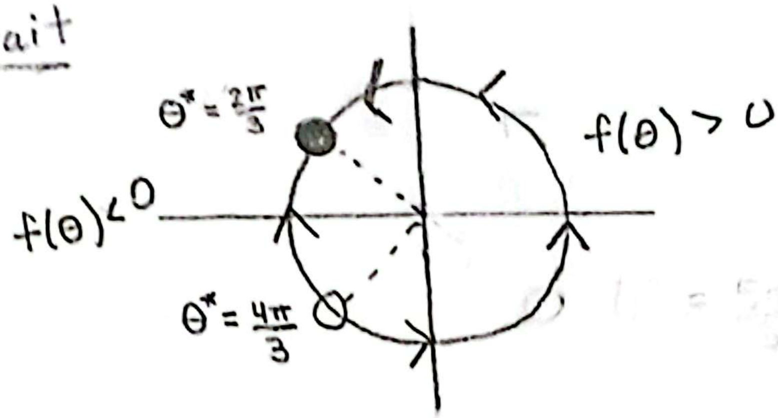
1) we have,

$$\dot{\theta} = 1 + 2\cos\theta = f(\theta) := 0$$

$$\longleftrightarrow \quad \cos\theta = -\frac{1}{2}, \quad \theta \in [0, 2\pi)$$

$$\longleftrightarrow \quad \theta^* = \frac{2\pi}{3}, \ \frac{4\pi}{3}$$

Phase portrait



So $\theta^* = \frac{2\pi}{3}$ is stable and $\theta^* = \frac{4\pi}{3}$ is unstable.

