Tensor electromagnetico:

$$F_{\mu\nu} = \begin{pmatrix} 0 & E_x & E_y & E_z \\ -E_x & 0 & -cB^z & cB^y \\ -E_y & cB^z & 0 & -cB^x \\ -E_z & -cB^y & cB^x & 0 \end{pmatrix} \qquad \eta_{\mu\nu} = \begin{pmatrix} -1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \end{pmatrix}$$

$\mathbb{E} = (E^x, E^y, E^z)$

$\mathbb{B} = (B^x, B^y, B^z)$

Observador 1:

$\mathbb{E} = (E^x, 0, 0)$

$\mathbb{B} = (0, 0, 0)$

Ob. 2:

¿Que campos ve?

$\underbrace{\beta = V^i \hat{i}}_{\text{otro observador.}}$

I

$$F_{\mu\nu} = \begin{pmatrix} 0 & E_x & 0 & 0 \\ -E_x & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \end{pmatrix}$$

El tensor $F_{\mu\nu}$ es invariante bajo la transformación de la metrica ($\eta_{\mu\nu}$)

$F_{\mu'\nu'} \rightarrow$ Tensor de campos del observador "2"

$$F_{\mu'\nu'} = \Lambda^{\mu}{}_{\mu'} \Lambda^{\nu}{}_{\nu'} F_{\mu\nu}$$

$$F_{\mu'\nu'} = \Lambda^{0}{}_{\mu'} \Lambda^{1}{}_{\nu'} F_{01} + \Lambda^{1}{}_{\mu'} \Lambda^{0}{}_{\nu'} F_{10}$$

$$F_{\mu\nu} = -F_{\nu\mu}$$

$$\downarrow$$

$$F_{\mu'\nu'} = \left(\Lambda^{0}{}_{\mu'} \Lambda^{1}{}_{\nu'} - \Lambda^{1}{}_{\mu'} \Lambda^{0}{}_{\nu'} \right) F_{01} \quad (1)$$

- Hay 6 Componentes que bajo la regla de antisimetria son iguales.

Expandiendo 1)

$$F^{01} = (\Lambda^0_0 \Lambda^1_0) F^{00} + (\Lambda^0_0 \Lambda^1_1) F^{01} + (\Lambda^0_1 \Lambda^1_0) F^{10} + (\Lambda^0_1 \Lambda^1_1) F^{11}$$

$$F^{01} = (\Lambda^0_0 \Lambda^1_0) F^{00} + (\Lambda^0_0 \Lambda^1_1 - \Lambda^0_1 \Lambda^1_0) F^{01} + (\Lambda^0_1 \Lambda^1_1) F^{11}$$

Reemplazando de la matriz de lorentz y de el tensor electromagnetico:

$$F^{01} = \underbrace{(\gamma\cdot(-\gamma\beta))\cdot 0}_{0} + (\gamma\cdot\gamma - (-\gamma\beta\cdot -\gamma\beta))E_x + \underbrace{(-\gamma\beta\cdot\gamma)0}_{0}$$

$$F^{01} : (\gamma^2 - (\gamma^2\beta^2))E_x = \gamma^2(1-V^2)E_x.$$

Realizando las demas expansiones de $F^{02}, F^{03}, F^{23}, F^{31}, F^{12}$. Las transformaciones quedarian:

$$F^{02} : (\Delta^0_0\,\Delta^1_2 - \Delta^1_0\,\Delta^0_2)F_{01}.$$

$$F^{02} : (\gamma\cdot 0 - \gamma 0)E_x = 0.$$

$$F^{03} : (\Delta^0_0\,\Delta^1_3 - \Delta^1_0\,\Delta^0_3)F_{01}$$

$$F^{03} = (\gamma\cdot 0 - 0\cdot 0)E_x = 0.$$

$$F^{23} = (\Delta^0_2\,\Delta^1_3 - \Delta^1_2\,\Delta^0_3)F_{01}$$

$$F^{23} = (-\gamma\beta\cdot 0 - \gamma\cdot 0)E_x = 0.$$

$$F^{31} = (\Delta^0_3\,\Delta^1_1 - \Delta^1_3\,\Delta^0_1)F_{10}$$

$$F^{31} = (0\cdot\gamma - 0\cdot -\gamma\beta)E_x = 0$$

$$F^{12} = (\Delta^0_1\,\Delta^1_2 - \Delta^1_1\,\Delta^0_2)F_{10}$$

$$F^{12} = (-\gamma\beta 0 - \gamma\cdot 0)E_x = 0$$

Reemplazando los valores en el tensor $F^{\mu'\nu'}$

$$F^{\mu'\nu'} = \begin{pmatrix} 0 & -(\gamma^2(1-V^2))E_x & 0 & 0 \\ (\gamma(1-V^2)E_x) & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \end{pmatrix}$$

De está forma queda demostrado que el segundo observado tambien ve unicamente el campo electrico la componente en x, pero el modulo de E se incrementa V, veces.

B) $\mu_0 = 4\pi \cdot 10^{-7} \left(\frac{H}{m}\right)$

$\nabla \times \mathbb{B} - \frac{\partial}{\partial t} \mathbb{E} = 4\pi J$

$\nabla \cdot \mathbb{E} = 4\pi \rho$ → se pueden expresar: $\frac{\partial}{\partial x^{\nu}} F^{\mu\nu} \equiv F^{\mu\nu} \quad \nu = 4\pi J^{\mu}$

$\mathbb{E} = (E^x, E^y, E^z)$, $\mathbb{B} = (B^x, B^y, B^z)$, $J^{\mu} = (c\rho, \mathbb{J})$ $\mathbb{J} = (J^x, J^y, J^z)$

$$F^{\mu\nu} = \begin{bmatrix} 0 & -E_x & E_y & E_z \\ -E_x & 0 & -cB^z & cB^y \\ -E_y & cB^z & 0 & -cB^x \\ -E_z & -cB^y & cB^x & 0 \end{bmatrix}$$

→ Tensor de Maxwell se puede calcular:

$$F^{\mu\nu} = \frac{\partial A_\nu}{\partial x_\mu} - \frac{\partial A_\mu}{\partial x_\nu} \begin{cases} \nu = \mu \to 0 \\ \nu \neq \mu \begin{cases} 1 \\ -1 \end{cases} \end{cases}$$

J → Representa la fuente del campo $F^{\mu\nu}$

Ya que el tensor de Maxwell, es antisimetrico, entonces,

(6) $\frac{\partial F^{\mu\nu}}{\partial x^{\nu}}$ → Divergencia de $F^{\mu\nu}$

$$\frac{\partial F^{\mu\nu}}{\partial x^{\nu}} = 4\pi J^{\mu}$$

Expandiendo $\mu = 1$

$$\frac{\partial F^{\mu\nu}}{\partial x^{\nu}} = \cancel{\frac{\partial F^{11}}{\partial x^1}}^{0} + \frac{\partial F^{12}}{\partial x^2} + \frac{\partial F^{13}}{\partial x^3} + \frac{\partial F^{14}}{\partial x^4} = 4\pi J^4$$

$$= \underbrace{0 + \frac{-\partial E_x}{\partial x^2} + \frac{-\partial E_y}{\partial x^3} + \frac{-\partial E^z}{\partial x^4}}_{\nabla \cdot E = 4\pi\rho} = 4\pi\rho$$

$\rightarrow \nabla = \left(\frac{\partial}{\partial x^1}, \frac{\partial}{\partial x^2}, \frac{\partial}{\partial x^3}\right)$ y $E = (E_x, E_y, E_z)$

Para obtener $\nabla \times \mathbb{B} - \frac{\partial \vec{E}}{\partial t} = 4\pi \vec{J}$, se va a usar el resultado anterior:

$$\frac{\partial F^{\mu\nu}}{\partial x^{\nu}} = \mu_0 J^{\mu}$$ pero ahora $\mu = 2,3,4$.

