AMATH 502

Nate Whybra

Homework 1

2.2.3

We have

$$\dot{x} = x - x^3 = f(x)$$

which has fixed points whenever

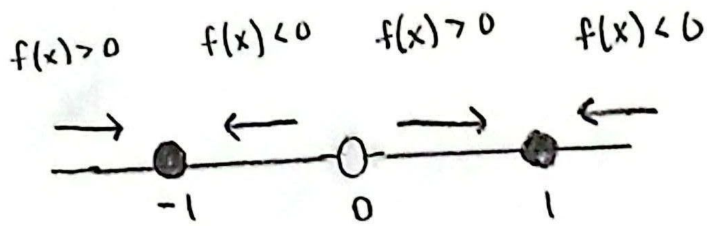
$$x - x^3 = 0$$

$$x(1 - x^2) = 0$$

$$x(1-x)(1+x) = 0$$

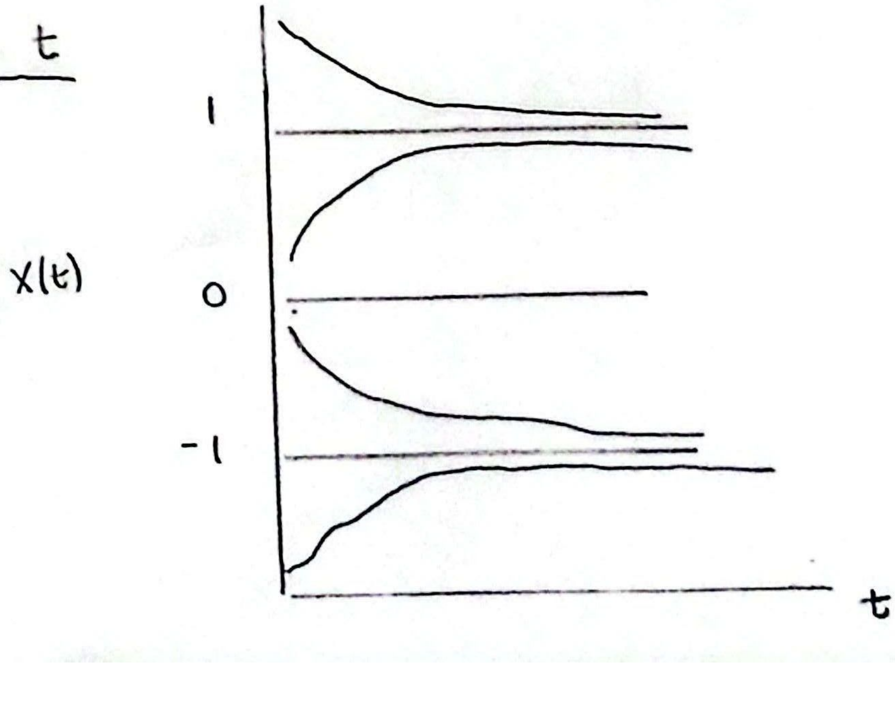
So the fixed points are $\{0, 1, -1\}$.

Phase Diagram



So $\{-1, 1\}$ are stable, and $\{0\}$ is unstable.

