1) a) We have

$$g(w) = \frac{1}{2\pi i} \oint_C \frac{t f'(t)}{f(t) - w} \, dt$$

Put $w = f(z)$, with z inside C

$$g(f(z)) = \frac{1}{2\pi i} \oint_C \frac{t f'(t)}{f(t) - f(z)} \, dt = I$$

As f is a bijection, $f(t) = f(z)$ precisely when $t = z$. Furthermore, as $f(z)$ is analytic, so the integrand above has a simple pole when $t = z$, and we can use the residue theorem to compute I, so is $z f'(z)$,

$$I = \frac{1}{2\pi i} \cdot 2\pi i \cdot \frac{z f'(z)}{f'(z)}$$

$$= z$$

Now consider $f(g(z))$, with z inside the codomain of f

$$= f\left[\frac{1}{2\pi i} \oint_C \frac{t f'(t)}{f(t) - z} \, dt \right]$$

Then as f is a bijection, there exists some t_0 in the domain of f such that $z = f(t_0)$, so the above

$$= f\left(\frac{1}{2\pi i} \oint_C \frac{t\, f'(t)}{f(t) - f(t_0)}\, dt \right)$$

where we have already computed this integral, it equals t_0, so the above is $f(t_0)$, which we defined to be z. We have both $g(f(z)) = z$ and $f(g(z)) = z$, so g is the inverse of f.

b) To compute the Taylor series for $g(w)$, we compute $g^n(0)$, firstly consider the function

$$h(w) = \frac{a}{b-w} = a(b-w)^{-1}$$

$$\frac{dh}{dw} = -a(b-w)^{-2}(-1) = a(b-w)^{-2}$$

$$\frac{d^2h}{dw^2} = -2a(b-w)^{-3}(-1) = 2a(b-w)^{-3}$$

$$\vdots$$

$$\frac{d^nh}{dw^n} = n!\, a(b-w)^{-n-1}$$

So $\dfrac{d^nh}{dw^n}(0) = n! \dfrac{a}{b^{n+1}}$

Now consider

$$\frac{d^n g}{dw^n} = \frac{1}{2\pi i} \frac{d^n g}{dw^n} \oint_C \frac{t f'(t)}{f(t) - w} \, dt$$

$$= \frac{1}{2\pi i} \oint_C \frac{d^n g}{dw^n} \frac{t f'(t)}{f(t) - w} \, dt$$

(as our integrand is/has continuous derivatives on the contour) Leibniz rule)

$$= \frac{1}{2\pi i} \oint_C \frac{n! \, t f'(t)}{(f(t) - w)^{n+1}} \, dt$$

Evaluating at $w = 0$. Take C to be any closed contour enclosing the origin

$$= \frac{1}{2\pi i} \oint_C \frac{n! \, t f'(t)}{(f(t))^{n+1}} \, dt$$

$$= \frac{n!}{2\pi i} \oint_C \frac{t^2 (t+1) e^t}{t^{n+1} e^{t(n+1)}} \, dt$$

$$= \frac{n!}{2\pi i} \oint_C \frac{(t+1) e^{-nt}}{t^{n-1}} \, dt$$

$$= \frac{n!}{2\pi i} \oint_C \frac{t e^{-nt}}{t^{n-1}} \, dt + \oint_C \frac{e^{-nt}}{t^{n-1}} \, dt$$

$$= \frac{n!}{2\pi i} \left[\oint_C \frac{e^{-nt}}{t^{n-2}} \, dt + \oint_C \frac{e^{-nt}}{t^{n-1}} \, dt \right]$$

$$= \frac{n!\, 2\pi i}{2\pi i} \left[\frac{1}{(n-3)!} \lim_{t\to 0} \frac{d^{n-3}}{dt^{n-3}} e^{-(n)t} + \frac{1}{(n-2)!} \lim_{t\to 0} \frac{d^{n-2}}{dt^{n-2}} e^{-nt} \right]$$

$$= n! \left[\frac{1}{(n-3)!} \lim_{t\to 0} (-1)^{n-3} (n)^{n-3} e^{-(nt)} + \frac{1}{(n-2)!} \lim_{t\to 0} (-1)^{n-2} (n)^{n-2} e^{-(nt)} \right]$$

$$= n! \left[\frac{(-1)^{n-3}}{(n-3)!} n^{n-3} + \frac{(-1)^{n-2}\, n^{n-2}}{(n-2)!} \right]$$