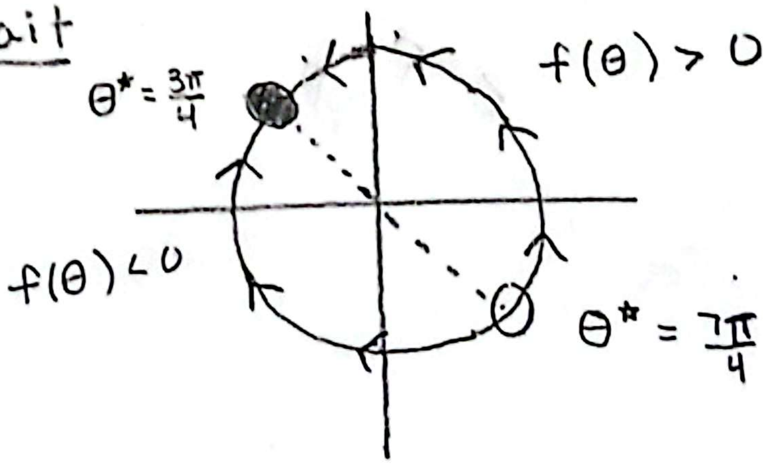
b) we have,

$$\dot{\theta} = \sin\theta + \cos\theta = f(\theta) := 0$$

$$\leftrightarrow \quad \sin\theta = -\cos\theta \qquad \theta \in [0, 2\pi)$$

$$\leftrightarrow \quad \theta^* = \frac{3\pi}{4}, \ \frac{7\pi}{4}$$

Phase portrait



So,

$\theta^* = \frac{3\pi}{4}$ is stable

and

$\theta^* = \frac{7\pi}{4}$ is unstable

c) we have,

$$\dot{\theta} = \sin(4\theta) = f(\theta) := 0$$

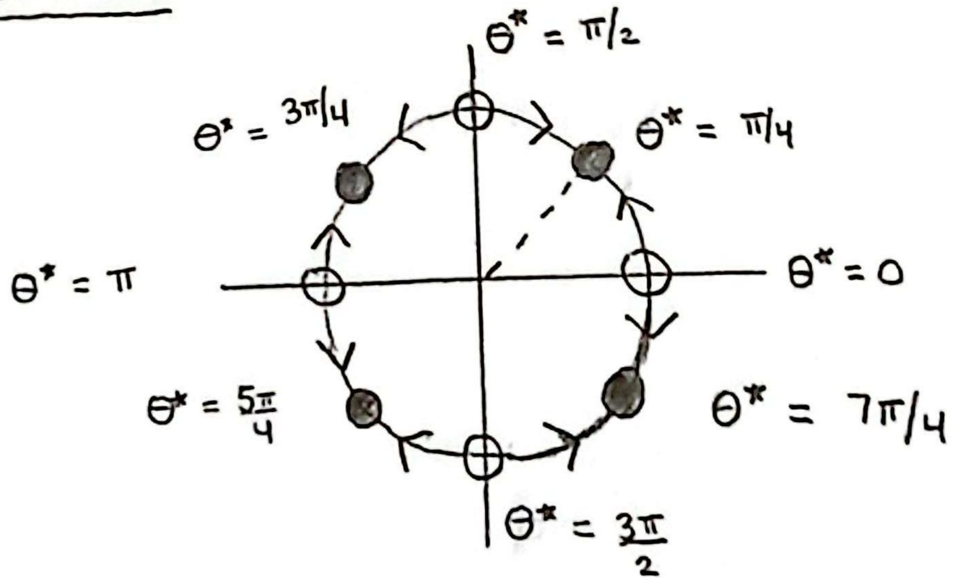
$$\longleftrightarrow \quad 4\theta = N\pi \quad \text{for } N \in \mathbb{Z}$$

$$\longleftrightarrow \quad \theta^* = \frac{N\pi}{4}$$

we must restrict our attention to when $\theta^* \in [0, 2\pi)$,
so when $N = 0, 1, 2, 3, 4, 5, 6, 7$, we get

$$\vec{\theta}^* = \begin{array}{c} \begin{bmatrix} 0 \\ \pi/4 \\ \pi/2 \\ 3\pi/4 \\ \pi \\ 5\pi/4 \\ 3\pi/2 \\ \frac{7\pi}{4} \end{bmatrix} \end{array} \begin{array}{l} N=0, \ \theta^*_0 \\ \vdots \\ \\ \\ \\ \\ \vdots \\ N=7, \ \theta^*_7 \end{array}$$

Phase portrait

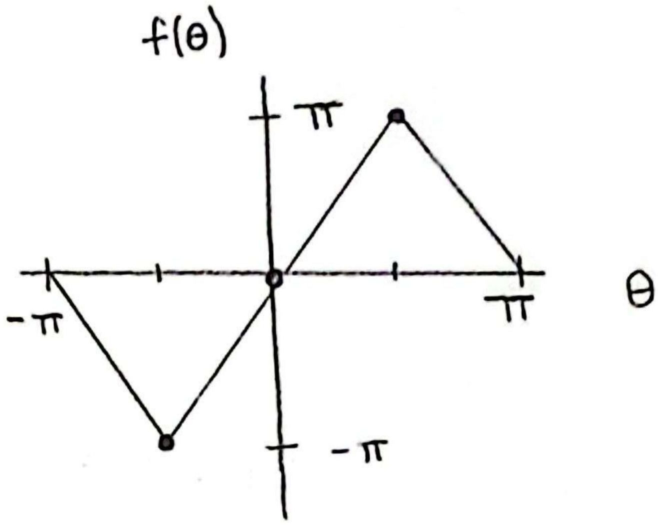


So $\theta^* \in \{ \pi/4, 3\pi/4, \frac{5\pi}{4}, 7\pi/4 \}$ are stable,

and $\theta^* \in \{ 0, \pi/2, \pi, \frac{3\pi}{2} \}$ are unstable.

