Global Positioning System

A GPS is much more than a fancy app on your phone, in fact, your phone is only a small part of the GPS system. GPS or Global Positioning System is a network of satellites, each of the satellites has a fairly accurate clock on board and is constantly sending signals to the ground. The signal travel close to the speed of light containing the time the signal was sent and the location of the GPS satellite sending the signal. The actually positioning is based on trilateration, which is a method of determining position by measuring distances to point at know coordinate. At a minimum, trilateration requires 3 ranges to 3 known points. GPS point positioning, on the other hand, requires 4 “pseudoranges” to 4 satellites.

The GPS segments

The GPS consist of three major segments: Space Segment, Control Segment, and User Segment. The Space Segment is where each satellite broadcast a Navigation Message with exact position, clock, status, orbit detail of itself, general position and health information of all other satellites. GPS satellites are watched by the Control Segment, five monitoring ground station constantly track the satellite transitions, clock, status, and orbit. The stations are located at Hawaii, Colorado Springs, Ascension Island, Diego Garcia, and Kwajalein. All faults and failures in the GPS system are detected and corrected by the Control Segment. Though it takes up to two hours before a satellite is found to be faulty and up to 12 hours to correct the situation. Finally, the User Segment which includes all users of GPS and it’s GPS-antenna, GPS-receiver, and GPS-navigator.

How do we know position of satellites?

As stated before a signal is transmitted from each satellite, the signal contained a “Navigation Message”, that can be read by the user’s GPS receiver. The Navigation Message contain orbit parameters, from which the receiver can computer satellite coordinate. The coordinates are in Cartesian form (XYZ) in a geocentric system, known as WGS-84, which has its origin at the Earth centre of mass, Z axis pointing towards the North Pole, X pointing towards the Prime Meridian (which crosses Greenwich), and Y at right angles to X and Z to form a right-handed orthogonal coordinate system.

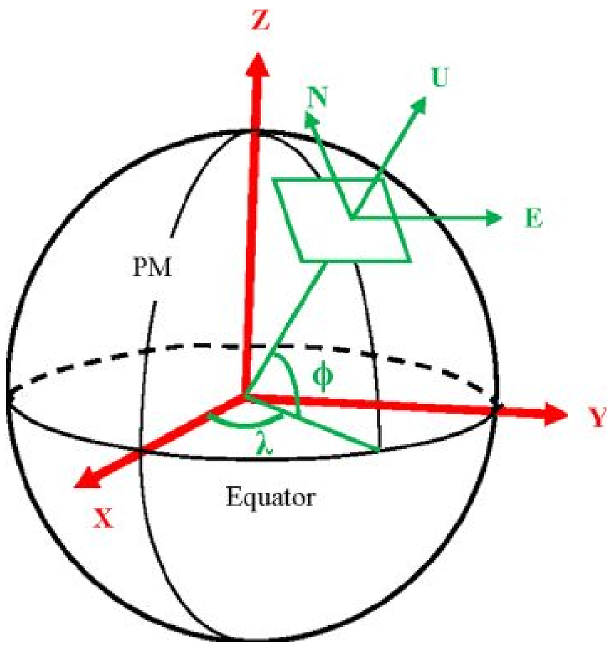

Fig1. WGS-84 (World Geodetic System 1984)

What are pseudoranges?

The time that the signal is transmitted from the satellite is also encoded on the signal, using the time according to a clock onboard the satellite. Time of signal reception is recorded by the receiver using a clock. A receiver measures difference in these times:

$p_{ri} = c * (t_{RX} - t_{TX})$

Where p_{ri} is the pseudorange measurements, c is the speed of light, t_{TX} is the time of transmission and t_{RX} is the time of reception. A pseudorange is a distance measurement from a satellite to a receiver that has not been correct for clock difference. Each pseudorange measurement represents a sphere of position for the receiver centered on the corresponding GPS satellite. The pseudorange measurement between the ith GPS satellite and the user's receiver is modeled as

$p_{ri} = [(x - x_i)^2 + (y - y_i)^2 + (z - z_i)^2]^{1/2} + R_C = \rho_{ri} + R_C$

Where R_C is the the receiver's clock offset expressed in units of length.

How do we correct for clock errors?

There are many algorithms to correct for clock error such the Krause's algorithm which is capable of processing only four measurements and Bancroft Algorithm which is capable of handling any amount of pseudorange measurement. However, Bancroft algorithm requires the inversion of the 4 x 4 matrix to compute the solution while Krause's algorithm only requires the inversion of two 2x2 matrices. The spherical-plane algorithm is a modification of both algorithms. The basis for this solution is the basic measure equation

$(p_{ri} - R_C)^2 = [(x-x_i)^2 + (y-y_i)^2 + (z-z_i)^2]$

A linear set of equations can be created by forming a set of (n-1) linearly independent difference between each of these measurements.

