Gaussian beam radius measurement with a knife-edge: a polynomial approximation to the inverse error function

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A method for approximating the inverse error function involved in the determination of the radius of a Gaussian beam is proposed. It is based on a polynomial inversion that can be developed to any desired degree, according to an *a priori* defined error budget. Analytic expressions are obtained and used to determine the radius of a TEM_{oo} He–Ne laser beam from intensity measurements experimentally obtained by using the knife edge method. The error and the interval of validity of the approximation are determined for polynomials of different degrees. The analysis of the theoretical and experimental errors is also presented. © 2013 Optical Society of America

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1. Introduction

Lasers have been used in many applications. In many cases, the parameters that describe the profile of a beam must be known; among them, the beam radius is probably the first to be considered. Many different methods have been developed to measure it. There are the mechanical methods, which make use of a scanning aperture and a power detector to sample the power distribution across the beam. For example, there are systems using a pinhole [1], a knife edge [2,3], a Ronchi grating [4], or a slit [5] as the aperture. Other optical beam profilers have also been demonstrated, such as photothermal deflection [6] and a thermographic technique [7], among others. Owing to its simplicity, however, the knife-edge method has gained popularity for the measurement of Gaussian beams and beams with arbitrary intensity distributions [8].

As is well known, the irradiance pattern $I(x,y)$ of a Gaussian beam is given by

$$I(x,y) = I_o \exp\left[-2\frac{(x-x_o)^2+(y-y_o)^2}{r_o^2}\right], \quad (1)$$

where I_o is the maximum irradiance of the beam at its center of symmetry, (x_o, y_o) are the coordinates of the center of the beam, and r_o is the radius of the beam measured at the point where the irradiance has a value of I_o/e^2.

In the knife-edge method, the opaque side of the knife blocks out a part of the beam, while the power of the transmitted part of the beam is measured by a detector. The transmitted power for different positions of the knife edge is obtained by scanning the knife edge transversely across the beam along the orthogonal direction to the edge. A plot of those data

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gives a characteristic S-shaped curve going from the total beam power P_T to 0. The transmitted power can be obtained by integration of the beam intensity [Eq. (1)] along the open semi-plane defined by the straight edge; the result is a well known expression [9] given by

$$P(x) = \frac{P_T}{2}\left[1 + \mathrm{erf}\left(\frac{\sqrt{2}(x - x_o)}{r_o}\right)\right], \tag{2}$$

where $P_T = (\pi/2)r_0^2 I_0$, x is the position of the knife-edge and $\mathrm{erf}(z)$ is the Gaussian error function defined as

$$\mathrm{erf}(z) = \frac{2}{\sqrt{\pi}}\int_0^z e^{-t^2}\mathrm{d}t. \tag{3}$$

Due to the presence of the error function, direct evaluation of r_o is a somewhat involved task. A common technique is to numerically differentiate the power measured by the knife-edge method and, by fitting the data to a Gaussian function, r_o is obtained. Recently, Veshapidze *et al.* [10] analyzed the problem and proposed to fit the directly measured power data to an analytical function that approximates Eq. (2). They showed that the error in evaluating r_o is reduced in comparison with the differentiation method.

Other methods [11,12] are based on a direct inversion of Eq. (2); solving for r_o gives

$$r_o = \frac{\sqrt{2}(x - x_o)}{\mathrm{erf}^{-1}\left(\frac{2P}{P_T} - 1\right)}, \tag{4}$$

where erf^{-1} means the inverse of the error function (IEF).

In principle, the problem of measuring the radius of the beam has been solved. In practice, however, the IEF is a complicated function that has no analytical expression; then, only approximated expressions for Eq. (4) can be obtained. For example, Díaz-Uribe *et al.* [9] proposed a linear approximation of the IEF, giving the approximate solution for the radius of the beam:

$$r_0 \approx \sqrt{\frac{8}{\pi}}\frac{(x - x_o)}{\left(\frac{2P}{P_T} - 1\right)}. \tag{5}$$

For this case, only one point (x, P) is necessary, but an accurate value for x_o must be known. Otherwise, when using Eq. (5) twice, for points (x_1, P_1) and (x_2, P_2), the dependence of x_o can be eliminated

$$r_o \approx \sqrt{\frac{2}{\pi}}\frac{P_T(x_2 - x_1)}{(P_2 - P_1)} = \sqrt{\frac{2}{\pi}}\frac{P_T}{m}, \tag{6}$$

where m is the slope of the straight line joining the two points. The approximation (6) is better when the selected points are closer to the center point, where $P = P_T/2$, and the selected points are symmetrical to the center.

