2) $\mathbb{P}(t)_4$ = Esp. Vect. $[-1,1]$

$$|f\rangle_t \rightarrow \sum_{n=0}^{4} a_n t^n$$

Producto interno: $\langle f|g\rangle = \int_{-1}^{1} f(t)g(t)\,dt.$

OPerador: $\mathbb{T} = e^{\mathbb{D}} \equiv \exp(\mathbb{D}) \qquad \mathbb{D} = \frac{d}{dt}$

Base: $\{1, t, t^2, t^3, t^4\}$ y $\{P_0, P_1, P_2, P_3, P_4\}$

Polinomio de Legendre

a) $|f\rangle_t \leftrightarrow f(t) = 5t + 3t^2 + 4t^3.$

Base: $\{1, x, \frac{1}{2}(3x^2-1), \frac{1}{2}(5x^3-3x), \frac{1}{8}(35x^4-30x^2+3)$

$|f\rangle = C_1(1) + C_2 x + \frac{1}{2}C_3(3x^2-1) + C_4\frac{1}{2}(5x^3-3x) + \frac{1}{8}C_5(35x^4-30x^2+3)$

¿C_n?

$$C_n = \frac{2n+1}{2}\int_{-1}^{1} f(x)P_n(x)\,dx \qquad n = 0,1,2,3,4.$$

$$C_0 = \frac{2(0)+1}{2}\int_{-1}^{1}(5t+3t^2+4t^3)(1)\,dx = 1$$

$$C_1 = \frac{2(1)+1}{2}\int(5t+3t^2+4t^3)(x)\,dx = \frac{37}{5}$$

$$C_2 = \frac{2(2)+1}{2}\int(5t+3t^2+4t^3)\left(\frac{1}{2}(3x^2-1)\right)dx = 2$$

$$C_3 = \frac{2(3)+1}{2}\int(5t+3t^2+4t^3)\left(\frac{1}{2}(5x^3-3x)\right)dx = \frac{8}{5}$$

$$C_4 = \frac{2(4)+1}{2}\int(5t+3t^2+4t^3)\left(\frac{1}{8}(35x^4-30x^2+3)\right)dx = 0$$

$$|f\rangle_t = 1 + \frac{37}{5}(f(t)) + 2(f(t)) + \frac{8}{5}f(t) + 0\,f(t).$$

Matriz de transformación

$$1 + C_1(5t+3t^2+4t^3) + C_2(5t+3t^2+4t^3) + C_3(5t+3t^2+4t^3) = (F(t))$$

$$1 + C_1 5t + C_1 3t^2 + C_1 4t^3 + C_2 5t + C_2 3t^2 + C_2 t^3 + C_3 5t + C_3 3t^2 + C_3 4t^3 = F(t)$$

$$\left(\begin{array}{ccccc|ccccc} 1 & 0 & -1/2 & 0 & 3/8 & 1 & 0 & 0 & 0 & 0 \\ 0 & 1 & 0 & -3/2 & 0 & 0 & 1 & 0 & 0 & 0 \\ 0 & 0 & \frac{3}{2} & 0 & -\frac{30}{8} & 0 & 0 & 1 & 0 & 0 \\ 0 & 0 & 0 & 5/2 & 0 & 0 & 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 0 & 35/8 & 0 & 0 & 0 & 0 & 1 \end{array}\right)$$

$$= \begin{pmatrix} 1 & 0 & 1/3 & 0 & 1/5 \\ 0 & 1 & 0 & 3/5 & 0 \\ 0 & 0 & 2/5 & 0 & 4/7 \\ 0 & 0 & 0 & 2/5 & 0 \\ 0 & 0 & 0 & 0 & 8/35. \end{pmatrix}$$

b) Proyector:

Matriz Proyección:

Base $P_2 = \{1, t, t^2\} \rightarrow \begin{pmatrix} 1 \\ 0 \\ 0 \end{pmatrix}, \begin{pmatrix} 0 \\ 1 \\ 0 \end{pmatrix}, \begin{pmatrix} 0 \\ 0 \\ 1 \end{pmatrix}$

$\vec{X} = \{0, 5, 3, 4\}$

$$\text{Proy}_x P_2 = P(P^T P)^{-1} P^T X$$

$$P = \begin{pmatrix} 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 0 \end{pmatrix} \quad P^T = \begin{pmatrix} 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 0 \end{pmatrix} \quad P^{-1} \begin{pmatrix} 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 0 \end{pmatrix}$$

c)

Tomando en cuenta la prop: $(\exp(\mathbb{D}))^{\dagger} = \exp(\mathbb{D}^{\dagger})$

$\langle g|\mathbb{D}|f\rangle = \langle f|\mathbb{D}^{\dagger}|g\rangle$ "Partiendo de $\langle g|\mathbb{D}|f\rangle$

$$\rightarrow \int_{-1}^{1} f'(t)g(t)\,dt = f(t)g(t)\Big|_{-1}^{1} - \int_{-1}^{1} f(t)g'(t)$$

→ Aplicando el operador asignado

$$= f(t)g(t)\Big|_{-1}^{1} - \langle f|\mathbb{D}|g\rangle$$

$$\rightarrow \int_{-1}^{1} f(t)(-g'(t))\,dt = \langle f|\mathbb{D}^{\dagger}|g\rangle$$

En conclusión el operador es unitario, porque,

$\mathbb{D}^{\dagger} = \mathbb{D}^{-1}$