$$= \frac{n!}{(n-3)!} n^{n-3} (-1)^{n-3} \left[1 - \frac{n}{(n-1)} \right]$$

$$= \frac{n!}{(n-3)!} n^{n-2} (-1)^{n-2} \left[-\frac{1}{(n-2)} \right]$$

$$= \frac{n!}{(n-2)!} n^{n-2} (-1)^{n-1}$$

So

$$g(w) = \sum_{n=1}^{\infty} \frac{(-1)^{n-1}\, n^{n-1}}{n!} w^n$$

To find the ROC, we use ratio test

$$\left| \frac{\frac{(-1)^n (n+1)^n}{(n+1)!}}{\frac{(-1)^{n-1} n^{n-1}}{n!}} w \right| = \left| \frac{(n+1)^n \, n!}{n^{n-1}(n+1)!} w \right|$$

$$= \left| \frac{(n+1)^n}{n^{n-1}(n+1)} \right| |w| < 1$$

$$= \left| \left(\frac{n+1}{n} \right)^{n-1} \right| |w| < 1$$

$$= \left(1 + \frac{1}{n} \right)^{n-1} |w| < 1$$

$$|w| < \left(1 + \frac{1}{n} \right)^{n-1} \quad \forall n$$

as $n \to \infty$

$$|w| < e$$

So instead of ROC being all of $\mathbb{R}$, we only get for when $|w| < e$

2) we have

$$\frac{1}{2\pi i}\oint_{\Gamma} (zI - A)^{-1} \vec{u}\, dz$$

From AMATH 584, $(zI - A)^{-1}$ has the same eigenvectors as A with eigenvalue $\frac{1}{z-\lambda}$.

So the above, (by definition of eigenvector)

$$= \frac{1}{2\pi i}\oint_{\Gamma} \frac{1}{z-\lambda} \vec{u}\, dz$$

scalar

$$= \left[\frac{1}{2\pi i}\oint_{\Gamma} \frac{1}{z-\lambda}\, dz\right] \vec{u}$$

residue theorem as λ is contained in C

$$= \left(\frac{1}{2\pi i}\, 2\pi i \cdot \lim_{z \to \lambda} 1\right) \vec{u}$$

$$= \vec{u}$$

3)

$$\| e^{-ct} e^{At} \|$$

$$= \left\| e^{-ct} \sum_{k=0}^{\infty} \frac{A^k t^k}{n!} \right\|$$

$$= \left\| \sum_{k=0}^{\infty} \frac{A^k t^k e^{-ct}}{k!} \right\|$$

always > 0

$$\leq \sum_{k=0}^{\infty} \frac{|t|^k e^{-ct} \|A\|^k}{k!}$$

$$= e^{-ct} e^{t\|A\|}$$ (dropping $|t|$ b/c as $t \to \infty$, $t > 0$)

$$= e^{-t(c - \|A\|)}$$

However, from class, there exists a constant S such that (with $S \geq 1$) (and $\rho = \max_{1 \leq i \leq n} |\lambda_i|$)

$$\rho \leq \|A\| \leq S \cdot \rho$$

So as $c > \rho$

$$c - \|A\| > \rho - \|A\| \geq 0$$

So $-t(c - \|A\|) < 0$ (strictly negative)

and as $t \to \infty$, $e^{-t(c - \|A\|)} \to 0$ as desired.

4)

a) $$L[y'](z) = \int_0^\infty y'(t)\, e^{-tz}\, dt = I$$

w/ integration by parts

$$u = e^{-tz}, \quad dv = y'(t)\, dt$$

$$du = e^{-tz}(-z)\,dt, \quad v = y(t)$$

$$I = \int_0^\infty u\,dv = uv\Big|_0^\infty - \int_0^\infty v\,du$$

$$= e^{-tz}\, y(t)\Big|_0^\infty + z\int_0^\infty y(t)\, e^{-tz}\, dt$$

$$= [0 - y(0)] + z\int_0^\infty y(t) e^{-tz}\, dz$$

$$= z\,L[y](z) - y(0), \text{ as desired.}$$

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b) we have, on one hand, as L is applied component-wise, from (a)

$$L[y'] = z\,L[y] - y[0] \quad 1$$

on the other hand, by just plugging into the formula

$$L[y'] = \int_0^\infty (Ay + b)\, e^{-tz}\, dt$$

$$= \int_0^\infty A y\, e^{-tz}\, dt + \int_0^\infty b\, e^{-tz}\, dt$$

$$= \int_0^\infty \sum_{j=1}^{n} \overset{\text{column of } A}{a_j}\, \overset{\text{component of } y}{y_j}\, e^{-tz}\, dt + L(b)$$

