AMATH 502

HOMEWORK 2

Nate Whybra

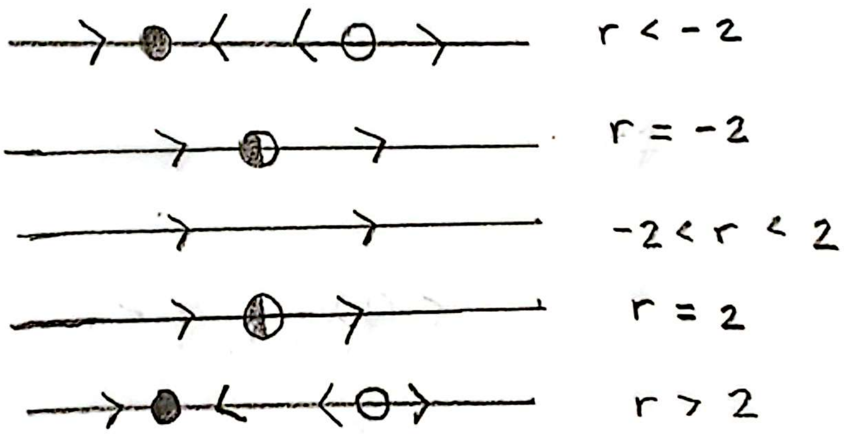
3.1.1

We have

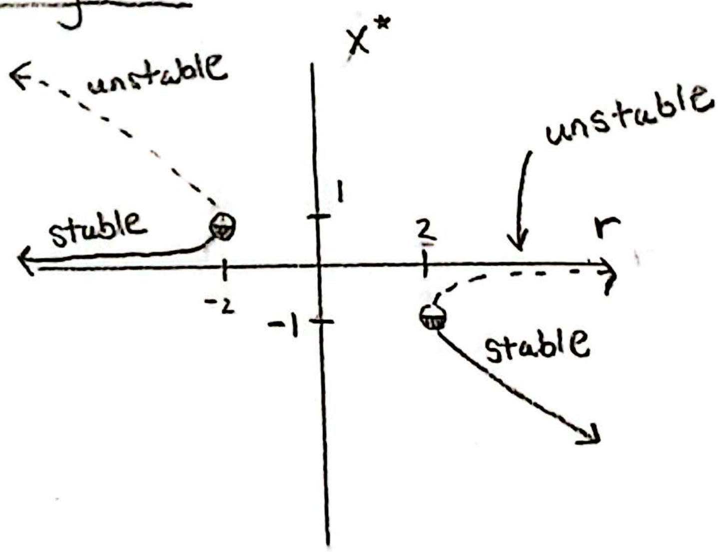
$$\dot{x} = x^2 + rx + 1 = f(x) := 0$$

$$\longleftrightarrow \quad x^* = \frac{1}{2}\left[-r \pm \sqrt{r^2-4}\,\right]$$

Vector fields



Bifurcation diagram



So there is a saddle-node bifurcation when $r = -2$ and when $r = 2$.

