The following problems, unless specifically noted, refer to the exercises in the book *Numerical Linear Algebra*, by Lloyd N. Trefethen and David Bau, III, SIAM 1997.

Homework 5

Reading: Lectures 12-13 and the supplementary material `stability.pdf` posted on Canvas (in the Modules folder). Lectures 20-21.

Problems: Exercises 12.1, 21.1.

Three additional problems:

A1. The number $\frac{8}{7}$ obviously has no exact representation in any decimal floating point system ($\beta = 10$) with finite precision t. Is there a finite floating point system (i.e., some finite integer base β and precision t) in which this number does have an exact representation? Answer the same question for the irrational number π.

A2. One can approximate the derivative of a function $f(x)$ by

$$f'(x) \approx \frac{f(x+h) - f(x)}{h}. \tag{1}$$

Using Taylor expansion, it can be easily verified that (if f is smooth enough):

$$\left| f'(x) - \frac{f(x+h) - f(x)}{h} \right| = O(h).$$

That is, it is expected to be a first order approximation with respect to the spacing h. Let $f(x) = \sin(x)$.

(a) Consider the problem of evaluating $f'(x)$ at $x_0 = 1.2$. Is the problem well-conditioned or ill-conditioned?

(b) Using (1) to approximate $f'(x)$ at $x_0 = 1.2$. Take `h = 10.^i; i = 0:-1:-16` and plot the absolute error $\left| f'(x_0) - \frac{f(x_0+h) - f(x_0)}{h} \right|$ vs. h using `loglog`. Does the plot behave as you expect?

(c) Suppose the only rounding errors made are in rounding $f(x_0+h)$ and $f(x_0)$, so that the computed values are $f(x_0+h)(1+\varepsilon_1)$ and $f(x_0)(1+\varepsilon_2)$, where $|\varepsilon_1|, |\varepsilon_2| \leq \varepsilon_{\text{machine}}$, then there will be a difference between the computed quotient and the true quotient

$$\frac{f(x_0+h)(1+\varepsilon_1) - f(x)(1+\varepsilon_2)}{h} = \frac{f(x_0+h) - f(x_0)}{h} + O\left(\frac{\varepsilon_{\text{machine}}}{h}\right).$$

Use this to explain your results in (b).

(d) By the trig identity $\sin(\alpha) - \sin(\beta) = 2\cos(\frac{\alpha+\beta}{2})\sin(\frac{\alpha-\beta}{2})$, one has

$$\frac{f(x_0+h) - f(x_0)}{h} = \frac{2\cos(x_0 + \frac{h}{2})\sin(\frac{h}{2})}{h}.$$

Using this formula to approximate $f'(x)$ at $x_0 = 1.2$. Make the same plot as in (b). Does the plot behave as you expect?

A3. In MATLAB, form a 100 by 100 matrix A with 1's on the main diagonal and in the last column, with -1's below the main diagonal, and with 0's everywhere else, as in (22.4) on page 165 in the textbook. Set a random vector $\mathbf{x}$ of length 100 by `x = randn(100,1)`. Set the vector $\mathbf{b}$ as `b = A*x`.

(a) Compute the 2-norm condition number of A using `cond(A)`. Is A well-conditioned or ill-conditioned?

(b) Solve the linear system $A\mathbf{x} = \mathbf{b}$ by `x_ge = A\b`. This is to use the Gaussian elimination with partial pivoting. Compute the relative 2-norm between the computed solution `x_ge` and the true solution `x`. Do you trust this solution?

(c) Solve the linear system $A\mathbf{x} = \mathbf{b}$ by the QR decomposition: `[Q,R] = qr(A,0); x_qr = R\(Q'b)`. Compute the relative 2-norm between the computed solution `x_qr` and the true solution `x`. Do you trust this solution?

(d) Find a code online or write one yourself which does the Gaussian elimination with *complete* pivoting. Use it to solve the linear system $A\mathbf{x} = \mathbf{b}$. Compute the relative 2-norm as above. Do you trust this solution?