1) They are all markov chains. We show this by computing the transition matrices whose rows sum to 1.

a.)

$$p(i,j) = \begin{cases} \frac{i}{6} & i = j \\ 1/6 & i < j \\ 0 & \text{else} \end{cases}$$

(given the largest roll is i, i still being the largest roll at $n+1$, can only happen if $1, 2, .., i$ is rolled, so $i/6$)

If the next roll is larger than the current, it has a $1/6$ chance of being rolled

$$\begin{bmatrix} 1/6 & 1/6 & 1/6 & 1/6 & 1/6 & 1/6 \\ & 2/6 & 1/6 & 1/6 & 1/6 & 1/6 \\ & & 3/6 & 1/6 & 1/6 & 1/6 \\ & & & 4/6 & 1/6 & 1/6 \\ & & & & 5/6 & 1/6 \\ & & & & & 1 \end{bmatrix}$$

The rows sum to 1, so this is a valid transition matrix.

b) If the nth roll isn't 6, $X_n = X_{n-1}$, if it is then $X_n = X_{n-1} + 1$. The first scenario happens w/ probability 5/6 and the second w/ probability 1/6, so

$$p(i,j) = \begin{cases} 5/6 & i=j \\ 1/6 & j=i+1 \\ 0 & \text{else} \end{cases}$$

The state space here is countable, but each row still sums to 1.

$$\begin{bmatrix} 5/6 & 1/6 & & & \\ & 5/6 & 1/6 & & \\ & & 5/6 & 1/6 & \\ & & & \ddots & \ddots \end{bmatrix}$$

c) Given that $X_n = i$ (the last time a 6 was rolled was), X_{n+1} can be either 0 (a 6 is rolled again w/ probability 1/6), or $i+1$ (a 6 is not rolled again w/ probability 5/6)

so

$$p(i,j) = \begin{cases} 1/6 & j = 0 \\ 5/6 & j = i+1 \\ 0 & \text{else} \end{cases}$$

$$\begin{bmatrix} 1/6 & 5/6 & & & \\ 1/6 & & 5/6 & & \\ 1/6 & & & 5/6 & \\ \vdots & & & & \ddots \end{bmatrix}$$

Again, the state space is countable, but the rows all sum to 1, so we have a valid transition matrix.

d) Given $X_n = 0$, the probability $X_{n+1} = 0$ (the next time a 6 is rolled is right now) is $1/6$. The probability $X_{n+1} = 1$ (a 6 is rolled on the next roll) is $\frac{5}{6} \cdot \frac{1}{6}$ (no 6 on first roll, 6 on second roll). The probability $X_{n+1} = 2$ is $\left(\frac{5}{6}\right)^2 \cdot \frac{1}{6}$ ... and the probability $X_{n+2} = j$ is $\left(\frac{5}{6}\right)^{j-1} \cdot \frac{1}{6}$.

This is a geometric series with

$$\frac{1}{6} \sum_{j=1}^{\infty} \left(\frac{5}{6}\right)^{j-1} = \frac{1}{6} \cdot \frac{1}{1 - 5/6} = \frac{1}{\cancel{6}} \cdot \frac{\cancel{6}}{1} = 1$$

So the first row takes the form $\frac{1}{6}\left(\frac{5}{6}\right)^{j-1}$.

If $X_n = 1$, a 6 is guaranteed next, so $P(1,0) = 1$, if $X_n = 2$, a 6 is guaranteed after the next role, so $P(2,1) = 1$, this pattern continues, so

$$P(i,j) = \begin{cases} \frac{1}{6}\left(\frac{5}{6}\right)^{j-1} & i = 0 \\ 1 & i = j+1 \\ 0 & \text{else} \end{cases}$$

$$\begin{bmatrix} 1/6 & \frac{5}{6}\cdot\frac{1}{6} & \left(\frac{5}{6}\right)^2\frac{1}{6} & \cdots \\ 1 & 0 & \cdots & \cdots \\ & 1 & 0 & \cdots \\ & & 1 & 0 & \cdots \\ & & & & \ddots \end{bmatrix}$$

we can see the rows sum to 1, so this is a transition matrix.

