AMATH 567 FALL 2024
HOMEWORK 9 — DUE NOVEMBER 25 ON GRADESCOPE BY 1:30PM

All solutions must include significant justification to receive full credit. If you handwrite your assignment you must either do so digitally or if it is written on paper you must *scan* your work. A standard photo is not sufficient.

If you work with others on the homework, you must name your collaborators.

1: From A&F: 4.1.2 only (i), i.e., only by computing residues inside.

2: From A&F: 4.2.1(b)

3: Existence and uniqueness of polynomial interpolants.

(a) Suppose $(z_i)_{i=1}^n$ are distinct points in $\mathbb{C}$ and suppose $f_i \in \mathbb{C}$ for $i = 1, \ldots, n$. Show that there is at most one polynomial $p(z)$ of degree $n-1$ such that $p(z_i) = f_i$ for $i = 1, \ldots, n$ using Liouville's theorem. Such a polynomial p is called an *interpolant*.

(b) Define the node polynomial $\nu(z) = \prod_{j=1}^n (z - z_i)$. Supposing that p is an interpolant, as above, express $p(z)/\nu(z)$ as a rational function. Find an expression for $p(z)$. This shows existence.

4: Bernstein interpolation formula. Suppose that $-1 \leq x_1 < x_2 < \cdots < x_n \leq 1$. And suppose that $f(z)$ is analytic in a region Ω that contains $[-1, 1]$. Show that for any simple contour C inside Ω with $[-1, 1]$ in its interior that

$$f(x) - p(x) = \frac{\nu(x)}{2\pi i} \int_C \frac{f(z)}{z - x} \frac{\mathrm{d}z}{\nu(z)}, \quad x \in [-1, 1],$$

where p is the degree $n-1$ polynomial interpolant satisfying $f(x_j) = p(x_j)$ for all j.

5: Chebyshev polynomial interpolants. Recall

$$\varphi(z) = z + \sqrt{z-1}\sqrt{z+1}, \quad z \in \mathbb{C} \setminus [-1, 1].$$

(a) Show that the polynomial

$$T_n(z) = \frac{1}{2}\left(\varphi(z)^n + \varphi(z)^{-n}\right),$$

has all of its roots $x_1 < x_2 < \cdots x_n$ within $[-1, 1]$.

(b) Consider $J(w) = 1/2(w + 1/w)$. Show that the image of the circle of radius $\rho > 1$ under J is an ellipse B_ρ that contains $[-1, 1]$ in its interior. Then show $\varphi(J(w)) = w$.

(c) Show that if f is analytic in a region that contains B_ρ and its interior, and $|f(z)| \leq M$ for z interior to B_ρ then for $-1 \leq x \leq 1$,

$$|f(x) - p(x)| \leq 2\frac{M|B_\rho|}{\pi}(\rho^n - \rho^{-n})^{-1}(\rho + \rho^{-1} - 1)^{-1} \leq 2\frac{M|B_\rho|}{\pi}\frac{\rho^{1-n}}{(\rho-1)^2}.$$

where $p(x_j) = f(x_j)$, i.e., that p is the degree $n-1$ interpolant of f at the roots of T_n. Here $|B_\rho|$ denotes the arclength of B_ρ. This shows that the exponential rate of convergence of Chebyshev interpolants is governed by the proximity of the nearest singularity of f.

6: Compute the following two integrals explicitly for $z \notin [-1, 1]$:

(a)

$$\frac{1}{\pi}\int_{-1}^{1} \frac{1}{\sqrt{1-x}\sqrt{1+x}} \frac{\mathrm{d}x}{x-z}.$$

(b)

$$\frac{2}{\pi}\int_{-1}^{1} \sqrt{1-x}\sqrt{1+x}\frac{\mathrm{d}x}{x-z}.$$