

A M A T H 567

Homework 4

Natc wonybra

1) c) Put $z = Re^{i\theta}$, then $\bar{z} = Re^{-i\theta}$, so

$$\frac{1}{\bar{z}} = \frac{1}{R} e^{i\theta}, \text{ so our integral becomes }$$

$$\int_C f(z) dz = \int_0^{2\pi} \frac{1}{R} e^{i\theta} \cdot \underbrace{iR e^{i\theta}}_{\text{derivative of } \frac{1}{\bar{z}}} d\theta$$

$$= i \int_0^{2\pi} e^{2i\theta} d\theta$$

$$\begin{aligned} &= i \left[\int_0^{\pi} \cos(2\theta) d\theta + i \int_0^{\pi} \sin(2\theta) d\theta \right] \\ &= i \left[\left. \frac{1}{2} \sin(2\theta) \right|_0^{\pi} + (-i) \left. \frac{1}{2} \cos(2\theta) \right|_0^{\pi} \right] \\ &= 0 \end{aligned}$$

e) By 1.2.19,

$$e^{\bar{z}} = \sum_{n=0}^{\infty} \frac{(\bar{z})^n}{n!}$$

Put $z = Re^{i\theta}$, then $\bar{z} = Re^{-i\theta}$, so the above

$$= \sum_{n=0}^{\infty} \frac{R^n}{n!} e^{-in\theta}$$

So,

$$\oint_C f(z) dz = \int_0^{2\pi} \sum_{n=0}^{\infty} \frac{R^n}{n!} e^{-in\theta} \cdot R i e^{i\theta} d\theta$$

$$= i \int_0^{2\pi} \sum_{n=0}^{\infty} \frac{R^{n+1}}{n!} e^{-(n-1)i\theta} d\theta$$

$$= i \sum_{n=0}^{\infty} \frac{R^{n+1}}{n!} \left[\int_0^{2\pi} e^{-(n-1)i\theta} d\theta \right]$$

$$= \int_0^{2\pi} \cos((n-1)\theta) d\theta - i \int_0^{2\pi} \sin((n-1)\theta) d\theta$$

$$(\text{assume } n \neq 1) = \frac{1}{n-1} \sin((n-1)\theta) \Big|_0^{2\pi} + \frac{i}{n-1} \cos((n-1)\theta) \Big|_0^{2\pi}$$

$$= 0$$

When $n=1$, the integral is

$$\int_0^{2\pi} e^{-(1-1)i\theta} d\theta = \int_0^{2\pi} d\theta = 2\pi,$$

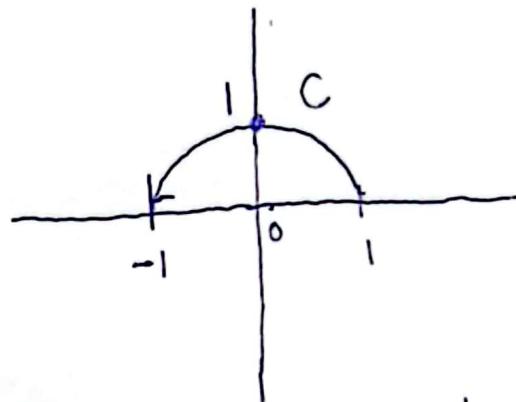
So every term in the sum vanishes except at $n=1$; so our final answer is

$$= \left[2\pi i \cdot \frac{R^2}{1!} \right] \\ = \boxed{2\pi i} \quad \text{as } R=1$$

2) a)

Note, that

$$\int_{-1}^1 \log(z) dz = - \int_1^{-1} \log(z) dz,$$



so to compute this

integral we choose a

half-circle contour starting at +1 and ending at -1. So put $z = e^{i\theta}$ for $0 \leq \theta < \pi$ and consider for $\epsilon > 0$ (note ϵ is approaching 0 from the right) to avoid branch

$$\int_{-1}^1 \log(z) dz = \textcircled{*} \int_0^{\pi-\epsilon} \log(e^{i\theta}) i e^{i\theta} d\theta$$

there should be $\lim_{\epsilon \rightarrow 0}$ in front

of these equalities, denote $\lim_{\epsilon \rightarrow 0}$ by $\textcircled{*}$

$$= \textcircled{*} i \int_0^{\pi-\epsilon} (\log(1) + i\theta) e^{i\theta} d\theta$$

$$= \textcircled{*} \int_0^{\pi-\epsilon} \theta e^{i\theta} d\theta$$

$$= \textcircled{*} - \left[\int_0^{\pi-\epsilon} \theta \cos(\theta) d\theta + i \int_0^{\pi-\epsilon} \theta \sin(\theta) d\theta \right]$$