4.3) Choose a state j other than K. As $j \to K$, $\exists$ a first integer $n_1 \geq 1$ such that $p_{n_1}(j,K) > 0$. Consider the events

$$E_1 = \overbrace{\{X_0 = j\}}^{A_1} \cap \overbrace{\{X_{n_1} = K\}}^{A_2}$$

we start at j and After n steps we are in state K

$$E_2 = \underbrace{\overbrace{\{X_0 = j\}}^{A_1}}_{\text{we start at } j} \cap \underbrace{\overbrace{\{X_n \text{ doesn't go back to } j\}}^{A_3}}_{\text{we never come back}}$$

for any $n \geq 0$

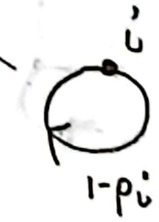
Since state K is an absorbing state $A_2 \subseteq A_3$, ie $E_1 \subseteq E_2$. So

$$0 < P(E_1) \leq P(E_2)$$

So the probability we never return to j is positive, hence j must be transient.

4.4 Suppose that $p_j = P(T_j < T_i \mid X_0 = i)$ and also $p_i = P(T_i < T_j \mid X_0 = j)$. It is clear that $p = p_i = p_j$ as they are defined to be the same ~~formulas with indices swapped~~. Put

$N =$ the number of times state j is visited before returning back to state i. Firstly, notice that



$$P(N=0) = 1 - p_j = 1 - p$$

$$P(N=1) = p_j p_i = p^2$$

$$P(N=2) = p_j(1-p_i)(p_i) = p^2(1-p)$$

$$\vdots$$

we see the pattern that $P(N=n) = p^2 \underbrace{}_{\text{leave and return to state } i} \underbrace{(1-p)^{n-1}}_{\text{repeat going to state } j}$ when $n \geq 1$, and $P(N=0) = 1-p$. So

$$E[N] = \sum_{n=0}^{\infty} n P(N=n)$$

$$= 0 \cdot (1-p) + \sum_{n=1}^{\infty} n \cdot p^2 (1-p)^{n-1}$$

$$= p^2 \frac{d}{dp} \sum_{n=0}^{\infty} -(1-p)^n$$ geometric series as $1-p < 1$

$$= p^2 \cdot \frac{d}{dp}\left(\frac{-1}{1-(1-p)}\right)$$

$$= p^2 \cdot \frac{d}{dp}\left(\frac{-1}{p}\right)$$

$$= p^2 \cdot \frac{1}{p^2}$$

$$= 1$$

As $E[N] = 1$, we are done.

5) a) we have

$$G_{n+1}(s) = E\left[s^{X_{n+1}}\right]$$

$$= E\left[E\left[s^{X_{n+1}} \mid X_n\right]\right]$$ by law of total probability

$$= \sum_{K=0}^{r} P(X_n=K)\, E\left[s^{X_{n+1}} \mid X_n=K\right]$$

the other terms are 0

$$= \sum_{K=0}^{r} P(X_n=K)\left[P(X_{n+1}=K+1 \mid X_n=K)s^{K+1} + P(X_{n+1}=K-1 \mid X_n=K)s^{K-1}\right]$$

the Ehrenfest transition probabilities

$$= \sum_{K=0}^{r} P(X_n=K)\left[\frac{r-K}{r}s^{K+1} + \frac{K}{r}s^{K-1}\right]$$

$$= \sum_{K=0}^{r} P(X_n=K)s^K\left[s\cdot\frac{(r-K)}{r} + \frac{1}{s}\cdot\frac{K}{r}\right]$$

$$\frac{1}{r}\left[\sum_{K=0}^{r} P(X_n=K)s^{K+1}(r-K) + \sum_{K=0}^{r} P(X_n=K)s^{K-1}\cdot K\right]$$