$(p_{ri} - R_C)^2 - (p_{rj+1} - R_C)^2 = [(x-x_i)^2 + (y-y_i)^2 + (z-z_i)^2] - [(x-x_{j+1})^2 + (y-y_{j+1})^2 + (z-z_{j+1})^2]$

This can simplify with quick arithmetic

$0 = a_j x + b_j x + c_j x + d_j R_C + e_j$

Where $a_j = 2(x_{j+1} + x_j)$, $b_j = 2(y_{j+1} + y_j)$, $c_j = 2(z_{j+1} + z_j)$, $d_j = 2(p_{rj} - p_{rj+1})$,

$e_j = (x_j^2 + y_j^2 + z_j^2) - (x_{j+1}^2 + y_{j+1}^2 + z_{j+1}^2) - (p_{rj}^2 - p_{rj+1}^2) = (\Omega_j - \Omega_{j+1}) + (p_{rj}^2 - p_{rj+1}^2)$

For the case where you have more than four pseudorange measure, then you would also have more than four linear equation this can simplify using a matrix

$(M^T M)s = H^T u$

Where $M = [a_1,\ b_1,\ c_1,\ d_1;\ a_2,\ b_2,\ c_2,\ d_2;\ \ldots,\ \ldots,\ \ldots,\ \ldots;\ a_{n-1},\ b_{n-1},\ c_{n-1},\ d_{n-1};\ ]$

And $u = [e_1;\ e_2;\ \ldots;\ e_{n-1}]$

For our case since we are only looking at four pseudorange measure, there are only three linearly independent equation and that be rewritten as a matrix as well

$[a_1,\ b_1,\ c_1;\ a_2,\ b_2,\ c_2;\ a_3,\ b_3,\ c_3;\ ][x;\ y;\ z] = -[d_1;\ d_2;\ d_3]R_C - [e_1;\ e_2; e_3]$

Or even more simplified via matrix division $[x;\ y;\ z] = [\Phi_1;\ \Phi_2;\ \Phi_3]R_C + [\Gamma_1;\ \Gamma_2; \Gamma_3]$

Amazingly once it's in this form it can become a simple polynomial

$0 = \alpha_2 x^2 + \alpha_1 x + \alpha_0$

Where $\alpha_2 = \Phi_1^2 + \Phi_2^2 + \Phi_3^2 - 1$

$\alpha_1 = 2(p_{ri} + \Phi_1(\Gamma_1 - x_i) + \Phi_2(\Gamma_2 - y_i) + \Phi_3(\Gamma_3 - z_i))$

And $\alpha_0 = (\Gamma_1 - x_i)^2 + (\Gamma_2 - y_i)^2 + (\Gamma_3 - z_i)^2 - p_{ri}^2$

Now we can use the quadratic formula to solve for the receiver's clock offset

$R_C = \frac{-\alpha_1 + \sqrt{\alpha_1^2 - 4 * \alpha_2 * \alpha_0}}{2\alpha_2}$

Plugging the R_C back into $[x;\ y;\ z] = [\Phi_1;\ \Phi_2;\ \Phi_3]R_C + [\Gamma_1;\ \Gamma_2; \Gamma_3]$ will solve for $[x;\ y;\ z]$ of the user.

The Example

For the case with only four measure, the R_C coordinate was chosen as the independent state variable and (x,y,z) was chosen as the dependent state variable. Using this table

ID	Satellite's X	Satellite's Y	Satellite's Z	Pseudorange measurement
GPS-1	16414028.668	660383.618	20932036.907	24658975.31743
GPS-2	16896800.648	-18784061.365	-7418318.856	22964286.41226
GPS-3	9339639.616	-14514964.658	20305107.161	21338550.64536
GPS-4	-18335582.591	-11640868.305	15028599.071	23606547.29359

The quadratic coefficient are

$[\alpha_2, \alpha_1, \alpha_0] = [-.929133.82910,\ 51262091.152897,\ 1848840181381.0]$

The two solutions are

$[x_{A,}\ Y_A,\ Z_A,\ R_{CA}] = [961333.82910,\ -5674076.36990,\ 2740537.66130,\ -36000.0000]$ and

$[x_{B,}\ Y_B,\ Z_B,\ R_{CB}] = [1072554.5379,\ 7595289.9354,\ -3094793.7428,\ 55173924.2593]$

Deciding which one of these solutions is the correct one would be decide plugging in our result back into $p_{ri} = [(x-x_i)^2 + (y-y_i)^2 + (z-z_i)^2]^{1/2} + R_C = \rho_{ri} + R_C$

We found the answer to be

$[x_{A,}\ Y_A,\ Z_A,\ R_{CA}] = [961333.82910,\ -5674076.36990,\ 2740537.66130,\ -36000.0000]$

Conclusion

In conclusion, A GPS is much more than a fancy app on your phone. It features three major segments: Space Segment, Control Segment, and User Segment. Using at least four satellites and the spherical-plane algorithm, we can set up a system of equation with unknowns. Solving this system for the unknowns results in the desired location of the GPS receiver.