3) a) we have,

$$f(z) = \frac{z}{e^z - 1} \longrightarrow \frac{1}{f(z)} = \frac{e^z - 1}{z}$$

$$= \frac{\sum_{n=0}^{\infty} \frac{z^n}{n!} - 1}{z}$$

$$= \frac{\sum_{n=1}^{\infty} \frac{z^n}{n!}}{z}$$

$$\overset{?}{=} \sum_{n=1}^{\infty} \frac{z^{n-1}}{n!}$$

$$= 1 + \frac{z}{2!} + \sum_{n=3}^{\infty} \frac{z^{n-1}}{n!}$$

at $z = 0 \rightarrow \frac{1}{f(0)} = 1$,

so we can define $f(0) = 1$ to remove the singularity at $z = 0$.

b) Since $f(z)$ has only one singularity, and it is removable, the Taylor series for $f(z)$ will have ∞ radius of convergence.

c)

$$\frac{z}{e^z - 1} = \sum_{n=0}^{\infty} \frac{B_n}{n!} z^n$$

$$z = (e^z - 1) \sum_{n=0}^{\infty} \frac{B_n}{n!} z^n$$

$$z = \sum_{m=1}^{\infty} \frac{z^m}{m!} \sum_{n=0}^{\infty} \frac{B_n}{n!} z^n$$

$$z = \sum_{m=0}^{\infty} \frac{z^{m+1}}{(m+1)!} \sum_{n=0}^{\infty} \frac{B_n}{n!} z^n \qquad \left.\begin{array}{c}\text{Cauchy} \\ \text{product} \\ \text{convolution} \\ \text{formula}\end{array}\right.$$

$$z = \sum_{n=0}^{\infty} \underbrace{\sum_{m=1}^{n} \frac{B_{n-m}}{(m)! \, (n-m)!}}_{// \ C_n} z^n$$

The LHS is always equal to z and the RHS is $\sum_{n=0}^{\infty} C_n z^n$, so $C_n = 0$ for $n \in \{0, 2, 3, 4, \dots\}$ and $C_n = 1$ for $n = 1$.

Put $n = 1$, then

$$1 = \frac{B_0}{1! \, 0!} = B_0$$

If $n > 1$

$$\sum_{m=1}^{n} \frac{B_{n-m}}{m!\,(n-m)!} = 0$$

$$\longleftrightarrow \sum_{m=1}^{n} \frac{1}{n!} \binom{n}{m} B_{n-m} = 0$$

pushing $1/n!$ into the other sum

$$\longleftrightarrow \sum_{m=1}^{n} \binom{n}{m} B_{n-m} = 0 \qquad \textcircled{*}$$

So $B_0 = 1$ and B_n for $n > 1$ can be chosen to satisfy the above recurrence relation.

d) ~~I wrote code to evaluate the recurrence we found in (c)...?~~

By repeatedly using the recurrence labeled $\textcircled{*}$, starting with $B_0 = 1$, we get

$B_1 = .1$ $B_8 = -1/30$

$B_2 = -1/2$ $B_9 = 0$

$B_3 = 0$ $B_{10} = 5/66$

$B_4 = -1/30$ $B_{11} = 0$

$B_5 = 0$ $B_{12} = -691/2730$

$B_6 = 1/42$

$B_7 = 0$

e) AS

$$f(z) = \sum_{n=0}^{\infty} \frac{B_n}{n!} z^n$$

For $B_{2n+1} = 0 \;\; \forall \, n \geq 1$ we need $f(z)$
to be an even function so that
the odd powers of z are removed
from the sum, but we must treat
the z^1 term seperately as $2n+1 \geq 3$

Notice

$$\sum_{\substack{n \neq 1 \\ n > 0}} \frac{B_n}{n!} z^n = \frac{z}{e^z} + \frac{z}{2}$$

$$= \frac{2z + z(e^z - 1)}{2(e^z - 1)}$$

$$= \frac{z + ze^z}{2 e^z - 1}$$

$$= \frac{z(e^z + 1)}{2(e^z - 1)}$$

Since this function is even, by our earlier discussion $B_{2n+1} = 0 \;\; \forall \, n \geq 1.$