which after integration by parts becomes

$$= \textcircled{*} - \left[\cos(\theta) + \theta \sin(\theta) \Big|_0^{\pi-\epsilon} + i (\sin(\theta) - \theta \cos(\theta)) \Big|_0^{\pi-\epsilon} \right]$$

$$= \textcircled{*} - \left[\cos(\pi-\epsilon) + (\pi-\epsilon) \sin(\pi-\epsilon) - 1 + i \left[\sin(\pi-\epsilon) - (\pi-\epsilon) \cos(\pi-\epsilon) \right] \right]$$

$$= (1 - \cos(\pi - \varepsilon) - (\pi - \varepsilon)\sin(\pi - \varepsilon)) + i((\pi - \varepsilon)\cos(\pi - \varepsilon) - \sin(\pi - \varepsilon))$$

The reason we introduced ε is to avoid the branch cut of the logarithm on the negative real axis. So as $\varepsilon \rightarrow 0$, our contour C approaches the branch cut, so as $\varepsilon \rightarrow 0$ the above reduces to

$$\Rightarrow (1 - \cos(\pi) - \pi\sin(\pi)) + i(\pi\cos(\pi) - \sin(\pi))$$

$$= 2 + i(-\pi) = 2 - i\pi$$

we have computed $+\int_{-1}^1 \log(z) dz$, so we need to negate our solution above, hence

$$\int_{-1}^1 \log(z) dz = \boxed{-2 + i\pi}$$

b) For this part, we use the same contour as in part (a) noting that

$$\int_{-1}^1 z^{1/2} dz = - \int_1^{-1} \bar{z}^{1/2} d\bar{z}$$

So if $z = Re^{i\theta} \rightarrow \bar{z}^{1/2} = R^{1/2} e^{i\theta/2}$, so (where e^{-sD} from the right to avoid branch)

$$\int_{-1}^1 z^{1/2} dz = \lim_{\epsilon \rightarrow 0} \int_0^{\pi-\epsilon} R^{1/2} e^{i\theta/2} \cdot R i e^{i\theta} d\theta$$

$$= \lim_{\epsilon \rightarrow 0} i \int_0^{\pi-\epsilon} e^{i \frac{3\theta}{2}} d\theta \quad (\text{as } R=1)$$

$$= \lim_{\epsilon \rightarrow 0} i \left[\int_0^{\pi-\epsilon} \cos(\frac{3\theta}{2}) d\theta + i \int_0^{\pi-\epsilon} \sin(\frac{3\theta}{2}) d\theta \right]$$

$$= \lim_{\epsilon \rightarrow 0} i \left[\frac{2}{3} \sin(\frac{3\theta}{2}) \Big|_0^{\pi-\epsilon} - i \left[\frac{2}{3} \cos(\frac{3\theta}{2}) \right]_0^{\pi-\epsilon} \right]$$

$$= \lim_{\epsilon \rightarrow 0} i \left[\frac{2}{3} \sin\left(\frac{3}{2}(\pi-\epsilon)\right) - \frac{2i}{3} \left[\cos\left(\frac{3(\pi-\epsilon)}{2}\right) - 1 \right] \right]$$

$$= \frac{2}{3} \left[\cos\left(\frac{3\pi}{2}\right) - 1 \right] + \frac{2}{3} i \sin\left(\frac{3\pi}{2}\right)$$

$$= -\frac{2}{3} - \frac{2}{3} i, \quad \text{so taking the negative we have}$$

$$\int_{-1}^1 \bar{z}^{1/2} d\bar{z} = \boxed{\frac{2}{3} + \frac{2}{3} i}$$

3) Firstly, to show

$$|f(z)| \leq \frac{1}{R^2 - a^2} \quad R > a$$

we must show

$$\frac{1}{|f(z)|} \geq R^2 - a^2$$

we have

$$\begin{aligned} \frac{1}{|f(z)|} &= |z^2 + a^2| = |(z-ia)(z+ia)| \\ &= |(z-ia)| |(z+ia)| \end{aligned}$$

By the triangle inequality, $|z-ia| \geq |z| - |ia|$
 $\stackrel{(1)}{=} |R-a| = R-a$ (as $R > a$). For $|z+ia|$,
we need to do more work

$$\begin{aligned} |z+ia| &= |x + (y+a)i| \\ &= \sqrt{x^2 + (y+a)^2} \\ &= \sqrt{x^2 + y^2 + a^2 + 2ya} \\ &\geq \sqrt{y^2 + 2ya + a^2} \quad \forall y \text{ on } C \end{aligned}$$

So pick $y = R$, then

$$\begin{aligned} &= \sqrt{R^2 + 2Ra + a^2} \\ &= \sqrt{(R+a)^2} \\ &= R+a \quad (2) \end{aligned}$$