3.1.5

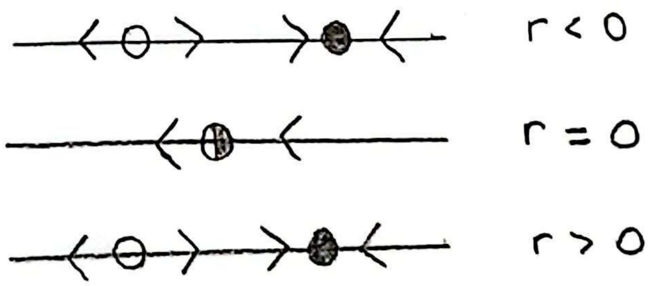
(a) We have

$$\dot{x} = r^2 - x^2 = f(x) := 0$$

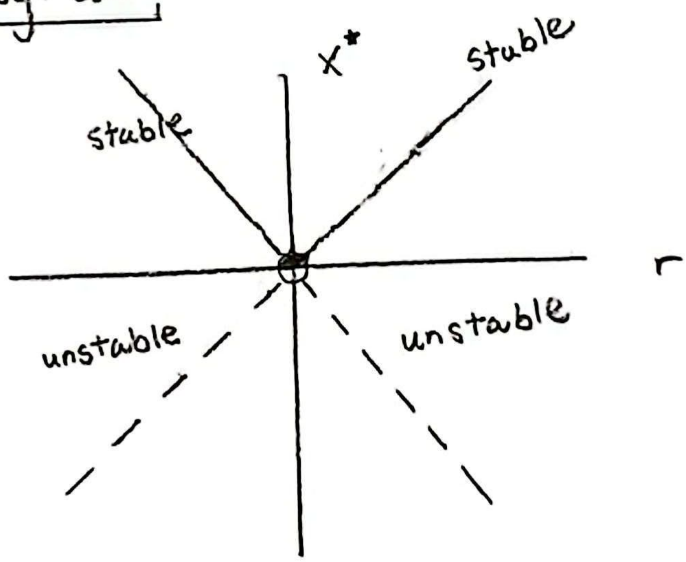
$$\longleftrightarrow (r-x)(r+x) = 0$$

$$\longleftrightarrow x^* = \pm r$$

vector field



Bifurcation diagram



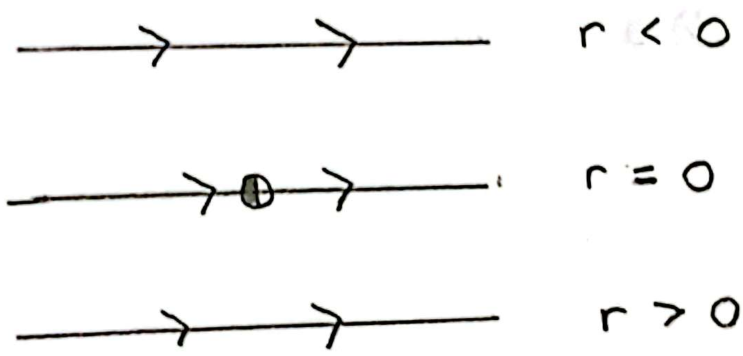
(b) we have

$$\dot{x} = r^2 + x^2 = f(x) := 0$$

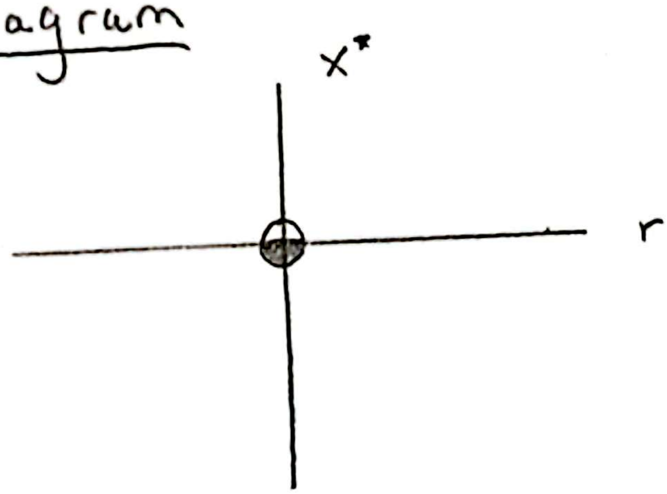
$$\longleftrightarrow \quad x^2 = -r^2$$

$$\longleftrightarrow \quad x^* = \pm i r$$

<u>vector field</u>



<u>Bifurcation diagram</u>



<u>Note</u>: complex fixed points ?

