# AMATH 569 - Homework 1

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March/April 2025

## Problem 1

As $f : \mathbb{R} \to \mathbb{R}$ is smooth, it has infinitely many continuous derivatives, meaning $f^k : \mathbb{R} \to \mathbb{R}$ exists for all $k \geq 0$ and is a continuous function. Now fix $x \in \mathbb{R}$ and let $g = f^k$. From the smoothness of $f$, it is clear that $g' = f^{k+1}$ exists and is continuous. The mean value theorem then guarantees that there exists some $c \in [0, x]$ such that:

$$g'(c) = \frac{g(x) - g(0)}{x - 0} = \frac{g(x) - g(0)}{x}$$

Or equivalently:

$$f^{k+1}(c) = \frac{f^k(x) - f^k(0)}{x}$$

$$f^{k+1}(c)x + f^k(0) = f^k(x)$$

We then proceed by integrating both sides from $0 \to x$ with respect to $x$:

$$\int_0^x f^{k+1}(c)x + f^k(0)\ dx = \int_0^x f^k(x)\ dx$$

$$\implies f^{k+1}(c)\frac{x^2}{2} + f^k(0)x = f^{k-1}(x) - f^{k-1}(0)$$

$$\implies f^{k+1}(c)\frac{x^2}{2} + f^k(0)x + f^{k-1}(0) = f^{k-1}(x)$$

From here, a clear pattern emerges. If we were to integrate both sides again (at total of $n = 2$ integrations), we'd recover this relationship:

$$\frac{f^{k+1}(c)}{3!}x^3 + \frac{f^k(0)}{2!}x^2 + \frac{f^{k-1}(0)}{1!}x + \frac{f^{k-2}(0)}{0!} = f^{k-2}(x)$$

After $k - 2$ more integrations (a total of $n = k$), we see:

$$\frac{f^{k+1}(c)}{(k+1)!}x^{k+1} + \frac{f^k(0)}{k!}x^k + \frac{f^{k-1}(0)}{(k-1)!}x^{k-1} + ... + \frac{f'(0)}{1!}x + \frac{f(0)}{0!} = f(x)$$

Or after rearranging:

$$f(x) = \sum_{i=0}^{k} \frac{f^i(0)}{i!}x^i + \frac{f^{k+1}(c)}{(k+1)!}x^{k+1}$$

As desired.

## Problem 2 (a)

First consider the case when both $a, b \neq 0$ and let $r = \frac{1}{p}$ and $q = \frac{1}{q}$. Then

$$\frac{a^p}{p} + \frac{b^q}{q} = ra^{\frac{1}{r}} + sb^{\frac{1}{s}} > 0$$

... so that it makes sense to consider the logarithm of this quantity.

$$\ln\left(ra^{\frac{1}{r}} + sb^{\frac{1}{s}}\right)$$

As the logarithm is a concave function (its second derivative is strictly negative), and $r + s = 1$, we see (with equality $\iff a^{\frac{1}{r}} = b^{\frac{1}{s}} \iff a^p = b^q$):

$$\ln\left(ra^{\frac{1}{r}} + sb^{\frac{1}{s}}\right) \geq r\ln\left(a^{\frac{1}{r}}\right) + s\ln\left(b^{\frac{1}{s}}\right) = \frac{r}{r}\ln a + \frac{s}{s}\ln b = \ln\left(ab\right)$$

Now as the logarithm is a strictly increasing function, applying the exponential to both sides, we get the desired inequality (with equality $\iff a^p = b^q$):

$$ab \leq \frac{a^p}{p} + \frac{b^q}{q}$$

If both $a, b = 0$ then the inequality is trivially satisfied. So we are done.

