AMATH 567 - Homework 1

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Problem 1 (b):

First realize that $-i = 0 + (-1)i$ which can be represented as the coordinate $(0, -1)$ in $\mathbb{R}^2$. To convert this into polar form, we need to find the length and direction (angle) of the vector $[0, -1]^T$. Firstly the length can be computed as:

$$R = \sqrt{0^2 + (-1)^2} = 1$$

The coordinate $(0, -1)$ is at angle $\theta = \frac{3\pi}{2}$ radians from the x-axis. So the number $-i$ represented in polar form is:

$$-i = R \cdot e^{i\theta} = 1 \cdot e^{i\frac{3\pi}{2}} = e^{i\frac{3\pi}{2}}$$

Problem 1 (e):

First realize that $\frac{1}{2} - \frac{\sqrt{3}}{2}i = \frac{1}{2} + (-\frac{\sqrt{3}}{2})i$ which can be represented as the coordinate $(\frac{1}{2}, \frac{\sqrt{3}}{2})$ in $\mathbb{R}^2$. To convert this into polar form, we need to find the length and direction (angle) of the vector $[\frac{1}{2}, -\frac{\sqrt{3}}{2}]^T$. Firstly the length can be computed as:

$$R = \sqrt{\left(\frac{1}{2}\right)^2 + \left(-\frac{\sqrt{3}}{2}\right)^2} = \sqrt{\frac{1}{4} + \frac{3}{4}} = 1$$

The coordinate $(\frac{1}{2}, -\frac{\sqrt{3}}{2})$ is in the fourth quadrant (positive real part and negative imaginary part). So to compute the angle, we can calculate $\arctan\left(\frac{\frac{\sqrt{3}}{2}}{\frac{1}{2}}\right) = \arctan\sqrt{3}$ which represents the angle between the x-axis and the coordinate going clockwise. To get the counterclockwise angle θ, we can subtract this angle from 2π. So $\theta = 2\pi - \arctan\sqrt{3} = 2\pi - \frac{\pi}{3} = \frac{5\pi}{3}$ and:

$$\frac{1}{2} - \frac{\sqrt{3}}{2}i = R \cdot e^{i\theta} = 1 \cdot e^{i\frac{5\pi}{3}} = e^{i\frac{5\pi}{3}}$$

Problem 2 (b):

$$\frac{1}{1+i} = \frac{1}{1+i} \cdot \frac{1-i}{1-i} = \frac{1-i}{1+i-i-i^2}$$

$$= \frac{1-i}{1-(-1)}$$

$$= \frac{1}{2} - \frac{1}{2}i$$

$$= \frac{1}{2} + \left(-\frac{1}{2}\right) i$$

So $a = \frac{1}{2}$ and $b = -\frac{1}{2}$.

Problem 2 (c):

By the binomial theorem:

$$(1+i)^3 = 1^3 + 3(1)(i^2) + 3(i)(1^2) + i^3$$

Note that $i^3 = i^2 \cdot i = -1 \cdot i = -i$, so the above can be simplified to:

$$= 1 - 3 + 3i - i$$

$$= -2 + 2i$$

So $a = -2$ and $b = 2$.

Problem 2 (d):

By definition of modulus:

$$|3+4i| = \sqrt{3^2 + 4^2} = \sqrt{25} = 5 = 5 + 0i$$

So $a = 5$ and $b = 0$.

Problem 3 (d):

Let $u = z^2$, then we have:

$$z^4 + 2z^2 + 2 = 0$$

$$u^2 + 2u + 2 = 0$$

We can solve this with the quadratic formula:

$$u = \frac{-2 \pm \sqrt{2^2 - 4(1)(2)}}{2(1)}$$

$$u = -1 \pm \frac{\sqrt{-4}}{2}$$

$$u = -1 \pm \frac{2i}{2}$$

$$u = -1 \pm i$$

Then, substituting back for u:

$$z^2 = -1 \pm i$$

$$z = \pm\sqrt{-1 \pm i}$$

To further simplify so that our final answers are in the form $Re^{i\theta}$, we can write both $-1 + i$ and $-1 - i$ in this way. For both numbers:

$$R = \sqrt{(-1)^2 + (1)^2} = \sqrt{(-1)^2 + (-1)^2} = \sqrt{2}$$

Now to compute the angles, note that $-1 + i$ is in the second quadrant of the complex plane and $-1 - i$ is in the third quadrant. We can compute the same reference angle for each number:

$$\theta_{reference} = \arctan\left(\frac{1}{1}\right) = \arctan 1 = \frac{\pi}{4}$$

