1) a) We have

$$I = \int_0^{\pi/2} \ln(\sin(x))\,dx = \int_0^{\pi/2} \ln(\cos(x))\,dx \quad \left(\text{by } u = \frac{\pi}{2} - x\right)$$

and

$$I = \int_0^{\pi/2} \ln(\cos(x))\,dx = \int_{\pi/2}^{\pi} \ln(\sin(x))\,dx \quad \left(\text{by } u = \frac{\pi}{2} + x\right)$$

So

$$2I = \int_0^{\pi/2} \ln(\sin(x))\,dx + \int_{\pi/2}^{\pi} \ln(\cos(x))\,dx$$

$$= \int_0^{\pi/2} \ln(\sin(x)\cos(x))\,dx = \int_0^{\pi/2} \ln\left(\tfrac{1}{2}\sin(2x)\right)dx$$

$$= \frac{\pi}{2}\ln\left(\tfrac{1}{2}\right) + \int_0^{\pi/2} \ln(\sin(2x))\,dx$$

$$= -\frac{\pi}{2}\ln(2) + \frac{1}{2}\int_0^{\pi} \ln(\sin(x))\,dx$$

$$= -\frac{\pi}{2}\ln(2) + \frac{1}{2}\left[\int_0^{\pi/2} \ln(\sin(x))\,dx + \int_{\pi/2}^{\pi} \ln(\sin(x))\,dx\right]$$

$$= -\frac{\pi}{2}\ln(2) + I$$

$$\rightarrow \boxed{I = -\frac{\pi}{2}\ln(2)}$$

(b) we have,

$$I = \int_0^{\pi} \frac{x\sin(x)}{1+\cos^2(x)}\,dx = \int_0^{\pi} \frac{(\pi-x)\sin(\pi-x)}{1+\cos^2(\pi-x)}\,dx$$

$$= \int_0^{\pi} \frac{(\pi-x)\sin(x)}{1+\cos^2(x)}\,dx = \pi\int_0^{\pi} \frac{\sin(x)}{1+\cos^2(x)}\,dx - I$$

$$\rightarrow \quad I = \frac{\pi}{2}\int_0^{\pi} \frac{\sin(x)}{1+\cos^2(x)}\,dx \rightarrow \quad \begin{aligned} u &= \cos(x) \\ du &= -\sin(x)\,dx \end{aligned}$$

$$= \frac{\pi}{2}\int_{-1}^{1} \frac{1}{1+u^2}\,du = \frac{\pi}{2}\left(\operatorname{atan}(1) - \operatorname{atan}(-1)\right)$$

$$= \frac{\pi}{2}\left(\frac{\pi}{4} + \frac{\pi}{4}\right) = \boxed{\frac{\pi^2}{4}}$$

c) we have,

$$I = \int_0^{\pi/2} \frac{1}{1+(\tan(x))^{\sqrt{2}}}\,dx = \int_0^{\pi/2} \frac{1}{1+\tan\left(\frac{\pi}{2}-x\right)^{\sqrt{2}}}\,dx$$

$$= \int_0^{\pi/2} \frac{1}{1+\cot(x)^{\sqrt{2}}}\,dx$$

So,

$$2I = \int_0^{\pi/2} \frac{1}{1+\tan(x)^{\sqrt{2}}}\,dx + \int_0^{\pi/2} \frac{1}{1+\cot(x)^{\sqrt{2}}}\,dx$$

$$= \int_0^{\pi/2} \frac{1+\cot(x)^{\sqrt{2}} + 1+\tan(x)^{\sqrt{2}}}{1+\tan(x)^{\sqrt{2}}+\cot(x)^{\sqrt{2}}+1}\,dx$$ combined terms

$$= \int_0^{\pi/2} 1\,dx = \frac{\pi}{2}$$

Hence, $\boxed{I = \frac{\pi}{4}}$

2)

(a) $1+i\sqrt{3}$ $\nearrow R = \sqrt{1^2+3} = 2$; $\searrow \theta = \operatorname{atan}\left(\frac{\sqrt{3}}{1}\right) = \pi/3$ $= 2e^{i\pi/3}$

So $(1+i\sqrt{3})^{11} = \left(2e^{i\pi/3}\right)^{11} = 2^{11}e^{i\frac{11\pi}{3}} = 2^{11}e^{i\frac{5\pi}{3}}$

$= 2^{11}\left(\cos\left(5/3\pi\right) + i\sin\left(5/3\pi\right)\right)$

$= 2^{11}\left(1/2 - i\frac{\sqrt{3}}{2}\right) = 2^{10} + i\left(-2^{10}\sqrt{3}\right)$

So $\boxed{a = 2^{10}}$ and $\boxed{b = -2^{10}\sqrt{3}}$

(b) The 5th roots of unity are $w_j = e^{i\frac{2\pi}{5}j}$ for $j \in \{0,1,2,3,4\}$. The principal root is