x(t) vs t

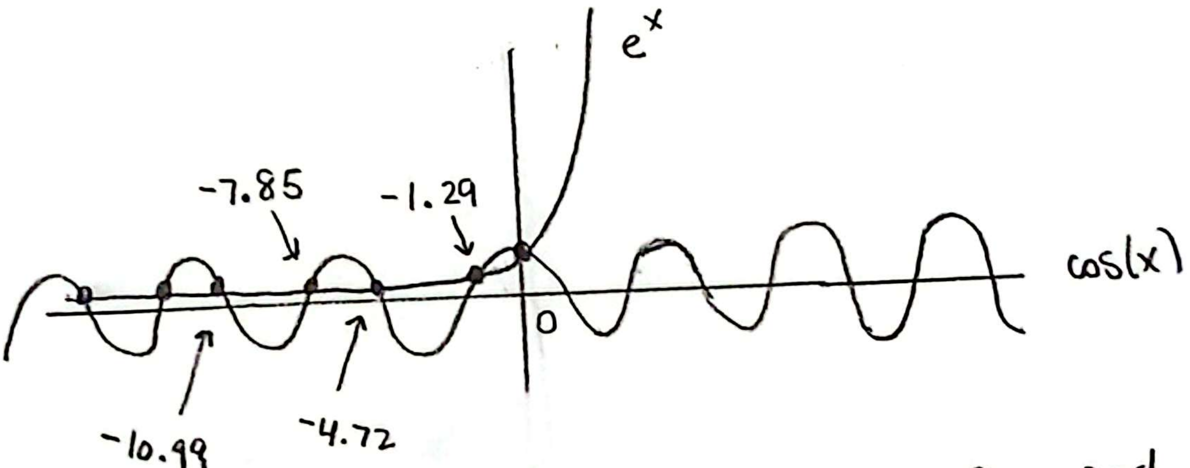


2.2.7

We have

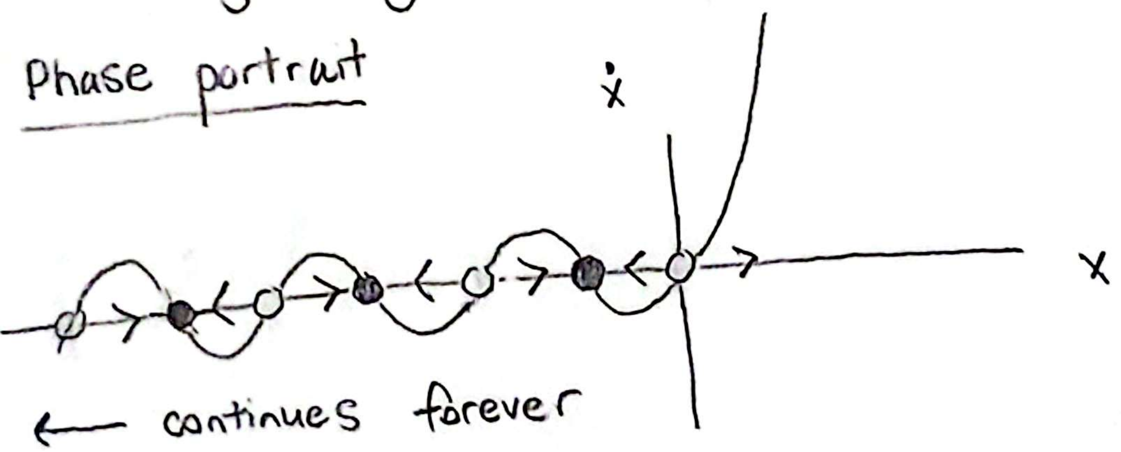
$$\dot{x} = e^x - \cos(x) = f(x)$$

which has fixed points when $e^x - \cos(x) = 0$



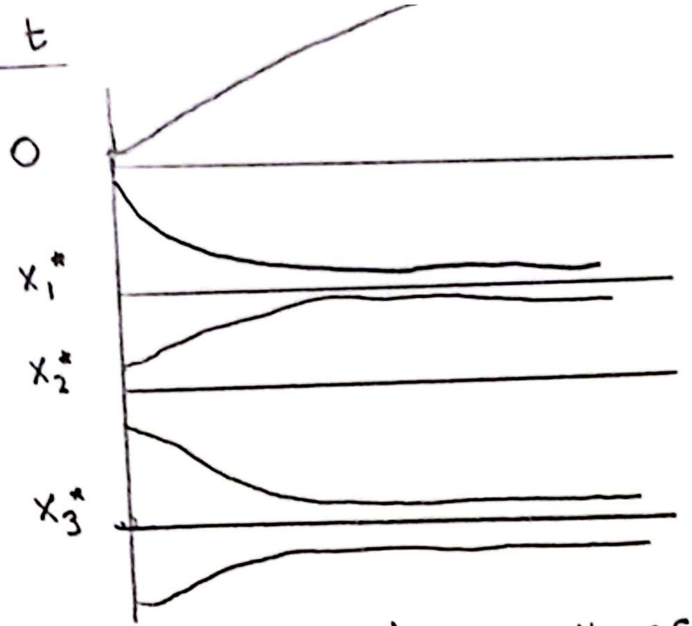
So there are fixed points when $x = 0$, and countably many points x that are negative.

Phase portrait



So $\{0\}$ is stable, the next is unstable, the next is stable, and that alternates for the remaining fixed points.