Putting ① and ② together we see

$$\frac{1}{|f(z)|} \geq |z-i\alpha||z+i\alpha| \geq (R-\alpha)(R+\alpha) \\ = R^2 - \alpha^2$$

Hence,

$$|f(z)| \leq \frac{1}{R^2 - \alpha^2} \quad \text{as desired}$$

Next we know the following bound

$$\left| \int_C f(z) dz \right| \leq \sup_{z \in C} |f(z)| \cdot L$$

where L is the arc length of C ,
since our contour is half a circle
of radius R , the arc length is half
the circumference, hence $L = \frac{2\pi R}{2} = \pi R$.

So finally

$$\left| \int_C f(z) dz \right| \leq \sup_{z \in C} |f(z)| \cdot L \leq \frac{1}{R^2 - \alpha^2} \cdot \pi R \\ = \frac{\pi R}{R^2 - \alpha^2}$$

which is the desired result.

4) Firstly,

$$\begin{aligned}|z^3 + 1| &= |z^3 - (-1)| \geq ||z^3| - |-1|| \\&= |R^3 - 1| \\&= R^3 - 1 \text{ as } R \rightarrow 1\end{aligned}$$

Also, the arclength of C , $L = \theta r = \frac{\pi}{3}R$.

$$\begin{aligned}\text{So } \left| \int_C \frac{1}{z^3+1} dz \right| &\leq \int_C \left| \frac{1}{z^3+1} \right| dz \leq \sup_{z \in C} \left| \frac{1}{z^3+1} \right| \cdot L \\&\leq \frac{1}{R^3-1} \cdot \frac{\pi}{3}R = \frac{\pi}{3} \left(\frac{R}{R^3-1} \right)\end{aligned}$$

Now, using the above

$$0 \leq \left| \int_{C_R} \frac{1}{z^3+1} dz \right| \leq \frac{\pi}{3} \left(\frac{R}{R^3-1} \right)$$

$$\lim_{R \rightarrow 0} 0 \leq \lim_{R \rightarrow 0} \left| \int_{C_R} \frac{1}{z^3+1} dz \right| \leq \lim_{R \rightarrow \infty} \frac{\pi}{3} \left(\frac{R}{R^3-1} \right)$$

$$0 \leq \lim_{R \rightarrow \infty} \left| \int_{C_R} \frac{1}{z^3+1} dz \right| \leq 0$$

So by the squeeze theorem, the modulus of the integral $\rightarrow 0$ as $R \rightarrow \infty$, so the integral itself must also approach 0 as $R \rightarrow \infty$, so we are done.

5) b) It is common knowledge that both $f_1(z) = e^z$ and $f_2(z) = z^2$ are both analytic on \mathbb{C} . On page 38 of the textbook it says the composition of analytic functions is also analytic, so $f(z) = e^{z^2}$ is also analytic, so $f(z) = e^{z^2}$ is also analytic on the whole plane \mathbb{C} . So by Cauchy's Theorem, we get for free that (since the unit circle is a closed contour in \mathbb{C})

$$\oint_C e^{z^2} dz = 0$$

c) we have

$$\begin{aligned} \oint_C \frac{1}{z^2+1} dz &= \frac{1}{2} \oint_C \frac{1}{z^2+1/2} dz \\ &= \frac{1}{2} \oint_C \frac{1}{(z-i/\sqrt{2})(z+i/\sqrt{2})} dz \end{aligned} \quad \textcircled{1}$$

Let's compute the partial fraction decomposition of the integrand

$$\frac{1}{(z-i/\sqrt{2})(z+i/\sqrt{2})} = \frac{A}{(z-i/\sqrt{2})} + \frac{B}{(z+i/\sqrt{2})}$$