Taking into account the location of each number in the complex plane we can compute:

$$\theta_{-1+i} = \pi - \frac{\pi}{4} = \frac{3\pi}{4}$$

$$\theta_{-1-i} = \pi + \frac{\pi}{4} = \frac{5\pi}{4}$$

So in polar form:

$$-1 + i = \sqrt{2}e^{i\frac{3\pi}{4}}$$

$$-1 - i = \sqrt{2}e^{i\frac{5\pi}{4}}$$

Now for each of these numbers, we need to find both the positive and negative roots. To do so, first notice that $-1 = e^{i\pi}$. Then for $-1+i$:

$$\sqrt{-1+i} = \left(\sqrt{2}e^{i\frac{3\pi}{4}}\right)^{\frac{1}{2}} = 2^{\frac{1}{4}}e^{i\frac{3\pi}{8}}$$

$$-\sqrt{-1+i} = e^{i\pi}2^{\frac{1}{4}}e^{i\frac{3\pi}{8}} = 2^{\frac{1}{4}}e^{i\frac{11\pi}{8}}$$

Similarly for $-1-i$:

$$\sqrt{-1-i} = \left(\sqrt{2}e^{i\frac{5\pi}{4}}\right)^{\frac{1}{2}} = 2^{\frac{1}{4}}e^{i\frac{5\pi}{8}}$$

$$-\sqrt{-1-i} = e^{i\pi}2^{\frac{1}{4}}e^{i\frac{5\pi}{8}} = 2^{\frac{1}{4}}e^{i\frac{13\pi}{8}}$$

So finally, we have 4 solutions: $\left[2^{\frac{1}{4}}e^{i\frac{3\pi}{8}}, 2^{\frac{1}{4}}e^{i\frac{11\pi}{8}}, 2^{\frac{1}{4}}e^{i\frac{5\pi}{8}}, 2^{\frac{1}{4}}e^{i\frac{13\pi}{8}}\right]$

Problem 4 (d):

Let $z = x + iy$, notice that:

$$\frac{z+\bar{z}}{2} = \frac{x+iy+x-iy}{2} = \frac{2x}{2} = x = Re(z)$$

So:

$$Re(z) = \frac{z+\bar{z}}{2} \leq \frac{1}{2}\left|z+\bar{z}\right| \leq \frac{1}{2}\left(|z|+|\bar{z}|\right) = \frac{1}{2}(2|z|) = |z|$$

Where the second inequality is an application of the triangle inequality, and the second to last equality is using the trivial fact that $|z| = |\bar{z}|$.

Problem 4 (f):

Let z_1, $z_2 \in \mathbb{C}$. We can write both numbers in polar form where $z_1 = R_1 e^{i\theta_1}$ and $z_2 = R_2 e^{i\theta_2}$ for some R_1, $R_2 \in \mathbb{R}$ and θ_1, $\theta_2 \in [0, 2\pi)$. Making this substitution, and noting that for any $x \in \mathbb{R}$ that $|e^{ix}| = 1$, we can say:

$$|z_1 z_2| = |R_1 e^{i\theta_1} R_2 e^{i\theta_2}| = |R_1||R_2||e^{i(\theta_1+\theta_2)}| = |R_1||R_2|$$

Next notice that:

$$|z_1||z_2| = |R_1 e^{i\theta_1}||R_2 e^{i\theta_2}| = |R_1||e^{i\theta_1}||R_2||e^{i\theta_2}| = |R_1||R_2|$$

Which when combing both lines gives us our desired result.

Problem 5:

For $a^b = i^i$, notice that we can write $a = i = e^{i\frac{\pi}{2}}$. So according to the formula:

$$i^i = e^{i \log i} = e^{i(\log 1 + i\pi/2)} = e^{-\frac{\pi}{2}}$$

The result is real, so the real part is $e^{-\frac{\pi}{2}}$ and the imaginary part is 0.

