AMATH 567 - Homework 2

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Problem 1:

According to the textbook, a circle on the sphere is the locus of points where some plane $AX + BY + CZ = D$ intersects the sphere with (X, Y, Z) being a point on the sphere. We considered a sphere with radius 1 who's south pole is touching $(0, 0, 0)$ and who's north pole is touching $(0, 0, 2)$. In the chapter, they developed a one-to-one correspondence between points on the sphere and points in the complex plane. So to show that a circle in the complex plane corresponds to a circle on the sphere, we can show that the points on a circle on the sphere simply map to a circle in the complex plane. To begin, from the textbook we have the following relationship between points on the sphere (X, Y, Z) and points on the plane $z = x + iy$:

$$X = \frac{4x}{4 + |z|^2}$$

$$Y = \frac{4y}{4 + |z|^2}$$

$$Z = \frac{2|z|^2}{4 + |z|^2}$$

We can plug these coordinates into the equation of the plane, as any points on the sphere that satisfy the plane equation are the points on the sphere we are interested in. Doing so we get:

$$\frac{1}{|z|^2 + 4} \cdot (4Ax + 4By + 2C|z|^2) = D$$

$$4Ax + 4By + 2C|z|^2 = D|z|^2 + 4D$$

$$(D - 2C)|z|^2 - 4Ax - 4By + 4D = 0$$

$$(D - 2C)(x^2 + y^2) - 4Ax - 4By + 4D = 0$$

If $D - 2C = 0$, then this is the formula of a line in the complex plane. If $D - 2C \neq 0$, then since the coefficients of x^2 and y^2 are the same, the above formula is the equation of some circle in the complex plane, and we are done (if they weren't the same it would be an ellipse).

Problem 2:

For this problem, we will use the limit definition of the derivative. We can rewrite $Re(z) = \frac{z+\bar{z}}{2}$ and that $Im(z) = \frac{z-\bar{z}}{2i}$. So:

$$(Re(z))' = \lim_{h\to 0} \frac{Re(z+h) - Re(z)}{h} = \lim_{h\to 0} \left(\frac{(z+h) + \overline{(z+h)} - (z+\overline{z})}{2h} \right)$$

$$= \lim_{h\to 0} \frac{z + h + \overline{z} + \overline{h} - z - \overline{z}}{2h}$$

$$= \lim_{h\to 0} \frac{h}{2h} + \frac{\overline{h}}{2h}$$

$$= \frac{1}{2} + \frac{1}{2} \lim_{h\to 0} \frac{\overline{h}}{h}$$

We can let $h = Re^{i\theta}$ so that the limit above becomes:

$$\lim_{h\to 0} \frac{\overline{h}}{h} = \lim_{R\to 0} \frac{Re^{-i\theta}}{Re^{i\theta}} = \lim_{R\to 0} e^{-2i\theta} = e^{-2i\theta}$$

Since this limit changes depending on the value of θ, this limit is undefined, and hence the function $f(z) = Re(z)$ is nowhere differentiable. Now for the case of $Im(z)$:

$$(Im(z))' = \lim_{h\to 0} \frac{Im(z+h) - Im(z)}{h} = \lim_{h\to 0} \left(\frac{(z+h) - \overline{(z+h)} - (z-\overline{z})}{2ih} \right)$$

$$= \lim_{h\to 0} \frac{z + h - \overline{z} - \overline{h} - z + \overline{z}}{2ih}$$

$$= \lim_{h\to 0} \frac{h}{2ih} - \frac{\overline{h}}{2ih}$$

$$= \frac{1}{2i} - \frac{1}{2i} \lim_{h\to 0} \frac{\overline{h}}{h}$$

And from the first part of this problem, we know this limit is not defined, hence the function $f(z) = Im(z)$ is nowhere differentiable.

Problem 3:

Let's begin by computing the integral:

$$I = \int_1^z \frac{1}{\sqrt{x^2-1}} dx$$

Let $x = \sec\theta$, then $dx = \sec\theta\tan\theta d\theta$. The lower bound of the integral changes from $x = 1$ to $\theta = 0$ and the upper bound changes from $x = z$ to $\theta = arcsec(z)$ and the integral becomes:

$$I = \int_0^{arcsec(z)} \frac{\sec\theta\tan(\theta)}{\sqrt{\sec^2(\theta)-1}} d\theta = \int_0^{arcsec(z)} \frac{\sec\theta\tan(\theta)}{\tan(\theta)} d\theta = \int_0^{arcsec(z)} \sec(\theta) d\theta$$