The RHS is an even function since
plugging $z = -z$

$$\rightarrow \frac{-z(e^{-z} + 1)}{2(e^{-z} \cdot 1)} = \frac{e^z}{e^{-z}} \cdot \frac{-z(e^z + 1)}{-(e^z - 1)2} = \frac{z(e^z + 1)}{2(e^z - 1)}$$

4) b) From (3), we can write

$$g(z) = \sum_{n=0}^{\infty} \frac{1}{n!} B_n z^n$$

Take $G(z)$

$$= \sum_{n=0}^{12} \frac{1}{n!} B_n z^n$$

where $B_{0:12}$ are from problem 3,

To show $\left| G(3^{-n}) - g(3^{-n}) \right| < 10^{-10} \quad \forall n$,

$g(z)$ can be written as $\sum_{n=1}^{\infty} \frac{z^{n-1}}{n!}$, so

$$\left| G(z) - g(z) \right| = \left| \sum_{n=0}^{12} \frac{1}{n!} B_n z^n - \sum_{n=1}^{\infty} B \frac{z^{n-1}}{n!} \right|$$

$$\leq \left| \frac{1}{(3^{-n})^n} - \sum_{n=1}^{\infty} \frac{(3^{-n})^{n-1}}{n!} \right|$$

$$= \left| \frac{1}{3^{-n^2}} - \frac{e^{3^{-n}} - 1}{3^{-n}} \right| < 10^{-10}$$

Since this function decreases with n growing

and with $n=1$ $\left| \frac{1}{\frac{1}{3}} - \frac{e^{1/3} - 1}{3^{-1/3}} \right| < 10^{-10}$

5) a) we use the ratio test,

$$\lim_{n \to \infty} \left| \frac{z^{n+1}}{z^n} \right| = \lim_{n \to \infty} \frac{|z|^{n+1}}{|z|^n} = \lim_{n \to \infty} |z| = |z|$$

So $\sum_{n=0}^{\infty} z^n$ is analytic for $|z| < 1$.

b)

$$\frac{1}{1-z} = \frac{1}{1 - \frac{1}{2} + \frac{1}{2} - z}$$

$$= \frac{1}{\frac{3}{2} - (z + \frac{1}{2})}$$

$$= \frac{2}{3} \cdot \frac{1}{1 - \frac{3(z + \frac{1}{2})}{2}}$$

$$= \frac{2}{3} \sum_{n=0}^{\infty} \frac{3^n}{2^n} (z + \frac{1}{2})^n$$

$$= \sum_{n=0}^{\infty} \frac{3^{n-1}}{2^{n-1}} (z + \frac{1}{2})^n$$

we now use the ratio test

$$\lim_{n \to \infty} \left| \frac{\frac{3^n}{2^n}(z + \frac{1}{2})^{n+1}}{\frac{3^{n-1}}{2^{n-1}}(z + \frac{1}{2})^n} \right| = \lim_{n \to \infty} \left| \frac{3}{2}(z + \frac{1}{2}) \right|$$

$$= \frac{3}{2} |z + \frac{1}{2}|$$

This limit is < 1 when $|z + \frac{1}{2}| < 2/3$

hence that is where this function $G(z)$ is analytic.

6) Let's use partial fractions

$$\frac{z^{n-1}}{(1-z^n)(1-z^{n+1})} = \frac{A(z)}{1-z^n} + \frac{B(z)}{1-z^{n+1}}$$

$\leftrightarrow z^{n-1} = A(1-z^{n+1}) + B(1-z^n)$

Put $B = -A$, then

$z^{n-1} = A(z^n - z^{n+1})$

$\frac{1}{A} = \frac{z^n - z^{n+1}}{z^{n-1}} = \frac{z^n}{z^{n-1}}(z - z^2) = z(1-z)$

$A = \frac{1}{z(1-z)}$ $B = \frac{-1}{z(1-z)}$

So we have

$$\sum_{n=1}^{\infty} \frac{z^{n-1}}{(1-z^n)(1-z^{n+1})} = \sum_{n=1}^{\infty} \frac{1}{z(1-z)}\left[\frac{1}{(1-z^n)} - \frac{1}{(1-z^{n+1})}\right]$$

