Test

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(ST516 exercises)

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Buffon's needle

In Buffon's needle problem dates back to 1733 where Georges-Louis Leclerc, Comte de Buffon in his *Sur le jeu de franc-carreau* put forward the needle problem stating:

"Suppose we have a floor made of parallel strips of wood, each the same width, and we drop a needle onto the floor. What is the probability that the needle will lie across a line between two strips?"

For the R-script we have following variables
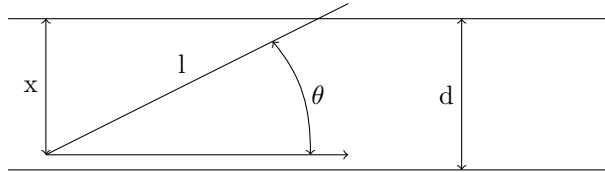
n: needles

l: length of needles

d: distance between lines in floor

dig: decimal places in resulting estimation of π

Let x be the shortest distance from the lower end of the needle to the closest line above and θ the angle between the needle and horizontal alignment. We suppose the placement of the needle - here described through the ends of the needle - is equally likely to occur anywhere between or on the lines. Any angles at which the needle could land is likewise assumed to be equally likely.

The angle θ lies in $(0, \pi]$. If $\theta > \pi$ then it's not the lower end. x lies in $[0, d]$.

Using that for a uniform distibution the p.d.f. is $\frac{1}{b-a}$ for $x \in [a, b]$, zero otherwise, we have $x \sim U(0, d)$ and thereby $f(x) = \frac{1}{d}, 0 \leq x \leq d$ and $\theta \sim U(0, \pi)$ and thereby $f(\theta) = \frac{1}{\pi}$

Since the position of the needle and the angle of the needle are independent, their joint probability function is $f(x, \theta) = \frac{1}{d \cdot \pi}$

The needle is on the line if $x \leq sin(\theta) \cdot l$, so the probability, that the needle will cross a line is

$$F(x,\theta) = \int\int_{X\,\Theta} f(x,\theta) = \int_0^\pi \int_0^{sin(\theta)\cdot l} \frac{1}{d\cdot\pi} dx d\theta = \frac{2l}{d\pi}$$

For a long needle, say $l = 2d$, we get $p = \frac{4d}{d\pi} = \frac{4}{\pi} > 1$, which is clearly wrong. We therefore need to reconsider the problem, when $d < l$.

The probability is the same as long as $sin(\theta)\cdot l \leq d \Rightarrow \theta \leq sin^{-1}(\frac{d}{l})$. If the angle is bigger than that, the needle crossed no matter what yielding a probability of 1 for a hit. We can therefore rewrite our function for $d < l$:

$$F(x,\theta) = 2\cdot\int\int_{X\,\Theta} f(x,\theta) = \frac{2}{\pi}\left(\int_0^{sin^{-1}(\frac{d}{l})} \int_0^{sin(\theta)\cdot l} \frac{1}{d} dx d\theta + \int_{sin^{-1}(\frac{d}{l})}^{\pi/2} 1 d\theta\right)$$

$$= \frac{2}{\pi}\left(\frac{l\left(1 - \sqrt{1 - \frac{d^2}{l^2}}\right)}{d} + \frac{\pi}{2} - sin^{-1}\left(\frac{d}{l}\right)\right)$$

Lazzarini's Lucky Approximation of π

Now recognize, that the theoretical probability can be expressed as the relative frequency of hits, $\frac{n}{n_{hit}} \approx \frac{2l}{d\pi}$, which means π can be estimated as $\hat{\pi} = \frac{2ln}{dn_{hit}}$.