For $a^b = (1+i)^i$, notice that we can write $a = 1 + i = \sqrt{2} e^{i\frac{\pi}{4}}$. So according to the formula:

$$(1+i)^i = e^{i \log i + 1} = e^{i(\log \sqrt{2} + i\frac{\pi}{4})} = e^{i(\frac{\log 2}{2} + i\frac{\pi}{4})} = e^{i\frac{\log 2}{2} - \frac{\pi}{4}} = e^{i\frac{\log 2}{2}} e^{-\frac{\pi}{4}}$$

$$= e^{-\frac{\pi}{4}} \cos\left(\frac{\log 2}{2}\right) + i \cdot e^{-\frac{\pi}{4}} \sin\left(\frac{\log 2}{2}\right)$$

So the real part is $e^{-\frac{\pi}{4}} \cos\left(\frac{\log 2}{2}\right)$ and the imaginary part is $e^{-\frac{\pi}{4}} \sin\left(\frac{\log 2}{2}\right)$.

Problem 6:

We have:

$$e(z_1 + z_2) = \sum_{n=0}^{\infty} \frac{(z_1 + z_2)^n}{n!}$$

By the binomial theorem, we can expand $(z_1 + z_2)^n$:

$$\sum_{n=0}^{\infty} \frac{(z_1 + z_2)^n}{n!} = \sum_{n=0}^{\infty} \sum_{k=0}^{n} \frac{1}{n!} \binom{n}{k} z_1^k z_2^{n-k}$$

To move forward from here we can rearrange the double sums (we can do this because the set of integer pairs $\{(n, k)\}$ we are summing over are the same either way).

$$= \sum_{k=0}^{\infty} \sum_{n=k}^{\infty} \frac{1}{n!} \binom{n}{k} z_1^k z_2^{n-k}$$

$$= \sum_{k=0}^{\infty} \sum_{n=k}^{\infty} \frac{1}{n!} \cdot \frac{n!}{k!(n-k)!} \cdot z_1^k z_2^{n-k}$$

$$= \sum_{k=0}^{\infty} \sum_{n=k}^{\infty} \frac{z_1^k}{k!} \cdot \frac{z_2^{n-k}}{(n-k)!}$$

The $\frac{z_1^k}{k!}$ term only depends on k, and can be separated into the sum indexed by k:

$$= \sum_{k=0}^{\infty} \frac{z_1^k}{k!} \sum_{n=k}^{\infty} \frac{z_2^{n-k}}{(n-k)!}$$

Now let $j = n - k$, then the sum on the RHS can be re-written solely in terms of j:

$$= \sum_{k=0}^{\infty} \frac{z_1^k}{k!} \sum_{j=0}^{\infty} \frac{z_2^j}{j!}$$

Notice how the sum on the LHS is the definition of $e(z_1)$ and the sum on the RHS is the definition of $e(z_2)$:

$$= e(z_1)e(z_2)$$

Which is the desired result. Now suppose we have some function:

$$E(z) = \sum_{n=0}^{\infty} a_n z^n$$

We want to find conditions for the sequence defined by a_n such that $E(z_1 + z_2) = E(z_1)E(z_2)$. Firstly, by repeating the same steps as above, we can get $E(z_1 + z_2)$ into the following form:

$$E(z_1 + z_2) = \sum_{n=0}^{\infty} a_n (z_1 + z_2)^n = \sum_{n=0}^{\infty} \sum_{k=0}^{n} a_n \binom{n}{k} z_1^k z_2^{n-k}$$

Now we can manipulate $E(z_1)E(z_2)$ into a similar form:

$$E(z_1)E(z_2) = \sum_{k=0}^{\infty} a_k z_1^k \sum_{j=0}^{\infty} a_j z_2^j$$

$$= \sum_{k=0}^{\infty} \sum_{j=0}^{\infty} a_k z_1^k \cdot a_j z_2^j$$

Now let $j = n - k$, then we have:

$$= \sum_{k=0}^{\infty} \sum_{n=k}^{\infty} (a_k z_1^k)(a_{n-k} \cdot z_2^{n-k})$$

