1) All we need to do is show that RHS is $\left|\frac{\partial f}{\partial y}\right|$ is bounded with $f(t, y(t)) = y'(t)$. So for $t \geq 1$ we have

$$f(t, y(t)) = \frac{1}{t^4 + 2y^2}$$

note this derivative exists as $t \geq 1$, never $(t=0, y=0)$

So,

$$\left|\frac{\partial f}{\partial y}\right| = \left|\frac{-4y}{(t^4+2y^2)^2}\right| = \frac{4|y|}{(t^4+2y^2)^2} \leq \frac{4y}{(t^4+2y^2)^2}$$

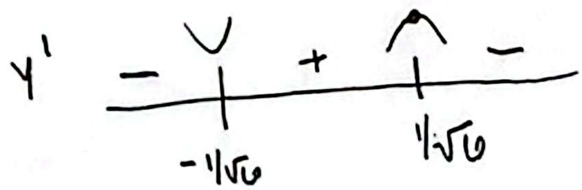
To show this function is bounded, we compute it's derivative, set it equal to 0, and find the maximum. As $t \geq 1$, the above is bounded above by $\frac{4y}{(1+2y^2)^2}$, so we can actually find the maximum of this instead,

$$\xrightarrow{\frac{d}{dy}} \frac{-2(4y)\cdot 4y}{(1+2y^2)^3} + \frac{4}{(1+2y^2)^2} = 0$$

$$\rightarrow \quad -32y^2 + 4(1+2y^2) = 0$$

$$\rightarrow \quad -24y^2 + 4 = 0$$

$$y^2 = 1/6 \rightarrow y = \pm\sqrt{1/6}$$



checking to see which is the maximum

So $\left|\frac{\partial f}{\partial y}\right|$ is bounded above by

$$C = \frac{4\sqrt{1/6}}{\left(1 + 2\sqrt{\frac{1}{6}}^{2}\right)^2}$$

$$= \frac{4}{\sqrt{6}} \frac{1}{\left(1+\frac{1}{3}\right)^2} = \frac{\cancel{4}}{\sqrt{6}} \frac{3^2}{4^{\cancel{2}}}$$

$$= \frac{9}{4\sqrt{6}}$$

. Hence f is Lipschitz, hence the system

$$y'(t) = f(t, y(t))$$

has a unique solution.

2) We have

$$\bullet \quad u'' = -Ku$$
$$u(t_0) = u_0$$
$$u'(t_0) = v_0$$

If we put $v = u'$, then the above becomes

$$\begin{aligned} u' &= v \\ v' &= -Ku \end{aligned} = \underbrace{\begin{pmatrix} 0 & 1 \\ -K & 0 \end{pmatrix}}_{A} \underbrace{\begin{pmatrix} u \\ v \end{pmatrix}}_{\vec{x}}$$

$$= A\vec{x} = f(\vec{x})$$

Now take $\vec{x}, \vec{y} \in \mathbb{R}^2$, then $\star$

$$\| f(\vec{x}) - f(\vec{y}) \| = \| A\vec{x} - A\vec{y} \|$$

$$= \|A\| \, \|\vec{x} - \vec{y}\|$$

So to satisfy the Lipschitz condition, we need

$$\|A\| \, \|\vec{x} - \vec{y}\| \leq C \|\vec{x} - \vec{y}\| \rightarrow C \geq \|A\|$$

Hence $C = \|A\|$ is the smallest Lipschitz constant of f, for a given norm.
The smallest constants for various norms are computed next. We have

(next page)

$C_1 = \|A\|_1 =$ "max absolute column sum"

$= \max(1, K)$

$C_2 = \|A\|_2 = \Big\{$ By noticing that A is Hermitian ($A = A^T$), the singular values are the absolute values of the eigenvalues of A. So

$$\det(A - \lambda I) = \det \begin{bmatrix} -\lambda & 1 \\ -K & -\lambda \end{bmatrix}$$

$$= \lambda^2 + K := 0$$

$\rightarrow \lambda_\pm = \pm i\sqrt{K}$, so that $\underset{\sigma_\pm}{\overset{}{|\lambda_\pm|}} = \sqrt{K}$

