1)

a) we have $x_0 = 1$, $x_1 = \frac{3}{2}$, $x_2 = 0$, $x_3 = 2$, and $y_0 = 2$, $y_1 = 6$, $y_2 = 0$, $y_3 = 14$. Firstly

- $$w_0 = \frac{1}{(x_0 - x_1)} \cdot \frac{1}{(x_0 - x_2)} \cdot \frac{1}{(x_0 - x_3)}$$

$$= \frac{1}{1 - \frac{3}{2}} \cdot \frac{1}{1 - 0} \cdot \frac{1}{1 - 2}$$

$$= -2 \cdot 1 \cdot -1 = 2$$

- $$w_1 = \frac{1}{(x_1 - x_0)} \cdot \frac{1}{(x_1 - x_2)} \cdot \frac{1}{(x_1 - x_3)}$$

$$= \frac{1}{\frac{3}{2} - 1} \cdot \frac{1}{\frac{3}{2} - 0} \cdot \frac{1}{\frac{3}{2} - 2}$$

$$= 2 \cdot \frac{2}{3} \cdot -2 = -8/3$$

- $$w_2 = \frac{1}{(x_2 - x_0)} \cdot \frac{1}{(x_2 - x_1)} \cdot \frac{1}{(x_2 - x_3)}$$

$$= \frac{1}{0 - 1} \cdot \frac{1}{0 - \frac{3}{2}} \cdot \frac{1}{0 - 2}$$

$$= -1 \cdot \cancel{-}\frac{2}{3} \cdot \frac{1}{\cancel{-}2} = -1/3$$

$$\bullet\; w_3 = \frac{1}{(x_3-x_0)} \cdot \frac{1}{(x_3-x_1)} \cdot \frac{1}{(x_3-x_2)}$$

$$= \frac{\cancel{1}^{\,1}}{2\cancel{-1}} \cdot \frac{1}{2-\frac{3}{2}} \cdot \frac{1}{2-0}$$

$$= 1 \cdot 2 \cdot \frac{1}{2} = 1$$

So we have

$$p(x) = \frac{\displaystyle\sum_{i=0}^{3} y_i \cdot \frac{w_i}{x-x_i}}{\displaystyle\sum_{i=0}^{3} \frac{w_i}{x-x_i}}, \quad \vec{w} = \begin{bmatrix} 2 \\ -8/3 \\ -1/3 \\ 1 \end{bmatrix}, \quad \vec{x} = \begin{bmatrix} 1 \\ 3/2 \\ 0 \\ 2 \end{bmatrix}, \quad \vec{y} = \begin{bmatrix} 2 \\ 6 \\ 0 \\ 14 \end{bmatrix}$$

$$= \frac{\frac{2 \cdot 2}{x-1} + \frac{6 \cdot -8/3}{x-3/2} + \frac{0 \cdot \cancel{-1/3}^{\,0}}{x\cancel{-0}} + \frac{14 \cdot 1}{x-2}}{\frac{2}{x-1} - \frac{8/3}{x-\frac{3}{2}} - \frac{1/3}{x-0} + \frac{1}{x-2}}$$

$$= \frac{\frac{4}{x-1} - \frac{16}{x-3/2} + \frac{14}{x-2}}{\frac{2}{x-1} - \frac{8}{3(x-3/2)} - \frac{1}{3x} + \frac{1}{x-2}}$$

(I simplify this more in part (c). As currently written, the function is not defined at $x = x_0, x_1, x_2, x_3$.)

b) Divided difference table

x	f			
1	2			
$3/2$	6	$\frac{6-2}{3/2-1} = 8$		
0	0	$\frac{0-6}{0-3/2} = 4$	$\frac{4-8}{0-1} = 4$	
2	14	$\frac{14-0}{2-0} = 7$	$\frac{7-4}{2-\frac{3}{2}} = 6$	$\frac{6-4}{2-1} = 2$

So, using (2.111)

$$p(x) = 2 + 8(x-1) + 4(x-\tfrac{3}{2})(x-1) + 2x(x-1)(x-3/2)$$

c) Let $p_1(x)$ be the function from (a) and let $p_2(x)$ be the function from (b). By combining fractions, p_1 can be written like (multiplying by $\frac{u}{u}$ w/ $u = 3x(x-1)(x-2)(x-3/2)$)

$$p_1(x) = \frac{12x(x-2)(x-3/2) - 48x(x-1)(x-2) + 42x(x-1)(x-3/2)}{6x(x-2)(x-3/2) - 8x(x-1)(x-2) - (x-1)(x-2)(x-3/2) + 3x(x-1)(x-3/2)}$$