$$1 = A(z+i/\sqrt{2}) + B(z-i/\sqrt{2})$$

$$z = i/\sqrt{2} \rightarrow A = \frac{1}{\sqrt{2}i}$$

$$z = -i/\sqrt{2} \rightarrow B = -\frac{1}{\sqrt{2}i}$$

So

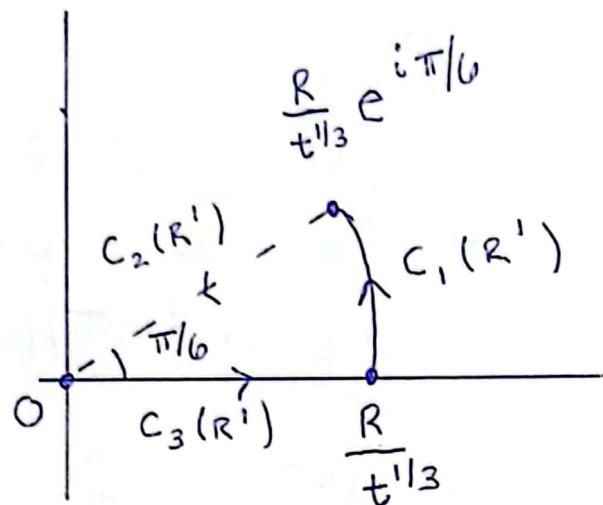
$$(1) = \frac{1}{2} \left[\frac{1}{\sqrt{2}i} \oint_C \frac{1}{z-i/\sqrt{2}} dz - \frac{1}{\sqrt{2}i} \oint_C \frac{1}{z+i/\sqrt{2}} dz \right]$$

By example 2.5.2, since both of the roots $z_1 = i/\sqrt{2}$, $z_2 = -i/\sqrt{2}$ lie inside the unit circle C (our closed contour), both integrals evaluate to $2\pi i$, so

$$(1) \frac{1}{2} \left[\frac{1}{\sqrt{2}i} \cdot 2\pi i - \frac{1}{\sqrt{2}i} \cdot 2\pi i \right]$$

$$= \boxed{0}$$

6) Consider the following contour



which is the circular sector with boundary points $R^t, 0, R^t e^{i\pi/6}$ where $R^t = \frac{R}{t^{1/3}}$ for $t > 0$. Following the ideas of exercise 2.5.5, first denote $C(R/t^{1/3}) = \sum_{i=1}^3 C_i(R/t^{1/3})$, which is the combination of the 3 segments shown in the diagram that creates the full sector. We are interested in the contour C_3 ,

$$\int_{C_3(R/t^{1/3})} e^{iz^3 t} dz = \int_{C(R^t)} e^{iz^3 t} dz - \int_{C_1(R^t)} e^{iz^3 t} dz - \int_{C_2(R^t)} e^{iz^3 t} dz$$

whole loop arc (2)

diagonal line (3)

we are interested in what happens as $R \rightarrow \infty$.
Firstly since $e^{iz^3 t}$ is analytic everywhere,
the loop integral ① vanishes for every
 $R > 0$. Secondly, we want to show that as
 $R \rightarrow \infty$, ② $\rightarrow 0$. To do so, put
 $z = \frac{R}{t^{1/3}} e^{i\theta}$ for $0 \leq \theta \leq \pi/6$. Then
 $z^3 = \frac{R^3}{t} e^{i3\theta}$ and $dz = \frac{iR}{t^{1/2}} e^{i\theta} d\theta$

$$\begin{aligned}
(2) &= \int_{C_1(R)} e^{iz^3 t} dz = \int_0^{\pi/6} e^{i(\frac{R^3}{t} e^{i3\theta}) \cdot t} \cdot \frac{iR}{t^{1/3}} e^{i\theta} d\theta \\
&= \frac{iR}{t^{1/3}} \int_0^{\pi/6} e^{i(\frac{R^3}{t} e^{i3\theta})} e^{i\theta} d\theta \\
&= \frac{iR}{t^{1/3}} \int_0^{\pi/6} [e^{i[R^3 \cos(3\theta) + iR^3 \sin(3\theta)]}] e^{i\theta} d\theta \\
&= \frac{iR}{t^{1/3}} \int_0^{\pi/6} e^{-R^3 \sin(3\theta)} e^{i(R^3 \cos(3\theta) + \theta)} d\theta
\end{aligned}$$

So,

$$\begin{aligned} |\textcircled{2}| &= \left| \frac{iR}{t^{1/3}} \int_0^{\pi/6} e^{-R^3 \sin(3\theta)} e^{i(R^3 \cos(3\theta) + \theta)} d\theta \right| \\ &\leq \frac{R}{t^{1/3}} \int_0^{\pi/6} \underbrace{|e^{-R^3 \sin(3\theta)}|}_{\text{this is } \geq 0} \cdot |e^{i(R^3 \cos 3\theta + \theta)}| d\theta \\ &= \frac{R}{t^{1/3}} \int_0^{\pi/6} e^{-R^3 \sin(3\theta)} d\theta \end{aligned}$$

