AMATH 569 Homework 4

Due: Monday, May 19 at Midnight.

Please submit your solutions to Gradescope. Typesetting your assignments using Latex is highly recommended but not required. If handing in written solutions please make sure they are readable. Show your work and describe all logical steps. If you are using other results or inequalities make sure to mention them clearly. You may also acknowledge collaborations with your classmates.

Note: The use of AI models such as ChatGPT is strictly prohibited and will result in a report to UW Academic Misconduct without warning.

1. (generalized Hölder's inequality) Let $p_i \in [0, +\infty]$ satisfy $\sum_{i=1}^N \frac{1}{p_i} = 1$ and let $f_i \in L^{p_i}(\Omega)$, $i = 1, \ldots, N$. Show that

$$\int_\Omega \prod_{i=1}^N |f_i| \mathrm{d}x \leq \prod_{i=1}^N \left(\int_\Omega |f_i|^{p_i} \mathrm{d}x \right)^{\frac{1}{p_i}}.$$

2. (McLean, Problem 3.6) Define $f : \mathbb{R} \to \mathbb{R}$ by

$$f(t) = \begin{cases} \exp(-1/t) \text{ if } t > 0, \\ 0 \text{ if } t \leq 0. \end{cases}$$

- Show that $f^{(j)}(t) \to 0$ as $t \to 0$ for all $j \in \mathbb{N}$.
- Deduce that $f \in C^\infty(\mathbb{R})$.
- Construct a C^∞ function $g : \mathbb{R} \to \mathbb{R}$ such that $g(t) > 0$ on $(-1, 1)$ and $\operatorname{supp} g = [-1, 1]$.
- Construct a function $\psi \in C_c^\infty(\mathbb{R}^n)$ satisfying

$$\int_{\mathbb{R}^n} \psi(x) \mathrm{d}x = 1, \qquad \psi \geq 0 \text{ on } \mathbb{R}^n, \qquad \psi(x) = 0 \text{ for } |x| > 1. \tag{1}$$

3. (Dilation of mollifier functions) Let $\psi \in C_c^\infty(\mathbb{R}^n)$ be a function satisfying (1). Now define the dilation of ψ as

$$\psi_\epsilon(x) = \frac{1}{\epsilon^n} \psi\left(\frac{x}{\epsilon}\right), \qquad x \in \mathbb{R}^n, \quad \epsilon > 0.$$

Show that $\psi_\epsilon \in C_c^\infty(\mathbb{R}^n)$ and that

$$\int_{\mathbb{R}^n} \psi_\epsilon(x) \mathrm{d}x = 1, \qquad \psi_\epsilon(x) \geq 0 \text{ on } \mathbb{R}^n, \qquad \psi_\epsilon(x) = 0 \text{ for } |x| > \epsilon.$$

4. Let A be a closed subset of $\mathbb{R}^n$. Show that for each $\epsilon > 0$ there exists a function $\chi_\epsilon \in C^\infty(\mathbb{R}^n)$ such that

$$\begin{aligned} &\chi_\epsilon(x) = 1 && \text{if } x \in A, \\ &0 \leq \chi_\epsilon \leq 1 \text{ and } |\partial^\alpha \chi_\epsilon(x)| \leq C\epsilon^{-|\alpha|} && \text{if } 0 < \operatorname{dist}(x, A) < \epsilon, \\ &\chi_\epsilon(x) = 0 && \text{if } \operatorname{dist}(x, A) \geq \epsilon \end{aligned}$$

where $C > 0$ is a constant independent of ϵ and x.

5. Show that the distributional derivative of the Heaviside function

$$H(x) = \begin{cases} 0 \text{ if } x < 0, \\ 1 \text{ if } x \geq 0 \end{cases}$$

is the Dirac delta function $\delta(\phi) = \phi(0)$.

6. (McLean, Problem 3.9) Let $\Omega \subset \mathbb{R}^n$ be bounded. Consider a linear functional $\ell : \mathcal{D}(\Omega) \to \mathbb{R}$. Show that ℓ is sequentially continuous (and hence a distribution on Ω) if and only if for each compact set $K \subseteq \Omega$ there exists an integer $m \geq 0$, and constants $C_{K,m} \geq 0$ such that

$$|\ell(\phi)| \leq C_{K,m} \sum_{|\alpha| \leq m} \sup_{x \in K} |\partial^\alpha \phi(x)|,$$

for all $\phi \in \mathcal{D}(\Omega)$ with supp $\phi \subseteq K$.

Hint: To prove the only if part, suppose, towards a contradiction, that there exists a set K such $m, C_{K,m}$ don't exist and deduce the existence of a sequence $\phi_j \to 0$ in $\mathcal{D}(K)$ such that $\ell(\phi_j) = 1$. Note, we say $\phi_j \to 0$ in $\mathcal{D}(K)$ if the ϕ_j are supported on K and their point values along with the point values of all of their partial derivatives converge to zero uniformly on K.

7. (Renardy and Rogers, Problem 5.41) Let $G : [0, \infty) \mapsto \mathcal{D}^\star(\mathbb{R}^n)$ be a continuous,distribution valued map, as defined in Lecture 18, which solves the parabolic PDE

$$\begin{cases} G_t(x,t) - \mathcal{L}G(x,t) = 0 & \text{in } \mathbb{R}^n \times (0, +\infty) \\ G(x,0) = \delta. & \end{cases}$$

For simplicity, we assume that $\mathcal{L}$ is a constant coefficient second order elliptic differential operator (the result still holds for non-constant coefficients and higher order operators as well). Verify that the function

$$u(x,t) = \int_{\mathbb{R}^n} G(x-y,t) u_0(y) dy + \int_0^t \int_{\mathbb{R}^n} G(x-y, t-s) f(y,s) dy ds$$

solves the PDE

$$\begin{cases} u_t - \mathcal{L}u = f, & \text{in } \mathbb{R}^n \times (0, +\infty) \\ u(x,0) = u_0(x) & x \in \mathbb{R}^n, \end{cases}$$

where f, u_0 are assumed to be smooth and compactly supported.

8. (Bonus problem) For $\epsilon > 0$ let u_ϵ solve the PDE

$$\Delta u_\epsilon = \psi_\epsilon \qquad \text{in } \mathbb{R}^n,$$

with the far field boundary conditions $|u(x)| \to 0$ as $|x| \to \infty$ and where ψ_ϵ is defined as in Problem 3. Show that $u_\epsilon \to G$ in the distributional sense, where G is the fundamental solution of the Laplace PDE. Can you derive a quantitative rate for the convergence in terms of ϵ?