1) we can write

$$Z(\omega) = \begin{cases} X(\omega) & \omega \in A \\ Y(\omega) & \omega \in A^c \end{cases}$$

To show Z is a random variable, we must show for every Borel set B

$$Z^{-1}(B) \in F \text{ (the } \sigma\text{-algebra)}$$

Since both X and Y are random variables, both $X^{-1}(B)$ and $Y^{-1}(B)$ are in the σ-algebra. Now take $\omega \in \Omega$, if $\omega \in A$ then either (disjointly)

$$① \ \omega \in A \cap X^{-1}(B) \quad \text{or} \quad \omega \in A \cap [X^{-1}(B)]^c \ ②$$

if $\omega \in A^c$, then either (disjointly)

$$③ \ \omega \in A^c \cap Y^{-1}(B) \quad \text{or} \quad \omega \in A^c \cap [Y^{-1}(B)]^c \ ④$$

where all 4 sets above are in F since countable intersections of sets in F are in F. In case ①, $Z(\omega) = X(\omega) \in B$. In case ②, $Z(\omega) = X(\omega) \notin B$. In case ③ $Z(\omega) = Y(\omega) \in B$. In case ④ $Z(\omega) = Y(\omega) \notin B$.

So $$Z^{-1}(B) = (X^{-1}(B) \cap A) \cup (Y^{-1}(B) \cap A^c)$$

which is in F, hence Z is a random variable.

2) a) Since g is continuous and increasing on $\mathbb{R}$, the range of g is some interval in $\mathbb{R}$, call it I, thus $g: \mathbb{R} \to I$ is onto.
we want to find $P(Y \leq y) = P(g(X) \leq y)$.
If $y \in I$, then if $g(X) \leq y$ then $X \leq g^{-1}(y)$ (since g is continuous and increasing in the interval).
If $y \notin I$, say $y < u \ \forall u \in I$, then $P(g(X) < y) = 0$.
If $y > u \ \forall u \in I$, then $P(g(X) < y) = 1$. So our distribution function is

$$F_Y(y) = P(Y \leq y) = \begin{cases} F_X(g^{-1}(y)) & \text{if } y \in I \\ 0 & \text{if } y < u \ \forall u \in I \\ 1 & \text{if } \underbrace{y > u \ \forall u \in I} \end{cases}$$

I chose to write it like this because the range of g could be $(-\infty, a)$ $(-\infty, \infty)$, (a, ∞), (a, b)

b) $$f_Y = F_Y' = \begin{cases} \frac{d}{dy} F_X(g^{-1}(y)) & \text{if } y \in I \\ 0 & \text{if } y \notin I \end{cases}$$

$$= \frac{d}{dy} \int_{-\infty}^{g^{-1}(y)} f_X(u)\, du = f_X(g^{-1}(y)) \cdot [g^{-1}(y)]'$$

since X is a continuous r.v.

by Fundamental Thm

$$\text{so } f_Y(y) = \begin{cases} f_X(g^{-1}(y)) \cdot [g^{-1}(y)]' & \text{for } y \in I \\ 0 & \text{for } y \notin I \end{cases}$$

3) a) $P(Y \le y) = P(X^2 \le y)$

$$= P(-\sqrt{y} \le X \le \sqrt{y})$$

$$= F_X(\sqrt{y}) - F_X(-\sqrt{y})$$

b) $P(Y \le y) = P(\sqrt{|X|} \le y)$

$$= P(|X| \le y^2)$$

$$= P(-y^2 \le \cancel{X} \le y^2)$$

$$= F_X(y^2) - F_X(-y^2)$$

c) $P(Y \le y) = P(\sin X \le y)$

d) ↓ ran out of time

4) a) The Borel σ-algebra on $[0,1]$ is generated by the intervals $(a,b] \subseteq [0,1]$. Let $(r_i)_{i \geq 1}$ be the rationals in $[0,1]$ other than 0 and 1. We can write

$$\{0\} = \bigcap_{K \geq 1} (0, \tfrac{1}{2^K}]$$

$$\{1\} = \bigcap_{K \geq 1} (1 - \tfrac{1}{2^K}, 1]$$

$$\{r_i\} = \bigcap_{K \geq K_i} (r_i - \tfrac{1}{2^K}, r_i + \tfrac{1}{2^K}]$$

where for every $i \geq 1$, K_i is chosen so that $(r_i - \frac{1}{2^{K_i}}, r_i + \frac{1}{2^{K_i}}] \subseteq [0,1]$ for each r_i. So

$$\mathbb{Q} \cap [0,1] = \{0\} \cup \{1\} \cup \left(\bigcup_{i \geq 1} \{r_i\} \right)$$

$$= \left[\bigcap_{K \geq 1} (0, \tfrac{1}{2^K}] \right] \cup \left[\bigcap_{K \geq 1} (1 - \tfrac{1}{2^K}, 1] \right] \cup \left[\bigcup_{i \geq 1} \bigcap_{K \geq K_i} (r_i - \tfrac{1}{2^K}, r_i + \tfrac{1}{2^K}] \right]$$

we have written $\mathbb{Q} \cap [0,1]$ as countable unions and intersections of intervals $(a,b] \subseteq [0,1]$, so $\mathbb{Q} \cap [0,1]$ is in $\mathcal{B}([0,1])$.

b) We can use problem 1 for this. Define random variables:

$$X_1(w) = 0 \quad \forall\, w \in [0,1]$$
$$X_2(w) = 1 \quad \forall\, w \in [0,1]$$

Then

$$X(w) = \begin{cases} 0 & \text{for } w \in Q \cap [0,1] \\ 1 & \text{for } w \in [Q \cap [0,1]]^c \end{cases}$$

$$= \begin{cases} X_1(w) & \text{for } w \in Q \cap [0,1] \\ X_2(w) & \text{for } w \in [Q \cap [0,1]]^c \end{cases}$$

From 4a) $Q \cap [0,1] \in B([0,1])$, so X is a random variable.

The distribution function is:

$$P(X \leq x) = \begin{cases} 0 & \text{if } x \in (-\infty, 0) \\ 1 & \text{if } x \in [0, \infty) \end{cases}$$

Since X only takes values $\{0,1\}$. The expectation is

$$E[X] = \int_\Omega X(w)\, P(dw) = \int_\Omega 1_{[Q\cap[0,1]]^c}\, dP$$
$$= P([Q \cap [0,1]]^c) = 1$$

4 b continued) X does not have a density function because no function when integrated from $(-\infty, x)$ returns the discrete function $1_{x \geq 0}$

X is a discrete random variable because the range of X is a finite set.