3.2.3

we have

$$\dot{x} = x - rx(1-x) = f(x) := 0$$

$$\leftrightarrow \quad x - rx + rx^2 = 0$$

$$\longleftrightarrow \quad rx^2 + (1-r)x = 0$$

$$r \neq 0 \quad \longleftrightarrow \quad x^2 + \frac{(1-r)}{r} x = 0$$

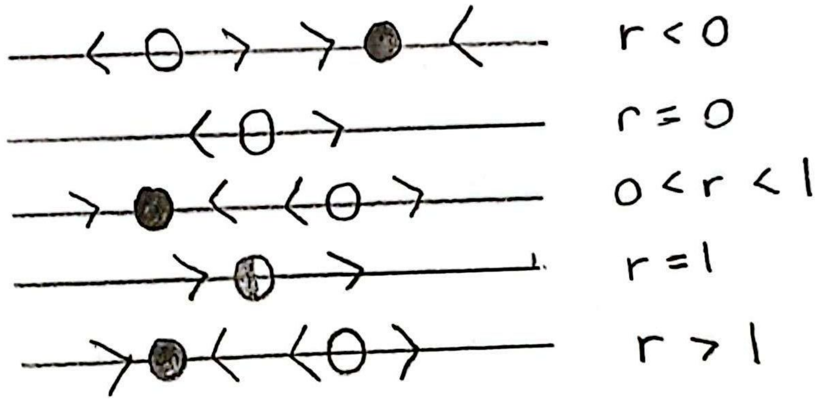
$$\longleftrightarrow \quad x^* = \frac{r-1}{2r} \pm \sqrt{\frac{(1-r)^2}{r^2}} \cdot \frac{1}{2}$$

$$= \frac{r-1}{2r} \pm \left| \frac{1-r}{2r} \right|$$

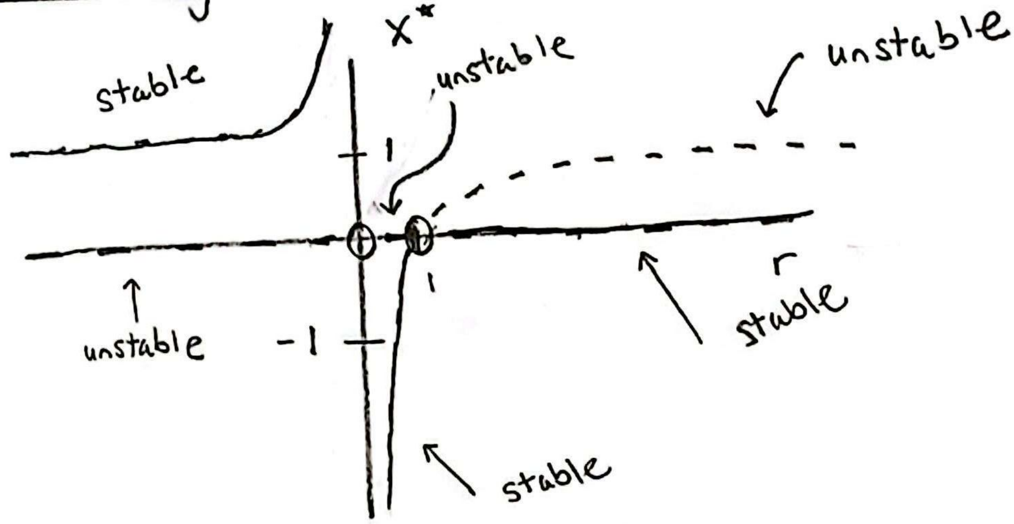
$$= \begin{cases} \frac{1}{2}\left[\frac{r-1}{r} \pm \left|\frac{r-1}{r}\right|\right] & \text{if } r \neq 0 \\ 0 & \text{if } r = 0 \end{cases}$$

$$= \begin{cases} \frac{1}{2}\left[\frac{r-1}{r} \pm \frac{r-1}{r}\right] = \left\{0, \frac{r-1}{r}\right\} & \text{if } r < 0 \\ 0 & \text{if } r = 0 \\ \frac{1}{2}\left[\frac{r-1}{r} \pm \frac{1-r}{r}\right] = \left\{0, \frac{r-1}{r}\right\} & \text{if } 0 < r < 1 \\ 0 & \text{if } r = 1 \\ \frac{1}{2}\left[\frac{r-1}{r} \pm \frac{r-1}{r}\right] = \left\{0, \frac{r-1}{r}\right\} & \text{if } r > 1 \end{cases}$$

vector fields



Bifurcation diagram



So a transcritical bifurcation occurs for $x^* = 0$ at the critical value of $r = 1$.