2)

(a)



b) we have,

$$\phi = \theta_s - \theta_f \longleftrightarrow \dot{\phi} = \dot{\theta}_s - \dot{\theta}_f$$

$$= \Omega - \omega - Af(\theta_s - \theta_f)$$

$$= \Omega - \omega - Af(\phi)$$

So, re-writing

$$\dot{\phi} = \frac{d\phi}{dt} = \Omega - \omega - Af(\phi)$$

Now put $\tau = At$, and $\mu = \frac{\Omega - \omega}{A}$, then

$$\frac{d\phi}{dt} = A\mu - Af\phi = A(\mu - f(\phi))$$

And $\frac{d\phi}{d\tau} = \frac{d\phi}{dt}\frac{dt}{d\tau} = \cancel{A}(\mu - f(\phi))\frac{1}{\cancel{A}} = \mu - f(\phi)$

So in dimensionless form, we have

$$\dot{\phi} = \frac{d\phi}{d\tau} = \mu - f(\phi) = F(\phi) \quad (*)$$

c) Explicitly writing out $(*)$, we have

$$\dot{\phi} = \begin{cases} \mu + 2(\pi - \phi), & -\pi \le \phi \le -\pi/2 \quad (1) \\ \mu - 2\phi, & -\pi/2 \le \phi \le \pi/2 \quad (2) \\ \mu - 2(\pi - \phi), & \pi/2 \le \phi < \pi \quad (3) \end{cases}$$

If they are phase-locked, then $\phi = 2\pi K$ and $\dot{\phi} = 0$ (they aren't changing out of phase lock) we get

$$\dot{\phi} = 0 = \mu - 2(2\pi K)$$

$$\rightarrow \quad \mu = 4\pi K \quad \text{for } K \in \mathbb{Z}$$

d) $$\frac{\Omega - \omega}{A} = 4\pi K \rightarrow \Omega = \omega + 4\pi A K$$

e) When $u = \pi$, we have

$$0 := \dot{\phi} = \pi - 2\phi \rightarrow \phi^* = \pi/2$$

when $\phi < \pi/2$, $\dot{\phi} > 0$ and when $\phi > \pi/2$, $\dot{\phi} > 0$, so we have a saddle node bifurcation that is stable coming from $\phi < \pi/2$.

Now when $u = -\pi$, we have

$$0 := \dot{\phi} = -\pi - 2\phi \rightarrow \phi^* = -\pi/2$$

which causes another saddle point w/ reversed stability.

f) we just solve

$$\dot{\phi} = u - 2\phi = 0 \longrightarrow \phi^* = \frac{u}{2} = \frac{4\pi K}{2} = 2\pi K$$

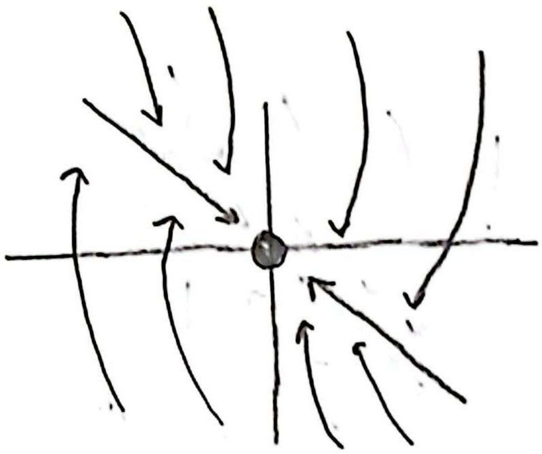
3) a) we have

$$\left.\begin{aligned}\dot{x} &= y \\ \dot{y} &= -2x-3y\end{aligned}\right\} \longrightarrow \dot{\vec{x}} = \underbrace{\begin{pmatrix} 0 & 1 \\ -2 & -3 \end{pmatrix}}_{A} \vec{x}$$

So $\Delta = 2$ and $\tau = -3$, and

$\tau^2 - 4\Delta = 9-8 = 1 > 0$, so the fixed point is a stable node.