In this paper, a better approximation to Eq. (4) is obtained by inverting the Taylor series of the error function using the method proposed by Arfken [13]; this is done in Section 2. Next, in Section 3, the experimental procedure is described, and in Section 4 the error and the interval of validity is determined for approximations of different degrees.

2. Polynomial Inversion

The Taylor series expansion of the error function can be found elsewhere [14],

$$\mathrm{erf}(\chi) = \frac{2}{\sqrt{\pi}}\sum_{n=0}^{\infty}\frac{(-1)^n\chi^{2n+1}}{n!(2n+1)} = \sum_{n=0}^{\infty}a_{2n+1}\chi^{2n+1}. \tag{7}$$

According to Arfken [13], it is possible to obtain the inverse series to any degree of approximation by solving a set of simultaneous algebraic equations with as many unknowns as the degree of the approximation. Explicit solutions to the seventh degree are listed by Dwight [14]. In addition, González-Cardel and Díaz-Uribe [15,16] deduced expressions for the eighth and ninth degree coefficients and showed how to find the interval of validity of the approximation, given a maximum acceptable error. In order to show how to use those results for the problem of finding an approximate expression for the beam radius, the third degree approximation will be described in some detail. Further approximations will be expressed later.

A. Third Degree Approximation

Using Eq. (7), the error function can be approximated to the third degree by

$$\mathrm{erf}(\chi) \approx \frac{2}{\sqrt{\pi}}\left\{\chi - \frac{\chi^3}{1!(3)}\right\} \equiv \xi. \tag{8}$$

The inverse function (IEF), to the same degree of approximation obtained by the Arfken inversion method, is given by

$$\chi(\xi) = \mathrm{erf}^{-1}(\xi) \approx \frac{\sqrt{\pi}}{2}\xi + \frac{\sqrt{\pi^3}}{24}\xi^3. \tag{9}$$

By substituting Eq. (9) in (4), the third degree approximation for the beam radius is obtained as

$$r_0 \approx \sqrt{\frac{8}{\pi}}\frac{x}{\xi}\left\{1 + \frac{\pi}{12}\xi^2\right\}^{-1}, \tag{10}$$

where $\xi = (2P/P_T - 1)$.

The interval of validity for the approximation is given by the inequality [15,16]

$$\varepsilon(\chi) \equiv |T_1\chi^3 + T_2\chi^4 + \cdots + T_6\chi^8| - \frac{e}{100} \leq 0, \tag{11}$$

where e is the error budget in percent and

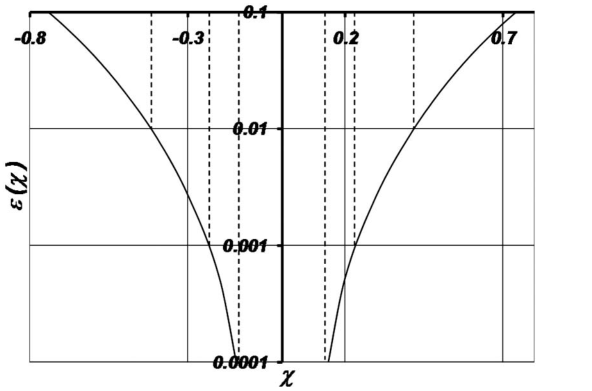

Fig. 1. Plot to calculate the solution interval of the function $\varepsilon(\chi)$ [see Eq. (11)].

$$
\begin{aligned}
T_1 &= \frac{5a_2^3 - 5a_1a_2a_3}{a_1^3}, \\
T_2 &= \frac{6a_2^4 + a_1a_2^2a_3 - 3a_1^2a_3^2}{a_1^4}, \\
T_3 &= \frac{2a_2^5 + 11a_1a_2^3a_3 - 7a_1^2a_2a_3^2}{a_1^5}, \\
T_4 &= \frac{6a_2^4a_3 + 3a_1a_2^2a_3^2 - 3a_1^2a_3^3}{a_1^5}, \\
T_5 &= \frac{6a_2^3a_3^2 - 3a_1a_2a_3^3}{a_1^5}, \\
T_6 &= \frac{2a_2^2a_3^3 - a_1a_3^4}{a_1^5}.
\end{aligned} \tag{12}
$$

