Assignment 7.

Due Wednesday, March 12, at 2:30pm PST.

Reading: Sec. 1.3-1.5 in ODE Notes.

1. Determine the coefficients b_0, b_1, b_2 for the third order, 2-step Adams-Moulton method:
$$y_{k+2} = y_{k+1} + h[b_0 f(t_k, y_k) + b_1 f(t_{k+1}, y_{k+1}) + b_2 f(t_{k+2}, y_{k+2})].$$
Do this in two different ways:
   (a) Using the theorem that for a general linear m-step method,
   $$\sum_{\ell=0}^{m} a_\ell y_{k+\ell} = h \sum_{\ell=0}^{m} b_\ell f(t_{k+\ell}, y_{k+\ell}), \quad a_m = 1,$$
   the LTE is of order $p \geq 1$ if and only if
   $$\sum_{\ell=0}^{m} a_\ell = 0 \quad \text{and} \quad \sum_{\ell=0}^{m} \ell^j a_\ell = j \sum_{\ell=0}^{m} \ell^{j-1} b_\ell, \quad j = 1, \ldots, p.$$
   (b) Using the relation
   $$y(t_{k+2}) = y(t_{k+1}) + \int_{t_{k+1}}^{t_{k+2}} f(s, y(s))\, ds,$$
   and replacing f in the integral by a quadratic polynomial $p(s)$ that takes the values $f(t_k, y_k)$, $f(t_{k+1}, y_{k+1})$, and $f(t_{k+2}, y_{k+2})$ at the points t_k, t_{k+1}, and t_{k+2}.

2. What is the order of the local truncation error for each of the following linear multistep methods, and which of these methods are *convergent*? Justify your answers.
   (a) $y_k - y_{k-2} = h[f(t_k, y_k) - 3f(t_{k-1}, y_{k-1}) + 4f(t_{k-2}, y_{k-2})]$.
   (b) $y_k - 2y_{k-1} + y_{k-2} = h[f(t_k, y_k) - f(t_{k-1}, y_{k-1})]$.
   (c) $y_k - y_{k-1} - y_{k-2} = h[f(t_k, y_k) - f(t_{k-1}, y_{k-1})]$.

3. (Computational Problem.) The following simple model describes the switching behavior for a muscle that controls a valve in the heart. Let $x(t)$ denote the position of the muscle at time t and let $\alpha(t)$ denote the concentration at time t of a chemical stimulus. Suppose that the dynamics of x and α are controlled by the following system of differential equations:
$$\frac{dx}{dt} = -\frac{x^3}{3} + x + \alpha$$
$$\frac{d\alpha}{dt} = -\epsilon x.$$
Here $\epsilon > 0$ is a parameter; its inverse estimates roughly the time that x spends near one of its rest positions.

(a) Taking $\epsilon = 1/100$, $x(0) = 2$, and $\alpha(0) = 2/3$, solve this system of differential equations using an explicit method of your choice. Integrate out to, say, $t = 400$, and turn in plots of $x(t)$ and $\alpha(t)$. Explain why you chose the method that you used and approximately how accurate you think your computed solution is and why. Comment on whether you seemed to need restrictions on the step size for stability or whether accuracy was the only consideration in choosing your stepsize.

(b) Solve the same problem using the backward Euler method:

$$y_{k+1} = y_k + hf(t_{k+1}, y_{k+1})$$

and solving the nonlinear equations at each step via Newton's method. Write down the Jacobian matrix for the system and explain what initial guess you will use. Were you able to take a larger time step using the backward Euler method than you were with the explicit method used in part (a)?

4. Show that the implicit midpoint method

$$y_{k+1} = y_k + hf(t_{k+1/2}, (y_k + y_{k+1})/2), \quad t_{k+1/2} = t_k + h/2$$

is A-stable.

5. Consider the A-stable trapezoidal method applied to the test problem $y' = \lambda y$:

$$y_{k+1} = y_k + \frac{h}{2}\lambda[y_k + y_{k+1}].$$

Of course, one can easily solve this linear equation for y_{k+1}, but suppose we used *fixed point iteration* to solve for y_{k+1}: Taking $w^{(0)}$ as an initial guess for y_{k+1}, set

$$w^{(j+1)} = y_k + \frac{h}{2}\lambda[y_k + w^{(j)}], \quad j = 0, 1, \ldots.$$

What conditions must $h\lambda$ satisfy so that $w^{(j)}$ will converge to y_{k+1}? [Note that if we use the trapezoidal rule but solve for y_{k+1} using fixed point iteration, we lose the benefits of A-stability!]