PP

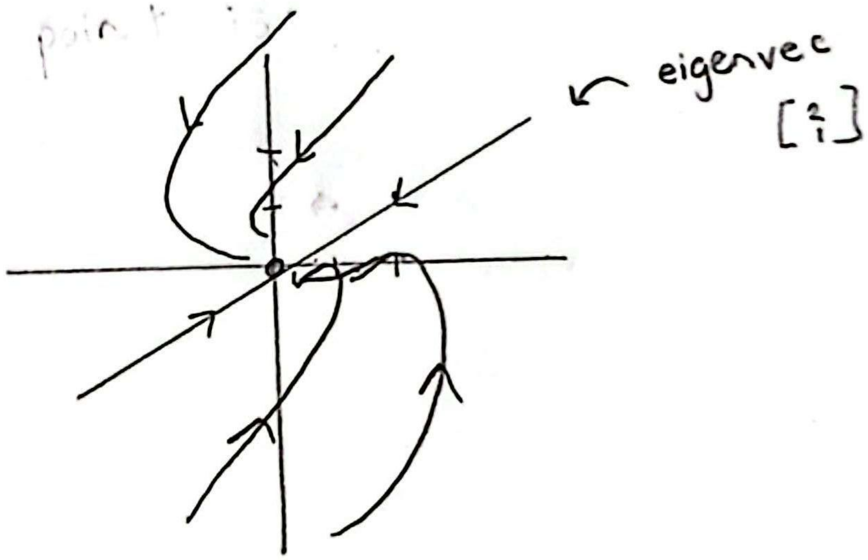


b) We have

$$\left.\begin{aligned}\dot{x} &= 3x - 4y \\ \dot{y} &= x - y\end{aligned}\right\} \rightarrow \begin{pmatrix} 3 & -4 \\ 1 & -1 \end{pmatrix} \vec{x} = \vec{\dot{x}}$$

So $\Delta = -3 + 4 = \underline{1}$, $\tau = 2$, and $\tau^2 - 4\Delta = \underline{0}$.

PP



computing the eigs specifically,

$$\det(A - \lambda I) = \det\begin{pmatrix} 3-\lambda & -4 \\ 1 & -1-\lambda \end{pmatrix}$$

$$\rightarrow -(3-\lambda)(1+\lambda) + 4 = 0$$

$$\rightarrow -(3 + 2\lambda - \lambda^2) + 4 = 0$$

$$\rightarrow \lambda^2 - 2\lambda + 1 = 0 \rightarrow (\lambda - 1)^2 = 0$$

$$\rightarrow \lambda = 1, 1$$

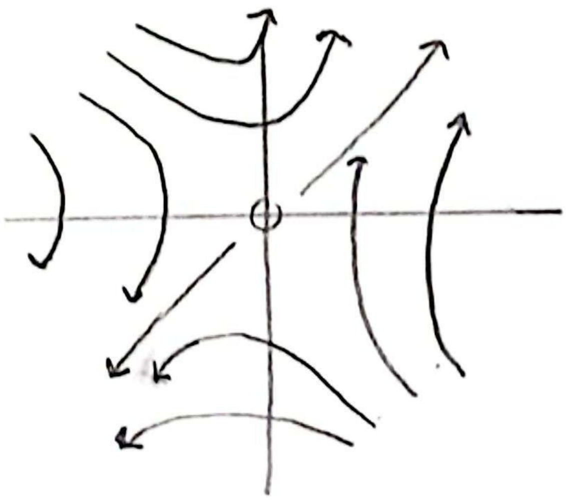
This is a rank 1 matrix, so its eigenspace is 1D, hence, our f.p must be a degenerate node,

c) we have $\ddot{x} = x - 2\dot{x}$. Put $y = \dot{x}$, then the system becomes

$$\left.\begin{aligned} \dot{x} &= y \\ \dot{y} &= x - 2y \end{aligned}\right\} \rightarrow \overbrace{\begin{pmatrix} 0 & 1 \\ 1 & -2 \end{pmatrix}}^{A} \vec{x} = \dot{\vec{x}}$$

So $\Delta = -1$, and the fixed point is a saddle point.