This experiment is caried out below. Here dig is number og decimal returned when estimating π.

```
buffon.needle = function(n, l, d, dig) {
  X = runif(n,0,d)
  t = runif(n,0,pi)
  hits = sum(ifelse(X<=sin(t)*l,1,0))
  if (d>=l) {
    est.pi = round(2*n*l/(d*hits),dig)
  } else {
    est.pi = round((2*n*(-l*sqrt((l^2-d^2)/(l^2))-d*asin(d/l)+l))/(d*hits-d*n),dig)
  }
  sprintf(format(pi,digits=dig+1))
  return(est.pi)
}
```

Simulation

In the simulation I keep l fixed. The plots can be seen in the appendix. As seen on the plots, the estimation seems to have the smallest error, when the ratio of the distance between lines and the length of is above 1 ($\approx \frac{6}{5}$), i.e. when $d < l0$ the estimate has an error of up to 0.35, approximately 11%.

```
sim = function(l) {
  l = l
  d = seq(0.01,3,0.01)
  n = c(1000,10000,100000)
  sim = matrix(nrow=length(d),ncol=length(n),dimnames = list(d,n))

  for (i in 1:length(n)) {
    for (j in 1:length(d)) {
      sim[j,i] = buffon.needle(n[i],l,d[j],6)
    }
  }

  for (i in 1:3) {
    plot(d,sim[,i],
         ylim=c(min(sim[,i])-0.1,max(sim[,i])+0.1),
         ylab="Estimated pi",
         xlab="Distance between lines in floor",
         main=paste("Simulation with l=",l ,n[i],"needles"))
    abline(a=pi,b=0, col="red")
  }
}
```
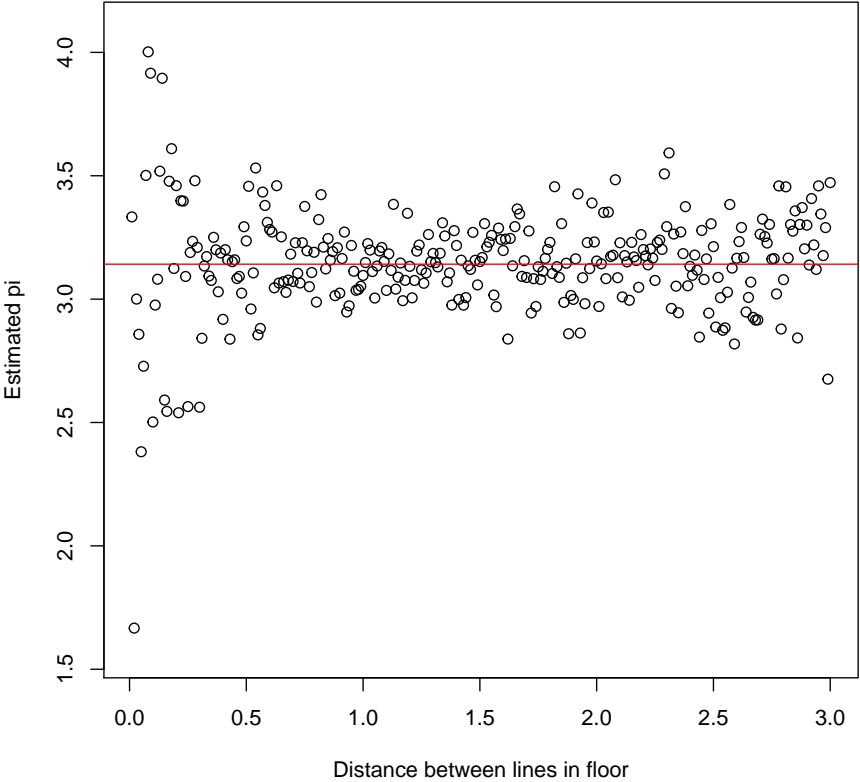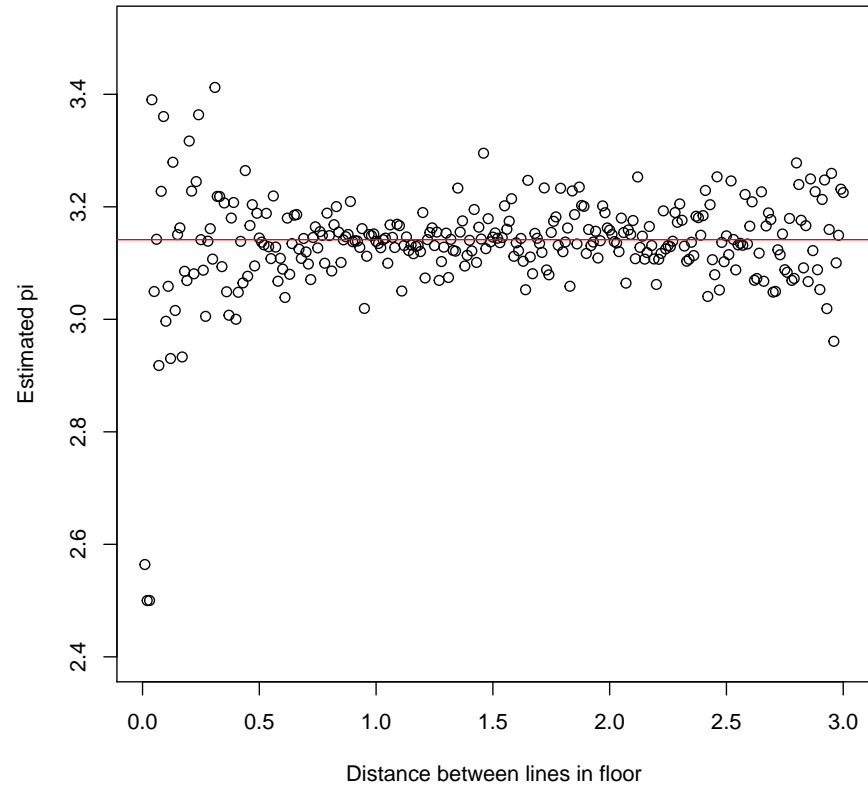
Appendix