Evaluating at x_0, x_1, x_2, x_3

x_2

$$p_1(0) = \frac{0}{-(-1)(-2)(-3/2)} = 0$$

x_1

$$p_1(3/2) = \frac{-48\cancel{(3/2)}\cancel{(3/2-1)}\cancel{(3/2-2)}}{-8\cancel{(3/2)}\cancel{(3/2-1)}\cancel{(3/2-2)}} = 6$$

$$p_1(\overset{x_0}{1}) = \frac{12(\cancel{1})\cancel{(1-2)}\cancel{(1-3/2)}}{6(\cancel{1})\cancel{(1-2)}\cancel{(1-3/2)}} = 2$$

$$p_1(\overset{x_3}{2}) = \frac{42}{3}\,\frac{\cancel{2}\cancel{(2-1)}\cancel{(2-3/2)}}{\cancel{2}\cancel{(2-1)}\cancel{(2-3/2)}} = 14$$

So $p_1(\vec{x}) = \begin{pmatrix} 2 \\ 6 \\ 0 \\ 14 \end{pmatrix}$

Now, evaluating for p_2, we have

$$p_2(\overset{x_0}{1}) = \underline{2}$$

$$p_2(\overset{x_1}{3/2}) = 2 + 8(3/2 - 1) = \underline{6}$$

$$p_2(\overset{x_2}{0}) = 2 + 8(0-1) + 4\left(0 - \tfrac{3}{2}\right)(0-1)$$

$$= 2 - 8 + \overset{2}{\cancel{4}} \cdot \frac{3}{\cancel{2}} \cdot 1 = -6 + 6 = \underline{0}$$

$$p_2(\overset{x_3}{2}) = 2 + 8(\overset{1}{\cancel{2-1}}) + 4(\overset{1/2}{2-3/2})(\overset{1}{2-1}) + 2 \cdot 2(\overset{1}{2-1})(\overset{1/2}{2-3/2})$$

$$= 2 + 8 + 2 + 2 = \underline{14}$$

So $p_2(\vec{x}) = \begin{pmatrix} 2 \\ 6 \\ 0 \\ 14 \end{pmatrix}$

we know that both p_1 and p_2 are cubic polynomials, so since p_1 and p_2 agree on 4 distinct points, it must be that $p_1 = p_2$. So indeed, the polynomials from (a) and (b) are the same.

2) a) From page 67 in the textbook, all we must check is that both

$$f = \frac{1}{1+25x^2} \in C[-1,1] \quad \text{and} \quad \{x^j\}_{j \geq 0} \in C[-1,1].$$

(1) (2)

functions forming basis

(2) is true as each x^j is a polynomial.

For (1), we compute

$$f' = -(1+25x^2)^{-2} \cdot 50x = -\frac{50x}{1+25x^2}$$

which is continuous on $[-1,1]$. So yes,

$$p_n \xrightarrow[L_2]{n \to \infty} f.$$

b) we have,

$$f(x) = \frac{1}{1+25x^2} \rightarrow f'(x) = -\frac{50x}{(1+25x^2)^2}$$

$$\rightarrow f''(x) = \frac{-50(1-75x^2)}{(1+25x^2)^3}$$

f'' is continuous on $[-1,1]$, so $f \in C^2[-1,1]$.
So from (2.127), where $\Delta \to 0$ as $n \to \infty$

$$\| f - S_{1,n} \|_\infty \leq \frac{1}{8} |\Delta|^2 \| f'' \|_\infty$$

$$= \frac{1}{8} |\Delta|^2 \underset{x \in [-1,1]}{\overset{(\sup)}{\max}} |f''(x)|$$

finding max in mathematic

$$= \frac{1}{8} |\Delta|^2 (25/2)$$

$$= \frac{25}{16} |\Delta|^2$$

So as $n \to \infty$, as $\Delta \to 0$, $f_1 \to S_{1,n}$ in L_∞, which means $f_1 \to S_{1,n}$ uniformly.

3) We have,

$$S(x) = \begin{cases} ax^2 + b(x-1)^3 = f_1(x), & x \in (-\infty, 1] \\ cx^2 + d = f_2(x), & x \in [1, 2] \\ ex^2 + f(x-2)^3 = f_3(x), & x \in [2, \infty) \end{cases}$$

For $S(x)$ to be a cubic spline, we need

$$\textcircled{\star} \begin{cases} \left.\begin{aligned} f_1(1) &= f_2(1) \\ f_2(2) &= f_3(2) \end{aligned}\right] \text{ match function} \\ \left.\begin{aligned} f_1'(1) &= f_2'(1) \\ f_2'(2) &= f_3'(2) \end{aligned}\right] \text{ match derivative} \\ \left.\begin{aligned} f_1''(1) &= f_2''(1) \\ f_2''(2) &= f_3''(2) \end{aligned}\right] \text{ match 2nd derivative} \end{cases}$$

