AMATH 561

HW 4

Note whybra

1)

a) $\sigma(X) = \{ A \subseteq \Omega : X(A) \in \mathcal{B}(\mathbb{R}) \}$

$$= \{ \underset{\substack{\uparrow \\ X \notin \{-1,1\}}}{\phi}, \underset{X=1}{\{a,b\}}, \underset{X=-1}{\{c,d\}}, \Omega \}$$

b)

$$\begin{bmatrix} \omega \\ X(\omega) \\ Y(\omega) \end{bmatrix} = \begin{bmatrix} a & b & c & d \\ 1 & 1 & -1 & -1 \\ 1 & -1 & 1 & -1 \end{bmatrix}$$

we can use the result introduced in Lecture 10. we have that $(\{a,b\}, \{c,d\})$ is a finite disjoint partition of Ω and our σ-algebra $\sigma(X) = \sigma(\{a,b\}, \{c,d\})$, so

$$E[Y|X](\omega) = E[Y|\sigma(X)](\omega) = \frac{\int_{\Omega_i} Y \, dP}{P(\Omega_i)} \quad \text{for } \omega \in \Omega_i$$

So

$$E[Y|X](a) = \frac{\int_{\{a,b\}} Y \, dP}{1/6 + 2/3} = \frac{Y(a)P(a) + Y(b)P(b)}{5/6}$$

$$= \left(1 \cdot 1/6 + (-1) \cdot 1/3\right) \cdot \frac{6}{5} = -\frac{1}{6} \cdot \frac{6}{5} = \underline{-1/5}$$

And for free we get

$$E[Y|X](b) = \underline{-1/5}$$

Now

$$E[Y|X](c) = \frac{\int_{\{c,d\}} Y \, dP}{1/4 + 1/4} = 2\left(Y(c)P(c) + Y(d)P(d) \right)$$

$$= 2(1 \cdot 1/4 - 1 \cdot 1/4) = \underline{0}$$

So also

$$E[Y|X](d) = \underline{0}$$

c)

$$\begin{bmatrix} \omega \\ Z(\omega) \end{bmatrix} = \begin{array}{c} \begin{array}{cccc} a & b & c & d \end{array} \\ \begin{bmatrix} 2 & 0 & 0 & -2 \end{bmatrix} \end{array} \leftarrow$$

just added the bottom 2 rows in the matrix from b) to get $Z = X + Y$

So $$E[Z|X](a) = \frac{\int_{\{a,b\}} Z \, dP}{5/6}$$

$$= \left(P(a)Z(a) + P(b)Z(b) \right) \cdot \frac{6}{5} = (1/6 \cdot 2 + 1/3 \cdot 0) \cdot \frac{6}{5}$$

$$= \frac{2}{6} \cdot \frac{6}{5} = \underline{\frac{2}{5}}$$

And also $E[Z|X](b) = \underline{\frac{2}{5}}$. Next

$$E[Z|X](c) = \frac{\int_{\{c,d\}} Z \, dP}{1/2} = 2\left(P(c)Z(c) + P(d)Z(d) \right)$$

$$= 2(1/4 \cdot 0 + 1/4 \cdot (-2))$$

$$= 2 \cdot -\frac{2}{4} = \underline{-1}$$

And also

$$E[Z|X](d) = \underline{-1}$$

2) a) We know by definition of $E[X|F]$

$$\int_\Omega X \, dP = \int_\Omega E[X|F] \, dP$$

The LHS $= E[X]$ and the RHS $= E[E[X|F]]$

So we are done.

b) Consider with $Y = E[X|F]$ and $Z = E(X|G)$

$$\left.\begin{aligned} &(X-Y)^2 + (Y-Z)^2 \quad (1) \\ &= (X^2 - 2XY + Y^2) + (Y^2 - 2YZ + Z^2) \end{aligned}\right\}$$

and also

$$(X-Z)^2 = X^2 - 2XZ + Z^2 \quad (2)$$

If we take expectations

$$E[(1)] = E[X^2] - 2E[XY] + E[Y^2] + E[Y^2] - 2E[YZ] + E[Z]^2$$

$$E[(2)] = E[X^2] - 2E[XZ] + E[Z^2]$$

In order to show the result we want, we need to show $-2(E[XY] - E[Y^2] + E[YZ])$
$= -2E[XZ] \leftrightarrow E[XZ] = E[XY] - E[Y^2] + E[YZ]$