PP



4)

(a) we have $L\ddot{I} + \dot{I}R + I/C = 0$, put $x = I$ and $y = \dot{I}$, then we get the system,

$$\dot{x} = y$$

$$\dot{y} = -\frac{1}{CL}x - \frac{R}{L}y$$

(b) The above system can be represented as

$$\dot{\vec{x}} = \underbrace{\begin{pmatrix} 0 & 1 \\ -1/_{CL} & -R/L \end{pmatrix}}_{A} \vec{x}$$

So $\Delta = \frac{1}{CL}$ and $\tau = -R/L$ and

$\tau^2 - 4\Delta = \frac{R^2}{L^2} - \frac{4}{CL}$. If $R > 0$, then $\tau < 0$ and $\Delta > 0$, so that the f.p. at $\vec{0}$ is asymptotically stable. If $R = 0$, then $\Delta > 0$, $\tau = 0$, and $\tau^2 - 4\Delta < 0$, so that the f.p at $\vec{0}$ is a center, which is neutrally stable.

(c) From (b), we have

$$\Delta = \frac{1}{CL}, \quad \tau = -\frac{R}{L}, \quad \overbrace{\tau^2 - 4\Delta}^{\omega} = \frac{R^2}{L^2} - \frac{4}{CL}$$

$$= \frac{1}{L}\left(R^2/L - 4/C \right)$$

$$= \frac{1}{L^2 C}\left[R^2 C - 4L \right]$$

As $\frac{1}{L^2C} > 0$, the sign of ω depends on the sign of $\underbrace{R^2C - 4L}_{\omega_0}$. If $\omega_0 > 0$, $\tau < 0$, and $\Delta > 0$, then the f.p. is a stable node. If $\omega_0 < 0$, $\tau < 0$, and $\Delta > 0$, then the f.p is a stable spiral. If $\omega_0 = 0$, $\tau < 0$, and $\Delta > 0$, to be sure we compute the eigenvalues of A,

$$\det(A - \lambda I) = \det\begin{pmatrix} -\lambda & 1 \\ -\frac{1}{CL} & -R/L - \lambda \end{pmatrix}$$

$$= \lambda\left(\frac{R}{L} + \lambda\right) + \frac{1}{CL} := 0 \longrightarrow \lambda^2 + \frac{R}{L}\lambda + \frac{1}{CL} = 0$$

$$\leftrightarrow \lambda = -\frac{R}{2L} \pm \frac{\sqrt{\frac{R^2}{L^2} - 4/CL}}{2} = \frac{1}{2L}\left[-R \pm \frac{1}{\sqrt{C}}\sqrt{\omega_0} \right]$$

If $\omega_0 = 0$, then $\longrightarrow \; = \frac{-R}{2L}, \frac{-R}{2L}$

So $\text{null}(A - \lambda I) = \text{null}\begin{pmatrix} R/2L & 1 \\ -1/CL & -R/2L \end{pmatrix}$

$$\rightarrow \begin{pmatrix} R/2L & 1 \\ -1/CL & -R/2L \end{pmatrix}\begin{pmatrix} a \\ b \end{pmatrix} = \begin{pmatrix} 0 \\ 0 \end{pmatrix}$$

$$\rightarrow \frac{Ra}{2L} + b = 0, \quad -\frac{a}{CL} - \frac{Rb}{2L} = 0$$

$$\longrightarrow b = -\frac{Ra}{2L}, \quad b = -\frac{a}{C\cancel{L}} \cdot \frac{2\cancel{L}}{R} = -\frac{2a}{CR}$$

which results in having 2 different eigenvectors. This means in this case, our f.p at $\vec{0}$ will be a <u>star</u>.

5) a) If R likes J up to some constant, more than J hates him, he will love J more. If R loves J, and J loves R, J will be inclined to hate R over time. a is like an "eagerness". If R is more eager, R likes J more, but J does not like it if J likes R and R is eager. If J hates R, and R hates J, then J will be more inclined towards R.

b) i) we have

$$\left.\begin{array}{l}\dot{R} = aR + J \\ \dot{J} = -R - aJ\end{array}\right\} \longrightarrow \dot{\vec{x}} = \underbrace{\begin{pmatrix} a & 1 \\ -1 & -a \end{pmatrix}}_{A} \vec{x}$$

we compute the eigenvalues of A.