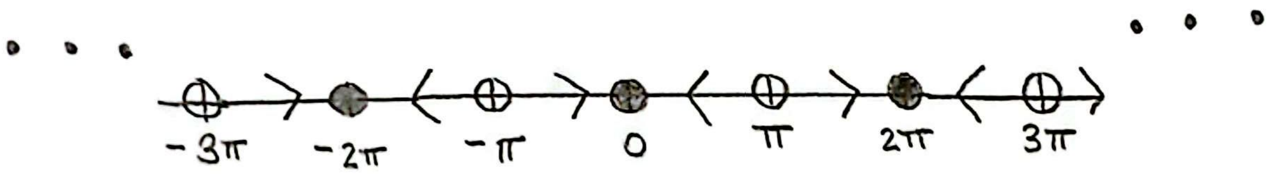
3.4.11

a) For $r=0$, we have $\dot{x} = -\sin(x) = f(x) := 0$

$\longleftrightarrow \quad \sin(x) = 0$

$\longleftrightarrow \quad x^* = n\pi$ for $n \in \mathbb{Z}$

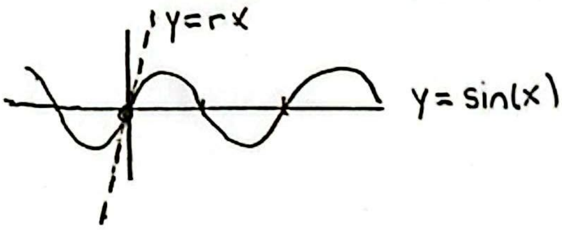
vector field



So the fixed points at $x^* = 2K$ for $K \in \mathbb{Z}$ are stable, and the ones at $x^* = 2K+1$ are unstable.

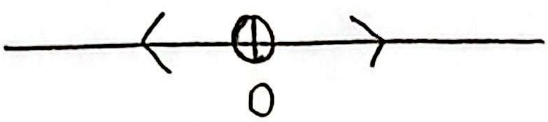
b) For $r>1$, we have $\dot{x} = rx - \sin(x) = f(x) := 0$

$\longleftrightarrow \quad \sin(x) = rx$



$\longleftrightarrow \quad x^* = 0$

vector field

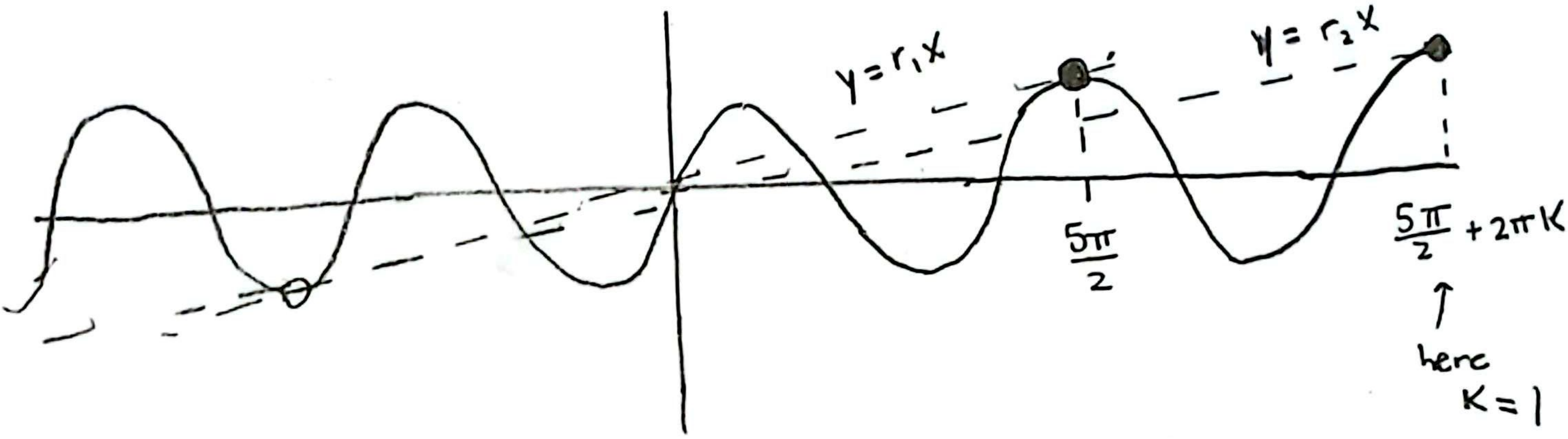


So there is only one fixed point at $x^* = 0$, and it is unstable.

