AMATH 561 - Homework 1

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Problem 1 (a):

To describe a probability space, $S = (\Omega, \mathcal{F}, P)$, we need the set of outcomes Ω, a sigma algebra $\mathcal{F}$, and a probability measure P. For this experiment, a biased coin is tossed 3 times, we can denote the set of outcomes as:

$$\Omega = \{X_1X_2X_3 : X_i \in \{H, T\}\}$$

Where $X_1X_2X_3$ represents a set of 3 consecutive coin flips. We can take the sigma algebra $\mathcal{F}$ to be the power set:

$$\mathcal{F} = 2^\Omega$$

Now since the coin is biased, assume the probability of flipping a heads is p and the probability of flipping a tails is $q = 1 - p$. Let k and l be the number of heads and tails flipped respectively and also let n be the total number of ways to achieve k heads and l tails. Then we can describe our probability measure with the following formula for an outcome $\omega \in \Omega$:

$$P(\omega) = np^kq^l = \binom{3}{k}p^kq^{3-k}$$

Since we are flipping 3 times, n is the number of ways to choose k heads out of 3 flips and l (the number of tails) is $3 - k$. This is a discrete probability space, so to show P is a probability measure, we just need to show $P(\omega) \geq 0$ for all $\omega \in \Omega$ and that $\sum_{\omega\in\Omega} P(\omega) = 1$. To show that $P(\omega) \geq 0$, note that p, q, and $\binom{3}{k}$ are all ≥ 0. Next:

$$\sum_{\omega\in\Omega} P(\omega) = \sum_{k=0}^{3} \binom{3}{k}p^kq^{3-k}$$

$$= p^3 + 3pq^2 + 3qp^2 + q^3$$

$$= p^3 + 3p(1-p)^2 + 3(1-p)p^2 + (1-p)^3$$

$$= p^3 - 3p - 6p^2 + 3p^3 + 3p^2 - 3p^3 + 1 + 3p^2 + 3p - p^3$$

$$= 1$$

So P is a probability measure, and $S = (\Omega, \mathcal{F}, P)$ is a probability space.

Problem 1 (b):

To describe a probability space, $S = (\Omega, \mathcal{F}, P)$, we need the set of outcomes Ω, a sigma algebra $\mathcal{F}$, and a probability measure P. For this experiment, two balls are drawn without replacement from an urn which originally contained 2 blue and 2 red balls, we can denote the set of outcomes as:

$$\Omega = \{X_1X_2 : X_i \in \{B, R\}\}$$

Where X_1X_2 represents a set of 2 balls chosen without replacement. We can take the sigma algebra $\mathcal{F}$ to be the power set:

$$\mathcal{F} = 2^{\Omega}$$

Since there are 4 total balls, with 2 red balls and 2 blue balls, the probability of choosing the first ball to be either red or blue is always $\frac{2}{4} = \frac{1}{2}$. The probability of choosing the second ball to be blue or red now depends on what ball was chosen first. If the first ball chosen was red, there is 1 remaining red ball and 2 blue balls, so the probability of choosing red is now $\frac{1}{3}$ and the probability of choosing blue is now $\frac{2}{3}$, and vice versa if the first ball chosen was blue. We can describe our probability measure with the following formula for $\omega = X_1X_2 \in \Omega$:

$$P(\omega) = \begin{cases} \frac{1}{6} & \text{if } X_1 = B \text{ and } X_2 = B \\ \frac{2}{6} & \text{if } X_1 = B \text{ and } X_2 = R \\ \frac{2}{6} & \text{if } X_1 = R \text{ and } X_2 = B \\ \frac{1}{6} & \text{if } X_1 = R \text{ and } X_2 = R \end{cases}$$

This is a discrete probability space, so to show P is a probability measure, we just need to show $P(\omega) \geq 0$ for all $\omega \in \Omega$ (which in this case is obvious) and that $\sum_{\omega \in \Omega} P(\omega) = 1$. So:

$$\sum_{\omega \in \Omega} P(\omega) = \frac{(1+2+2+1)}{6} = \frac{6}{6} = 1$$

Which establishes P as a probability measure and S as a probability space.