$$\det(A - \lambda I) = \det\begin{pmatrix} a-\lambda & 1 \\ -1 & -a-\lambda \end{pmatrix}$$

$$= -(a-\lambda)(a+\lambda) + 1 = 0$$

$$\rightarrow -(a^2 - \lambda^2) + 1 = 0 \rightarrow \lambda^2 - a^2 + 1 = 0$$

$$\rightarrow \lambda^2 = a^2 - 1 \rightarrow \lambda = \pm\sqrt{a^2-1}$$

when $a > 1$, we have 2 distinct real eigenvalues, with 1 positive and 1 negative.

we now compute the eigenvectors

$$\begin{pmatrix} a+\sqrt{a^2-1} & 1 \\ -1 & -a+\sqrt{a^2-1} \end{pmatrix}\begin{pmatrix} u \\ v \end{pmatrix} = \begin{pmatrix} 0 \\ 0 \end{pmatrix}$$

$$(a+\sqrt{a^2-1})u + v = 0 \rightarrow v = -(a+\sqrt{a^2-1})u$$

$$-u - (a+\sqrt{a^2-1})v = 0$$

$$\hookrightarrow v = \frac{-u}{a+\sqrt{a^2-1}}$$

So the (-) solution eigenvector is,

$$\vec{v}_{(-)} = \begin{pmatrix} 1 \\ -(a+\sqrt{a^2-1}) \end{pmatrix}$$

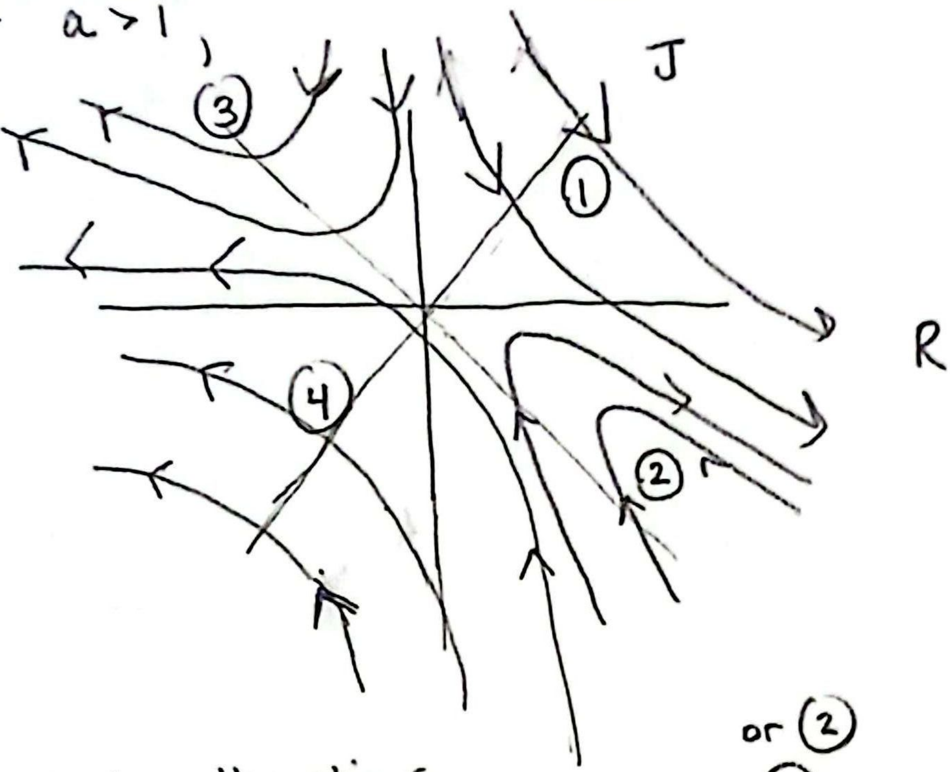
So by orthogonality of eigenvectors, it must be that

$$\vec{v}_{(+)} = \begin{pmatrix} a+\sqrt{a^2-1} \\ 1 \end{pmatrix}$$

So when $a>1$, the f.p is a saddle node where the stable manifold is the line spanned by $\vec{v}_{(-)}$ and the unstable manifold is the line spanned by $\vec{v}_{(+)}$. Since it is a saddle node, the f.p. is unstable and unattracting.