For the same reason as earlier we can rearrange the sums:

$$= \sum_{n=0}^{\infty} \sum_{k=0}^{n} (a_k z_1^k)(a_{n-k} \cdot z_2^{n-k})$$

$$= \sum_{n=0}^{\infty} \sum_{k=0}^{n} (a_k \cdot a_{n-k})(z_1^k \cdot z_2^{n-k})$$

We now want to look for conditions on a_n such that $E(z_1 + z_2) = E(z_1)E(z_2)$:

$$E(z_1 + z_2) = \sum_{n=0}^{\infty} \sum_{k=0}^{n} a_n \binom{n}{k} z_1^k z_2^{n-k} = \sum_{n=0}^{\infty} \sum_{k=0}^{n} (a_k \cdot a_{n-k})(z_1^k \cdot z_2^{n-k}) = E(z_1)E(z_2)$$

The sum on the LHS is exactly the same as the sum on the RHS other than the terms involving members of the sequence a_n. The only way for these two sums to be equal in general is if the sequence defined by a_n satisfies the following relation for $0 \leq k \leq n$:

$$a_n \binom{n}{k} = a_k \cdot a_{n-k}$$

So any sequence defined by a_n that satisfies the above relationship will generate a function $E(z)$ such that $E(z_1 + z_2) = E(z_1)E(z_2)$. Notice what happens if we set $k = 1$:

$$a_n \binom{n}{1} = a_1 \cdot a_{n-1}$$

$$a_n \cdot n = a_1 \cdot a_{n-1}$$

$$a_n = \frac{a_1}{n} \cdot a_{n-1}$$

We get a recurrence relation for a_n which we can repeatedly apply:

$$a_n = \frac{a_1}{n} \cdot a_{n-1}$$

$$a_n = \frac{a_1}{n} \cdot \frac{a_1}{n-1} \cdot a_{n-2}$$

$$a_n = \frac{a_1}{n} \cdot \frac{a_1}{n-1} \cdot \frac{a_1}{n-2} \cdot a_{n-3}$$

$$\dots$$

$$a_n = \frac{a_1}{n} \cdot \frac{a_1}{n-1} \dots \frac{a_1}{1} \cdot a_0 = a_0 \cdot \frac{a_1^n}{n!}$$

We can now plug this formula for a_n into our definition of $E(z)$ to show that this expression will lead to $E(z_1 + z_2) = E(z_1)E(z_2)$:

$$E(z_1 + z_2) = \sum_{n=0}^{\infty} \sum_{k=0}^{n} a_0 \cdot \frac{a_1^n}{n!} \binom{n}{k} z_1^k z_2^{n-k}$$

$$= \sum_{n=0}^{\infty} \sum_{k=0}^{n} a_0 \cdot a_1^n \cdot \frac{z_1^k}{k!} \cdot \frac{z_2^{n-k}}{(n-k)!}$$

$$E(z_1)E(z_2) = \sum_{n=0}^{\infty} \sum_{k=0}^{n} (a_k \cdot a_{n-k})(z_1^k \cdot z_2^{n-k})$$

$$= \sum_{n=0}^{\infty} \sum_{k=0}^{n} a_0 \cdot \frac{a_1^k}{k!} \cdot a_0 \cdot \frac{a_1^{n-k}}{(n-k)!} \cdot (z_1^k \cdot z_2^{n-k})$$

$$= \sum_{n=0}^{\infty} \sum_{k=0}^{n} (a_0)^2 \cdot a_1^n \cdot \frac{z_1^k}{k!} \cdot \frac{z_2^{n-k}}{(n-k)!}$$

So if we compare both sums, $E(z_1 + z_2) = E(z_1)E(z_2)$ exactly if $a_0 = (a_0)^2$, which happens when $a_0 = 0$ (trivial degenerate case) or $a_0 = 1$.