Where we have used the fact that $\sec^2\theta = \tan^2\theta + 1$. From Calculus 2, the anti-derivative of $\sec\theta$ is $\log|\sec\theta + \tan\theta| + C$, so the above integral can be simplified further:

$$I = \log|\sec(arcsec(z)) + \tan(arcsec(z))| - \log|\sec 0 + \tan 0|$$

$$I = \log|z + \tan(arcsec(z))|$$

To simplify $\tan(arcsec(z))$, let $u = arcsec(z)$, then $\sec(u) = z$, which by drawing a right triangle implies $\tan(u) = \sqrt{z^2-1}$, so:

$$I = \log|z + \sqrt{(z^2-1)}|$$

The inside of the logarithm is positive for $z > 1$, so we can drop the absolute value to have $I = \log(z + \sqrt{(z^2-1)})$. If $\phi(z) = z + \sqrt{(z^2-1)}$, we have shown that $\log\phi(z) = z + \sqrt{(z^2-1)}$ for $z > 1$ as desired.

Problem 4:

We can begin by looking for zeros of $\tan z$:

$$\tan z = -i \cdot \frac{e^{iz} - e^{-iz}}{e^{iz} + e^{-iz}} = 0$$

$$e^{iz} - e^{-iz} = 0$$

$$e^{iz} = e^{-iz}$$

$$\cos(z) + i\sin(z) = \cos(z) - i\sin(z)$$

$$\sin(z) = -\sin(z)$$

$$2\sin(z) = 0$$

$$z = n\pi \text{ for } n \in \mathbb{Z}$$

So the zeroes of $\tan(z)$ are the integer multiples of π. Now for $\tanh z$:

$$\tanh z = \frac{\sinh z}{\cosh z} = -i \cdot \frac{\sin iz}{\cos iz} = -i\tan iz = 0$$

The zeroes of $\tan(u)$ are when $u = n\pi$ for $n \in \mathbb{Z}$, but in our case $u = iz$. So $iz = n\pi$, or $z = -ni\pi$. In other words, the zeros of the function $\tanh z$ are the integer multiples of π on the imaginary axis as opposed to $\tan z$ whose zeroes were the integer multiples of π on the real axis.

Problem 5:

To plot $|f_\epsilon(z)|$ for various values of ϵ, I used Python. I made a $2D$ mesh to represent the complex plane, and then plotted $|f_\epsilon(z)|$ as a surface $S(x, y)$ in $\mathbb{R}^3$. Here are the results for 5 values of ϵ in the center $0.5x0.5$ square of the complex plane:

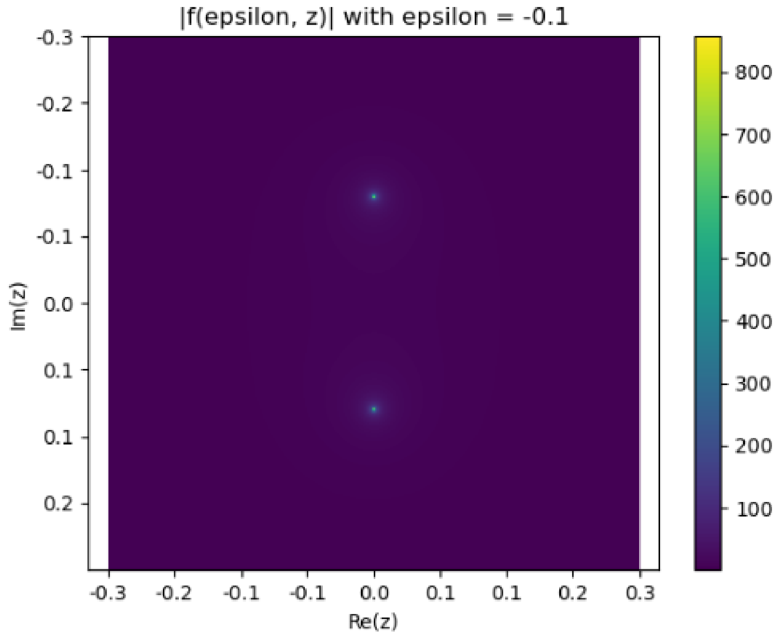

Figure 1: Plot of $f_{-0.1}(z)$

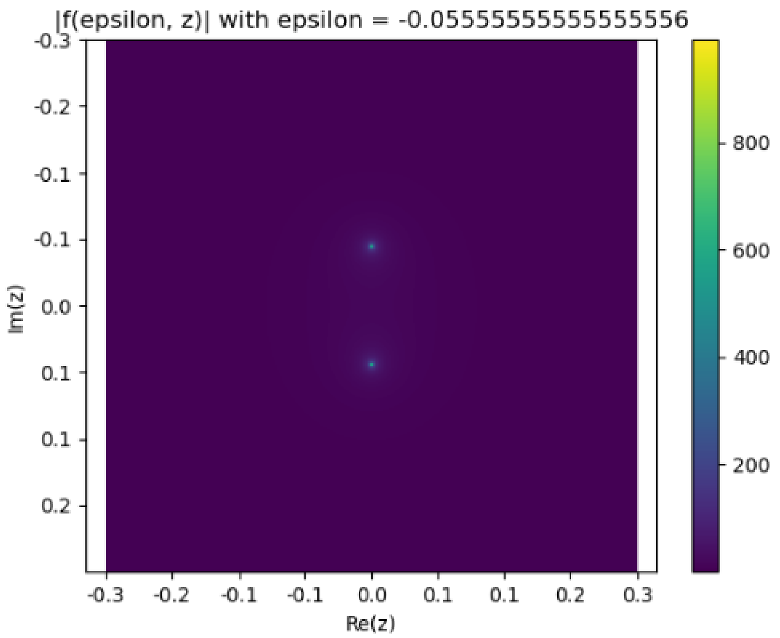

Figure 2: Plot of $f_{-0.055}(z)$

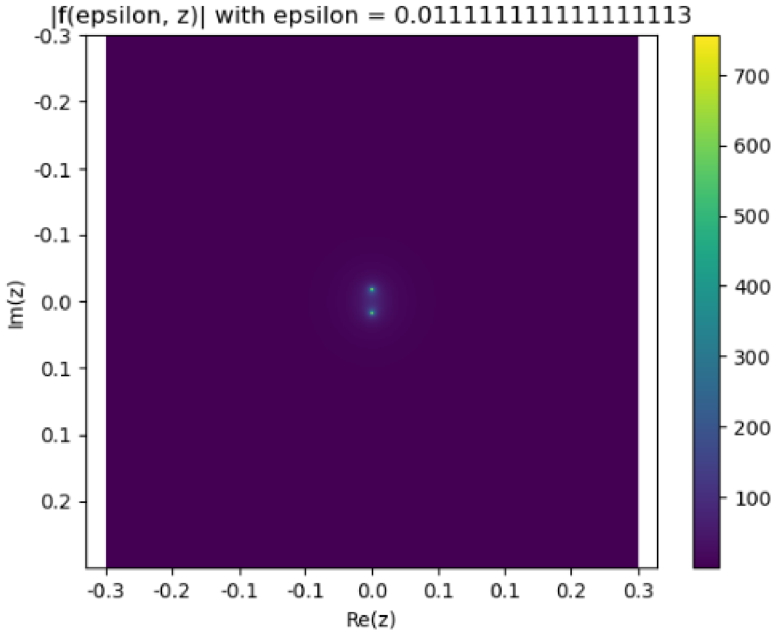

Figure 3: Plot of $f_{0.01}(z)$

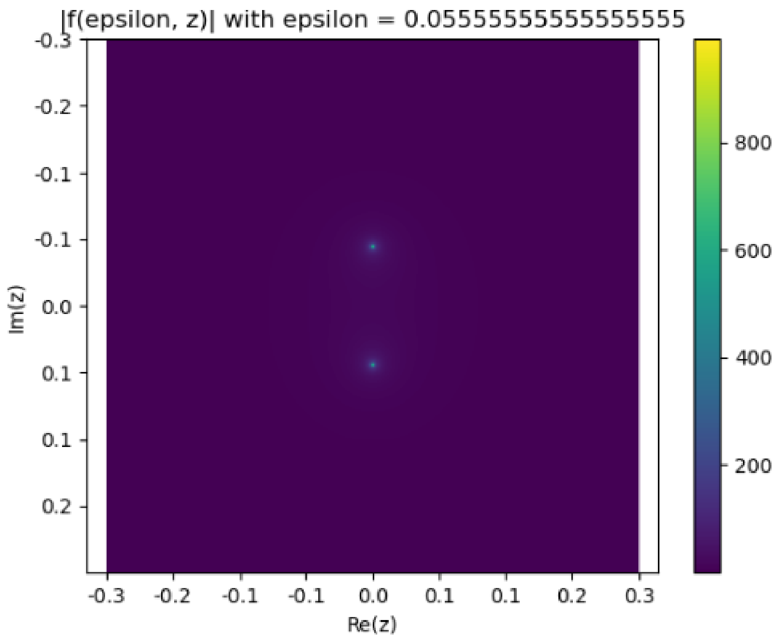

Figure 4: Plot of $f_{0.055}(z)$

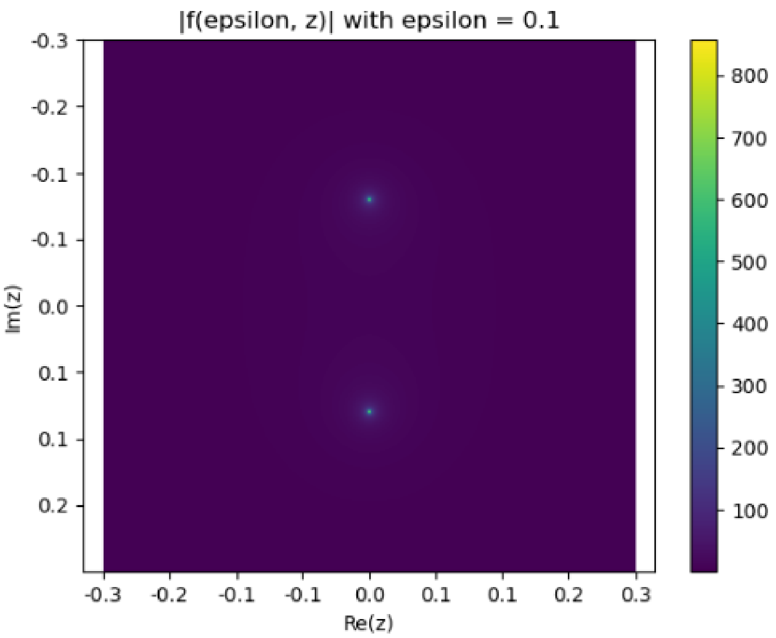

Figure 5: Plot of $f_{0.1}(z)$

You can see the 2 singularities caused by $z = \pm i\epsilon$ as bright peaks, so as $\epsilon \to 0$ you can see the singularities approaching the real line where $Re(z) = 0$ and $Im(z) = 0$ and kind of combining together into 1 singularity. As ϵ gets larger again, you can see the 2 singularities separating. What I gather from this is that if you can write a real function as the limit of complex valued functions, then the singularities of the complex valued functions could approach a singularity of the real valued function. Now we compute:

$$\begin{aligned} I = \int_{-\infty}^{\infty} f_\epsilon(x)dx &= \int_{-\infty}^{\infty} \frac{\epsilon}{\epsilon^2 + x^2}dx \\ &= \frac{1}{\epsilon}\int_{-\infty}^{\infty} \frac{1}{1 + \left(\frac{x}{\epsilon}\right)^2}dx \end{aligned}$$

If we let $u = \frac{x}{\epsilon}$ then $dx = \epsilon du$, and the bounds of the integral don't change. So the integral becomes:

$$I = \int_{-\infty}^{\infty} \frac{1}{1 + u^2}du$$

From Calculus 2, the anti-derivative of this function is $\arctan(u) + C$, so the integral is:

$$I = \lim_{u \to \infty} \arctan u - \lim_{u \to -\infty} \arctan u$$

$$= \frac{\pi}{2} - \frac{-\pi}{2}$$

$$= \pi$$

Problem 6(a):

I have plotted the plots below:

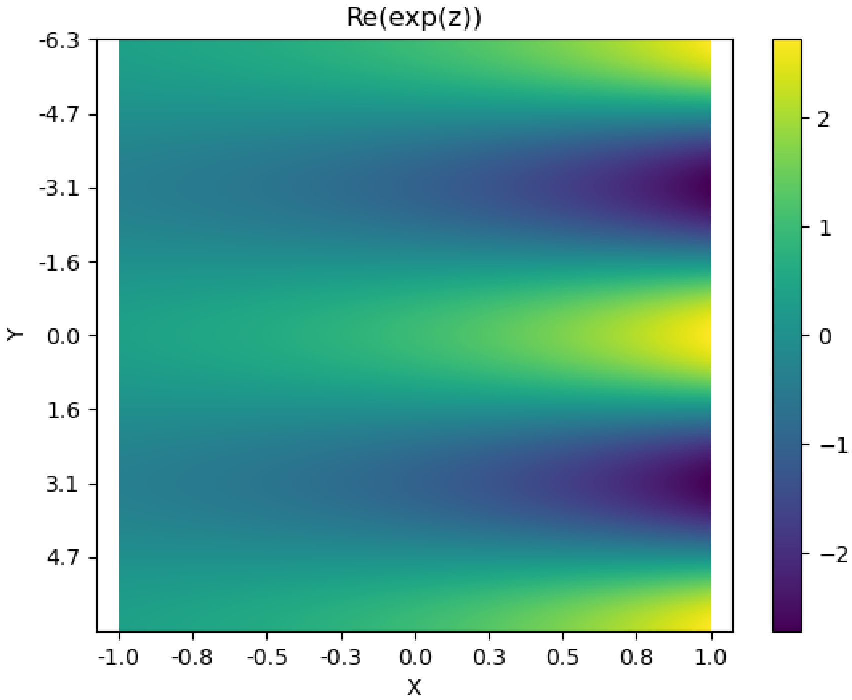

Figure 6: Plot of $Re(e^z)$.

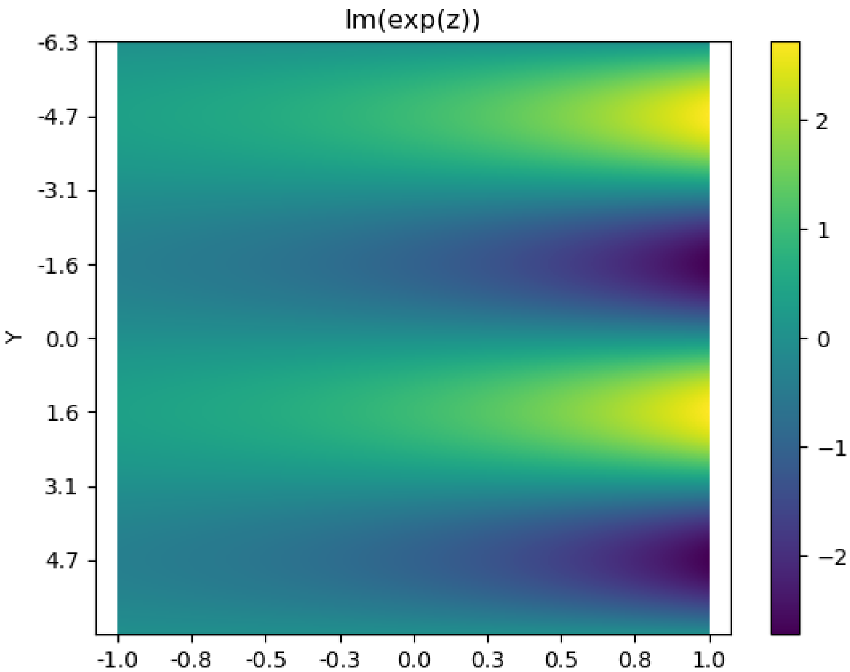

Figure 7: Plot of $Im(e^z)$.

The plots of $Re(e^z)$ and $Im(e^z)$ seem identical up to a phase shift, which makes sense since $\sin(y)$ and $\cos(y)$ are the same functions just shifted by $\frac{\pi}{2}$.