(see next page)

we have

$$E[XZ] = E[X\,E[X|G]]$$
$$= E[E[X\,E[X|G]]\,|G]$$
$$= E[E[X|G]\,E[X|G]]$$
$$= E[E[X|G]^2] \quad (A)$$

Similarly

$$E[XY] = E[E[X|F]^2] \quad (B)$$

Now

$$E[YZ] = E[E[X|F]\,E[X|G]]$$
$$= E[E[E[X|F]\,E[X|G]]\,|\,G]$$
$$= E[E[X|G] \cdot E[E[X|F]|G]]$$

and since $G \subset F$

$$= E[E[X|G] \cdot E[X|G]]$$
$$= E[E[X|G]^2] \quad (C) = (A)$$

and

$$E[Y^2] = E[E[X|F]^2] \quad (D) = (B)$$

So $E[XZ] = E[E[X|G]^2] = (A)$

and $E[XY] - E[Y^2] + E[YZ]$

$$= \cancel{(B)} - \cancel{(D)} + (C)$$

$= (C) = (A)$, ie they are equal as desired, which completes our proof,

3) From 2b)

$$E[(X - E[X|F])^2] + E[(E[X|F] - E[X|G])^2] = E[(X - E[X|G])^2]$$

Put $G = \{\phi, \Omega\}$, then $E[X|G] = E[X]$, so we get

$$\overbrace{E[(X - E[X|F])^2]}^{①} + \overbrace{E[(E[X|F] - E[X])^2]}^{②} = E[(X - E[X])^2]$$

The RHS is the definition of $Var(X)$.

$$① = E[X^2 - 2X\,E[X|F] + E[X|F]^2]$$

$$= E[X^2] + E[X|F]^2 - 2E[X\,E[X|F]]$$

from 2b)

$$= E[X^2] + E[X|F]^2 - 2E[E[X|F]^2]$$

$$② = E[E[X|F]^2 + E[X]^2 - 2E[X]\,E[X|F]]$$

$$= E[E[X|F]^2] + E[X]^2 - 2E[X]^2$$

$$① + ② = \overbrace{E[X|F]^2 - E[E[X|F]^2]}^{Ⓐ} + \underbrace{E[E[X|F]^2] - E[E[X|F]]^2}_{Ⓑ}$$

just cancelling, and rearranging; splitting the term with -2

(see next page)

B, since $E[X|F]^2$ is F measurable, we can say

$$B = E[E[X|F]^2|F] - E[E[X|F]]^2$$

$$= Var[E[X|F]]$$

For A, let's compute $E[var(X|F)]$

$$E[var(X|F)] = E[E[X|F]^2|F] - E[E[X|F]|F]^2$$

$$= E[E[X|F]^2] - E[E[X|F]]^2$$

$$= A$$

So $A + B = var(X)$ and we are done.

4) From 3), we have

$$\text{Var}(X) = E[\text{var}(X|N)] + \text{var}(E[X|N])$$

As $X = \sum_{i=1}^{n} Y_i$ with the Y_i's being i.i.d,

$$\text{Var}(X|N) = \sum_{i=1}^{N} \text{Var}(Y_i) = N\sigma^2, \text{ so}$$

$$E[\text{var}(X|N)] = E[N\sigma^2] = \sigma^2 E[N]$$

Now $E[X|N] \overset{\text{by linearity}}{=} \sum_{i=1}^{N} E[Y_i] = N\omega$ and

$\text{Var}(E[X|N]) = \text{var}(N\omega) = \omega^2 \text{Var}(N)$, so we get

$$\text{Var}(X) = \sigma^2 E[N] + \omega^2 \text{Var}(N)$$ as desired.