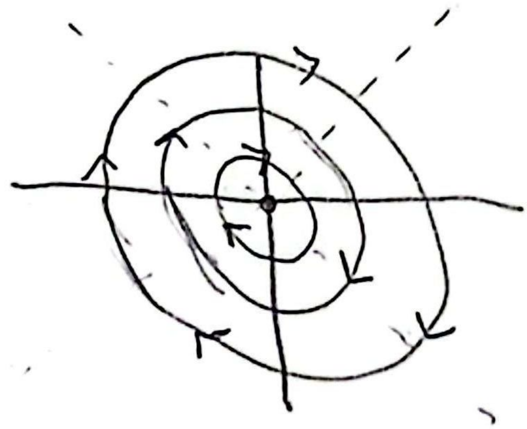
When $a<1$, λ_+ and λ_- are (but distinct) complex, and so is $\vec{v}_-$ and $\vec{v}_+$. They are purely imaginary, as $\lambda_\pm = \pm i\sqrt{1-a^2}$, so the f.p must be a center which is unattracting.

ii and c) If $a>1$,

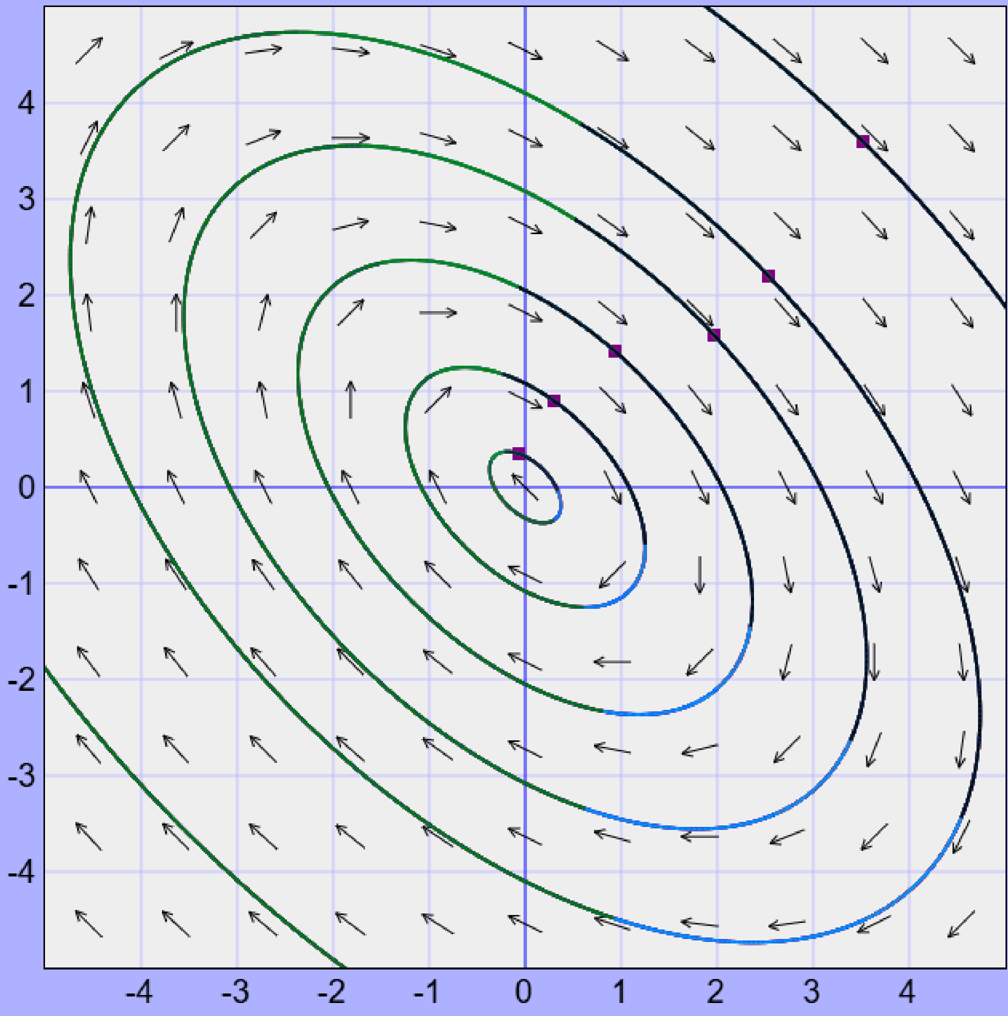


If the initial attractions are in 1 or 2 Romeo and Juliet will hate each