Now as $0 \leq \theta \leq \pi/6$, $0 \leq 3\theta \leq \pi/2$, so we can use the inequality mentioned in the problem

$$\sin(3\theta) \geq \frac{2(\sin 3\theta)}{\pi} = \frac{6\theta}{\pi}$$

$$\Leftrightarrow -\sin(3\theta) \leq -\frac{6\theta}{\pi}$$

So the above is

$$\begin{aligned} &\leq \frac{R}{t^{1/3}} \int_0^{\pi/6} e^{-\frac{R^3 \cdot 6\theta}{\pi}} d\theta = \frac{-\pi R}{6R^3 t^{1/3}} \cdot \left[e^{-\frac{R^3 \cdot 6\theta}{\pi}} \right]_0^{\pi/6} \\ &= \frac{-\pi}{6R^2 t^{1/3}} \left[e^{-R^3} - 1 \right] = \frac{\pi}{6R^2 t^{1/3}} [1 - e^{-R^3}] \end{aligned}$$

$$\leq \frac{\pi}{6R^2 t^{1/3}}. \text{ So as } R \rightarrow \infty,$$

$$\lim_{R \rightarrow \infty} 0 \leq \lim_{R \rightarrow \infty} |②| \leq \lim_{R \rightarrow \infty} \frac{\pi}{6R^2 t^{1/3}} = 0$$

which implies $\textcircled{2} \rightarrow 0$ as $R \rightarrow \infty$ as desired.

Now let's take a look at $\textcircled{3}$. We

can put $z = \frac{R}{t^{1/3}} e^{i\pi/6}$ so that

$$z^3 = \frac{R^3}{t} e^{i\pi/2} = \frac{R^3}{t} \cdot i \quad \text{and} \quad dz = \frac{dr}{t^{1/3}} \cdot e^{i\pi/6}$$

This is technically a parametrization for $-C_2(R)$, so after computing the next integral, we need to remember to negate the result, so

$$\textcircled{3} = - \int_0^R e^{i(r^3/t) \cdot i \cdot t} \cdot \frac{1}{t^{1/3}} e^{i\pi/6} dr$$

$$= - \frac{e^{i\pi/6}}{t^{1/3}} \int_0^R e^{-r^3} dr.$$

$$\text{So as } R \rightarrow \infty, \textcircled{3} \rightarrow - \frac{e^{i\pi/6}}{t^{1/3}} \int_0^\infty e^{-r^3} dr$$

So finally,

$$\int_0^\infty e^{iz^3 t} dz = \lim_{R \rightarrow \infty} \int_{C_3(R^1)} e^{iz^3 t} dz$$

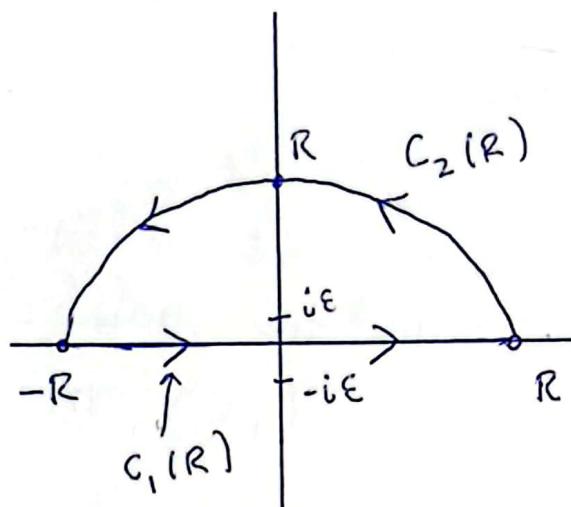
$$= \lim_{R \rightarrow \infty} \textcircled{1} - \textcircled{2} - \textcircled{3}$$

$$= - \left(-\frac{e^{i\pi/6}}{t^{1/3}} \int_0^\infty e^{-r^3} dr \right)$$

$$= \boxed{\frac{e^{i\pi/6}}{t^{1/3}} \int_0^\infty e^{-r^3} dr}$$

which completes our computation.

7) consider the following contour



which is a half circle in the upper-half plane. You can see the whole contour $C(R) = C_1(R) + C_2(R)$ where $C_1(R)$ is the horizontal line from $-R$ to R and $C_2(R)$ is the arc of the circle. We are interested in the contour $C_1(R)$.

Notice

$$\int_{C_1(R)} \frac{\epsilon}{z^2 + \epsilon^2} dz = \int_{C(R)} \frac{\epsilon}{z^2 + \epsilon^2} dz - \int_{C_2(R)} \frac{\epsilon}{z^2 + \epsilon^2} dz$$

we are interested in what happens as $R \rightarrow \infty$. Firstly, notice that

$$\frac{\epsilon}{z^2 + \epsilon^2} = \epsilon \frac{1}{(z-i\epsilon)(z+i\epsilon)} = \frac{\epsilon}{2i\epsilon} \left[\frac{1}{z-i\epsilon} - \frac{1}{z+i\epsilon} \right]$$

So

$$\textcircled{1} = \frac{1}{2i} \int_{C(R)} \frac{1}{z-i\varepsilon} - \frac{1}{z+i\varepsilon} dz$$