So as $\|A\|_2$ is the largest singular value of A,

$C_2 = \|A\|_2 = \sqrt{K}$

$C_\infty = \|A\|_\infty =$ "max absolute row sum"

$= \max(1, K)$

3) Firstly, by Taylor's theorem

$$f_{n+1} = f(t_{n+1}, y_{n+1}) = f(t_n + h, y_{n+1}) = f_n + h \overbrace{\frac{df}{dt}\Big|_{(t_n, y_{n+1})}}^{g_n} + O(h^2)$$

So our step is (numerical solution)

$$y_{n+1} = y_n + h\left[\theta f_n + (1-\theta) f_{n+1}\right]$$

$$= y_n + h\left[\theta f_n + (1-\theta)\left[f_n + h g_n + O(h^2)\right]\right]$$

$$= y_n + h\left[f_n + (1-\theta) h g_n + O(h^2)\right]$$

The actual solution is

$$y(t_{n+1}) = y(t_n + h) = y(t_n) + h y'(t_n) + \frac{h^2}{2} y''(t_n) + O(h^3)$$

$$= y(t_n) + h f_n + \frac{h^2}{2} g_n + O(h^3)$$

So the local truncation error is (replacing $y(t_j)$ w/ y_j)

$$\frac{y_{n+1} - y(t_{n+1})}{h} = \frac{1}{h}\left(\cancel{y_n} + \cancel{h f_n} + h^2(1-\theta) g_n - \cancel{y_n} - \cancel{h f_n} - \frac{h^2}{2} g_n + O(h^3)\right]$$

$$= \frac{1}{h}\left[\frac{(1/2 - \theta)}{2} h^2 g_n + O(h^3)\right]$$

$$= \overbrace{\frac{1}{2} h g_n (1/2 - \theta)}^{*} + O(h^2)$$

So when $\theta = 1/2$ it is $O(h^2)$ as $* = 0$, and when $\theta \neq 1/2$ it is $O(h)$ as $* \neq 0$.

4) The numerical solution is

$$y_{n+1} = y_n + \frac{h}{2}\left[f(t_{n+1}, \tilde{y}_{n+1}) + f_n \right]$$

$$= y_n + \frac{h}{2}\left[\underbrace{f(t_n+h,\ y_n + hf_n)}_{A} + f_n \right] \qquad (*)$$

Using Taylor's thm, centered at (t_n, y_n), A becomes

$$= f_n + h\left[\underbrace{\overbrace{f_t + f f_y}^{a_n}}_{=\, y''(t_n, y_n)} \right] + \frac{h^2}{2}\left[\underbrace{\overbrace{f_{tt} + 2 f f_{ty} + f_n^2 f_{yy}}^{b_n}}_{y'''(t_n, y_n)} \right] + O(h^3)$$

So

$$(*) = y_n + hf_n + \frac{h^2}{2} a_n + \frac{h^3}{4} b_n + O(h^4)$$

The exact solution is

$$y(t_{n+1}) = y(t_n + h)$$

$$= y(t_n) + hf_n + \frac{h^2}{2} a_n + \frac{h^3}{6} b_n + O(h^4)$$

So the local truncation error is (putting $y_n = y(t_n)$)

$$\frac{y_{n+1} - y(t_{n+1})}{h} = \frac{\left[\cancel{y_n} + \cancel{hf_n} + \cancel{\frac{h^2}{2}} a_n + \frac{h^3}{4} b_n - \cancel{y_n} - \cancel{hf_n} - \cancel{\frac{h^2}{2}} a_n - \frac{h^3}{6} b_n + O(h^4) \right]}{h}$$

$$= \left(\frac{1}{4} - \frac{1}{6} \right) h^2 b_n + O(h^3)$$

So the error is $O(h^2)$ with leading term

$$\frac{1}{12} b_n(t_n, y_n) = \underline{\frac{1}{12} y'''(t_n)}$$