17 + 10 + 20 + 25 = 72

AMATH 561

HW 6

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1) Let $X = \text{Binom}(n, U)$ with $U \sim [0,1]$. The density function of $U$ is $f_u(x) = 1$ for $x \in [0,1]$.

Now,

$$P(X = K) = E[\mathbb{1}_{\{X = K\}}]$$

$$= \int \mathbb{1}_{\{X(p) = K\}} \, dP$$

$$= \int_0^1 P(X = K \mid p) \, f_u(p) \, dP$$

$$= \int_0^1 \binom{n}{K} p^K (1-p)^{n-K} \, dp$$

$$= \binom{n}{K} \int_0^1 p^{(K+1)-1} (1-p)^{(n-K+1)-1} \, dP$$

where B is the Beta function from wikipedia

$$= \binom{n}{K} B(K+1, n-K+1)$$

$$= \binom{n}{K} \frac{\Gamma(K+1)\,\Gamma(n-K+1)}{\Gamma(n+2)}$$

$$= \frac{n!}{(n-K)!\,K!} \cdot \frac{(K)!\,(n-K)!}{(n+1)!}$$

$$= \frac{1}{n+1}$$

So the generating function

$$G_X(s) = \sum_{k=0}^{\infty} P(X=k)\, s^k$$

(-8) Should not add to infinity

$$= \sum_{k=0}^{\infty} \frac{s^k}{n+1}$$

$$= \frac{1}{n+1} \sum_{k=0}^{\infty} s^k$$

$$= \frac{1}{n+1} \cdot \frac{1}{1-s}$$

$$= \frac{1}{(n+1)(1-s)}$$

2) we have

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$$Z_{n+1} - Y_{n+1} = \sum_{i=1}^{Z_n} \varepsilon_i^{n+1}$$

From Lecture 15, the generating functions of both sides are

Not the same process

the r.v's are independent

$$G_{Z_{n+1}-Y_{n+1}}(s) = \underbrace{G_\varepsilon(s) \circ \dots \circ G_\varepsilon(s)}_{n+1 \text{ times}}$$

$$\boxed{G_{Z_{n+1}}(s)\, G_{-Y_{n+1}}(s)} = G_{Z_n}(s) \circ G_\varepsilon(s)$$

$$\boxed{G_{Z_{n+1}}(s)\, G_{Y_{n+1}}(1/s)} = G_{Z_n}(G_\varepsilon(s))$$

$$\rightarrow \quad G_{Z_{n+1}}(s)\, G_Y(1/s) = G_{Z_n}(G_\varepsilon(s))$$

$$\rightarrow \quad G_{Z_{n+1}}(s) = \frac{G_{Z_n}(G_\varepsilon(s))}{G_Y(1/s)}$$

Now

$$G_{Z_2}(s) = \frac{G_{Z_1}(G_\varepsilon(s))}{G_Y(1/s)} = \frac{G_\varepsilon(G_\varepsilon(s))}{G_Y(1/s)}$$

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3) a)

$$E[e^{itX}] = \int_\Omega e^{itX}\, dP$$

$$= \int_0^\infty e^{itx}\, \lambda e^{-\lambda x}\, dx$$

$$= \lambda \int_0^\infty e^{-\lambda x}\, [\cos(tx) + i\sin(tx)]\, dx$$

$$= \lambda \Big[ \underbrace{\int_0^\infty e^{-\lambda x} \cos(tx)\, dx}_{1} + i \underbrace{\int_0^\infty e^{-\lambda x} \sin(tx)\, dx}_{2} \Big]$$

To find antiderivatives for the integrands, we must use integration by parts twice,

for 1

$$\int e^{-\lambda x}\cos(tx)\,dx = \frac{1}{t} e^{-\lambda x}\sin(tx) - \frac{\lambda}{t^2} e^{-\lambda x}\cos(tx) - \frac{\lambda^2}{t^2}\int e^{-\lambda x}\cos(tx)\,dx$$

$$\leftrightarrow \int e^{-\lambda x}\cos(tx)\,dx = \frac{e^{-\lambda x}}{\left(1+\frac{\lambda^2}{t^2}\right)}\left[\frac{1}{t}\sin(tx) - \frac{\lambda}{t^2}\cos(tx)\right]$$

as $x \to \infty$, the term on the right w/ sin/cos is bounded above by $\frac{1}{|t|} + \frac{\lambda}{t^2}$ by triangle inequality and $e^{-\lambda x} \to 0$. when $x = 0$, we get $-\frac{\lambda}{t^2} \cdot \frac{1}{\left(1+\frac{\lambda^2}{t^2}\right)}$

So $1 = 0 - \left(-\frac{\lambda}{t^2+\lambda^2}\right) = \frac{\lambda}{\lambda^2+t^2}$

For ②, we again use integration by parts twice

$$\int e^{-\lambda t} \sin(tx)\, dx = e^{-\lambda x}\left(-\frac{\cos(tx)}{t} + \frac{\lambda \sin(tx)}{t^2}\right) - \frac{\lambda^2}{t^2}\int \sin(tx)\, e^{-\lambda x}\, dx$$

$$\leftrightarrow \int e^{-\lambda t} \sin(tx)\, dx = \frac{e^{-\lambda x}}{\left(1+\frac{\lambda^2}{t^2}\right)}\left(\frac{\lambda \sin(tx)}{t^2} - \frac{\cos(tx)}{t}\right)$$