$C(R)$

By examples 2.5.2 and 2.5.3 in the textbook
since $z = i\varepsilon$ is inside the half circle $C(R)$
and $z = -i\varepsilon$ is outside $C(R)$, the above
equals

$$2\pi i \cdot \frac{1}{2i} \cdot 1 = \pi \quad \text{for any } R > 0$$

Now let's take care of \textcircled{2}. Problem
3 (2.4.7) exactly applies here, so we
have

$$0 \leq |\textcircled{2}| = \left| \int_{C_2(R)} \frac{\varepsilon}{z^2 + \varepsilon^2} dz \right| \leq \frac{\varepsilon \pi R}{R^2 - \varepsilon^2}$$

$$\lim_{R \rightarrow 0} 0 \leq \lim_{R \rightarrow \infty} \left| \int_{C_2(R)} \frac{\varepsilon}{z^2 + \varepsilon^2} dz \right| \leq \lim_{R \rightarrow \infty} \frac{\varepsilon \pi R}{R^2 - \varepsilon^2} = 0$$

So as $R \rightarrow \infty$, \textcircled{2} $\rightarrow 0$

Putting this all together

$$\int_{-\infty}^{\infty} \frac{\epsilon}{z^2 + \epsilon^2} dz = \lim_{R \rightarrow \infty} \int_{C_1(R)} \frac{\epsilon}{z^2 + \epsilon^2} dz$$

$$= \lim_{R \rightarrow \infty} (1) - (2)^0$$

$$= (\boxed{\pi})$$

confirming with real integration

$$\int_{-\infty}^{\infty} \frac{\epsilon}{x^2 + \epsilon^2} dx = \int_{-\infty}^{\infty} \frac{\epsilon/\epsilon^2}{x^2/\epsilon^2 + 1} dx$$

$$= \frac{1}{\epsilon} \int_{-\infty}^{\infty} \frac{1}{(x/\epsilon)^2 + 1} dx \quad \text{put } u = \frac{x}{\epsilon} \quad du = \frac{1}{\epsilon} dx \rightarrow dx = \epsilon du$$

$$= \frac{1}{\epsilon} \int_{-\infty}^{\infty} \frac{1}{u^2 + 1} du$$

$$= \left. \arctan(u) \right|_{-\infty}^{\infty} = \pi/2 - (-\pi/2) = \boxed{\pi}$$

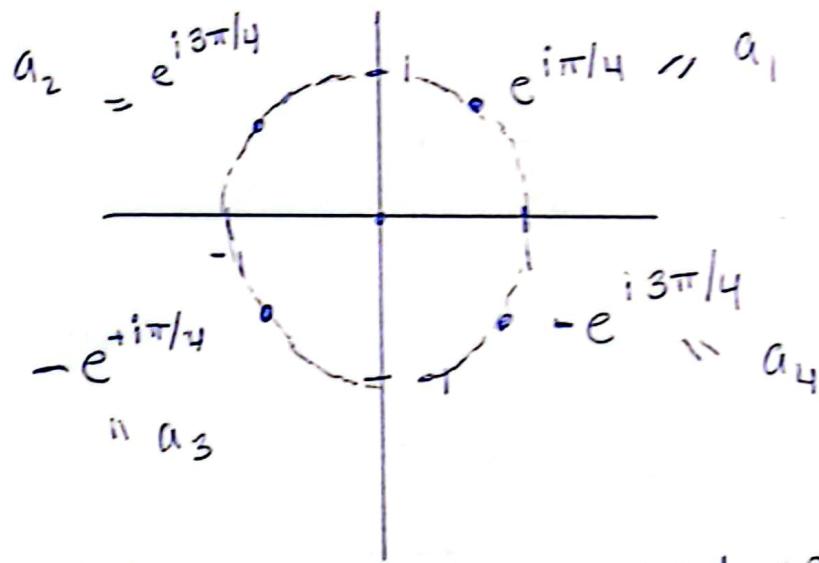
So both computations give the same result.

8) Let's start by examining the function

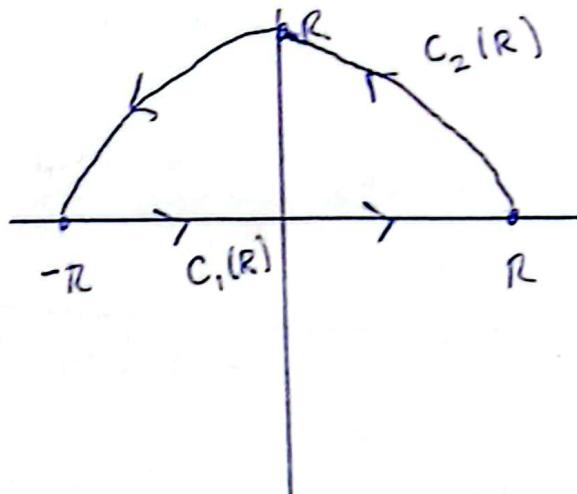
$$f(z) = \frac{1}{1+z^4}, \text{ we have}$$