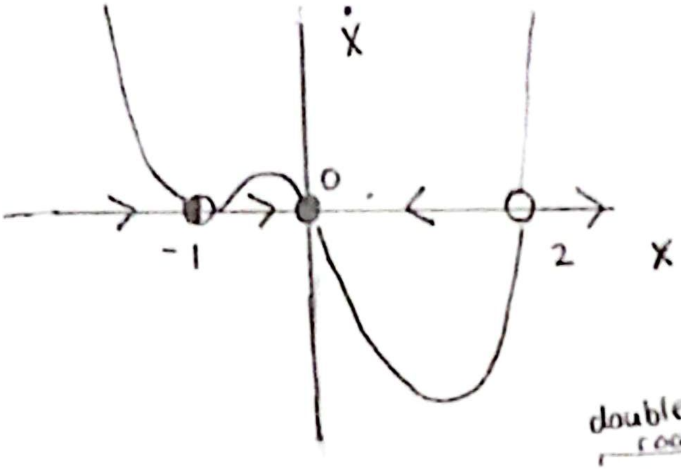
$x(t)$ vs t



⋮ ↓ continues

2.2.8

The phase portrait could look like this



double root at -1

single roots at 0, 2

which is satisfied by $f(x) = (x+1)^2 (x-2) x = \dot{x}$.

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2.2.10

a) $f(x) = 0$, is zero $\forall\ x \in \mathbb{R}$

b) $f(x) = \sin(\pi x)$, has zeroes when $x \in \mathbb{Z}$

c) This is not possible, as it would violate the mean-value theorem. To have 3 stable fixed points, we would need $f(x)$ to be negative before and positive after each fixed point, there could not be an odd number.

d) $f(x) = 5$, as $5 \neq 0\ \forall\ x \in \mathbb{R}$

e) $f(x) = \prod_{i=1}^{100} (x-i)$, has fixed points when $x = i$ for $1 \leq i \leq 100$.

2.2.13

a) We have,

$$m\frac{dv}{dt} = mg - Kv^2$$

Assuming $m \neq 0$,

$$\frac{dv}{dt} = g - \frac{K}{m}v^2$$

$$\int \frac{1}{g - \frac{K}{m}v^2}\, dv = \int dt$$

$$\frac{1}{g}\int \frac{1}{1 - \frac{K}{mg}v^2}\, dv = t + c \quad , \quad \text{put } \alpha = \frac{K}{mg}$$

$$\frac{1}{g}\int \frac{1}{1-\alpha v^2}\, dv = t + c$$

$u = \sqrt{\alpha}\, v \rightarrow du = \sqrt{\alpha}\, dv \rightarrow dv = \frac{du}{\sqrt{\alpha}}$, so

$$\rightarrow \frac{1}{g\sqrt{\alpha}}\int \frac{1}{1-u^2}\, du = t + c$$

$$\rightarrow \frac{1}{2g\sqrt{\alpha}} \ln\left|\frac{1+u}{1-u}\right| = t + c$$

$$\rightarrow \quad \frac{1}{2g\sqrt{\alpha}} \ln \left| \frac{1+\sqrt{\alpha}v}{1-\sqrt{\alpha}v} \right| = t + C$$

as $v(0) = 0 \rightarrow \quad C = \frac{1}{2g\sqrt{\alpha}} \ln|1| = 0$

So,

$$\ln \left| \frac{1+\sqrt{\alpha}v}{1-\sqrt{\alpha}v} \right| = 2g\sqrt{\alpha}\,t$$

$$\frac{1+\sqrt{\alpha}v}{1-\sqrt{\alpha}v} = e^{2g\sqrt{\alpha}t}$$

$$\downarrow$$

$$\frac{1+\sqrt{\alpha}v}{1-\sqrt{\alpha}v} = e^{2g\sqrt{\alpha}t}$$

$$1+\sqrt{\alpha}v = e^{2g\sqrt{\alpha}t} - \sqrt{\alpha}v e^{2g\sqrt{\alpha}t}$$

$$1 - e^{2g\sqrt{\alpha}t} = -\sqrt{\alpha}v\left(1+e^{2g\sqrt{\alpha}t}\right)$$

$$\rightarrow \quad \boxed{v = \frac{1}{\sqrt{\alpha}} \frac{e^{2g\sqrt{\alpha}t} - 1}{e^{2g\sqrt{\alpha}t} + 1}}$$

b) As $t \to \infty$, $v(t) \to \frac{1}{\sqrt{\alpha}} = \frac{1}{\sqrt{\frac{K}{mg}}} = \sqrt{\frac{mg}{K}}$

