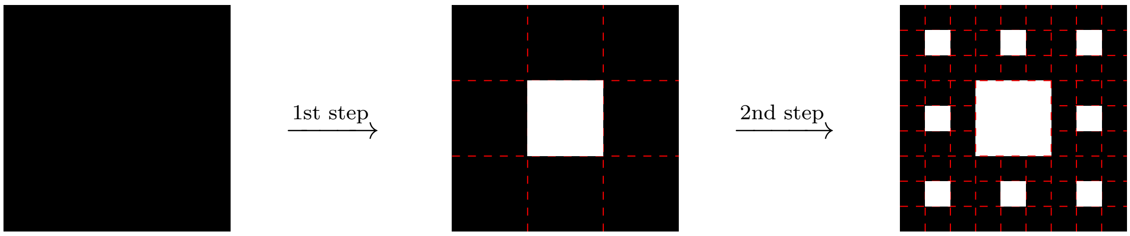
Homework #8

All of your work must be justified by calculations or an explanation.

1. In this problem we will construct and analyze the Sierpinski Carpet. Consider the closed unit box, $[0, 1] \times [0, 1]$. Divide it into nine equal boxes and then remove the middle box. Continue this process on the remaining 8 subboxes, removing the middle box each time. The first two steps of this process is visualized below.



   (a) Show that the (Lebesgue) measure of the resulting fractal is 0. Justify your work.

   (b) Find the similarity dimension of the limiting fractal. Show and explain your work.

   (c) Show that the box-counting dimension of this fractal is the same as the similarity dimension. (You may not be able to complete this until box-counting dimension is defined in class).

   (d) Show that there are uncountably many points on the interior of the limiting fractal (i.e. infinitely points without $x = 0$, $x = 1$, $y = 0$, or $y = 1$).
   Hint: Try showing this in just one dimension, e.g., show that there are uncountably many points (x, y) with a fixed value of y, perhaps $y = 0.5$ is a good choice.

2. In this problem we construct what is called a middle-halves Cantor set. Consider the following Cantor set construction. Start with the interval $[0, 1]$, then remove the middle half. Continue this process for each sub-interval.

   (a) Draw S_1 and S_2.

   (b) Find the similarity dimension of the set.

   (c) Find the measure of the set.

3. In this problem we construct what is called a *fat fractal*. After working through this problem you should see why it is called fat. Consider the following Cantor set construction. Start with the interval $[0, 1]$. Instead of removing an interval of the same length each time, in this construction we vary the size of the interval we are removing at each step. Begin by removing the middle two-sevenths of the interval $[0, 1]$. Then in the following steps, remove subintervals of width $(2/7)^n$ from the middle of each of the remaining intervals. e.g. for the second step, remove subintervals of length $(2/7)^2 = 4/49$.

   (a) Find the (Lebesgue) measure of this Cantor set. Show your work.

   (b) Is this fractal self similar? Justify your answer.

   Hint: Can you find the similarity dimension of this set? What happens when you try?

   Note: You find this part to be difficult. If you are struggling with it, you may want to skip it for now and come back to it later.

   (c) You don't need to answer anything here, this is just an observation. Fat fractals answer a fascinating question about the logistic map. Farmer (1985) has shown numerically that the set of parameter values for which chaos occurs in the logistic map is a fat fractal. In particular, if r is chosen at random between r_∞ and $r = 4$ there is about an 89% chance that the map will be chaotic. Farmer's analysis also suggests that the odds of making a mistake (calling an orbit chaotic when it's actually periodic) are aboute one in a million, using double-precision arithmetic!

4. The tent map on the interval $[0, 1]$ is defined by $x_{n+1} = f(x_n)$, where

$$f(x) = \begin{cases} rx, & 0 \le x \le \frac{1}{2} \\ r(1-x), & \frac{1}{2} < x \le 1 \end{cases}$$

   Assume that $r > 2$. Then some points get mapped outside of the interval $[0, 1]$. If $f(x_0) > 1$ then we say that x_0 has "escaped" after one iteration. Similarly, if $f^n(x_0) > 1$ for some finite n and n is the smallest integer for which this is true, then we say x_0 has escaped after n iterations.

Hint: You should plot the function $f(x)$ and think about orbits of different initial conditions when you are solving this problem, it will make your work much easier. Use these plots to guide your analysis, the plots alone are not enough.

(a) Find the set of initial conditions x_0 that escape after one iteration.

(b) Find the set of initial conditions x_0 that escape after two iterations.

(c) Describe the set of x_0 that never escape. This is called the **invariant set**.

Hint: First look at what happens for $r = 3$. Does this look like a set you recognize?

(d) Find the box dimension of the invariant set (for general r, not $r = 3$).

(e) This is just a note, you don't need to answer anything here. After completing this problem, you will have shown that the invariant set of this chaotic map forms a fractal. Cool! This invariant set is called a *strange repeller* because it is a fractal set that repels all nearby points that are not in the set and points in the set are part of a chaotic orbit.