Equation (7) defines the a coefficients for odd indices as

$$
a_{2n+1} = \frac{2}{\sqrt{\pi}} \frac{(-1)^n}{n!(2n+1)}, \tag{13}
$$

where, for even indices, the coefficients are zero $a_{2n} = 0$. Then, the odd T coefficients are zero and the even coefficients are

$$
\begin{aligned}
T_2 &= -3\left(\frac{a_3}{a_1}\right)^2 = -\frac{1}{3} \\
T_4 &= -3\left(\frac{a_3}{a_1}\right)^3 = \frac{1}{9}, \\
T_6 &= -\left(\frac{a_3}{a_1}\right)^4 = -\frac{1}{81}.
\end{aligned} \tag{14}
$$

For an error budget equal to 1%, the plot shown in Fig. 1 is obtained for the function $\varepsilon(\chi)$ defined by Eq. (11). It can be seen that the interval of validity of the third degree approximation of the IEF is the interval where the curve is below the x axis. Numerically, the interval is $-0.422 \le \chi \le 0.422$ or, in terms of the power of the transmitted beam, $0.289P_T \le P \le 0.711P_T$.

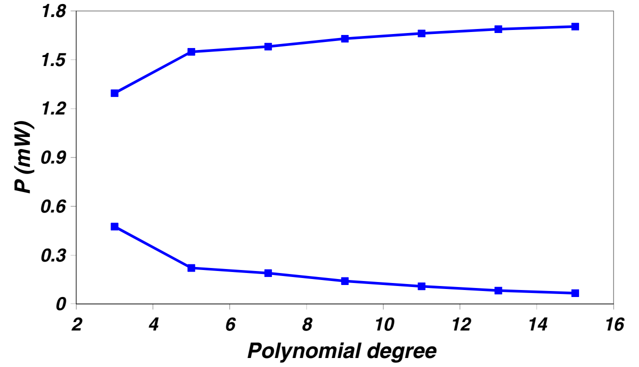

Fig. 2. Plot of the limits of the interval of validity for different degrees of the approximating polynomial.

B. Higher Degree Approximations

Doing further calculations, it is easy to find approximations to the IEF to a higher degree than 3. The general expression is given by

$$
\chi_{2N}(\xi) = \mathrm{erf}^{-1}(\xi) \approx \frac{\sqrt{\pi}}{2}\xi \sum_{n=0}^{N} s_{2n}\xi^{2n}, \tag{15}
$$

where $s_0 = 1$, $s_2 = \pi/12$, $s_4 = 7\pi^2/480$, etc.

Significantly, Eq. (15) is the same expression used by Mathematica [17], but with the full development given here it is possible to find the interval of validity for a given error budget or the amount of error in some calculation given the measured power. In Fig. 2, the limits of the interval of validity are plotted as a function of the degree of the approximating polynomial for 1% of error. It is worth recognizing here that there is not a specific rule for computing the s coefficients. In addition, the T coefficients for computing the interval of validity are different for every degree of approximation.

Then, the beam radius can be expressed at any desired degree of approximation as

$$
r_o \approx \sqrt{\frac{8}{\pi}} \frac{x}{\xi} \left\{ \sum_{n=0}^{N} s_{2n}\xi^{2n} \right\}^{-1}. \tag{16}
$$

In Table 1 the upper and lower limits for the intervals of validity for approximations between 3 to 15 deg are listed; in every case the limits are computed for a unit error budget e. Each calculation was done with number of points listed in the fourth

Table 1. Intervals of Validity for Every Approximation

Degree	Low Limit	Top Limit	Data	Set Equations
3	0.475	1.295	11	55
5	0.221	1.549	21	210
7	0.189	1.581	23	253
9	0.140	1.630	25	300
11	0.108	1.662	27	351
13	0.082	1.688	30	435
15	0.066	1.704	32	496

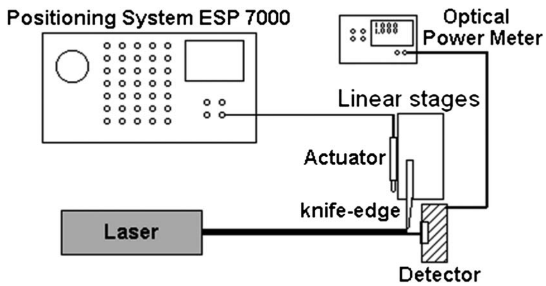

Fig. 3. Experimental device.

column (Data), the number of solved equations is listed on the fifth column (Set Equations).