c) Looking at the formula as a D.S

$$\dot{v} = g - \frac{K}{m} v^2 := 0$$

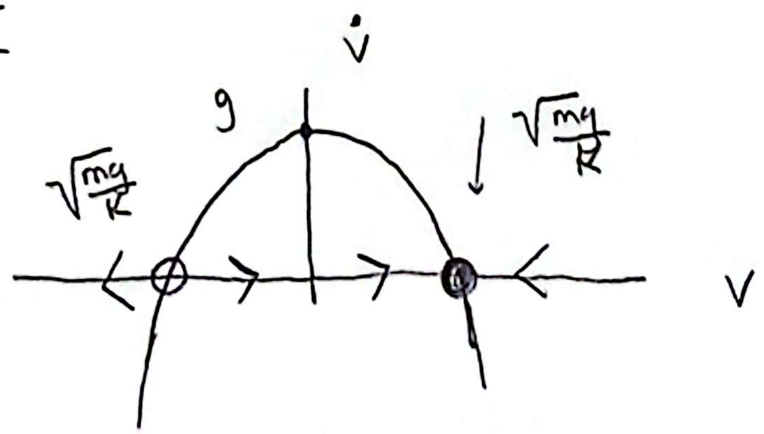
$$\to \quad g = \frac{K}{m} v^2$$

$$\frac{mg}{K} = v^2$$

$$v = \pm \sqrt{\frac{mg}{K}}$$

we see there are fixed points when $v = \pm \sqrt{\frac{mg}{K}}$

Phase portrait



we see that $\sqrt{\frac{mg}{K}}$ is the only stable fixed point. So $V_{terminal} = \sqrt{\frac{mg}{K}}$.

d)

$$V_{avg} = \frac{(31{,}400 - 2{,}100)\ ft}{116\ sec} = \boxed{252.6\ ft/sec}$$

2.3.2)