$(1+i\sqrt{3})^{1/5} = 2^{1/5}e^{i\pi/15}$

So all solutions are given by

$\underline{s_j = 2^{1/5}e^{i\pi/15}e^{i\frac{2\pi}{5}j}}$, $j \in \{0,1,2,3,4\}$

(c)

$$w^{\frac{4}{3}} = -2i$$

$$w^4 = 8i$$

cube both sides

$$\rightarrow \quad w^4 = 2^3 e^{i\pi/2} \quad (\star)$$

The 4th roots of unity are $w_j = e^{i\frac{2\pi}{4}j} = e^{i\frac{\pi}{2}j}$ for $j \in \{0,1,2,3\}$. The principal root of $(\star)$ satisfies

$$w = 2^{3/4} e^{i\pi/8}$$

So all the solutions are given by

$$\underline{w_j^* = 2^{3/4} e^{i\pi/8} e^{i\frac{\pi}{2}j}} \quad \text{for } j \in \{0,1,2,3\}$$

3)

(a) $1 + 10^{-2} + 10^{-4} + \ldots = \sum_{i=0}^{\infty} \left(\frac{1}{10}\right)^{2i}$

$$= \sum_{i=0}^{\infty} \left(\frac{1}{100}\right)^{i} = \frac{1}{1-\frac{1}{100}} = \frac{1}{\frac{99}{100}} = \boxed{\frac{100}{99}}$$

(b) $376.\overline{376} = 376 + 376 \cdot 10^{-3} + 376 \cdot 10^{-6} + \ldots$

$$= 376 \cdot (1 + 10^{-3} + 10^{-6} + \ldots)$$

$$= 376 \sum_{K=0}^{\infty} \left(\frac{1}{10}\right)^{3K}$$

$$= 376 \sum_{K=0}^{\infty} \left(\frac{1}{1000}\right)^{K} = 376 \left(\frac{1}{1-\frac{1}{1000}}\right)$$

$$= 376 \left(\frac{1000}{999}\right) = \boxed{\frac{376{,}000}{999}}$$

(c) $0.\overline{9} = 9 \cdot \frac{1}{10} + 9 \cdot \frac{1}{10^2} + \frac{9}{10^3} + \ldots$

$$= 9 \sum_{K=1}^{\infty} \left(\frac{1}{10}\right)^{K} = 9 \sum_{K=0}^{\infty} \left(\frac{1}{10}\right)^{K+1}$$

$$= \frac{9}{10} \sum_{K=0}^{\infty} \left(\frac{1}{10}\right)^{K} = \frac{9}{10} \frac{1}{1-\frac{1}{10}} = \frac{9}{10} \cdot \frac{10}{9}$$

$$= \boxed{1}$$

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4)

(a) consider $f(x) = x^3 - 1.1$. If $x = (1.1)^{1/3}$, then $f(x) = 0$. Let $x_0 = 0$ and $x_1 = 1$, then the secant method update is

$$x_n = x_{n-1} - f(x_{n-1}) \frac{x_{n-1} - x_{n-2}}{f(x_{n-1}) - f(x_{n-2})}$$

So

$$\underline{x_2} = 1 - \frac{(1^3 - 1.1)(1-0)}{(1^3 - 1.1) - (0^3 - 1.1)}$$

$$= 1 - \frac{(-0.1)}{(-0.1) + 1.1} = 1 + \frac{0.1}{1} = 1.1 = 11/10$$

$\left(\frac{11}{10}\right)^2$

$1.1(1.1^2 - 1)$

and

$$\underline{x_3} = 1.1 - \frac{(1.1^3 - 1.1)(1.1 - 1)}{(1.1^3 - 1.1) - (1^3 - 1.1)}$$

$$= 1.1 - \left[\frac{\frac{11}{10}\left(\frac{121}{100} - \frac{100}{100}\right)}{\frac{11}{10}\left(\frac{121}{100} - \frac{100}{100}\right) + \frac{1}{10}} \right] \cdot \frac{1}{10}$$

$210 + 21 = 231$

$$= \frac{11}{10} - \frac{1}{10}\left[\frac{\frac{11}{10}\,\frac{21}{100}}{\frac{11 \cdot 21}{1000} + \frac{1}{10}} \right] = \frac{11}{10} - \frac{1}{10}\left[\frac{11 \cdot 21}{11 \cdot 21 + 100} \right]$$

