AMATH 561 Autumn 2024
Problem Set 4

Due: Wed 10/30 at 10am

Note: Submit electronically to Canvas.

1. Let $\Omega = \{a, b, c, d\}$ and let $\mathcal{F} = 2^\Omega$. We define a probability measure P as follows:

$$P(a) = 1/6, \ \ , P(b) = 1/3, \ \ P(c) = 1/4, \ \ P(d) = 1/4.$$

Next, define three random variables:

$$X(a) = 1, \ \ X(b) = 1, \ \ X(c) = -1, \ \ X(d) = -1,$$
$$Y(a) = 1, \ \ Y(b) = -1, \ \ Y(c) = 1, \ \ Y(d) = -1,$$

and $Z = X + Y$. (a) List the sets in $\sigma(X)$. (b) Calculate $E(Y|X)$. (c) Calculate $E(Z|X)$.

2. (a) Prove that $E(E(X|\mathcal{F})) = EX$.
(b) Show that if $\mathcal{G} \subset \mathcal{F}$ and $EX^2 < \infty$ then

$$E(\{X - E(X|\mathcal{F})\}^2) + E(\{E(X|\mathcal{F}) - E(X|\mathcal{G})\}^2) = E(\{X - E(X|\mathcal{G})\}^2)$$

3. An important special case of the previous result (2b) occurs when $\mathcal{G} = \{\emptyset, \Omega\}$. Let $\text{var}(X|\mathcal{F}) = E(X^2|\mathcal{F}) - E(X|\mathcal{F})^2$. Show that

$$\text{var}(X) = E(\text{var}(X|\mathcal{F})) + \text{var}(E(X|\mathcal{F})).$$

4. Let $Y_1, Y_2, ...$ be i.i.d. (independent and identically distributed) random variables with mean μ and variance σ^2, N an independent positive integer valued random variable with $EN^2 < \infty$ and $X = Y_1 + ... + Y_N$. Show that $\text{var}(X) = \sigma^2 EN + \mu^2 \text{var}(N)$. (To understand and help remember the formula, think about the two special cases in which N or Y is constant.)