Problem 7:

We have:

$$f(z) = z^{\frac{1}{2}}$$

Since both z and $f(z)$ are complex numbers, we can say $z = x_1 + ix_2$ and $f(z) = y_1 + iy_2$ for x_1, x_2, y_1, $y_2 \in \mathbb{R}$. With this we can say:

$$f(z) = y_1 + iy_2 = (x_1 + ix_2)^{\frac{1}{2}} = z^{\frac{1}{2}}$$

We'd like to solve for y_1 and y_2, so we can begin by squaring both sides:

$$(y_1 + iy_2)^2 = (x_1 + ix_2)$$

$$y_1^2 + 2iy_1y_2 - y_2^2 = x_1 + iy_2$$

$$y_1^2 - y_2^2 + i(2y_1y_2) = x_1 + iy_2$$

By equating both the real and imaginary parts of both sides of the equation, we see that:

$$x_1 = y_1^2 - y_2^2 \text{ and } x_2 = 2y_1y_2$$

We need to solve for y_1 and y_2, so we can start by manipulating the equation on the RHS:

$$x_2^2 = 4y_1^2y_2^2$$

We also have that:

$$y_1^2 = x_1 + y_2^2$$

So combining:

$$x_2^2 = 4(x_1 + y_2^2)y_2^2$$

$$x_2^2 = 4x_1y_2^2 + 4y_2^4$$

$$y_2^4 + x_1y_2^2 - \frac{1}{4}x_2^2 = 0$$

$$(y_2^2)^2 + x_1(y_2^2) - \frac{1}{4}x_2^2 = 0$$

Which gives us a quadratic equation in y_2^2, which we can solve with the quadratic formula:

$$y_2^2 = \frac{-x_1 \pm \sqrt{x_1^2 - 4(1)(-\frac{1}{4}x_2^2)}}{2}$$

$$y_2^2 = \frac{-x_1 \pm \sqrt{x_1^2 + x_2^2}}{2}$$

$$y_2 = \pm\sqrt{\frac{-x_1 \pm \sqrt{x_1^2 + x_2^2}}{2}}$$

We can now substitute the above expression for y_2 into one of our original equations to solve for y_1:

$$y_1^2 = x_1 + y_2^2$$

$$y_1^2 = x_1 + \frac{-x_1 \pm \sqrt{x_1^2 + x_2^2}}{2} = \frac{x_1 \pm \sqrt{x_1^2 + x_2^2}}{2}$$

$$y_1 = \pm\sqrt{\frac{x_1 \pm \sqrt{x_1^2 + x_2^2}}{2}}$$

The way things are written right now, it looks like we have 4 solutions, but we must throw 2 of them away. We know that y_1 and y_2 have to be real numbers, so if any solutions lead to complex results, we know we should get rid of them. For instance, $|z| = \sqrt{x_1^2 + x_2^2}$, but from problem 4 we know that $x_1 \leq |z|$, so in y_1, under the outer square root, the solution with the minus sign will always lead to negative numbers being square-rooted, which generates a complex number, so we must throw that solution away. Similarly for y_2, we must also throw away the solution with the minus sign under the outer square root for the same reason. In order to satisfy the constraint that $x_2 = 2y_1y_2$, we must also restrict y_1 and y_2 so that they have the same sign outside the outer square root. If we allowed y_1 and y_2 to have opposite signs, we could have $2y_1y_2 = -x_2$ which is not correct. So in the end our solutions are:

$$\left(y_1 = \sqrt{\frac{x_1 + \sqrt{x_1^2 + x_2^2}}{2}}, y_2 = \sqrt{\frac{-x_1 + \sqrt{x_1^2 + x_2^2}}{2}}\right)$$

$$\left(y_1 = -\sqrt{\frac{x_1 + \sqrt{x_1^2 + x_2^2}}{2}}, y_2 = -\sqrt{\frac{-x_1 + \sqrt{x_1^2 + x_2^2}}{2}}\right)$$

Problem 8 (a):