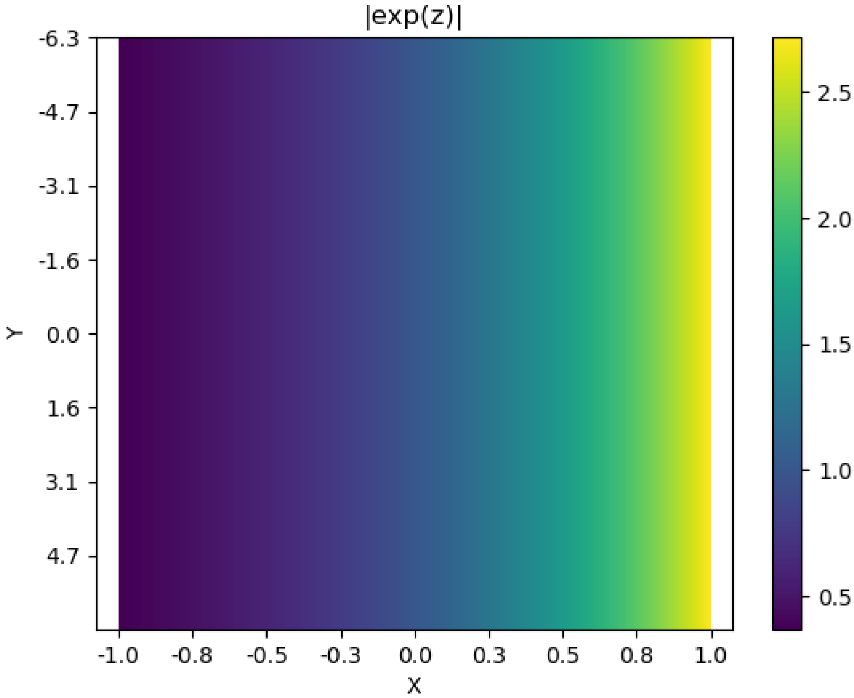

Figure 8: Plot of $|e^z|$.

The modulus looks like a kind of superposition between the real and imaginary parts where the function grows larger as the real part of z increases and smaller as the real part of z decreases. It looks like the function gets, for the lack of better words, filled in when taking the modulus.

Problem 6(b):

Here is the phase plot of e^z:

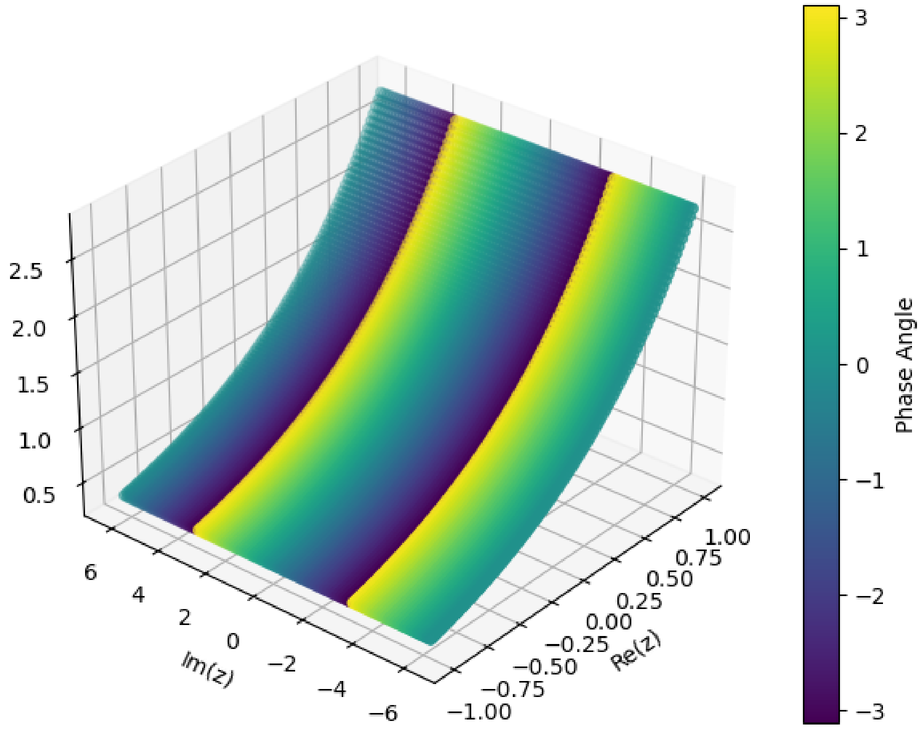

Figure 9: Plot of $|e^z|$ where the color map describes the phase of the function. I computed the phase as $\phi = atan2(Im(e^z), Re(e^z))$ so that the phase ranges from $-\pi \to \pi$.

Problem 7(a):

We can write:

$$f(x, y) = (x + 1) + i(-y) = u(x, y) + iv(x, y)$$

The Cauchy-Riemann equations require that $u_x = v_y$ and that $v_x = -u_y$. However, notice that $u_x = 1$ and that $v_y = -1$ which are never equal. So $f(x, y)$ is nowhere analytic.