## Problem 2 (b)

First, define the notation $\left(\int_0^1 |h(x)|^s \, dx\right)^{\frac{1}{s}} = \|h\|_s$. Let $a = \frac{|f|}{\|f\|_p}$ and $b = \frac{|g|}{\|g\|_q}$ with $\frac{1}{p} + \frac{1}{q} = 1$. Then from part (a):

$$ab \leq \frac{a^p}{p} + \frac{b^q}{q}$$

$$\frac{|f||g|}{\|f\|_p \|g\|_q} \leq \frac{|f|^p}{p\|f\|_p^p} + \frac{|g|^q}{q\|g\|_q^q}$$

Now, integrating both sides from $0 \to 1$, we see (this is allowed since both sides are strictly non-negative):

$$\frac{1}{\|f\|_p \|g\|_q} \int_0^1 |f||g| \, dx \leq \frac{1}{p\|f\|_p^p} \int_0^1 |f|^p \, dx + \frac{1}{q\|g\|_q^q} \int_0^1 |g|^q \, dx$$

$$= \frac{1}{p}\frac{\|f\|_p^p}{\|f\|_p^p} + \frac{1}{q}\frac{\|g\|_q^q}{\|g\|_q^q}$$

$$= \frac{1}{p} + \frac{1}{q}$$

$$= 1$$

Now rearranging, we get the desired result:

$$\int_0^1 |f||g| \, dx \leq \|f\|_p \|g\|_q$$

This all holds when both $\|f\|_p$ and $\|g\|_q > 0$. Without loss of generality, if $\|f\|_p = 0$, then it must be that $f = 0$ almost everywhere. This would cause the LHS of the inequality above to be 0 (as well as the RHS), so the inequality still holds.

## Problem 3 (a) and (b)

The ODE satisfies the characteristic equation $r^2 + \lambda = 0 \iff r = \pm i\sqrt{\lambda}$. The general solution should then satisfy (as $\lambda > 0$):

$$u(x) = C_1 e^{i\sqrt{\lambda}x} + C_2 e^{-i\sqrt{\lambda}x}$$

The initial conditions suggest that:

$$u(0) = 0 = C_1 + C_2 \implies C_1 = -C_2$$

$$u'(1) = 0 = i\sqrt{\lambda}C_1 e^{i\sqrt{\lambda}} - i\sqrt{\lambda}C_2 e^{-i\sqrt{\lambda}} = i\sqrt{\lambda}C_1(e^{i\sqrt{\lambda}} + e^{-i\sqrt{\lambda}}) = 2i\sqrt{\lambda}C_1 \cos\sqrt{\lambda}$$

So either $C_1 = C_2 = 0$ suggesting a trivial solution $u(x) = 0$, or there are solutions when $\cos\sqrt{\lambda} = 0 \implies \sqrt{\lambda} = \frac{(2n+1)\pi}{2} \implies \lambda_n = \frac{(2n+1)^2\pi^2}{4}$ for $n \geq 0$. Putting this together, we see:

$$u(x) = C_1(e^{i\sqrt{\lambda_n}x} - e^{-i\sqrt{\lambda_n}x}) = -2iC_1 \sin(\sqrt{\lambda_n}x) = A\sin(\sqrt{\lambda_n}x)$$

Where $A = -2iC_1$ can be any real number. Thus when $\lambda \neq \lambda_n$ we can only have $u(x) = 0$ as a solution, and when $\lambda = \lambda_n$ the solutions take the above form, as desired.

## Problem 4

First, assume the solution takes the form $u(x,y) = X(x)Y(y)$. Then we have:

$$u_{xx} + u_{yy} = X''(x)Y(y) + X(x)Y''(y) = X''Y + XY'' = 0$$

$$\implies \frac{X''}{X} = \frac{-Y''}{Y}$$

When $X, Y \neq 0$. The LHS of the above equation only depends on $x$ and the RHS only depends on $y$, so it must be that both expressions are constant (say equal to $C$) for all $x, y$. This leads to the following system of ODEs:

$$X'' - CX = 0 \text{ ...and... } Y'' + CY = 0$$

First assume $C \geq 0$. For the $X$ equation, the characteristic equation is $r^2 - C = 0 \iff r = \pm\sqrt{C}$ so that $X(x) = A_1 e^{\sqrt{C}x} + A_2 e^{-\sqrt{C}x}$. For the $Y$ equation, the characteristic equation is $r^2 + C = 0 \iff r = \pm i\sqrt{C}$ so that $Y(y) = A_3 e^{i\sqrt{C}y} + A_4 e^{-i\sqrt{C}y}$. The first couple boundary conditions imply:

$$u(0,y) = 0 \implies X(0) = 0 \implies A_1 + A_2 = 0 \implies A_1 = -A_2$$

$$u(1,y) = 0 \implies X(1) = 0 \implies 2A_1 \sinh(\sqrt{C}) = 0$$

Where we have ignored $Y(y)$, as $Y(y) = 0$ means both $A_3$ and $A_4$ are 0, which leads to a trivial solution. So either $A_1 = A_2 = 0$ or $C = 0$, which in both cases leads to a trivial solution, and will never satisfy the boundary conditions $u(x,0) = u(x,1) = \sin(2\pi x)$. This means that it suffices to consider $C < 0$. Let $\lambda = -C > 0$, then in this case, for the $X$ equation, the characteristic equation is $r^2 + \lambda = 0 \iff r = \pm i\sqrt{\lambda}$ so that $X(x) = A_1 e^{i\sqrt{\lambda}x} + A_2 e^{-i\sqrt{\lambda}x}$. For the $Y$ equation, the characteristic equation is $r^2 - \lambda = 0 \iff r = \pm\sqrt{\lambda}$ so that $Y(y) = A_3 e^{\sqrt{\lambda}y} + A_4 e^{-\sqrt{\lambda}y}$. The first couple boundary conditions imply (again ignoring $Y(y)$ for the same reason as above):

$$u(0,y) = 0 \implies X(0) = 0 \implies A_1 + A_2 = 0 \implies A_1 = -A_2$$

$$u(1,y) = 0 \implies X(1) = 0 \implies -2iA_1 \sin(\sqrt{\lambda}x) = 0$$

Where we used that $A_1 = -A_2$ to to get the expression in the second equation. For the second equation to be true, either $A_1 = 0$ or $\sqrt{\lambda} = n\pi \implies \lambda = n^2\pi^2$ for $n \geq 0$. Now absorb $-2i$ into $A_1$ so that the solution $X(x) = A_1 \sin(\sqrt{\lambda}x)$ for $n \geq 0$. Then, the next boundary condition implies:

$$u(x,0) = X(x)Y(0) = A_1 \sin(\sqrt{\lambda}x)(A_3 + A_4) = \sin(2\pi x)$$

Looking at both sides of the equation, we need $A_1(A_3 + A_4) = 1$ and $\sqrt{\lambda} = 2\pi \implies n = 2$, so that the only value of $\lambda$ we need to consider is $\lambda = 4\pi^2$. The final boundary condition gives:

$$u(x,1) = X(x)Y(1) = A_1 \sin(\sqrt{\lambda}x)(A_3 e^{\sqrt{\lambda}} + A_4 e^{-\sqrt{\lambda}}) = \sin(2\pi x)$$

$$\implies A_1(A_3 e^{\sqrt{\lambda}} + A_4 e^{-\sqrt{\lambda}}) = 1$$

By distributing, and redefining $B_1 = A_1 A_3$ and $B_2 = A_1 A_4$, we get the system:

$$B_1 + B_2 = 1 \implies B_2 = 1 - B_1$$

$$B_1 e^{\sqrt{\lambda}} + B_2 e^{-\sqrt{\lambda}} = 1$$

Solving the system, we recover (with $\sqrt{\lambda} = 2\pi$):

$$B_1 = \frac{1 - e^{-\sqrt{\lambda}}}{e^{\sqrt{\lambda}} - e^{-\sqrt{\lambda}}}$$

$$B_2 = \frac{e^{\sqrt{\lambda}} - 1}{e^{\sqrt{\lambda}} - e^{-\sqrt{\lambda}}}$$

In conclusion, we see:

$$u(x,y) = X(x)Y(y) = A_1 \sin(\sqrt{\lambda}x)(A_3 e^{\sqrt{\lambda}y} + A_4 e^{-\sqrt{\lambda}y})$$

$$= \sin(\sqrt{\lambda}x)(A_1 A_3 e^{\sqrt{\lambda}y} + A_1 A_4 e^{-\sqrt{\lambda}y})$$

$$= \sin(2\pi x)(B_1 e^{2\pi y} + B_2 e^{-2\pi y})$$