$= \lim_{M\to\infty} \sum_{n=1}^{M} \frac{1}{z(1-z)}\left[\frac{1}{1-z^n} - \frac{1}{1-z^{n+1}}\right]$

This sum telescopes, so

$= \lim_{M\to\infty} \frac{1}{z(1-z)}\left[\frac{1}{1-z} - \frac{1}{1-z^{M+1}}\right]$

$= \frac{1}{z(1-z)} \lim_{M\to\infty}\left[\frac{1}{1-z} - \frac{1}{1-z^{M+1}}\right]$

If $|z|<1$, the limit converges to $\frac{1}{1-z}$, meaning the sum

$= \frac{1}{z(1-z)}\left[\frac{1}{1-z} - 1\right]$

$= \frac{1}{z(1-z)} \cdot \frac{1-(1-z)}{1-z} = \frac{z}{1-z}$

Simplifies to $\dfrac{1}{z(1-z)} \cdot \dfrac{z}{(1-z)} = \dfrac{1}{(1-z)^2}$

If $|z| > 1$, then the limit converges to

$\dfrac{1}{1-z}$, and the sum becomes $\dfrac{1}{z(1-z)} \cdot \dfrac{1}{(1-z)}$

$= \dfrac{1}{z(1-z)^2}$ which is the desired result,

This function does not violate analytic
continuation because at $z = 1$, both functions
are undefined, ie this function (these 2
functions) are not the same on the (unit circle
functions) are not the functions are boundary,
So it's okay that the functions are
different inside and outside the boundary
(the unit circle).

7) Consider the sequence of functions

$$\hat{f}_n(z) = \int_0^n e^{izx} f(x)\, dx . \quad \text{①}$$

Since e^{izx} and $f(x)$ are continuous, the integrand is continuous, and if $\hat{f}(x)$ exists

$\lim_{n\to\infty} \hat{f}_n(z) = \hat{f}(z)$, Let's show $\hat{f}(z)$ is bounded

$$|\hat{f}(z)| = \left| \int_0^\infty e^{izx} f(x)\, dx \right|$$

$$\leq \int_0^\infty |e^{izx}|\,|f(x)|\, dx$$

$$\leq M \int_0^\infty |e^{ix(a+ib)}|\, dx$$

$$= M \int_0^\infty |e^{ixa}\, e^{-xb}|\, dx$$

$$= M \int_0^\infty e^{-xb}\, dx \quad (\text{as } x \text{ and } b > 0)$$

$$= M \left[-\frac{1}{b} e^{-xb} \right]_0^\infty$$

$$= -\frac{M}{b} \left[\lim_{x\to\infty} \frac{1}{e^{bx}} - 1 \right] = \frac{q}{b} = \frac{M}{b}$$

So the integral exists.

The integral ① is the same as

$$\hat{f}_n(x) = \int_0^\infty g_n(z,x)\,dx \quad ; \quad g_n(z,x) = \begin{cases} g(z,x) & x \in (0,n] \\ 0 & \text{else} \end{cases}$$

where $g(z,x) = f(z)e^{izx}$. $g_n \to g$ uniformly on z (the limit is independent of z), since the bound doesn't depend on z), so from previous homework $\hat{f}_n \to f$ uniformly. If we can show $\hat{f}_n$ is analytic $\forall n$, then we can conclude our limit function f is analytic as well. $g_n(z,x)$ is analytic since it is bounded and continuous _____, it's loop integral will vanish because of Cauchy's theorem, and morera's theorem then tells us $g_n(z,x)$ is analytic. Since $g_n(z,x)$ is analytic, we can use Theorem 3.11 from Bernard's notes, to say $\hat{f}_n(z)$ is analytic, Since $g_n(z,x)$ is continuous for all $x \in (0,n)$, which by our earlier discussion. Concludes the argument that $\hat{f}(z)$ is analytic,