$$\begin{aligned} f(z) &= \frac{1}{z^4+1} = \frac{1}{(z^2-i)(z^2+i)} = \frac{1}{(z-\sqrt{i})(z+\sqrt{i})(z-\sqrt{-i})(z+\sqrt{-i})} \\ &= \frac{1}{(z-e^{i\pi/4})(z+e^{i\pi/4})(z-e^{i3\pi/4})(z+e^{i3\pi/4})} \end{aligned}$$

Let's see where the roots of the denominator are in the plane



we can repeat the same procedure as in problem 7, with the following contour



$$C(R) = C_1(R) + C_2(R)$$

we want to focus our attention on $C_1(R)$
as $R \rightarrow \infty$

$$\int_{C_1(R)} f(z) dz = \int_{C(R)} f(z) dz - \int_{C_2(R)} f(z) dz$$

To compute ① we must compute the
partial fraction for $f(z)$ (where the
 a_i 's are the roots)

$$f(z) = \frac{1}{(z-a_1)(z-a_2)(z-a_3)(z-a_4)}$$

$$= \frac{A}{z-a_1} + \frac{B}{z-a_2} + \frac{C}{z-a_3} + \frac{D}{z-a_4}$$

$$\rightarrow 1 = A(z-a_2)(z-a_3)(z-a_4) + B(z-a_1)(z-a_3)(z-a_4) \\ + C(z-a_1)(z-a_2)(z-a_4) + D(z-a_1)(z-a_2)(z-a_3)$$

$$z = a_2 \rightarrow B = \frac{1}{(a_2-a_1)(a_2-a_3)(a_2-a_4)}$$

$$z = a_1 \rightarrow A = \frac{1}{(a_1-a_2)(a_1-a_3)(a_1-a_4)}$$

$$z = a_3 \rightarrow C = \frac{1}{(a_3-a_1)(a_3-a_2)(a_3-a_4)}$$

$$z = a_4 \rightarrow D = \frac{1}{(a_4-a_1)(a_4-a_2)(a_4-a_3)}$$

Again referencing examples 2.5.2 and 2.5.3
 the contributions from the "C" and "D"
 fractions will integrate to be 0, so
 we only need to find A and B.

$$\begin{aligned}\frac{1}{A} &= (e^{i\pi/4} - e^{i3\pi/4})(2e^{i\pi/4})(e^{i\pi/4} + e^{i3\pi/4}) \\ &= 2e^{i\pi/4} \cdot e^{i\pi/4}(1-i) \cdot e^{i\pi/4}(1+i) \\ &= 4e^{i3\pi/4} \\ A &= \frac{1}{4}e^{-i3\pi/4}\end{aligned}$$

$$\begin{aligned}\frac{1}{B} &= (e^{i3\pi/4} - e^{i\pi/4})(e^{i3\pi/4} + e^{i\pi/4})(2e^{i3\pi/4}) \\ &= e^{i\pi/4}(i-1)e^{i\pi/4}(i+1) \cdot 2e^{i3\pi/4} \\ &= -4e^{i5\pi/4}\end{aligned}$$

$$B = -\frac{1}{4}e^{-5\pi/4}$$

So,

$$\begin{aligned}① &= \frac{1}{4}e^{-i3\pi/4} \int_{C(R)} \frac{1}{(z-e^{i\pi/4})} dz - \frac{1}{4}e^{-5\pi/4} \int_{C(R)} \frac{1}{(z-e^{i3\pi/4})} dz \\ &\quad + C \int_{C(R)} \frac{1}{(z+e^{i\pi/4})} dz + D \int_{C(R)} \frac{1}{(z+e^{i3\pi/4})} dz\end{aligned}$$

$$= \frac{1}{4} e^{-\frac{i3\pi}{4}} \cdot 2\pi i - \frac{1}{4} e^{-\frac{5\pi}{4}} \cdot 2\pi i \quad (\star)$$

$$= \frac{\pi i}{2} [e^{5\pi/4i} - e^{3\pi/4i}]$$

$$= \frac{\pi}{2} i \left[-\frac{1}{\sqrt{2}} - \frac{1}{\sqrt{2}} i - \left(-\frac{1}{\sqrt{2}}\right) + i \cdot \frac{1}{\sqrt{2}} \right]$$