3.4.11

c) When $r > 1$, $x^* = 0$ is an unstable f.p.

When, $0 < r < 1$



we have $\sin(x) \approx rx$ roughly whenever ~~when~~ $x = \frac{5\pi}{2} + 2\pi K$.
$\sin(x) = 1$ with $x > 0$,

So we have

$$\sin\left(\frac{5\pi}{2} + 2\pi K\right) = r\left(\frac{5\pi}{2} + 2\pi K\right)$$

$$1 = r\left(\frac{5\pi}{2} + 2\pi K\right)$$

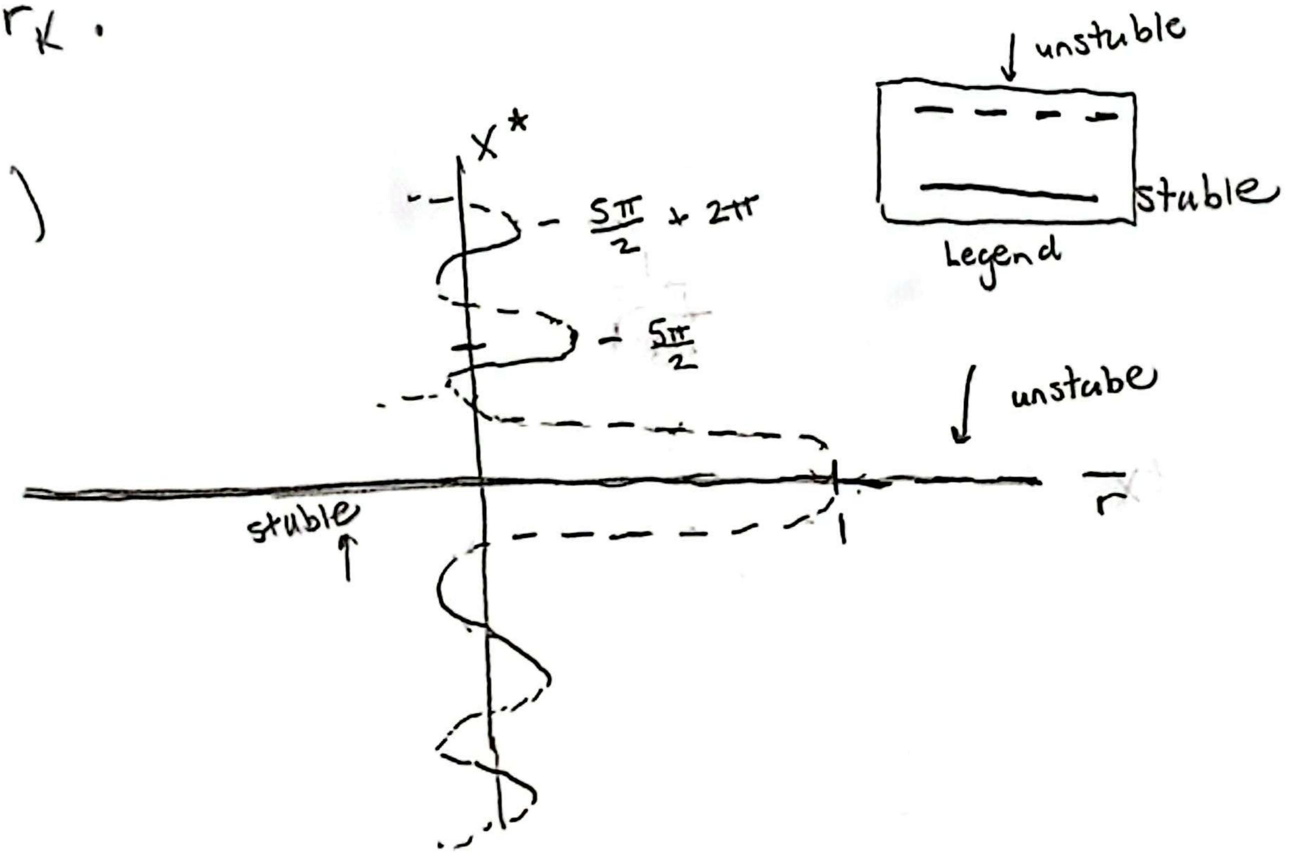
$$\circledast \quad r_K = \frac{1}{\frac{5\pi}{2} + 2\pi K}$$