Problem 7(b):

We can write:

$$f(x, y) = (y^3 + -3x^2y) + i(x^3 - 3xy^2 + 2) = u(x, y) + iv(x, y)$$

The Cauchy-Riemann equations require that $u_x = v_y$ and that $v_x = -u_y$. We have:

$$u_x = -6xy$$

$$v_y = -6xy$$

$$v_x = 3x^2 - 3y^2$$

$$-u_y = -(3y^2 - 3x^2) = 3x^2 - 3y^2$$

So the Cauchy-Riemann equations are satisfied for all $z = x + iy \in \mathbb{C}$. We can thus compute the derivative as:

$$f^{'}(x, y) = u_x + iv_x = -6xy + i \cdot 3(x^2 - y^2)$$

Problem 7 (c):

We can write:

$$f(x, y) = (e^y \cos(x)) + i(e^y \sin(x)) = u(x, y) + iv(x, y)$$

The Cauchy-Riemann equations require that $u_x = v_y$ and that $v_x = -u_y$. We have:

$$u_x = -e^y \sin(x)$$

$$v_y = e^y \sin(x)$$

$$v_x = e^y \cos(x)$$

$$-u_y = -(e^y \cos(x)) = -e^y \cos(x)$$

So we must check for the values of x and y such that the Cauchy-Riemann equations hold:

$$u_x = v_y$$

$$2e^y \sin x = 0$$

$$\sin x = 0$$

$$x = n\pi \text{ for } n \in \mathbb{Z}$$

The first CR equation is satisfied for any value of y when x is an integer multiple of π. For the second equation:

$$v_x = -u_y$$

$$2e^y \cos(x) = 0$$

$$\cos(x) = 0$$

$$x = \frac{k\pi}{2} \text{ for } k \in \mathbb{Z}_{odd}$$

The second CR equation is satisfied for any value of y when x is an odd integer multiple of $\frac{\pi}{2}$. The conditions on (x, y) from the first equation and second equation do not overlap, meaning there is no set of coordinates where both CR equations hold, so this function is analytic nowhere.

Problem 8:

As $w = f(z) = f(x + iy)$ and $w = u + iv$ (w is a function of x and y), we can write Ω in terms of $w(x, y) = u(x, y) + iv(x, y)$. To make this problem easier to type I will be using subscript notation for the partial derivatives of functions. So:

$$\Omega(x, y) = \phi(u(x, y), v(x, y)) + i\psi(u(x, y), v(x, y))$$

Now, it simply follows from the multivariate chain-rule that:

$$\phi_x = u_x\phi_u + v_x\phi_v$$

Which is the first desired result. Now, if we differentiate both sides again with respect to x:

$$\frac{\partial^2\phi}{\partial x^2} = \frac{\partial}{\partial x}\phi_x = u_x\phi_u + v_x\phi_v$$

$$= \frac{\partial}{\partial x}u_x\phi_u + \frac{\partial}{\partial x}v_x\phi_v$$

We must now apply the product rule to each piece:

$$= u_x\phi_{xu} + \phi_u u_{xx} + v_x\phi_{xv} + \phi_v v_{xx}$$

Now, by the equality of mixed partials, we can say that $\phi_{xu} = \phi_{ux}$ and that $\phi_{xv} = \phi_{vx}$. Writing this way, we substitute the formula we got for ϕ_x above and apply the product rule again:

$$\phi_{xu} = \phi_{ux} = \frac{\partial}{\partial u}\phi_x = \frac{\partial}{\partial u}(u_x\phi_u + v_x\phi_v) = u_x\phi_{uu} + \phi_u u_{ux} + v_x\phi_{uv} + \phi_v v_{ux}$$

$$= u_x\phi_{uu} + v_x\phi_{uv} + \phi_v v_{ux}$$

And similarly for ϕ_{xv}:

$$\phi_{xv} = \phi_{vx} = \frac{\partial}{\partial v}\phi_x = \frac{\partial}{\partial v}(u_x\phi_u + v_x\phi_v) = u_x\phi_{vu} + \phi_u u_{vx} + v_x\phi_{vv} + \phi_v v_{vx}$$

$$= u_x\phi_{vu} + \phi_u u_{vx} + v_x\phi_{vv}$$

So making these substitutions:

$$\phi_{xx} = u_x\phi_{xu} + \phi_u u_{xx} + v_x\phi_{xv} + \phi_v v_{xx}$$

$$= u_x(u_x\phi_{uu} + v_x\phi_{uv} + \phi_v v_{ux}) + \phi_u u_{xx} + v_x(u_x\phi_{vu} + \phi_u u_{vx} + v_x\phi_{vv}) + \phi_v v_{xx}$$

$$= (u_x)^2\phi_{uu} + u_x v_x\phi_{uv} + u_x\phi_v v_{ux} + \phi_u u_{xx} + v_x u_x\phi_{uu} + v_x\phi_u\phi_{vv} + (v_x)^2\phi_{vv} + \phi_v v_{xx}$$