other. If they start in 3 or 4, the will eventually both love each other.

If $a<1$,



Then R and J will fall in a cycle of love and hate, unless they feel nothing about eachother at the beginning, in which case they will remain neutral.



4
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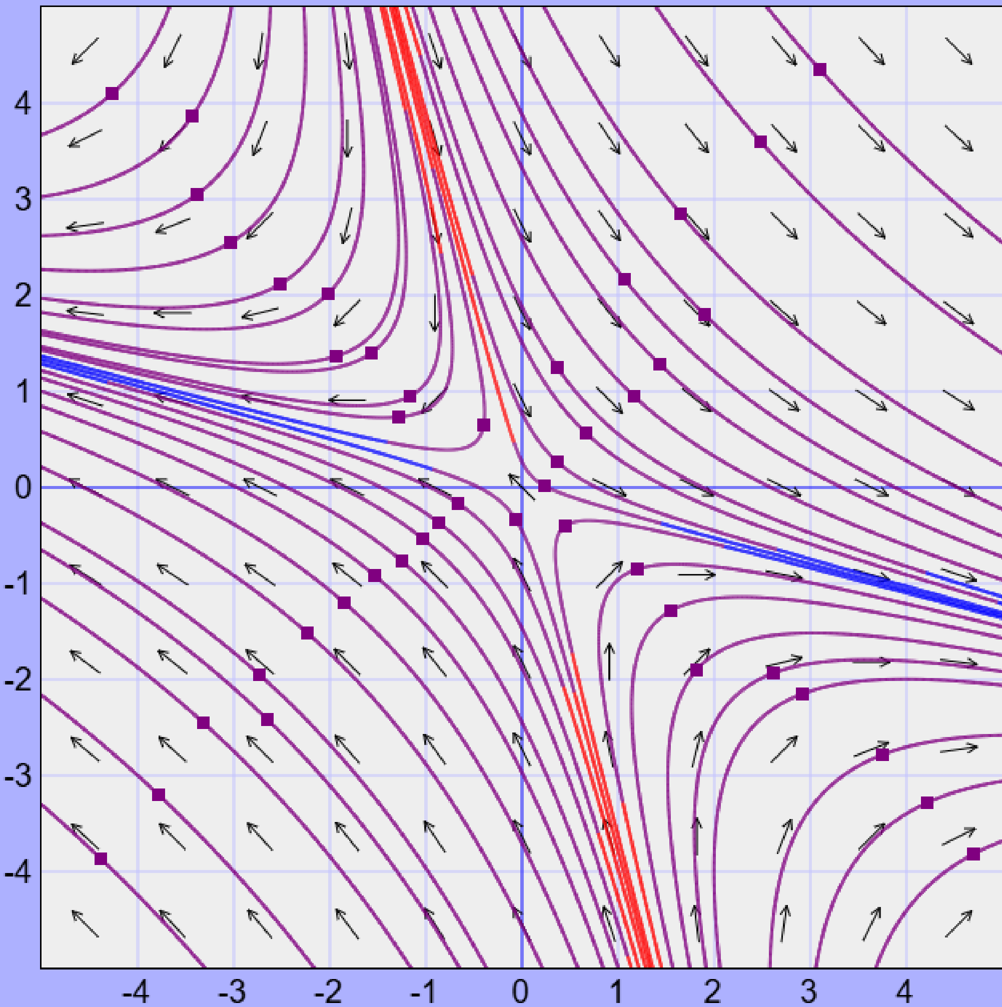
4



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