Suppose we have the equation:

$$x^3 + ax^2 + bx + c = 0$$

Let $x = y - \frac{a}{3}$, then we have:

$$\left(y - \frac{a}{3}\right)^3 + a\left(y - \frac{a}{3}\right)^2 + b\left(y - \frac{a}{3}\right) + c = 0$$

$$\left(y^3 + 3y \cdot \frac{a^2}{9} - 3y^2 \cdot \frac{a}{3} - \frac{a^3}{27}\right) + a\left(y^2 - 2y \cdot \frac{a}{3} + \frac{a^2}{9}\right) + b\left(y - \frac{a}{3}\right) + c = 0$$

$$y^3 - ay^2 + ay^2 + \frac{1}{3}a^2y - \frac{2}{3}a^2y + by - \frac{1}{27}a^3 + \frac{1}{9}a^3 - \frac{1}{3}ab + c = 0$$

$$y^3 + \left(b - \frac{1}{3}a^2\right)y + \left(\frac{2}{27}a^3 - \frac{1}{3}ab + c\right) = 0$$

$$y^3 + py + q = 0$$

Where $p = b - \frac{1}{3}a^2$ and $q = \frac{2}{27}a^3 - \frac{1}{3}ab + c$.

Problem 8 (b):

Now let $y = u + v$, then:

$$y^3 + py + q = 0$$

$$(u+v)^3 + p(u+v) + q = 0$$

$$u^3 + 3uv^2 + 3u^2v + v^3 + pu + pv + q = 0$$

$$u^3 + 3uv(v+u) + v^3 + p(u+v) + q = 0$$

$$u^3 + v^3 + (3uv + p)(u+v) + q = 0$$

Problem 8 (c):

We have the following two equations:

$$u^3v^3 = -\frac{1}{27}p^3$$

$$u^3 + v^3 = -q$$

Using equation 2, we can write $v^3 = -q - u^3$, so we can substitute this into equation 1:

$$u^3(-q - u^3) = -\frac{1}{27}p^3$$

$$-qu^3 - u^6 + \frac{1}{27}p^3 = 0$$

$$u^6 + qu^3 - \frac{1}{27}p^3 = 0$$

$$(u^3)^2 + q(u^3) - \frac{1}{27}p^3 = 0$$

Where the above is quadratic in u^3. Again using equation 2, we can write $u^3 = -q - v^3$, so we can substitute this into equation 1 again:

$$(-q - v^3)v^3 = -\frac{1}{27}p^3$$

$$-qv^3 - v^6 + \frac{1}{27}p^3 = 0$$

$$v^6 + qv^3 - \frac{1}{27}p^3 = 0$$

Where the above is quadratic in v^3.

Problem 8 (d):

To solve for u and v we need to solve the quadratic formulas from (c) in terms of u^3 and v^3. To do so, we can use the quadratic formula:

$$u^3 = \frac{-q \pm \sqrt{q^2 - 4(1)(-\frac{1}{27})p^3}}{2(1)}$$

$$u^3 = \frac{-q \pm \sqrt{q^2 + \frac{4}{27}p^3}}{2}$$

$$u = \sqrt[3]{\frac{-q \pm \sqrt{q^2 + \frac{4}{27}p^3}}{2}}$$

Where by $\sqrt[3]{\mathrm{z}}$, I mean the principal cube-root. The other solutions for u can be found by multiplying the principal u above by the roots of unity $w_1 = e^{\frac{2\pi i}{3}}$ and $w_2 = e^{\frac{4\pi i}{3}}$. At the end of the day, there are technically 6 solutions for u.

Similarly for v, since it solves the same quadratic:

$$v = \sqrt[3]{\frac{-q \pm \sqrt{q^2 + \frac{4}{27}p^3}}{2}}$$

Again, just like u, we can get the other solutions for v by multiplying by the roots of unity w_1 and w_2, so v will also technically have 6 solutions.