3. Experimental Procedure

With the above results, it seems that it is good enough to measure the power for only one given position x of the knife edge, inside the interval of validity of the inversion polynomial, to find the radius of the laser beam. It is worth noting here that using an x value within the interval of validity only ensures that the corresponding value of χ has an error smaller than the error budget ε.

To check this assumption, the transmitted power P for several positions x of the knife edge were experimentally measured using the setup shown in Fig. 3. A stabilized laser, Spectra Physics Model 117A with a total power of 1.7703 mW, was used as a Gaussian beam source. The accurate positioning of the knife edge was made with a ILS100CC linear stage controlled by the ESP 7000 system by Newport, with a nominal accuracy of 1 μm. The laser beam power was measured with a 2862-C Newport power meter and the 498-UV Si detector at 547 mm from the exit aperture of the laser head. The power meter accuracy depends on the scale used for each measurement.

A total of 75 different values for P and x were measured, starting from a knife-edge position where the laser beam was not blocked at all, so the measured power was equal to the total power. The final position was chosen at a point where the beam was totally blocked out and the measured power was negligible (about 0.06% of P_T). The total displacement was 2.220 mm; each position step was 30 μm long. Figure 4

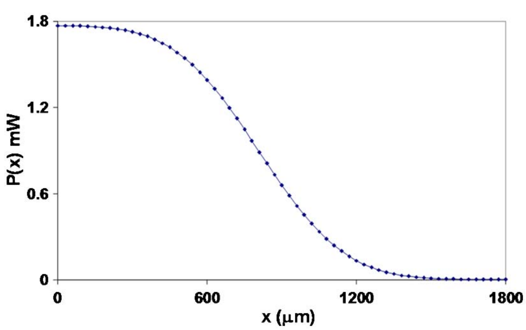

Fig. 4. Plot of optical power versus position, where the S-shaped curve is observed.

shows a plot of these data, with the characteristic S-shaped curve of the error function.

For the third degree approximation [Eq. (10)], the average beam radius is 539 μm, with a standard deviation of 8 μm; these values were obtained by solving 55 different sets of simultaneous two-variable equations obtained for a total of 11 different data points lying within the interval of validity. To get an alternative measurement of these results, power measurements after a knife edge were performed at the exit aperture of the laser head. the beam radius was determined from the points I_o/e^2 [12], finding a value of $r_o = 223\ \mu\text{m} \pm 0.005\ \mu\text{m}$. The beam radius at $z = 547$ mm can be computed through the equation [18]

$$w = w_0 \sqrt{1 + \left(\frac{\lambda z}{\pi w_0^2}\right)^2}, \tag{17}$$

where w_o and w are the beam radius at the beam waist and distance z from it, respectively, and λ is the beam wavelength. The beam radius obtained in this way gives a very close value to the one obtained through Eq. (10): $w = 542\ \mu\text{m} \pm 40\ \mu\text{m}$. The difference between both results is only 0.5%.

On the other hand, when using the fifteenth degree approximation, in Eq. (4), we have two unknowns, r_o and x_o. With the experimental data, we can derive $C_{32}^2 = 496$ different sets of simultaneous two-variable equations obtained for a total of 38 different data points lying within the interval of validity. The average beam radius value obtained is $r_o = 541\ \mu\text{m}$, with a standard deviation $\sigma = 14\ \mu\text{m}$. For this case, the difference from the value obtained through the I_o/e^2 points is ±0.2%.

4. Error Analysis

To find the error in the determination of the radius of the laser beam for the method used in this work, the main error sources are the truncation error, the reversion error, and the uncertainty in measuring the position of the knife edge and the beam power. These sources must be analyzed through Eq. (4) or its approximation given by Eq. (16).