$$= \sum_{j=1}^{n} a_j \int_0^\infty y_j\, e^{-tz}\, dz + L(b)$$

$$= \sum_{j=1}^{n} a_j\, L(y_j) + L(b)$$

$$= A\,L(y) + L(b) \quad 2$$

So ① - ②

$$= (zI - A) L(y) - (y[0] + L(b)) = 0$$

$$= (zI - A) L(y) = (y[0] + L(b))$$

and if $z \neq \lambda$, $zI - A$ has an inverse as A has an inverse, so

$$L(y) = (zI - A)^{-1} (y[0] + L(b))$$

as desired.

c) From Bernard's notes, we can apply the inverse Laplace transform on both sides of (b)

$$\rightarrow L^{-1}(L(y)) = L^{-1}\left((zI - A)^{-1} (y[0] + L[b]) \right)$$

$$y = \frac{1}{2\pi i} \int_{c-i\infty}^{c+i\infty} e^{tz} (zI - A)^{-1} (y[0] + L[b](z)) \, dz$$

where we choose $c > \max_{1 \le i \le n} |\lambda_i| = \rho$ (eigs of A) to ensure we are away from the singularities of the integrand.

5) Put $h(x) = \dfrac{\log(p(x))}{\sqrt{1-x^2}}$, then

$$h'(x) = \frac{\dfrac{p'(x)}{p(x)}\sqrt{1-x^2} + \dfrac{x\log(p(x))}{\sqrt{1-x^2}}}{1-x^2}$$

So on $(-1,1)$, as $p(x) > 0$, the derivative of h exists and is continuous, so h is continuously differentiable on $(-1,1)$, now consider (principal branch)

$$\log(G(z)) = \frac{-\omega(z)}{2\pi} \int_{-1}^{1} \frac{h(x)}{x-z}\, dx$$

$$= -i\,\omega(z) \cdot \underbrace{\left[\frac{1}{2\pi i} \int_{-1}^{1} \frac{h(x)}{x-z}\, dx \right]}_{\Phi(z)}$$

as h is cont. diff.

By Plemelj, with $L = (-1,1)$

$$\Phi^{\pm}(t) = \pm\frac{1}{2} h(t) + \overbrace{\frac{1}{2\pi i} \fint_L \frac{h(x)}{x-t}\, dx}^{I}, \quad t \text{ on } L$$

So $\log(G_+(t)) + \log(G_-(t))$ (pulling the limit from I)

$(\star)$

$$\lim_{\varepsilon \to 0} -i\omega(t+i\varepsilon)\left[\frac{1}{2}h(t) + I_\varepsilon\right] - i\omega(t-i\varepsilon)\left[-\frac{1}{2}h(t) + I_\varepsilon\right]$$

In the uhp, $\omega(z) = i\sqrt{1-z}\sqrt{1+z}$

In the lhp, $\omega(z) = -i\sqrt{1-z}\sqrt{1+z}$

In the limit $I_\varepsilon \to I$, we never cross the branch cut of $w(z)$, so we have $w(t+i\varepsilon)$ and $w(t-i\varepsilon)$

$\to i\overbrace{\sqrt{1-t}\sqrt{1+t}}^{\sqrt{1-t^2}}$ and $-i\overbrace{\sqrt{1-t}\sqrt{1+t}}^{\sqrt{1-t^2}}$

So as $\varepsilon \to 0$ we get

$$(\star) = \overbrace{\sqrt{1-t^2}}^{s(t)}\left(\frac{1}{2}\frac{\log(p(t))}{\sqrt{1-t^2}} + I\right) - \sqrt{1-t^2}\left(-\frac{1}{2}\frac{\log(p(t))}{\sqrt{1-t^2}} + I\right)$$

$$= \frac{1}{2}\log(p(t)) + \cancel{s(t)}\, I + \frac{1}{2}\log(p(t)) - \cancel{s(t)}\, I$$

$$= \log(p(t)) = (1)$$

Taking the exponential on both sides,

$$\log(G_+(t)) + \log(G_-(t)) = \log(p(t))$$

$$e^{\log(G_+(t))}\, e^{\log(G_-(t))} = p(t)$$

$$= G_+(t)\, G_-(t) = p(t), \quad t \in (-1,1)$$

as desired.