Problem 2:

For the purpose of contradiction, suppose that there is a probability measure P on the integers $\mathbb{Z}$ with the discrete σ-algebra $2^{\mathbb{Z}}$ with the translation-invariance property $P(E+n) = P(E)$ for every event $E \in 2^{\mathbb{Z}}$ and every integer n. Take the sequence of events $(E_i)_{i\in\mathbb{Z}} \subset 2^{\mathbb{Z}}$ such that:

$$E_i = \{i\}$$

Note the E_i's are disjoint. By countable additivity we have:

$$P\left(\bigcup_{i\in\mathbb{Z}} E_i\right) = P\left(\bigcup_{i\in\mathbb{Z}}\{i\}\right) = \sum_{i\in\mathbb{Z}} P(\{i\})$$

For every integer i, $\{i\} = \{0+i\}$, so by translation invariance,
$P(\{i\}) = P(\{0+i\}) = P(\{0\})$, the above sum equals:

$$= \sum_{i\in\mathbb{Z}} P(\{0\})$$

Since $P(\{0\})$ is constant, the above sum can only converge if $P(\{0\}) = 0$.
This cannot be the case since if $P(\{0\}) = P(\{i\}) = 0$, then:

$$1 = P(\Omega) = P\left(\bigcup_{i\in\mathbb{Z}}\{i\}\right) = \sum_{i\in\mathbb{Z}} P(\{i\}) = 0$$

Which is a contradiction, as $1 \neq 0$.

Problem 3:

For the purpose of contradiction, suppose that there is a probability measure P on the reals $\mathbb{R}$ with the Borel σ-algebra $B(\mathbb{R})$ with the translation-invariance property $P(E+x) = P(E)$ for every event $E \in B(\mathbb{R})$ and every real x. Take the sequence of events $(E_i)_{i\geq 0} \subset B(\mathbb{R})$.:

$$E_i = (-i-1, -i] \cup (i, i+1]$$

Note the E_i's are disjoint. As stated each $E_i \in B(\mathbb{R})$ since $B(\mathbb{R})$ is generated by the intervals $(a, b]$ for a, $b \in \mathbb{R}$ implying any unions of intervals of that form will be in $B(\mathbb{R})$. By countable additivity and since the union of all E_i's is the whole space $\mathbb{R}$, we have:

$$1 = P(\mathbb{R}) = P\left(\bigcup_{i\geq 0} E_i\right) = P\left(\bigcup_{i\geq 0}(-i-1,-i] \cup (i,i+1]\right)$$
$$= \sum_{i\geq 0} P[(-i-1,-i] \cup (i,i+1]]$$
$$= \sum_{i\geq 0} P((-i-1,-i]) + P((i,i+1])$$

The translation-invariance property gives us that each term in each of the sums is equal, call this value p. Then we have that the above sum:

$$= \sum_{i\geq 0} 2p$$

Which only converges if $p = 0$. Which implies the sum is 0, which leads to another contradiction since $1 \neq 0$.

Problem 4:

We must show that $(1) : \mathcal{F}$ is a σ-algebra and $(2) : P$ is a probability measure on $\mathcal{F}$. To show (1), we need to show that if $A \in \mathcal{F}$ then $A^c \in \mathcal{F}$ and that if $(A_i)_{i\geq 0}$ is a countable sequence of sets in $\mathcal{F}$, then their union is also in $\mathcal{F}$. So suppose $A \in \mathcal{F}$, then either A is countable or A^c is countable by the definition of $\mathcal{F}$. If A is countable, then $(A^c)^c = A$ is in $\mathcal{F}$, hence A^c is also in $\mathcal{F}$. If A is not countable, then A^c is countable which means $A^c \in \mathcal{F}$. Now take a countable sequence $(A_i)_{i\geq 0} \subset \mathcal{F}$. It follows that some of the $A_i's$ could be countable and some of the $A_i's$ could be uncountable. So the union of the $A_i's$ can be written:

$$\bigcup_{i\geq 0} A_i = \left(\bigcup_{j\geq 0 \text{ with } A_j \text{ countable}} A_j\right) \cup \left(\bigcup_{k\geq 0 \text{ with } A_k \text{ not countable}} A_k\right)$$