A. Truncation Error in IEF

For the error function [Eq. (3)], approximated by the Taylor polynomial to the $2n+1$ degree, the residual of Taylor is given by

$$R_{2n+1}(\xi) = \frac{2}{\sqrt{\pi}}\left|\frac{\xi^{2n+3}}{n!(2n+3)}\right|.$$

Dividing the previous equation for the polynomial of Taylor that approximates the error function, we obtain the relative error of the approximation,

$$\delta f \le \left|\frac{\frac{2}{\sqrt{\pi}}\frac{\xi^{2n+3}}{n!(2n+3)}}{\frac{2}{\sqrt{\pi}}\left\{\sum_{i=0}^{n}\frac{(-1)^i \xi^{2i+1}}{i!(2i+1)}\right\}}\right| = \left|\frac{\frac{\xi^{2n+3}}{n!(2n+3)}}{\left\{\sum_{i=0}^{n}\frac{(-1)^i \xi^{2i+1}}{i!(2i+1)}\right\}}\right|.$$

B. Error in the Polynomial Reversion

A reversed polynomial was developed with the condition of having a previously selected error [16] of 1%. The interval of validity for the third degree is $z \in (-0.4631, 0.4631)$, or equivalently $P \in (0.275P_T, 0.749P_T)$. For other degrees, the interval can be deduced from the plots in Fig. 2.

C. Instrumental Uncertainty in the Experimental Data

1. *Uncertainty in the Position*

The instrumental error in x depends on the measuring system; for the positioning system ESP 7000 of Newport, the absolute error corresponds to 5×10^{-7} m. Therefore, the relative error of the measurement in position is

$$\delta_x = \frac{5 \times 10^{-7}\ \mathrm{m}}{x},$$

where x is expressed in meters. For all the measured distances the error is smaller than 2%.

2. *Uncertainty in the Measured Power*

The second instrumental error is due to the device used for the measurement of the power. The Dual-Channel Optical Power Meter model 2832-C from Newport was used, which has an absolute associated uncertainty of 1 μW for the used scale. Then, the relative error is

$$\delta_P = \frac{1\ \mu\mathrm{W}}{P},$$

where P is expressed in microwatt; for all the measured power values the error is smaller than 5%.

The propagation of uncertainty on the beam radius evaluation, due to P, can be obtained by deriving partially for P from Eq. (4) [19]; the relative error is, up to the fifth degree, as follows:

$$\delta r_P = \frac{\frac{7\pi^{5/2}}{96}\xi^5 + \frac{\pi^{3/2}}{4}\xi^3 + \sqrt{\pi}\xi}{\frac{7\pi^{5/2}}{960}\xi^5 + \frac{\pi^{3/2}}{24}\xi^3 + \frac{\sqrt{\pi}}{2}\xi}\delta P,$$

and deriving partially for x,

$$\delta r_x = \left(\frac{\sqrt{2}}{\left\{\frac{7\pi^{5/2}}{960}\left\{\frac{2P}{P_T}-1\right\}^5 + \frac{\pi^{3/2}}{24}\left\{\frac{2P}{P_T}-1\right\}^3 + \frac{\sqrt{\pi}}{2}\left\{\frac{2P}{P_T}-1\right\}\right\}}\right)\delta x,$$

In a similar way, the approximation to degree 15 is obtained.

D. Total Error

Therefore, we can say that the error of the radius of the laser beam is given by

$$\delta r = 0.01 + \delta r_x + \delta r_P + \delta f,$$

or a general expression,

$$\delta r = 0.01 + \left|\frac{1}{\chi_{2n}(\xi)}\right| \left\{\left|\sum_{i=0}^{(n-1)/2} \frac{(2z^{2i+1})}{(2i)!}\left(\frac{d^{(2i+1)}}{dz^{(2i+1)}}\chi_{2n}(\xi)\bigg|_{z=0}\right)\right|\delta P + \sqrt{2}\delta x + \frac{\xi^{2n+3}}{(2n+3)n!}\right\},$$

where n is the degree of the Taylor polynomial used in the approximation [20]. With

$$\chi_{2n}(\xi) = \frac{2}{\sqrt{\pi}}\left\{\sum_{i=0}^{n}\frac{(-1)^i\xi^{2i+1}}{i!(2i+1)}\right\},$$

$$\xi = \frac{2P}{P_T} - 1.$$

If x is expressed in meters, for a polynomial of degree 3, this satisfies $0.2684P_T < P < 0.7315P_T$.

In our case, the uncertainty in the determination of the radius of the laser beam is smaller than 3%.