$$= \frac{1}{r}\left[sr\sum_{K=0}^{r} P(X_n=K)s^K - s^2\sum_{K=0}^{r} P(X_n=K)Ks^{K-1} + \sum_{K=0}^{r} P(X_n=K)Ks^{K-1}\right]$$

$$= s\sum_{K=0}^{r} P(X_n=K)s^K + \frac{(1-s^2)}{r}\sum_{K=0}^{r} P(X_n=K)Ks^{K-1}$$

$$= sG_n(s) + \frac{(1-s^2)}{r}\frac{dG_n(s)}{ds}$$ by definition of $G_n(s)$

b) From (a) we have

$$G_{n+1}(s) = s\,G_n(s) + \frac{(1-s^2)}{r}\frac{dG_n(s)}{ds}$$

As $n \to \infty$, we see

$$G(s) = s\,G(s) + \frac{(1-s^2)}{r}\frac{dG(s)}{ds}$$

$$\to (1-s)G(s) = \frac{(1-s)(1+s)}{r}\frac{dG(s)}{ds}$$

$$\to G(s) = \frac{(1+s)}{r}\frac{dG(s)}{ds}$$

$$\to \frac{r}{1+s}\,ds = \frac{dG(s)}{G(s)}$$

Integrating both sides (w/ c a constant)

$$r\ln(1+s) + c = \ln(G(s))$$

$$\to e^c e^{r\ln(1+s)} = G(s)$$

$$\to C[1+s]^r = G(s)$$ (replacing e^c w/ c)

ie $G(s) = c[1+s]^r$

$$= c\sum_{k=0}^{r}\binom{r}{k}s^k$$

binomial theorem

when $s=1$, $\sum_{k=0}^{r}\binom{r}{k} = 2^r$ (a well known fact)

As $G(s)$ is the limit of a sequence of Generating functions, it too is a generating function, so $G(1)$ should equal 1. Hence the constant C must be $\frac{1}{2^r}$, so

$$G(s) = \frac{1}{2^r} \sum_{k=0}^{r} \binom{r}{k} s^k$$

c) With the transition matrix P from Lecture 17, we'd like to find a distribution π such that

$$\pi P = \pi$$

or

$$P^T \pi^T = \pi^T$$

$$\downarrow$$

$$\begin{array}{cc} & r+1 \\ r+1 & \begin{bmatrix} 0 & 1/r & & & \\ 1 & 0 & 2/r & & \\ & \frac{r-1}{r} & 0 & 3/r & \\ & & \frac{r-2}{r} & \ddots & 1 \\ & & & \frac{1}{r} & 0 \end{bmatrix} \end{array} \begin{bmatrix} \pi_0 \\ \pi_1 \\ \vdots \\ \pi_r \end{bmatrix} = \begin{bmatrix} \pi_0 \\ \pi_1 \\ \vdots \\ \pi_r \end{bmatrix}$$

P^T is tridiagonal, so based off the Tridiagonal matrix algorithm (wikipedia), the π_k's must satisfy the following recurrence

$$a_i x_{i-1} + b_i x_i + c_i x_{i+1} = d_i \qquad 0 \le i \le r$$

where $a_i = \begin{cases} 0 & i=0 \\ \frac{r-(i-1)}{r} & 1 \le i \le r \end{cases}$ $b_i = 0, \ 0 \le i \le r$

and $c_i = \begin{cases} \frac{i+1}{r} & 0 \le i \le r-1 \\ 0 & i = r \end{cases}$ and $x_i = d_i = \pi_i, \ \forall i$