We have,

$f_1(x) = ax^2 + b(x-1)^3$

$f_2(x) = cx^2 + d$

$f_3(x) = ex^2 + f(x-2)^3$

$f_1'(x) = 2ax + 3b(x-1)^2$

$f_2'(x) = 2cx$

$f_3'(x) = 2ex + 3f(x-2)^2$

$f_1''(x) = 2a + 6b(x-1)$

$f_2''(x) = 2c$

$f_3''(x) = 2e + 6f(x-2)$

So, our conditions become

$$(*)\quad \begin{cases} a = c + d \\ 4c + d = 4e \\ 2a = 2c \rightarrow \underline{a = c} \\ 4c = 4e \rightarrow \underline{c = e} \\ 2a = 2c \rightarrow a = c \\ 2c = 2e \rightarrow c = e \end{cases}$$

which reduces to

$$\begin{cases} a = c = e := \alpha_1 \\ d = 0 \\ b := \alpha_2 \\ f := \alpha_3 \end{cases}$$

So, if $(\alpha_1, \alpha_2, \alpha_3) \in \mathbb{R}^3$, then

$$S(x) = \begin{cases} \alpha_1 x^2 + \alpha_2 (x-1)^3, & \text{if } x \in (-\infty, 1] \\ \alpha_1 x^2, & \text{if } x \in [1, 2] \\ \alpha_1 x^2 + \alpha_3 (x-2)^3, & \text{if } x \in [2, \infty) \end{cases}$$

is a valid cubic spline. We would need to define more boundary conditions to find a unique solution.

4) We want to define a function

$$S(x) = \begin{cases} a + bx + cx^2 + dx^3, & \text{if } x \in [0,1] \\ e + fx + gx^2 + hx^3, & \text{if } x \in [1,2] \end{cases}$$

$$= \begin{cases} f_1(x), & \text{if } x \in [0,1] \\ f_2(x), & \text{if } x \in [1,2] \end{cases}$$

such that

$$(*) \quad \begin{cases} \left.\begin{array}{l} f_1(1) = f_2(1) \\ f_1'(1) = f_2'(1) \\ f_1''(1) = f_2''(1) \end{array}\right] \text{continuity matching} \\ \left.\begin{array}{l} f_1(0) = f(0) = 0^4 = 0 \\ f_1(1) = f(1) = 1^4 = 1 \\ f_2(2) = f(2) = 2^4 = 16 \end{array}\right] \text{matching with } f(x) = x^4 \\ \left.\begin{array}{l} f_1''(0) = 0 \\ f_2''(2) = 0 \end{array}\right] \text{Natural spline conditions} \end{cases}$$

We have,

$f_1(x) = a + bx + cx^2 + dx^3$

$f_2(x) = e + fx + gx^2 + hx^3$

$f_1'(x) = b + 2cx + 3dx^2$

$f_2'(x) = f + 2gx + 3hx^2$

$f_1''(x) = 2c + 6dx$

$f_2''(x) = 2g + 6hx$

So, our conditions become

$$\circledast \begin{cases} a+b+c+d = e+f+g+h \\ b+2c+3d = f+2g+3h \\ 2c+6d = 2g+6h \\ \underline{a=0} \\ a+b+c+d=1 \\ e+2f+4g+8h = 16 \\ 2c = 0 \rightarrow \underline{c=0} \\ 2g+12h = 0 \rightarrow \underline{g=-6h} \end{cases}$$

Which can be further reduced to

$$\begin{cases} 1 = e+f-5h \\ b+3d = f-9h \\ 6d = -6h \rightarrow \underline{d=-h} \\ b+d=1 \rightarrow \underline{b=1-d = 1+h} \\ e+2f-16h = 16 \end{cases}$$

Reducing further to

$$\begin{cases} e = 1-f+5h \\ 1+h-3h = f-9h \rightarrow 1-2h = f-9h \rightarrow f = 1+7h \\ e+2f-16h = 16 \end{cases}$$

Reducing one more time

$$e = 1 - (1+7h) + 5h = -2h$$

$$e + 2f - 16h = -2h + 2 + 14h - 16h = 16$$

$$\Rightarrow -4h = 14 \rightarrow \underline{h = -7/2}$$

So, our coefficients are

$a = 0$

$b = 1 + h = \frac{2}{2} - \frac{7}{2} = \frac{-5}{2}$

$c = 0$

$d = -h = 7/2$

$e = 7$

$f = \frac{2}{2} - \frac{49}{2} = -47/2$

$g = -6h = -\frac{6 \cdot -7}{2} = 21$

$h = -7/2$

And,

$$s(x) = \begin{cases} -\frac{5}{2}x + \frac{7}{2}x^3 & \text{if } x \in [0,1] \\ 7 - \frac{47}{2}x + 21x^2 - \frac{7}{2}x^3 & \text{if } x \in [1,2] \end{cases}$$