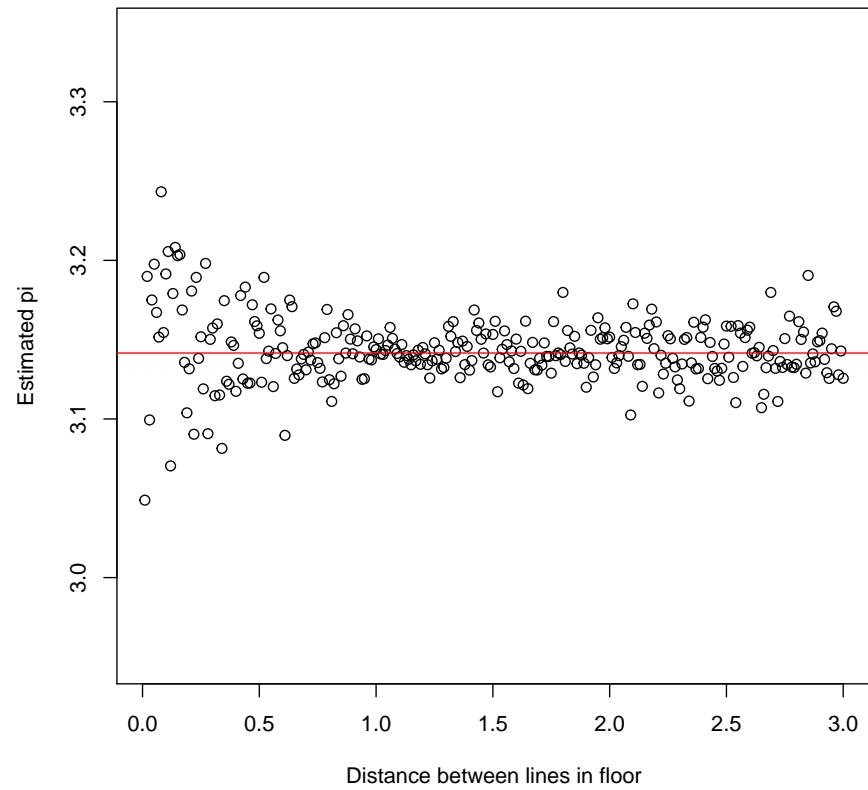**Simulation of needle length 1**

```
sim(1)
```