Now since Ω is analytic, we can use the CR equations to simplify the above expression further, namely we can say $v_x = -u_y$ so that the above simplifies to:

$$= u_{xx}\phi_u - u_{xy}\phi_v + (u_x)^2\phi_{uu} - 2u_x u_y \phi_{uv} + (u_y)^2\phi_{vv}$$

As desired. Now to write an expression for ϕ_y we can again simply use the multivariate chain rule:

$$\phi_y = u_y\phi_u + v_y\phi_v$$

To get an expression for ϕ_{yy} we can realize that since Ω is analytic, that ϕ satisfies Laplace's equation, meaning $\phi_{xx} + \phi_{yy} = 0$. So $\phi_{yy} = -\phi_{xx}$ which we already computed above. There is another part to this problem but it is currently 1:17 pm on Monday, so I'm going to turn in the homework as it is. This is the only problem I didn't quite finish haha...

Problem 9:

To show the derivative of $f(z) = f(x+iy) = |z|^2 = x^2 + y^2 = (x^2+y^2) + i(0) = u(x,y) + iv(x,y)$ is defined at $z = 0$ but nowhere else, we will use the Cauchy-Riemann equations. The CR equations require the $u_x = v_y$ and $v_x = -u_y$. So computing:

$$u_x = 2x$$

$$v_y = 0$$

$$v_x = 0$$

$$-u_y = -2y$$

$u_x = 2x = 0 = v_y$ only happens when $x = 0$. $v_x = 0 = -2y = -u_y$ only happens when $y = 0$. The only overlapping coordinate that satisfies both relationships is the point $(x = 0, y = 0)$, or when $z = 0$ as desired. The derivative at this point is:

$$f'(z) = u_x + iv_x = 2x + i(0) = 2x$$

Which is 0 when $z = 0$.

Problem 10:

From (2.1.9) in the textbook, we have the following differential relationships:

$$\frac{\partial}{\partial x} = \cos(\theta)\frac{\partial}{\partial r} - \frac{\sin\theta}{r}\frac{\partial}{\partial \theta}$$
$$\frac{\partial}{\partial y} = \sin(\theta)\frac{\partial}{\partial r} + \frac{\cos\theta}{r}\frac{\partial}{\partial \theta}$$

The Cartesian version of the CR equations say that we need $\frac{\partial u}{\partial x} = \frac{\partial v}{\partial y}$ and $\frac{\partial v}{\partial x} = -\frac{\partial u}{\partial y}$. So if we apply the above operators to the CR equations, we get the following 2 equations:

$$\cos(\theta)\frac{\partial u}{\partial r} - \frac{\sin\theta}{r}\frac{\partial u}{\partial \theta} = \sin(\theta)\frac{\partial v}{\partial r} + \frac{\cos\theta}{r}\frac{\partial v}{\partial \theta}$$
$$\sin(\theta)\frac{\partial u}{\partial r} + \frac{\cos\theta}{r}\frac{\partial u}{\partial \theta} = -\cos(\theta)\frac{\partial v}{\partial r} + \frac{\sin\theta}{r}\frac{\partial v}{\partial \theta}$$

If we multiply the first equation by $\cos(\theta)$ on both sides, multiply the second equation by $\sin(\theta)$ on both sides, and then add the 2 equations together, we get (using that $\cos^2(\theta) + \sin^\theta = 1$ and that $\sin(\theta)\cos(\theta) - \sin(\theta)\cos(\theta) = 0$):

$$1 \cdot \frac{\partial u}{\partial r} + \left(\frac{0}{r}\right)\frac{\partial u}{\partial \theta} = 0 \cdot \frac{\partial v}{\partial r} + \frac{1}{r}\frac{\partial v}{\partial \theta}$$
$$\frac{\partial u}{\partial r} = \frac{1}{r}\frac{\partial v}{\partial \theta}$$
$$u_r = \frac{1}{r}v_\theta$$

Which is one of the results we are looking for. If we now multiply the first equation by $\sin(\theta)$ on both sides, multiply the second equation by $-\cos(\theta)$ on both sides, and then add the 2 equations together, we get:

$$0 \cdot \frac{\partial u}{\partial r} + \left(\frac{1}{r}\right)\frac{\partial u}{\partial \theta} = 1 \cdot \frac{\partial v}{\partial r} + \left(\frac{0}{r}\right)\frac{\partial v}{\partial \theta}$$
$$-\frac{1}{r}\frac{\partial u}{\partial \theta} = \frac{\partial v}{\partial r}$$
$$-\frac{1}{r}u_\theta = v_r$$

Which is the second desired result.