Firstly as soon a $r < 1$, 0 remains a f.p but 2 new f.p's come into existance. This is a subcritical pitchfork bifurcation at $r = 1$. Now in $\circledast$ for all $K \geq 0$, 2 saddle node bifurcations happen as 2 fp's are generated to the right of 0 and 2 are generated to the left.

d) See the result in part (c).

e) The same formula holds from part (c) with $K \le -2$, so for each K in this range, there will similarly still be 2 new saddle node bifurcations at each r_K.

f)



3.5.8

we have,

$$\frac{du}{dt} = \dot{u} = au + bu^3 - cu^5$$

Put $x = \frac{u}{U} \rightarrow u = Ux$ and $\tau = \frac{t}{T} \rightarrow t = \tau T$.

Then

$$\dot{u} = \frac{du}{dt} = U\frac{dx}{dt}$$

And

$$au + bu^3 - cu^5 = aUx + bU^3x^3 - cU^5x^5$$

So we have,

$$U\frac{dx}{dt} = aUx + bU^3x^3 - cU^5x^5$$

$\xrightarrow{U \neq 0}$

$$\frac{dx}{dt} = ax + bU^2x^3 - cU^4x^5$$

Now, by the chain rule

$$\frac{dx}{d\tau} = \frac{dx}{dt}\frac{dt}{d\tau}$$

And $\frac{dt}{d\tau} = T$

CamScanner

So

$$\frac{dx}{d\tau} = Tax + bTU^2x^3 - cU^4Tx^5$$

Now, we'd like

$$\begin{cases} Ta = r & \rightarrow \quad T = \frac{r}{a} \\ bTU^2 = 1 & \rightarrow \quad T = \frac{1}{U^2 b} \\ cTU^4 = 1 & \rightarrow \quad T = \frac{1}{cU^4} \end{cases}$$

So $\bullet$ $\frac{r}{a} = \frac{1}{U^2 b} \rightarrow \underline{r} = \frac{a}{U^2 b} \rightarrow \underline{r} = \frac{a}{\frac{b}{c} \cdot b} = \frac{ac}{b^2}$

$\bullet$ $\frac{1}{U^2 b} = \frac{1}{cU^4} \rightarrow \underline{U^2} = \frac{b}{c} \rightarrow \underline{U} = \sqrt{\frac{b}{c}}$ as $b, c > 0$

$\underline{T} = \frac{1}{cU^4} = \frac{1}{\cancel{c}\,\frac{b^2}{c^{\cancel{2}}}} = \frac{c}{b^2}$

So if $(r, U, T) = \left(\frac{ac}{b^2}, \sqrt{\frac{b}{c}}, \frac{c}{b^2}\right)$, we have the desired result that

$$\frac{dx}{d\tau} = rx + x^3 - x^5$$

3.7.3

a) we have

$$\dot{N} = rN\left(1-\frac{N}{K}\right) - H = rN - \frac{r}{K}N^2 - H$$

Put $\underbrace{x = \frac{N}{K}}$ $\rightarrow$ $N = Kx$ and $t = \frac{\tau}{rK}$ $\rightarrow$ $\underbrace{\tau = rK\tau}$