Simulation with l= 1 1000 needles

Estimated pi

Distance between lines in floor

5

Simulation with l= 1 10000 needles

Distance between lines in floor

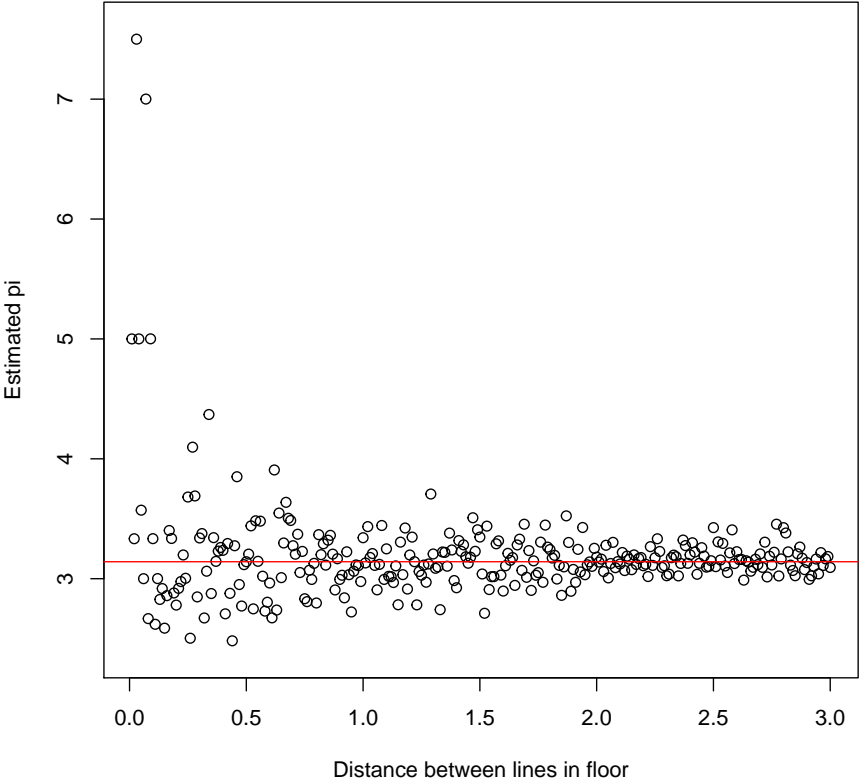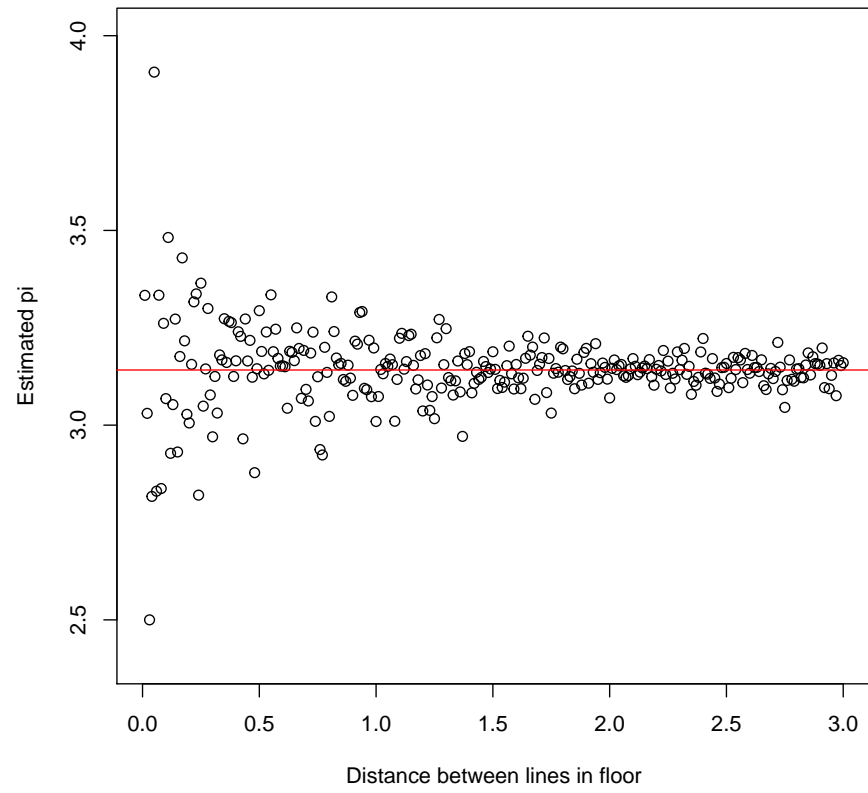**Simulation with l= 1 1e+05 needles**

Simulation of needle length 2

```
sim(2)
```

Simulation with l= 2 1000 needles

Simulation with l= 2 10000 needles

Estimated pi

Distance between lines in floor

9

Simulation with l= 2 1e+05 needles