So we get

$$\pi_0 = \frac{1}{r}\pi_1 \quad , \quad i=0$$

$$\frac{1}{r}\pi_{r-1} = \pi_r \quad , \quad i=r$$

and for $1 \le i \le r-1$

$$\frac{r-(i+1)}{r}\pi_{i-1} + \frac{i+1}{r}\pi_{i+1} = \pi_i$$

I claim $\pi_i = \frac{1}{2^r}\binom{r}{i}$ solves the recurrence, which I will prove by strong induction. Suppose

$$\pi_0 = \frac{1}{2^r}\binom{r}{0} = \frac{1}{2^r} \ , \quad \text{then } \pi_1 = r\pi_0$$

$$= \frac{r}{2^r} = \frac{1}{2^r}\binom{r}{1}$$

Follows the pattern. Rearranging

$$\pi_{i+1} = \frac{r\pi_i - (r-i+1)\pi_{i-1}}{i+1}$$

So $\pi_2 \stackrel{?}{=} \frac{r\frac{r}{2^r} - \frac{r}{2^r}}{2} = \frac{1}{2^r}\left(\frac{r^2-r}{2}\right) = \frac{1}{2^r}\left(\frac{r(r-1)}{2}\right)$

$\frac{1}{2^r}\binom{r}{2}$ $\|$ as desired.

Suppose this holds for $\pi_1, \pi_2, \pi_3, \pi_4, \dots \pi_{K-1}, \pi_K$ we want to show the recurrence gives us π_{K+1}

$$\rightarrow \pi_{K+1} \stackrel{?}{=} \frac{r \frac{1}{2^r} \binom{r}{K} - (r-K+1) \binom{r}{K-1} \frac{1}{2^r}}{K+1}$$

$$= \frac{1}{2^r} \frac{1}{K+1} \left(r \frac{r!}{K!(r-K)!} - \frac{r!}{(K-1)!(r-K+1)!} (r-K+1) \right)$$

$$= \frac{1}{2^r} \left(\frac{r\, r!}{(K+1)!\,(r-K)!} - \frac{r!\,K}{(K+1)(K-1)!\,(r-K)!\,K} \right)$$

$$= \frac{1}{2^r} \left(\frac{r r! - r! K}{(K+1)!\,(r-K)!} \right)$$

$$= \frac{1}{2^r} \left(\frac{r!\,(r-K)}{(K+1)!\,(r-K)!} \right)$$

$$= \frac{1}{2^r} \left(\frac{r!}{(K+1)!\,(r-K-1)!} \right)$$

$$= \frac{1}{2^r} \binom{r}{K+1}$$

As desired.

This holds for all $K \geq 0$, where we have assumed the value of π_0 to be $\frac{1}{2^r}$. As π must sum to 1, and $\frac{1}{2^r} \sum_{K=0}^{r} \binom{r}{K} = 1$.

4.2) a) Put $Y_n = X_{2n}$. As X is a random walk on the line, and we are skipping every 2 steps, given a starting state j, we can only end up in 4 scenarios. We can step forward twice, backwards twice, forwards once and backwards once, and backwards once and forwards once. If we step forward with probability p, and backwards with probability $q = 1-p$, then our transition matrix takes the form

$$\overset{Y_n \; X_{n+1}}{p(i,j)} = \begin{cases} p^2 & j = i+2 \\ q^2 & j = i-2 \\ 2pq & j = i \\ 0 & \text{else} \end{cases}$$

Each row has the sum $p^2 + 2pq + q^2 = (p+q)^2 = 1^2 = 1$, so this is a valid transition matrix.

b) Put $Y_n = X_{2n}$. From the lecture notes

$$P(X_{n+1} = j \mid X_n = i)$$

$$= P(\varepsilon_1 + \varepsilon_2 + \dots + \varepsilon_i = j)$$

we have since the ε_i are iid

$$G_{\varepsilon_1 + \varepsilon_2 + \dots + \varepsilon_i}(s)$$

$$= (G(s))^i$$

So by the definition of generating function

$$P(i,j) =$$

$$P(\varepsilon_1 + \varepsilon_2 + \dots + \varepsilon_i = j) = \left[\frac{1}{j!} \frac{d^j (G(s))^i}{ds^j} \right] \text{evaluated at } s=0$$

For $Y_n = X_{2n}$

$P(X_{2n} = j \mid X_{2n-2} = i)$ can be thought as a branching after 2 branches, so the above formula can be modified like so

$$P(i,j) = \left[\frac{1}{j!} \frac{d^j (G(G(s)))^i}{ds^j} \right] \text{evaluated at } s=0$$