As $x \to \infty$, the right term is bounded above by $\frac{\lambda}{t^2} + \frac{1}{|t|}$ and $e^{-\lambda x} \to 0$, so the limit is 0,

when $x = 0$, we get $\frac{1}{\left(1+\frac{\lambda^2}{t^2}\right)}\left(-\frac{1}{t}\right) = \frac{-1}{t+\frac{\lambda}{t}} = \frac{-t}{t^2+\lambda}$

So $② = 0 - -\frac{t}{\lambda+t^2} = \frac{t}{\lambda+t^2}$

So,

$$① + i② = \frac{\lambda}{\lambda^2+t^2} + \frac{it}{\lambda+t^2}$$

$$= \frac{\lambda}{(\lambda-it)(\lambda+it)} + \frac{it}{\lambda+t^2}$$

$$= \frac{\lambda(\lambda+t^2) + it(\lambda-it)(\lambda+it)}{(\lambda+t^2)(\lambda-it)(\lambda+it)}$$

$$= \frac{\lambda^2 + \lambda t^2 + it(\lambda^2 - i\cancel{\lambda t} + i\cancel{\lambda t} + t^2)}{(\lambda+t^2)(\lambda-it)(\lambda+it)}$$

$$= \frac{\lambda^2 + \lambda t^2 + it\lambda^2 + it^3}{(\lambda+t^2)(\lambda-it)(\lambda+it)} = \frac{(\lambda+\cancel{it})(\cancel{\lambda+t^2})}{(\cancel{\lambda+t^2})(\lambda-it)(\lambda+\cancel{it})}$$

$$= \frac{1}{\lambda - it}$$

multiplying in the $\lambda$ out front we get the desired result, $E[e^{itX}] = \dfrac{\lambda}{\lambda - it}$.

b) $E[e^{itx}] = \int_\Omega e^{itx} \, dP$

$$= \int_{-\infty}^{\infty} e^{itx} \cdot \frac{1}{2} e^{-|x|} \, dx$$

splitting the absolute value

$$= \frac{1}{\cancel{2}} \cdot \cancel{2} \cdot \int_0^\infty e^{itx} \cdot e^{-x} \, dx$$

(-5) Not symmetric

However we computed this in (a) with $\lambda = 1$, so we get

$$E[e^{itx}] = \frac{1}{1 - it}$$

4) we want to find $P(N=n)$. So there are $n$ coin flips, and in the first $n-1$ flips, there are $\binom{n-1}{k-1}$ ways to flip $k-1$ heads.

The probability of getting $k-1$ heads is $p^{k-1}$ and the probability of getting $(n-1)-(k-1)$ tails is $(1-p)^{n-k}$. To make $k$ heads, the probability of getting heads on the last coin flip is $p$.

So putting everything together (w/ $q=1-p$)

$$P(N=n) = \binom{n-1}{k-1} p^k q^{n-k}$$

The generating function is

$$G_N(s) = \sum_{n=k}^{\infty} \binom{n-1}{k-1} p^k q^{n-k} \cdot s^n$$

since there need to be at least $k$ flips to get $k$ heads. The above can be written as

$$\sum_{j=0}^{\infty} \binom{j+k-1}{k-1} p^k q^j s^{j+k}$$

$$= p^k s^k \sum_{j=0}^{\infty} \binom{j+k-1}{k-1} (sq)^j \cdot 1^{k-1}$$

generalized binomial theorem

$$= p^k s^k (1-sq)^{-k}$$

$$= \left(\frac{ps}{1-sq}\right)^k$$

Let $s = e^{itN}$, then

$$\phi_N(t) = G_N(e^{it}) = \left(\frac{pe^{it}}{1-qe^{it}}\right)^K$$

we'd like to use the continuity theorem. So put $X = 2Np$, then

$$\phi_X(t) = E[e^{itX}] = E[e^{it2Np}] = \phi_N(2pt)$$

$$= \left(\frac{pe^{i2pt}}{1-qe^{i2pt}}\right)^K$$

By a (the same) similar argument as example 6 in the (Lecture 16) notes, we can replace $p$ with $\frac{1}{n}$ and see what happens as $n \to \infty$ (which defines a sequence of characteristic functions which we hope converges to the characteristic function of a $\Gamma$ distribution)

$$\to \left(\frac{\frac{1}{n}e^{i\frac{2t}{n}}}{1-(1-\frac{1}{n})e^{i\frac{2t}{n}}}\right)^K$$

$$= \left(\frac{e^{\frac{i2t}{n}}}{n-(n-1)e^{\frac{i2t}{n}}}\right)^K$$

$$\left(\frac{(1+\frac{i2t}{n}+\dots)}{n-(n-1)(1+\frac{2it}{n},\dots)}\right)^K$$

this is the same as the lecture notes in example 6 with $\lambda = 1$ and with $2it$

And as $n \to \infty$

$$\to = \left(\frac{1}{1-2it}\right)^K$$

(Alpha)

Now from Wolfram, the characteristic function of the $\Gamma$ distribution is $(Y \sim \Gamma(\lambda, r))$

$$\phi_Y(t) = \frac{1}{(1 + i\lambda t)^K}$$

So as $p \to 0$

the r.v

$\phi_X(t)$ is that of a $\Gamma$ distribution $Y$ with $Y \sim \Gamma(-2, r)$ by the continuity theorem.