Homework #7

1. A powerful tool for numerically finding the roots of an equation $g(x) = 0$ is *Newton's method*. Newton's method says to construct a map $x_{n+1} = f(x_n)$, where

$$f(x_n) = x_n - \frac{g(x_n)}{g'(x_n)}.$$

   (a) A simple root of the function $g(x)$ is defined as a value x for which $g(x) = 0$ and $g'(x) \neq 0$. Show that the simple roots of $g(x)$ are fixed points of the Newton map.

   Note: if you are not used to writing proofs, you may fall into a common trap here. Our goal is to show that a simple root of $g(x)$ is a fixed point. That means you start with the assumption that you have a simple root of $g(x)$ and then work to show that it is a fixed point. Do not start with fixed points of the Newton map and show that they are simple roots, that is not the correct direction.

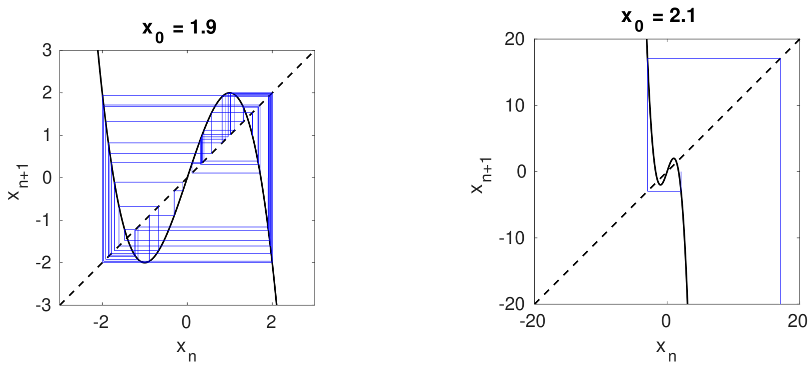
   (b) Show that these fixed points are *superstable*, which means that the linear stability analysis shows *zero* growth for perturbations ($f'(x^*) = 0$).

   Note: see note above. Start with fixed point, then show that they are superstable.

2. Consider the map $x_{n+1} = 3x_n - x_n^3$. This well-studied map is an example of a cubic map and is known to exhibit chaos.

   (a) Find all the fixed points and classify their stability.

   (b) In the figure 1, you are given the coweb diagrams for $x_0 = 1.9$ and $x_0 = 2.1$.



Show analytically that if $|x| \leq 2$, then $|f(x)| \leq 2$, where $f(x) = 3x - x^3$. Then show that if $|x| > 2$, $|f(x)| > |x|$. **Hint: recall how you found maxima and minima in calculus.** Use this to explain the behavior in cobweb diagrams for $x_0 = 1.9$ and $x_0 = 2.1$.

(c) Show that $(2, -2)$ (repeating) is a 2 cycle. This 2 cycle is analogous to a boundary that we defined when we were doing phase-plane analysis. What would you call this 2-cycle? (Not a limit cycle or a periodic orbit).

3. Consider a 1D ODE

$$\dot{x} = f(x), \qquad x \in \mathbb{R}. \tag{1}$$

The most basic method for solving this ODE numerically is to use the Forward Euler method,

$$x_{n+1} = x_n + hf(x_n), \tag{2}$$

where $h > 0$ is a chosen step size. This method comes from discretizing the derivative, as discussed in class.

(a) Show that fixed points of the ODE (1) correspond to fixed points of the Forward Euler map (2).

(b) Show that the stability of fixed points of the ODE (1) does not necessarily agree with the stability of the fixed points of the the Forward Euler map (2).

(c) Give a condition which guarantees stability of fixed points of the Forward Euler map (2). Comment on this condition: how must we generally choose the step size h in order to find equilibrium solutions of the ODE (1) using the Forward Euler method?

(d) It is common to see the Forward Euler solution oscillating about the true solution when solving numerically. Give a condition involving $f'(x)$ and h for which the numerical solution oscillates about a fixed point of the ODE (1) (hint: when did we have oscillations for the linear discrete-time dynamical systems?). Given this condition, why is it common to see oscillations in the Forward-Euler solution (hint: see above problem)?

(e) Consider a linear ODE,

$$\dot{x} = kx, \;\; k \in \mathbb{R}. \tag{3}$$

Give a condition on h and k for which 2-cycles (the non-fixed point 2 cycles) exist for the Forward-Euler map when solving this ODE. Show that these 2 cycles are neutrally stable. Comment on your results (in particular, when h and k match your condition, what happens to the numerical solution for any initial condition you use?).