AMATH 563 - HOMEWORK 4 (COMPUTATIONAL REPORT)

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1. Introduction

We aim to use the graph Laplacian to approximate the eigenfunctions of the Laplace operator on both regular and irregular domains. Rather than using a mesh, we sample points randomly in the domain and connect nearby points using a kernel-based weighting scheme. This gives a weighted graph from which we construct the Un-Normalized Graph Laplacian. The eigenvectors of this graph Laplacian serve as discrete approximations to the true eigenfunctions of the continuous Laplace operator. These eigenfunctions describe important physical and geometric properties of the domain and appear in the solutions to many partial differential equations, such as the heat and wave equations. We study both a regular domain (a square) and an irregular, L-shaped domain. In each case, we visualize the leading eigenvectors of the graph Laplacian and also the true eigenfunctions. Our results demonstrate how graph-based methods can be used to approximate PDE solutions on complex geometries.

2. Methods

We consider the domain $\Omega = [0,1]^2 \subset \mathbb{R}^2$ and generate m uniformly distributed random points $\{x_1, \dots, x_m\} \subset \Omega$. These points define the vertex set X of a weighted graph $G = (X, W)$, where the weight matrix $W \in \mathbb{R}^{m \times m}$ is constructed via the ε-neighborhood kernel:

$$w_{ij} = \kappa_\varepsilon(\|x_i - x_j\|_2), \quad \kappa_\varepsilon(t) = \begin{cases} (\pi\varepsilon^2)^{-1} & \text{if } t \le \varepsilon \\ 0 & t > \varepsilon \end{cases}$$

The scale parameter ε controls local connectivity and is chosen as:

$$\varepsilon = \frac{C \log(m)^{3/4}}{\sqrt{m}},$$

With constant $C = 1$ used throughout. The Un-Normalized Graph Laplacian is then defined as $L = D - W$, where D is the diagonal degree matrix with entries $D_{ii} = \sum_j w_{ij}$.

To approximate the spectrum of the continuous Laplace operator $\mathcal{L}f = -\left(\frac{\partial^2 f}{\partial x_1^2} + \frac{\partial^2 f}{\partial x_2^2}\right)$ under Neumann boundary conditions, we compute the first four eigenvectors $\{q_0, q_1, q_2, q_3\}$ of the Graph Laplacian L using the "eigsh" function from `scipy.sparse.linalg`. These are then compared to the analytic eigenfunctions:

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$$\psi_j(x) = \cos(n\pi x_1)\cos(k\pi x_2)$$

Where $(n, k, j) \in \{(0,0,0), (1,0,1), (0,1,2), (1,1,3)\}$ and the ψ_j's are evaluated at the sampled points X and normalized with the l^2 norm to yield the vectors $\{\psi_1, \psi_2, \psi_3, \psi_4\}$.

To quantify the similarity between the subspaces spanned by $\{q_1, q_2, q_3, q_4\}$ and $\{\psi_1, \psi_2, \psi_3, \psi_4\}$ we define the projection operators $P_Q = QQ^\top$ and $P_\Psi = \Psi\Psi^\top$, where $Q, \Psi \in \mathbb{R}^{m\times 4}$ are the matrices whose columns are the corresponding eigenvectors. We then compute the "projection error" as defined by:

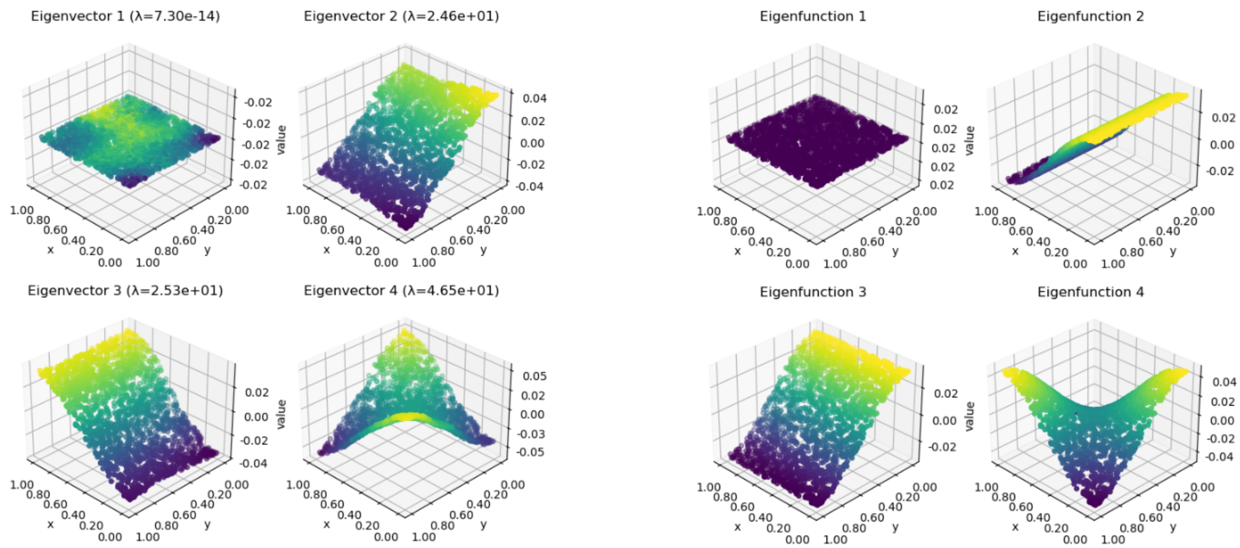
$$\text{error}(m) := \|P_Q P_\Psi - P_\Psi P_Q\|_F$$