Problem 8 (e):

We know:

$$y = u + v$$

$$x + \frac{a}{3} = u + v$$

$$x = u + v - \frac{a}{3}$$

Where u and v are the solutions to the previous section of this problem. Since there are 6 choices for u and 6 choices for v, there are in total $6 \cdot 6 = 36$ possible combinations of u and v, however, we know that a cubic polynomial should only have 3 solutions, so there are some redundancies. Since u and v share the exact same 6 solutions, and $y = u + v$, we can restrict the principal values of u and v (call them u_0 and v_0) such that:

$$u_0 = \sqrt[3]{\frac{-q + \sqrt{q^2 + \frac{4}{27}p^3}}{2}}$$

$$v_0 = \sqrt[3]{\frac{-q - \sqrt{q^2 + \frac{4}{27}p^3}}{2}}$$

Basically we let u capture the sum under the outer cube root, and v capture the difference under the outer cube root. Under this restriction, there are 3 choices for u and 3 choices for v, so a total of 9 possibilities. If we consider one of our constraints from an earlier section, namely that $3uv = -p$, we can restrict our solutions even further, because otherwise this constraint will not be satisfied. Without writing out all the algebra, I found the final 3 solutions for x to be:

$$x_1 = u_0 + v_0 - \frac{a}{3}$$

$$x_2 = u_0 w_1 + v_0 w_2 - \frac{a}{3}$$

$$x_3 = u_0 w_2 + v_0 w_1 - \frac{a}{3}$$

Problem 8 (e):

We have:

$$x^3 + 3x^2 + 6x + 8 = 0$$

From an earlier section of this problem we can compute:

$$p = b - \frac{1}{3}a^2 = 6 - \frac{1}{3} \cdot 9 = 3$$

$$q = \frac{2}{27}a^3 - \frac{1}{3}ab + c = \frac{2}{27} \cdot 27 - \frac{1}{3}(3)(6) + 8 = 4$$

So:

$$u_0 = \sqrt[3]{\frac{-4 + \sqrt{4^2 + \frac{4}{27}3^3}}{2}}$$

$$u_0 = \sqrt[3]{-2 + \sqrt{5}}$$

And similarly:

$$v_0 = \sqrt[3]{-2 - \sqrt{5}}$$

So by plugging into the formulas in the previous section of the problem, our solutions are:

$$x_1 = \sqrt[3]{-2 + \sqrt{5}} + \sqrt[3]{-2 - \sqrt{5}} - 1$$

$$x_2 = \left(\sqrt[3]{-2 + \sqrt{5}}\right) e^{\frac{2\pi i}{3}} + \left(\sqrt[3]{-2 - \sqrt{5}}\right) e^{\frac{4\pi i}{3}} - 1$$

$$x_3 = \left(\sqrt[3]{-2 + \sqrt{5}}\right) e^{\frac{4\pi i}{3}} + \left(\sqrt[3]{-2 - \sqrt{5}}\right) e^{\frac{2\pi i}{3}} - 1$$

Problem 8 (f):

We have:

$$x^3 - 15x - 4 = 0$$

This cubic equation is already depressed (no x^2 term) so we can see p and q directly, where $p = -15$ and $q = -4$. So:

$$u_0 = \sqrt[3]{\frac{4 + \sqrt{(-4)^2 + \frac{4}{27}(-15)^3}}{2}}$$

$$u_0 = \sqrt[3]{2 + 11i}$$

$$= 2 + i$$

And similarly for v_0:

$$v_0 = \sqrt[3]{2 - 11i}$$

$$= 2 - i$$

Where the facts that $\sqrt[3]{2 - 11i} = 2 - i$ and $\sqrt[3]{2 + 11i} = 2 + i$ follow by just cubing $(2 - i)$ and $(2 + i)$ respectively. So our solutions are:

$$x_1 = 2 + i + 2 - i = 4$$

$$x_2 = (2 + i)e^{i\frac{2\pi}{3}} + (2 - i)e^{i\frac{4\pi}{3}} = (2 + i)\left(-\frac{1}{2} + i\frac{\sqrt{3}}{2}\right) + (2 - i)\left(-\frac{1}{2} - i\frac{\sqrt{3}}{2}\right)$$

$$= -2 - \sqrt{3}$$

$$x_3 = (2 + i)e^{i\frac{4\pi}{3}} + (2 - i)e^{i\frac{2\pi}{3}} = (2 + i)\left(-\frac{1}{2} - i\frac{\sqrt{3}}{2}\right) + (2 - i)\left(-\frac{1}{2} + i\frac{\sqrt{3}}{2}\right)$$

$$= \sqrt{3} - 2$$

After simplifying our solutions, we notice there are only real solutions that are listed above (so the imaginary parts are all 0).