$$= \frac{\pi}{2} i \left[\cancel{-\frac{1}{\sqrt{2}}} - \frac{1}{\sqrt{2}} i + \cancel{\frac{1}{\sqrt{2}}} - \frac{i}{\sqrt{2}} \right]$$

$$= \frac{\pi}{2} i \left[-\frac{2i}{\sqrt{2}} \right]$$

$$= \boxed{\frac{\pi}{\sqrt{2}}}$$

I circled
this but
we aren't done

where at (1) we have used examples 2.5.2 and 2.5.3 to say the integrals should compute to $2\pi i$, since the corresponding roots lie inside $C(R)$. Now for (2), notice that

$$|z^4 + 1| = |z^4 - (-1)|$$

$$\geq ||z^4|| - 1$$

$$= R^4 - 1 \quad \text{for } R > 1$$

arc length
of C
 \parallel

$$\text{So} \left| \oint_{C_2(R)} \frac{1}{z^4+1} dz \right| \leq \oint_{C_2(R)} \left| \frac{1}{z^4+1} \right| dz \leq \sup_{z \in C_2(R)} \left| \frac{1}{z^4+1} \right| \cdot L$$

$$\leq \frac{\pi R}{R^4 - 1}$$

So as $R \rightarrow \infty$

$$\lim_{R \rightarrow \infty} 0 \leq \lim_{R \rightarrow \infty} |(2)| \leq \lim_{R \rightarrow \infty} \frac{\pi R}{R^4 - 1} = 0$$

which implies (2) $\rightarrow 0$ as $R \rightarrow \infty$,
wrapping this up, as $R \rightarrow \infty$

$$\int_{-\infty}^{\infty} f(z) dz = \lim_{R \rightarrow \infty} \int_{C_1(R)} f(z) dz$$

$$= \lim_{R \rightarrow \infty} (1) - (2)$$

$$= \boxed{\pi/\sqrt{2}}$$

and we are done.

9) a) since $f(z) = \sin(z)$ is analytic everywhere in \mathbb{C} , we can use Cauchy's integral formula with $z=0$

$$\begin{aligned} \oint \frac{\sin(\varphi)}{\varphi} d\varphi &= 2\pi i f(z=0) \\ &= 2\pi i \sin(0) \\ &= \boxed{0} \end{aligned}$$

e) we have

$$\begin{aligned} f(z) &= e^{z^2} \left(\frac{1}{z^2} - \frac{1}{z^3} \right) = \frac{e^{z^2}}{z^2} - \frac{e^{z^2}}{z^3} \\ &= \underbrace{\sum_{n=0}^{\infty} \frac{z^{2n}}{n!}}_{z^2} - \underbrace{\sum_{n=0}^{\infty} \frac{z^{2n}}{n!}}_{z^3} \\ &= \sum_{n=0}^{\infty} \frac{z^{2n-2}}{n!} - \sum_{n=0}^{\infty} \frac{z^{2n-3}}{n!} \\ &= \left(\frac{1}{z^2} + 1 + \underbrace{\sum_{n=2}^{\infty} \frac{z^{2n-2}}{n!}}_{(1)} \right) - \left(\frac{1}{z^3} + \frac{1}{z^1} + \underbrace{\sum_{n=2}^{\infty} \frac{z^{2n-3}}{n!}}_{(2)} \right) \end{aligned}$$

The terms (1) and (2) are analytic everywhere in \mathbb{C} because they are polynomials. So if we loop integrate them, they will vanish. So simplifying things, we can write

$$\oint_C f(z) dz = \oint_C e^{z^2} \left(\frac{1}{z^2} - \frac{1}{z^3} \right) dz$$

$$= \oint_C \frac{1}{z^2} - \frac{1}{z^3} - \frac{1}{z} dz$$

$$= \oint_C \frac{1}{z^2} dz - \oint_C \frac{1}{z^3} dz - \oint_C \frac{1}{z} dz$$

Since the function $g(z) = 1$ is analytic everywhere, Cauchy's integral formula tells us

$$\oint_C \frac{1}{z} dz = 2\pi i g(0) = 2\pi i$$

Theorem 2.6.2

tells us $\oint_C \frac{1}{z^2} dz = \frac{2\pi i}{1!} \underbrace{g'(0)}_{=0} = 0$

(since $g(z) = 1$ is constant)

And Theorem 2.6.2 gives us again

$$\oint_C \frac{1}{z^3} dz = \frac{2\pi i}{2!} \underbrace{g''(0)}_{=0}$$

So finally,

$$\begin{aligned}\oint_C f(z) dz &= \oint_C e^{z^2} \left(\frac{1}{z^2} - \frac{1}{z^3} \right) dz \\ &= 0 - 0 = 2\pi i \\ &= \boxed{-2\pi i}\end{aligned}$$