$3310 + 331$

$$= \frac{11}{10} - \frac{1}{10}\left[\frac{231}{331} \right] = \frac{331 \cdot 11 - 231}{3310} = \frac{3410}{3310}$$

$$= \boxed{\frac{341}{331}}$$

So this is an approximation of $(1.1)^{1/3}$

(b) Let $f(x) = x^2 - 8.5$. Then $f(x) = 0$ when $x = \pm\sqrt{8.5}$. To ensure a positive approximation, choose $x_0 = 0$, $x_1 = 3$, then the secant update rule is

$$x_n = x_{n-1} - f(x_{n-1}) \frac{x_{n-1} - x_{n-2}}{f(x_{n-1}) - f(x_{n-2})}$$

So

$$x_2 = 3 - \frac{(9-8.5)(3-0)}{(9-8.5) - (0-8.5)}$$

$$= 3 - \frac{\frac{1}{2} \cdot 3}{\frac{1}{2} + \frac{17}{2}} = 3 - \frac{\frac{3}{\cancel{2}}}{\frac{18}{\cancel{2}}} = 3 - \frac{1}{6} = \boxed{\frac{17}{6}}$$

approximates $\sqrt{8.5}$

5) For this problem, the multinomial theorem is helpful

$$(x_1 + \ldots + x_m)^n = \sum_{k_1 + \ldots + k_m = n} \binom{n}{k_1, \ldots, k_m} x_1^{k_1} x_2^{k_2} \ldots x_m^{k_m}$$

where $\binom{n}{k_1, k_2, \ldots, k_m} = \dfrac{n!}{k_1!\, k_2! \ldots k_m!}$

(a) we have $(x+y+z)^7$ and want the term when $k_1 = k_2 = 2$ and $k_3 = 3$, so by the above, the coefficient of $x^2 y^2 z^3$ is $\binom{7}{2,2,3} = \dfrac{7!}{2!\,2!\,3!} = 7 \cdot 6 \cdot 5 = \boxed{210}$

(b) Here $k_1 = 3$, $k_2 = 0$, and $k_3 = 4$. So the coefficient of $x^3 z^4$ is

$$\binom{7}{3,0,4} = \frac{7!}{3!\,0!\,4!} = \frac{7 \cdot \cancel{6} \cdot 5}{\cancel{3} \cdot \cancel{2} \cdot 1} = \boxed{35}$$

(6) a) We have $(x + 2y - 3z + 2w + 5)^{16}$.

Put $K_1 = 2$, $K_2 = 3$, $K_3 = 2$, $K_4 = 5$, $K_5 = 4$ (so they add to 16). The multinomial theorem states the coefficient of

$$x^2 (2y)^3 (-3z)^2 (2w)^5 (5)^4 \quad \text{is} \quad \binom{16}{2,3,2,5,4}$$

So the coefficient of $x^2 y^3 z^2 w^5$ must be,

$$\frac{2^8 3^2 5^4 \cdot 16!}{2!\,3!\,2!\,5!\,4!} = \frac{2^8 3^2 5^4\, 16!}{\underline{2} \cdot \underline{3 \cdot 2} \cdot \underline{2} \cdot \underline{5 \cdot 2^2 \cdot 3 \cdot 2} \cdot \underline{2^2 \cdot 3 \cdot 2}}$$

$$= \frac{2^8 3^2 5^4\, 16!}{2^9 3^3 5} = \boxed{\frac{5^3 \cdot 16!}{6}}$$

7) (a) Let $a = -5$ and $b = 8$. Then

$$\underbrace{a}_{}(\overset{x}{\overset{\shortparallel}{8n+3}}) + b(\overset{y}{\overset{\shortparallel}{5n+2}}) = -5(8n+3) + 8(5n+2)$$

$$= -\cancel{40n} - 15 + \cancel{40n} + 16 = 1$$

So $\exists\ a, b \in \mathbb{Z}$ s.t $ax + by = 1$, so $8n+3$ and $5n+2$ are relatively prime.

(b)

$$250 = 2 \cdot 111 + 28$$
$$111 = 3 \cdot 28 + 27$$
$$28 = 1 \cdot 27 + 1$$
$$27 = 27 \cdot 1 + 0$$

Euclid's algorithm

So $\boxed{\gcd(250, 111) = 1}$, and back-substition now gives

$$250 - 2 \cdot 111 = 28 \rightarrow 250 - 2 \cdot 111 = \frac{111 - 27}{3}$$

$$\rightarrow 3 \cdot 250 - 7 \cdot 111 = -27$$

$$\rightarrow 3 \cdot 250 - 7 \cdot 111 = 1 - 250 + 2 \cdot 111$$

$$\rightarrow \boxed{4 \cdot 250 - 9 \cdot 111 = 1}$$

8) $980220 = 2 \cdot 490110$

$= 2^2 \cdot 245055$

$= 2^2 \cdot 3 \cdot 81685$

$= 2^2 \cdot 3 \cdot 5 \cdot 16337$

$= 2^2 \cdot 3 \cdot 5 \cdot 17 \cdot 961$

$$= \boxed{2^2 \cdot 3 \cdot 5 \cdot 17 \cdot 31^2}$$

These are all primes, so we are done.

$$\begin{array}{r} 961 \\ 17 \overline{)16337} \\ 153 \\ \hline 103 \\ -102 \\ \hline 17 \\ -17 \\ \hline 0 \end{array}$$

$$\begin{array}{r} 31 \\ \times 31 \\ \hline 31 \\ 930 \\ \hline 961 \end{array}$$