Then

$$\frac{dN}{dt} = K\frac{dx}{dt} = rKx - rKx^2 - H$$

Since

$$\frac{dx}{d\tau} = \frac{dx}{dt}\frac{dt}{d\tau} \quad \text{and} \quad \frac{dt}{d\tau} = \frac{1}{rK}$$

we get

$$\frac{dx}{d\tau} = \frac{1}{rK}\left(rKx - rKx^2 - H\right)$$

$$= x - x^2 - \frac{H}{rK}$$

$$= x(1-x) - h, \quad \text{where } \underbrace{h = \frac{H}{rK}}$$

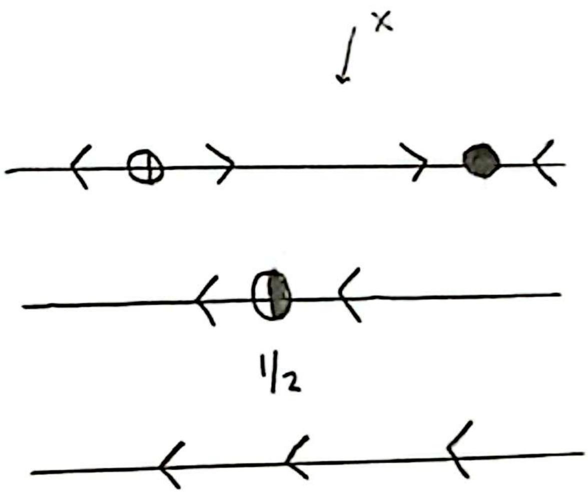
b) we have $\dot{x} = x(1-x) - h = f(x) := 0$

$$\leftrightarrow \quad -x^2 + x - h = 0$$

$$\leftrightarrow \quad x^2 - x + h = 0$$

$$\leftrightarrow \quad x^* = \frac{1}{2}\left[1 \pm \sqrt{1-4h}\right]$$

<u>vector fields</u>

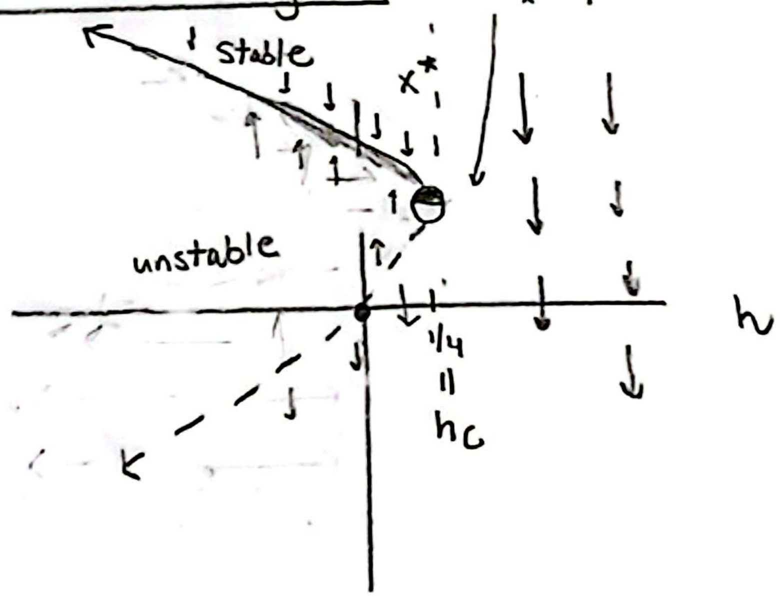


2 fp, 1 stable, 1 unstable — $h < \frac{1}{4}$

1 fp, semi stable — $h = 1/4$

$h > 1/4$

c) <u>Bifurcation diagram</u>



So there is a saddle-node bifurcation at $(x^* = 1/2,\ h_c = 1/4)$

d) From the bifurcation diagram, if $h > h_c$, the fish population declines towards being negative (going extinct). If $h < h_c$ and $x_0 < \frac{1}{2}[1 - \sqrt{1-4h}]$, then similarly the fish population trends to extinction. If $h < h_c$ and $x_0 > \frac{1}{2}[1 - \sqrt{1-4h}]$ then the fish population stabilizes to some equilibrium value.

2.6.1

The system $m\ddot{x} = -Kx$ is not a 1D system, but rather a 2D system. Put $x_1 = x$ and $x_2 = \dot{x}_1$, then the above system can be expressed like

$$\begin{cases} \dot{x}_1 = x_2 \\ \dot{x}_2 = -\frac{K}{m} x_1 \end{cases}$$

so that there is no paradox.