This error is computed for $m \in \{2^7, 2^8, 2^9, 2^{10}\}$ and averaged over 30 independent trials for each m. Lastly, to explore non-standard domains, we approximate the spectrum of $\mathcal{L}$ on the L-shaped region:

$$\Omega = [0,1]^2 \cup ([1,2]\times[0,1]) \cup ([0,1]\times[1,2])$$

Which corresponds to a square $[0,2]\times[0,2]$ with the top-right quadrant removed. We uniformly sample $m = 2^{13}$ points over this domain, construct the associated graph Laplacian, and extract the eigenvectors $\{q_6, q_7, q_8, q_9\}$ for visualization and analysis. We also plot the corresponding eigenfunctions $\{\psi_6, \psi_7, \psi_8, \psi_9\}$ and compute the associated subspace error as done earlier but for $(n, k, j) \in \{(2,1,6), (1,2,7), (2,2,8), (3,0,9)\}$. Our results are visualized in the next section:

3. Results



(A) Eigenvectors $0, 1, 2, 3$ of Graph Laplacian (labels shifted by 1).

(B) Eigenfunctions $0, 1, 2, 3$ of Laplace Operator on Box (labels shifted by 1).

Figure 1. Comparing eigenvectors of graph Laplacian to eigenfunctions of Laplace operator.

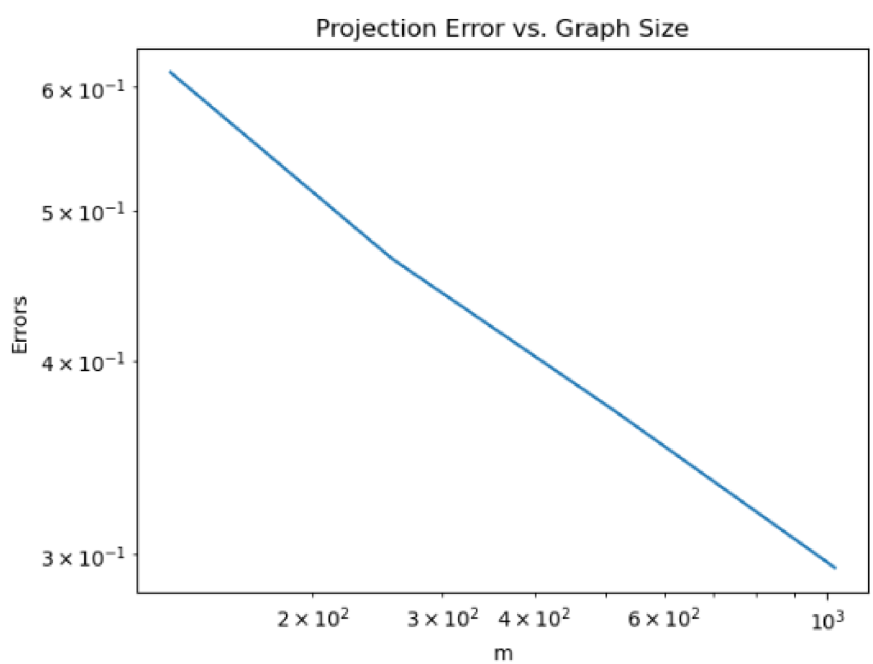


Figure 2. The average projection error as a function of graph nodes m for the square $[0, 2] \times [0, 2]$ domain.

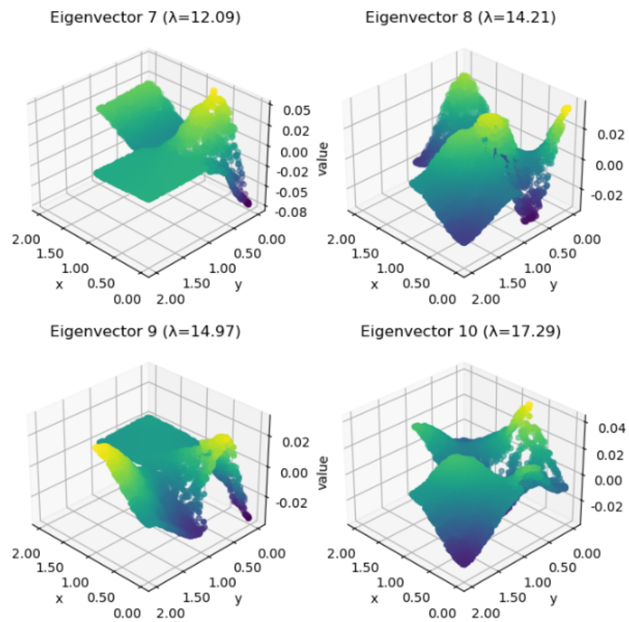


FIGURE 3. Eigenvectors 6, 7, 8, 9 for L shaped domain (labels shifted by 1).

4. Summary and Conclusions

Our results show that the first four eigenvectors of the Graph Laplacian closely resemble the first four eigenfunctions of the continuous Laplace operator, up to a rotation or reflection. This is expected, as eigenvectors are only defined up to scalar multiples and orientation. As the number of graph nodes increases, the projection error between the subspaces spanned by the numerical and analytical eigenfunctions decreases, though the convergence is relatively slow. Nonetheless, this demonstrates that our graph-based method can effectively approximate Laplace eigenfunctions, even with randomly sampled points and no explicit mesh. We also applied the method to an L-shaped domain, observing coherent structure in the higher-index eigenvectors. Although we do not have closed-form expressions for the true eigenfunctions on this irregular domain, the results suggest that the Graph Laplacian continues to provide meaningful spectral information in more complex geometries. Future work could involve comparing these numerical approximations to known analytic or finite element solutions on the L-shaped domain, if available, to better assess accuracy.

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References

[1] B. Hosseni. Lecture notes for AMATH 563: Lecture Notes. Lecture notes distributed in class, 2025. University of Washington, unpublished lecture notes.

[1]