5. Simulation

With the objective of analyzing the stability of the method, we realized several simulations in which a known profile was proposed, and simultaneously

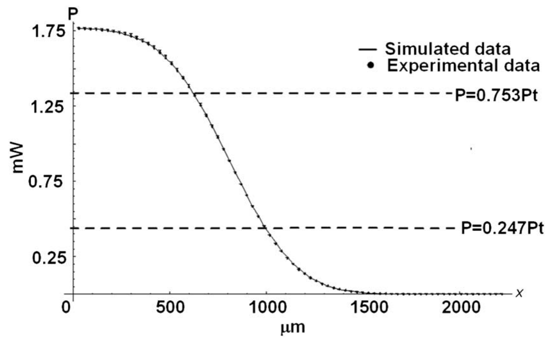

Fig. 5. Simulated and experimental laser beam intensity distribution curve versus the knife-edge position at 54.7 cm. Dashed lines represent the valid interval in which the series expansion is valid.

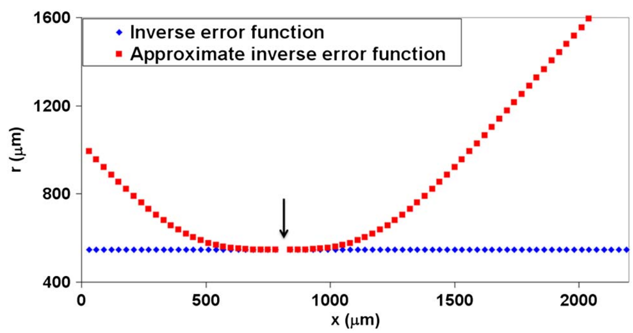

Fig. 6. Calculated beam width of simulated data using Eqs. (4) and (6). We can observe that there exists an indeterminate value, which appears when the intensity is a half of its maximum intensity.

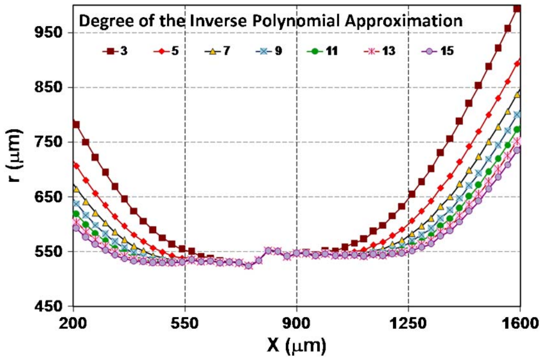

Fig. 7. Plot that shows the radius of beams for all experimental data, with approximations to different degrees.

we added to the values of position and power random errors between 0% and 2%. Using, for the evaluation of the radius, a polynomial of degree 15 with 32 data points in the interval of validity, $C_{32}^2 = 496$ different sets of simultaneous two-variable equations were formed.

For both simulated and experimental results, we use the Spectra-Physics He–Ne laser model 117A that has a waist diameter $w_0 = 0.5$ mm. At a distance of 54.7 cm, the measured total power was 1.7703 mW and the calculated spot size at this position was 0.546 mm. In both cases, the distance in which we displaced the knife edge was of 0.03 m. In Fig. 5 we show the simulated and experimental laser beam intensity distribution curve. We can see that both transmitted powers agree at all of the points.

To obtain the width beam for both simulated and experimental data we use Eqs. (4) and (6). Fig. 6 shows the calculation of simulated data. We can observe an indetermination that is due to the fact that, at the center of the beam, $P = P_T/2$, giving $\xi = 0$; by direct substitution of this value in Eqs. (5), (10), or (16) the indeterminacy appears.

This occurs when the knife edge is exactly at the center of Gaussian beam.

6. Results

We worked with the experimental data, with approximations to different degrees. Using Eq. (4), the radius of the beam was determined for every experimental value; the results are shown in Fig. 7.

It can be seen that a common interval exists where the results are the same for different degrees of the polynomial. It is necessary to mention that evaluating the equation in $x = x_0$ the function has a singularity.

We find the intervals of validity for every approximation; we take the number of data point in every interval of validity,

Solving all the different sets of simultaneous two-variable equations, and averaging the radii obtained for every approximation, we obtain the results on Table 2. The results present a variation smaller than 1%. It is, therefore, possible to conclude that the radius obtained has an error less than 3%.