8) From previous homework, both
functions have a branch cut on $[-1,1]$,
There was a theorem presented in lecture
Saying that if on the domain both functions
are defined, if they are equal for ω
Sequence of points in the domain, and this
Sequence of points has an accumulation point
inside the domain, then both functions are
equal on the whole domain, So consider the
Sequence of points $z_n = 2 + \frac{1}{n}$, then

$$\sqrt{z_n - 1}\sqrt{z_n + 1} = \sqrt{1 + \frac{1}{n}} \cdot \sqrt{3 + \frac{1}{n}}$$

$$= \sqrt{\frac{1 + \frac{1}{n}}{1 + \frac{1}{n}}} \cdot \sqrt{3 + \frac{1}{n}} \sqrt{1 + \frac{1}{n}}$$

$$= \left(1 + \frac{1}{n}\right)\sqrt{\frac{3 + \frac{1}{n}}{1 + \frac{1}{n}}}$$

$$= \sqrt{z_n - 1} \cdot \sqrt{\frac{z_n + 1}{z_n - 1}}$$

where the manipulations above are justified be—cause of real numbers,
above are justified be—cause of real numbers,
$z_n \to 2$ as $n \to \infty$, and $2 \notin [-1,1]$. So
we get $\sqrt{z-1}\sqrt{z+1} = (z-1)\sqrt{\frac{z+1}{z-1}}$ where

they are defined,

Now put $z = \frac{1}{t}$, as $t \to 0$, $z \to \infty$, so
because the above 2 functions are equal

$$\sqrt{z-1} \ \sqrt{z+1} = \sqrt{(z-1)(z+1)}$$

$$= \sqrt{z^2 - 1}$$

$$= \sqrt{\frac{1}{t^2} - 1}$$

$$= \frac{1}{t} \sqrt{1-t} \quad \circledast$$

The binomial expansion then gives

$$\circledast = \frac{1}{t} \left(1 - \frac{t^2}{2} + \frac{t^4}{8} + O(t^5) \right) \quad (\text{as } t \to 0)$$

$$= z \left(1 - \frac{1}{z^2 \cdot 2} + \frac{1}{z^4 \cdot 8} + O\left(\frac{1}{z^5}\right) \right) \quad (\text{as } z \to \infty)$$

$$= z - \frac{1}{2} \frac{1}{z^2} + \frac{1}{8} \cdot \frac{1}{z^4} + O\left(\frac{1}{z^5}\right)$$

$$= z + 0 - \frac{1}{2} z^{-2} + O(z^{-3}) \quad \text{as} \quad (\text{as } z \to \infty)$$

and $b_1 = 0$.

So $b_0 = 0$, $b_2 = -\frac{1}{2}$,

(∂)

1) (3.3.2)

a) If $|z| < a$, then $\dfrac{|z|^2}{a^2} < \dfrac{a^2}{a^2} = 1$, so

$$\frac{z}{a^2 - z^2} = \frac{z}{a^2} \cdot \frac{1}{1 - \dfrac{z^2}{a^2}}$$

$$= \frac{z}{a^2} \sum_{n=0}^{\infty} \left(\frac{z^2}{a^2}\right)^n$$

$$= \sum_{n=0}^{\infty} \frac{z^{2n+1}}{a^{2n+2}}$$

b) If $|z| > a$, then $\dfrac{1}{|z|} < \dfrac{1}{a}$ and $\dfrac{a^2}{|z|^2} < \dfrac{a^2}{a^2} = 1$,

so,

$$\frac{z}{a^2 - z^2} = \frac{-z}{z^2} \frac{1}{1 - \dfrac{a^2}{z^2}}$$

$$= -\frac{1}{z} \sum_{n=0}^{\infty} \left(\frac{a^2}{z^2}\right)^n$$

$$= - \sum_{n=0}^{\infty} \frac{a^{2n}}{z^{2n+1}}$$

2) (3.3.5) we can use Theorem 3.12 to

Say

$$J_n(t) = \frac{1}{2\pi i} \oint_C \frac{e^{t/2 \, (z - 1/z)}}{z^{n+1}} \, dz$$

our singularity is at $z = 0$, as $\frac{1}{z}$ is not defined there. So for convenience let's take our annulus to be $1/2 \le |z| \le 3/2$, and choose our contour C to be the unit circle. Put $z = e^{i\theta}$. Then $z - \frac{1}{z} = e^{i\theta} - e^{-i\theta}$ $= 2i \sin(\theta)$, and $dz = i e^{i\theta} d\theta$. we can also take $-\pi \le \theta < \pi$, then our integral becomes

