1)

a) We need $a \oplus z = z \oplus a = a \quad \forall a \in \mathbb{Z}$.

So

(1) $a \oplus z = a + z - 1 = a \rightarrow z = 1$

(2) $z \oplus a = z + a - 1 = a \rightarrow z = 1$

So $\boxed{z = 1}$ is the additive identity.

b) We need $\forall a \in \mathbb{Z}$, $\exists b \in \mathbb{Z}$ such that
(1) $a \oplus b = 1$ and (2) $b \oplus a = 1$. So:

(1) $a \oplus b = a + b - 1 = 1 \rightarrow a + b = 2 \rightarrow b = 2 - a$

(2) $b \oplus a = b + a - 1 = 1 \rightarrow a + b = 2 \rightarrow b = 2 - a$

So for every $a \in \mathbb{Z}$, the additive inverse is $\boxed{b = 2 - a}$.

c) To show that $\oplus$ commutes, $\forall a, b \in \mathbb{Z}$ we have

$a \oplus b = a + b - 1 = b + a - 1 = b \oplus a$ ✓

(d) we want a $u \in \mathbb{Z}$ such that $u \neq 1$, and $a \odot u = u \odot a = a$, $\forall a \in \mathbb{Z}$.
To that effect, we have

① $a \odot u = a + u - a \cdot u := a$

$\rightarrow u(1-a) = 0$

$\rightarrow u = 0$ or $a = 1$

② $u \odot a = u + a - u \cdot a := a$

$\rightarrow u(1-a) = 0$

$\rightarrow u = 0$ or $a = 1$

So $\boxed{u = 0}$ is the multiplicative identity.

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2)

a) By re-arranging, we see

$$T_n = \underbrace{n\underbrace{\left(T_{n-1} - 4T_{n-2} + 4T_{n-3}\right)}_{A_{n-1}}}_{A_n} + \underbrace{2\underbrace{\left(2T_{n-1} - 4T_{n-3}\right)}_{B_{n-1}}}_{B_n}$$

So that $A_n = n A_{n-1}$ with $A_3 = 3 \cdot (T_2 - 4T_1 + 4T_0) = 3(6 - 4 \cdot 3 + 4 \cdot 2) = 6 = 3!$, this is a well known recurrence for $A_n = n!$

Also, $B_n = 2B_{n-1}$ with $B_3 = 2 \cdot (2T_2 - 4T_0) = 2(2 \cdot 6 - 4 \cdot 2) = 8 = 2^3$, this is a well known recurrence for $B_n = 2^n$.

There fore

$$T_n = A_n + B_n, \quad n \geq 0$$

$$= n! + 2^n, \quad n \geq 0$$

b) We have

$$T_0 = 2 = 0! + 2^0 = 1+1 \checkmark$$

$$T_1 = 3 = 1! + 2^1 = 1+2 \checkmark$$

$$T_2 = 6 = 2! + 2^2 = 2+4 \checkmark$$

$$T_3 = (3+4)\cdot 6 - 4\cdot 3(3) + (4\cdot 3 - 8)\cdot 2$$

$$= 42 - 36 + 8 = 14 = 3! + 2^3 = 14 \checkmark$$

So for $0 \le n \le 3$, the formula $T_n = n! + 2^n$ holds,
Now suppose for all $n \le K$ the formula holds, then

$$T_{K+1} = ((K+1)+4)T_K - 4(K+1)T_{K-1} + (4(K+1)-8)T_{K-2}$$

$$= (K+1+4)(K! + 2^K) - 4(K+1)((K-1)! + 2^{K-1}) + 4(K-1)((K-2)! + 2^{K-2})$$

$$= (K+1)K! + \cancel{4K!} + (K+5)2^K - 4\cancel{K(K-1)!} - \cancel{4}(K-1)! - 4(K+1)2^{K-1} + 4(K-1)\cancel{(K-2)!} + 4(K-1)2^{K-2}$$

$$= (K+1)! + (K+5)2^K - (K+1)2^{K+1} + (K-1)2^K$$

$$= (K+1)! + 2^{K+1}\Big(\overset{\cancel{\frac{1}{2}K} + \frac{5}{2}}{\tfrac{1}{2}(K+5)} - \overset{-\cancel{K}-1}{(K+1)} + \overset{+\cancel{\frac{1}{2}K}-\frac{1}{2}}{\tfrac{1}{2}(K-1)}\Big)$$