Table 2. Radii Average

Degree	Average Radius (μm)
3	539
5	544
7	541
9	540
11	540
13	541
15	541

7. Conclusions

A different approach has been proposed for determining the radius of a Gaussian beam from the knife-edge method measurements of beam power and knife edge positions. The method is based on the reversion of the polynomial approximation to the error function, to obtain a semi-analytical expression to the IEF. This approximation can be made to an arbitrary degree of approximation. In the present work, explicit expressions of results for the third degree are deduced, and numerical results are obtained for the fifteenth degree, showing the improvement of the calculations by reducing the numerical errors.

This work allows an approximation of higher degree to be found; in some works linear approximations are only used. An expression is reported for the error of the mentioned approximation. Some calculation programs are also capable of estimating approximations to a high degree, but they do not report the approximation error. In addition, an interval of validity can be found for a fixed error. The radius of the beam can be measured with a maximum error of 3%.

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References

1. P. J. Shayler, “Laser beam distribution in the focal region,” Appl. Opt. **17**, 2673–2674 (1978).
2. Y. Suzaki and A. Tachibana, “Measurement of the μm sized radius of Gaussian laser beam using the scanning knife-edge,” Appl. Opt. **14**, 2809–2810 (1975).
3. M. S. Scholl, “Measured spatial properties of the CW Nd:YAG laser beam,” Appl. Opt. **19**, 3655–3659 (1980).
4. A. K. Cherri and M. S. Alam, “Accurate measurement of small Gaussian laser beam diameters using various rulings,” Opt. Commun. **223**, 255–262 (2003).
5. R. L. McCally, “Measurement of Gaussian beam parameters,” Appl. Opt. **23**, 2227 (1984).
6. A. Rose, Y.-X. Nie, and R. Gupta, “Laser beam profile measurement by photothermal deflection technique,” Appl. Opt. **25**, 1738–1741 (1986).
7. T. Baba, T. Arai, and A. Ono, “Laser beam profile measurement by a thermographic technique,” Rev. Sci. Instrum. **57**, 2739–2742 (1986).
8. A. E. Siegman, M. W. Sasnett, and T. F. Johnston, “Choice of clip levels for beam width measurements using knife-edge techniques,” IEEE J. Quantum Electron. **27**, 1098–1104 (1991).

9. R. Díaz-Uribe, M. Rosete-Aguilar, and R. Ortega-Martínez, "Position sensing of a Gaussian beam with a power meter and knife edge," Rev. Mex. Fís. **39**, 484–492 (1992).
10. G. Veshapidze, M. L. Trachy, M. H. Shah, and B. D. De Paola, "Reducing the uncertainty in laser beam size measurement with a scanning edge method," Appl. Opt. **45**, 8197–8199 (2006).
11. J. M. Khosrofian and B. A. Garetz, "Measurement of a Gaussian laser beam diameter through the direct inversion of knife-edge data," Appl. Opt. **22**, 3406–3410 (1983).
12. M. A. C. de Araujo, R. Silva, E. de Lima, D. P. Pereira, and P. C. de Oliveira, "Measurement of Gaussian laser beam radius using the knife-edge technique: improvement on data analysis," Appl. Opt. **48**, 393–396 (2009).
13. G. Arfken, *Mathematical Methods for Physicists* (Academic, 1981).
14. B. H. Dwight, *Tables of Integrals and Other Mathematical Data*, 4th ed. (Macmillan, 1961), p. 136.
15. M. F. González-Cardel, "Determinación de los coeficientes de asfericidad de una superficie óptica rápida (aspheric coefficients determination for a fast optical surface)," M. Sc. dissertation (Facultad de Ciencias, UNAM, México, 2003).
16. M. F. González-Cardel and R. Díaz-Uribe, "An analysis on the inversion of polynomials," Rev. Mex. Fís. E **52**, 163–171 (2006).
17. Wolfram Research, "Inverse error function," (http://functions.wolfram.com/GammaBetaErf/InverseErf/06/01/01/01/0001/).
18. H. Kogelnik and T. Li, "Laser beams and resonators," Appl. Opt. **5**, 1550–1567 (1966).
19. D. C. Baird, *Experimentation: An Introduction to Measurement Theory and Experimental Design* (Prentice Hall, 1962).
20. M. Spivak, *Calculus* (W. A. Benjamín, 1967), pp. 345–352.