta)\right) = 0$$

$$\Leftrightarrow r^2 = \frac{1}{1 + \sin^2 \theta} = S(\theta)$$

But $1/2 \leq S(\theta) \leq 1$, so for $(r, \theta)$ to be a f.p, we'd need $1/2 \leq r^2 \leq 1$, or $\frac{1}{\sqrt{2}} \leq r \leq 1$, but $\dot{r} = 0$ when $r = \frac{1}{\sqrt{2}}, 1$. So it must be that on $\sqrt{1/2} + \varepsilon \leq r \leq 1 - \varepsilon$, there are no fixed points. Hence we have a limit cycle in the region.