$$= (K+1)! + 2^{K+1}\cdot 1$$

$$= (K+1)! + 2^{K+1} \checkmark$$

So by strong induction our formula works,

4)

a) Suppose that $\forall x \in \mathbb{R}$, that

$$a \cdot 1 + b \cdot (1-x) + c \cdot (1-x)^2 = 0$$

Then if $x = 1 \rightarrow a = 0$, so

$$(1-x) \cdot [b + c(1-x)] = 0$$

So if $x = 0$, we see

$$b = -c$$

So

$$b[(1-x) - \underbrace{(1-x)^2}_{}] = 0 \qquad -(1-2x+x^2)$$

$$\rightarrow b[-x^2 + x] = 0, \quad \forall x \in \mathbb{R}$$

$$\rightarrow b = 0 \rightarrow c = 0$$

As $a, b, c = 0$, it must be that $\{1, (1-x), (1-x)^2\}$ are linearly dependent. To show they span P^2, we show for some other tuple (a^*, b^*, c^*) that we can make (uniquely)

$$a + b(1-x) + c(1-x)^2 = a^* + b^* x + c^* x^2$$

Expanding, we see

$a+b-bx+c-2xc+cx^2$

$$= \underbrace{(a+b+c)}_{\underset{a^*}{\shortparallel}} - \underbrace{(2c+b)}_{\underset{b^*}{\shortparallel}}x + \underbrace{c}_{\underset{c^*}{\shortparallel}}x^2$$

So $c = c^*$, $-2c-b = b^*$

$\rightarrow b = -2c^* - b^*$

and $a = a^* - b - c$

$= a^* + 2c^* + b^* - c^*$

$= a^* + b^* + c^*$

So any vector $a^* + b^*x + c^*x^2 \in P^2(\mathbb{R})$ can be written as a combination

$a \cdot 1 + b(1-x) + c(1-x)^2$

Therefore $\{1, (1-x), (1-x)^2\}$ spans $P^2(\mathbb{R})$, and forms a basis for $P^2(\mathbb{R})$.

4) b)

Let $V_1 = \dfrac{1}{\sqrt{\int_{-1}^{1} 1\, dt}} = \boxed{\dfrac{1}{\sqrt{2}}}$

Then by G.S

$$V_2 = t - \text{proj}_{1/\sqrt{2}}(t) = t - \frac{1}{\sqrt{2}}\int_{-1}^{1} t\, dt \cdot \frac{1}{\sqrt{2}} = t$$

Normalizing

$$V_2 = \frac{t}{\sqrt{\int_{-1}^{1} t^2}} = \frac{t}{\sqrt{\frac{2}{3}}} = \boxed{\sqrt{\frac{3}{2}}\, t}$$

To get V_3

$$V_3 = t^2 - \text{proj}_{1/\sqrt{2}}(t^2) - \text{proj}_{\sqrt{\frac{3}{2}}t}(t^2)$$

$$= t^2 - \left[\frac{1}{\sqrt{2}}\int_{-1}^{1} t^2\, dt\right]\frac{1}{\sqrt{2}} - \left[\frac{\sqrt{3}}{2}\int_{-1}^{1} \cancel{t^3}^{\,0}\, dt\right]\frac{\sqrt{3}}{2}t$$

$$= t^2 - \frac{2}{3\sqrt{2}}\frac{1}{\sqrt{2}} = t^2 - 1/3$$

Normalizing

$$V_3 := \frac{t^2 - 1/3}{\sqrt{\int_{-1}^{1} (t^2 - 1/3)^2\, dt}} = \frac{t^2 - 1/3}{\sqrt{2\int_{0}^{1} t^4 - \frac{2}{3}t^2 + \frac{1}{9}\, dt}}$$

$$= \frac{t^2 - 1/3}{\sqrt{2\left[1/5 - \frac{2}{9} + \frac{1}{9}\right]}} = \frac{t^2 - 1/3}{\sqrt{\frac{2}{5} - \frac{2}{9}}}$$

$$= \frac{t^2 - \frac{1}{3}}{\sqrt{\frac{8}{45}}} = \sqrt{45/8}\left(t^2 - 1/3\right)$$

So the orthonormal set is

$$\left\{ 1/\sqrt{2},\ \sqrt{\tfrac{3}{2}}\,t,\ \sqrt{45/8}\,(t^2 - 1/3) \right\}$$