The union can be written as the union of all the countable sets union-ed with the union of all the uncountable sets. We will show that the complement of this union is in $\mathcal{F}$. So:

$$\left(\bigcup_{i\geq 0} A_i\right)^c = \left(\left(\bigcup_{j\geq 0 \text{ with } A_j \text{ countable}} A_j\right) \cup \left(\bigcup_{k\geq 0 \text{ with } A_k \text{ not countable}} A_k\right)\right)^c$$

$$= \left(\bigcup_{j\geq 0 \text{ with } A_j \text{ countable}} A_j \right)^c \cap \left(\bigcup_{k\geq 0 \text{ with } A_k \text{ not countable}} A_k \right)^c$$

$$= \left(\bigcap_{j\geq 0 \text{ with } A_j \text{ countable}} A_j^c \right) \cap \left(\bigcap_{k\geq 0 \text{ with } A_k \text{ not countable}} A_k^c \right)$$

The intersection depending on the index j may or may not be countable, but the intersection depending on the index k must be countable since it is the intersection of countable sets (since $A_k \in \mathcal{F}$ and A_k is not countable, A_k^c must be countable). The intersection of a countable set with an uncountable set is a countable set. Thus $\bigcup_{i\geq 0} A_i \in \mathcal{F}$ as its compliment is countable. For completion purposes, if all the $A_i's$ are countable, then since countable unions of countable sets are countable, the union would still be in $\mathcal{F}$. Also if all the $A_i's$ are not countable the same argument used above will work again to show that the complement of their union is in $\mathcal{F}$. Thus $\mathcal{F}$ is a σ-algebra. We must now show (2). To do so we must show that $P(\emptyset) = 0$, $P(\Omega) = 1$, and that countable additivity holds. By the definition of P, if A is countable then $P(A) = 0$ and $P(A^c) = 1$, and if A^c is countable then $P(A) = 1$ and $P(A^c) = 0$. Since $\emptyset$ is countable, $P(\emptyset) = 0$, and $1 = P(\emptyset^c) = P(\Omega)$ by definition of P. Now take $(A_i)_{i\geq 0} \subset \mathcal{F}$ all disjoint. To show countable additivity, we want to show that $P\left(\bigcup_{i\geq 0} A_i\right) = \sum_{i\geq 0} P(A_i)$. If all the $A_i's$ are countable, then the union is countable. By the definition of P, $P(A_i) = 0$ for all $i \geq 0$ and also $P\left(\bigcup_{i\geq 0} A_i\right) = 0$ as desired. Now we will show that no 2 uncountable sets inside $\mathcal{F}$ can be disjoint. Suppose for purpose of contradiction that $B, C \in \mathcal{F}$ are disjoint and are uncountable. Then:

$$B \cap C = \emptyset$$

$$(B \cap C)^c = \mathbb{R}$$

$$B^c \cup C^c = \mathbb{R}$$

$$P(B^c \cup C^c) = P(\mathbb{R}) = 1$$

But since B^c and C^c are countable from the definition of $\mathcal{F}$, their union is countable, and by definition of P it must be that $P(B^c \cup C^c) = 0$ which leads to a contradiction where $0 = 1$. Hence, no 2 uncountable sets in $\mathcal{F}$ can be disjoint. So from the countable sequence of disjoint $A_i's$, at most 1 can be uncountable and the rest have to be countable. So since the union of all the

$A_i's$ would be uncountable in this case, we'd have $P\left(\bigcup_{i\geq 0} A_i\right) = 1$. We'd also have that $P(A_i) = 0$ for all i such that A_i is countable, and $P(A_i) = 1$ for the single A_i that is uncountable. So $\sum_{i\geq 0} P(A_i) = 0 + 0 + \cdots + 0 + 1 + 0 + \cdots$. Thus, in all cases countable additivity holds and P is a probability measure. We have shown $(\mathbb{R}, \mathcal{F}, P)$ defines a probability space.