$$\frac{1}{2\pi i} \int_{-\pi}^{\pi} \frac{e^{t/2 \cdot 2i \sin\theta}}{e^{i(n+1)\theta}} \cdot i \, e^{i\theta} \, d\theta$$

$$= \frac{i}{2\pi i} \int_{-\pi}^{\pi} e^{t/2 \, (2i \sin\theta) - ni\theta} \, d\theta$$

$$= \frac{1}{2\pi} \int_{-\pi}^{\pi} e^{-i(n\theta - t\sin\theta)} \, d\theta \qquad \checkmark$$

$$= \frac{1}{2\pi} \int_{-\pi}^{\pi} \cos(n\theta - t\sin\theta)\, d\theta \; - \; \frac{1}{2\pi} i \int_{-\pi}^{\pi} \sin(n\theta - t\sin\theta)\, d\theta$$

(and over $-a \leq \theta \leq a$)

Since $\sin(x)$ is an odd function, the second integral is 0, and since $\cos(x)$ is an even function, the above can be written as

$$= \frac{1}{2\pi} \int_{0}^{\pi} 2\cos(n\theta - t\sin\theta)\, d\theta$$

$$= \frac{1}{\pi} \int_{0}^{\pi} \cos(n\theta - t\sin\theta)\, d\theta \qquad \checkmark$$

as desired.

```python
# Show data in an imagesc type plot.
def plot(x, y, xlabel='', ylabel='', title=''):
    plt.plot(x, y)
    plt.xlabel(xlabel)
    plt.ylabel(ylabel)
    plt.title(title)
    plt.show()


def f(z):
    return z / (np.exp(z) - 1)


def g(z):
        return (np.exp(z) - 1) / z


def G(z):
    factorials = np.array([1 /math.factorial(n) for n in range(13)])
    z_powers = np.array([z ** n for n in range(13)])
    b = np.array([1, -1/2, 1/6, 0, -1/30, 0, 1/42, 0, -1/30, 0, 5/66, 0, -691/2730])
    return 1 / np.dot(b * factorials, z_powers)


def problem_4_a():
    eps = np.array([3^(-n) for n in range(1, 53)])
    errors = np.abs(g(eps) - 1)
    indices = np.arange(1, 53)
    return indices, errors


def problem_4_b():
    eps = np.array([3^(-n) for n in range(1, 53)])
    errors = np.abs(G(eps) - g(eps))
    indices = np.arange(1, 53)
    return indices, errors
```
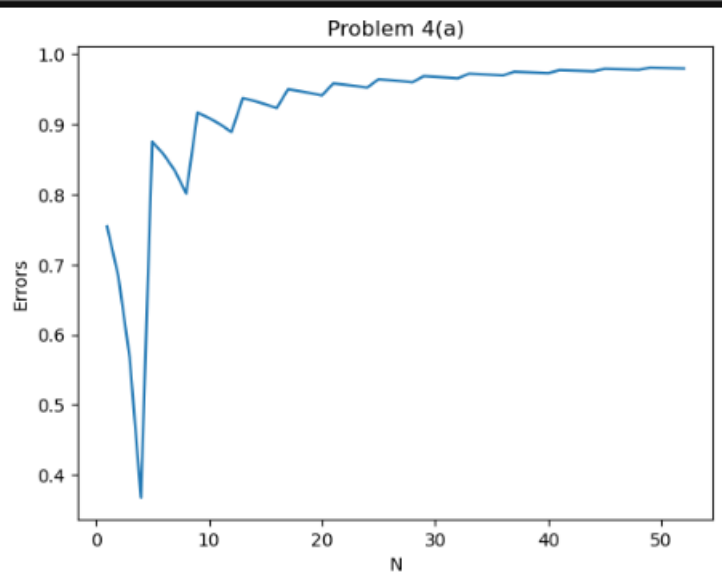
```
: indices, errors = problem_4_a()
  plot(x=indices, y=errors, xlabel='N', ylabel='Errors', title='Problem 4(a)')
```


Problem 4(a)

```
: indices, errors = problem_4_b()
  plot(x=indices, y=errors, xlabel='N', ylabel='Errors', title='Problem 4(b)')
```


Problem 4(b)