a) we have

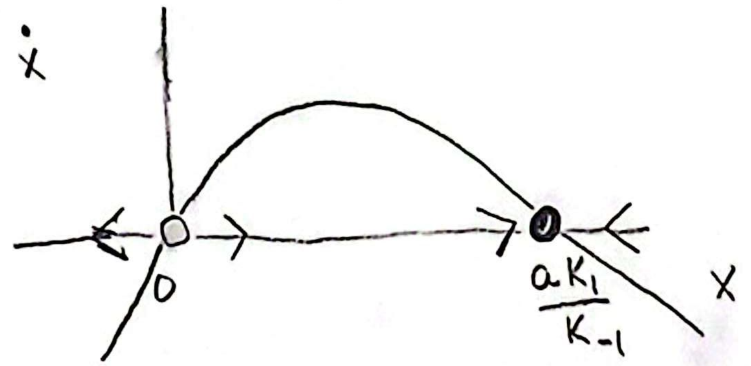
$$\dot{x} = K_1 a x - K_{-1} x^2 := 0$$

$$\rightarrow \quad x(K_1 a - K_{-1} x) = 0$$

$$\rightarrow \quad x = 0, \quad x = aK_1 / K_{-1} > 0$$

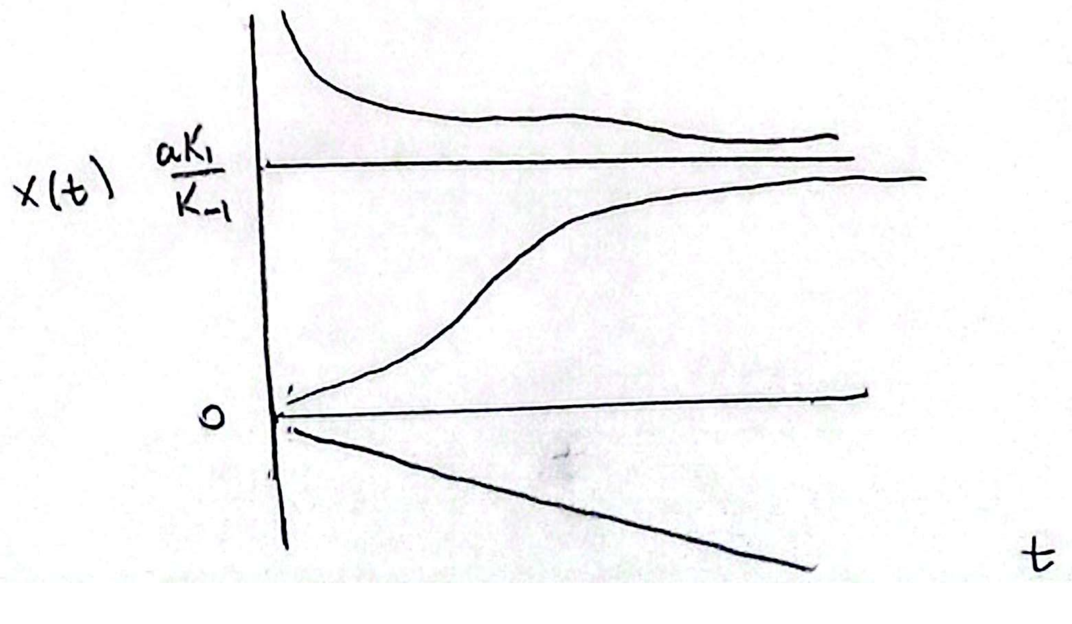
are fixed points

Phase diagram



So $\{0\}$ is unstable and $\{\frac{aK_1}{K_{-1}}\}$ is stable,

b) x(t) vs. t



2.3.6

a) we have,

$$\dot{x} = s(1-x)x^a - (1-s)x(1-x)^a := 0$$

Firstly notice that $\boxed{x=0}$ and $\boxed{x=1}$ are fixed points. The third must satisfy,

$$s(1-x)x^a = (1-s)x(1-x)^a$$

$$\rightarrow \frac{x^{a-1}}{(1-x)^{a-1}} = \frac{1-s}{s}$$

$$\rightarrow \left(\frac{x}{1-x}\right)^{a-1} = \frac{1-s}{s}$$

$$\rightarrow \frac{x}{1-x} = \sqrt[a-1]{\frac{1-s}{s}} := C_a$$

$$\rightarrow x = C_a - C_a x$$

$\rightarrow x(1+C_a) = C_a \rightarrow$ $\boxed{x = \frac{C_a}{1+C_a}}$ is the third fixed point.

b) We have

$$f(x) = s(1-x)x^a - (1-s)x(1-x)^a$$

So by product rule,

$$f'(x) = s(1-x)ax^{a-1} - x^a - (1-s)\left[-xa(1-x)^{a-1} + (1-x)^a\right]$$

$$= sax^{a-1}(1-x) - x^a + (1-s)\left[xa(1-x)^{a-1} - (1-x)^a\right]$$

And

$f'(0) = s-1 < 0 \rightarrow$ meaning 0 is stable

$f'(1) = -1 < 0 \rightarrow$ meaning 1 is stable

c) As previously discussed in 2.2.10, we cannot have exactly 3 fixed points that are all stable (if our function is smooth), so as $\{0, 1\}$ are stable, it must be that $\{x^*\}$, $0 < x^* < 1$, is unstable.

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2.4.7

we have,

$$\dot{x} = ax - x^3 = f(x)$$

The fixed points must satisfy,

$$ax - x^3 = 0$$

$$\rightarrow \quad x(a - x^2) = 0$$

so, $x = \pm\sqrt{a}$ are fixed points

$\rightarrow$ $x = 0$ or

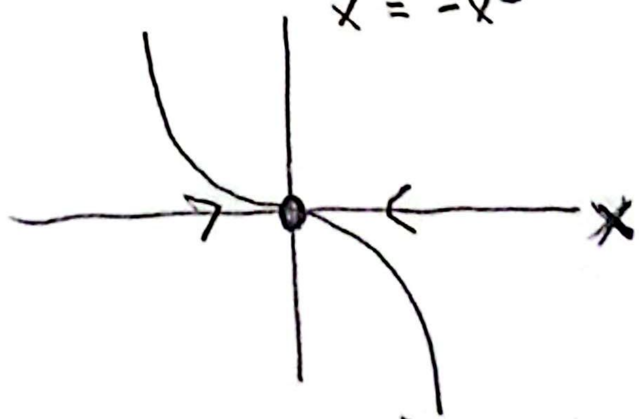
Notice that,

$$f'(x) = a - 3x^2$$

So $f'(0) = a$, $f'(\pm\sqrt{a}) = -2a$

when $a > 0$, $\{0\}$ is unstable ($f'(0) > 0$) and $\{\pm\sqrt{a}\}$ are stable ($f'(\pm\sqrt{a}) < 0$). When $a = 0$, $f'(x^*) = 0 \ \forall x^*$, so we must use traditional methods

$$\dot{x} = -x^3$$



ie, in this scenario $x^* = 0$ is the only fixed point and it is stable.

$f'(0) < 0$

When $a < 0$, $\{0\}$ is stable and $\{\pm\sqrt{a}\}$ are unstable $(f'(\pm\sqrt{a}) > 0)$.