$\mu = 2$

$$\frac{\partial F^{21}}{\partial x^1} + \frac{\partial F^{22}}{\partial x^2} + \frac{\partial F^{23}}{\partial x^3} + \frac{\partial F^{24}}{\partial x^4} = \mu_0 J^2$$

$$\rightarrow -\frac{\partial E_x}{\partial x^1} + 0 + \left(-c\frac{\partial B^z}{\partial x^3}\right) + \left(c\frac{\partial B^y}{\partial x^4}\right) = \mu_0 J^x$$

$\mu = 3$

$$\frac{\partial F^{31}}{\partial x^1} + \frac{\partial F^{32}}{\partial x^2} + \frac{\partial F^{33}}{x^3} + \frac{\partial F^{34}}{\partial x^4} = \mu_0 J^3$$

$$-\frac{\partial E_y}{\partial x^1} + c\frac{\partial B^z}{\partial x^2} + 0 + \left(-c\frac{\partial B^x}{\partial x^4}\right) = \mu_0 J^y$$

$\mu = 4$

$$\frac{\partial F^{41}}{\partial x^1} + \frac{\partial F^{42}}{\partial x^2} + \frac{\partial F^{43}}{\partial x^3} + \frac{\partial F^{44}}{\partial x^4} = \mu_0 J^4$$

$$-\frac{\partial E_z}{\partial x^1} - c\frac{\partial B^y}{\partial x^2} + c\frac{\partial B^x}{\partial x^3} + 0 = \mu_0 J^z$$

$$x = (x^1, x^2, x^3, x^4) = (t, x^2, x^3, x^4)$$

$$-\left(\frac{\partial E^x}{\partial t} + \frac{\partial E^y}{\partial t} + \frac{\partial E^z}{\partial t}\right) = -\frac{\partial}{\partial t}\mathbb{E}$$

$$c\left(-\frac{\partial B^z}{\partial x^3} + \frac{\partial B^y}{\partial x^4} + \frac{\partial B^z}{\partial x^2} - \frac{\partial B^x}{\partial x^4} - \frac{\partial B^y}{\partial x^2} + \frac{\partial B^x}{\partial x^3}\right) = c(\nabla \times \mathbb{B})$$

Por tanto;

$$(\nabla \times \mathbb{B}) - \frac{\partial}{\partial t}\mathbb{E} = 4\pi J^{\mu}$$

Así queda demostrado que las dos leyes se pueden expresar de forma comprimida

C) Identidad de Bianchi:

$$\frac{\partial F_{\mu\nu}}{\partial x^{\gamma}} + \frac{\partial F_{\nu\gamma}}{\partial x^{\mu}} + \frac{\partial F_{\gamma\mu}}{\partial x^{\nu}} = \partial_{\gamma} F_{\mu\nu} + \partial_{\mu} F_{\nu\gamma} + \partial F_{\gamma\mu}$$

a) $\nabla \cdot \mathbb{B} = 0$

b) $\nabla \times \mathbb{E} + \frac{\partial}{\partial t} \mathbb{B} = 0$

Usando:

$\mathbb{E} = (E^x, E^y, E^z)$

$\mathbb{B} = (B^x, B^y, B^z)$

$\nabla = \left(\frac{\partial}{\partial x} x, \frac{\partial}{\partial y} y, \frac{\partial}{\partial z}\right)$

$J^{\alpha} = (c\rho, \mathbb{J}) \quad \mathbb{J} = (J^x, J^y, J^z)$

$$\frac{\partial F_{\mu\nu}}{\partial x^{\gamma}} + \frac{\partial F_{\nu\gamma}}{\partial x^{\mu}} + \frac{\partial F_{\gamma\mu}}{\partial x^{\nu}} = 0$$

$\alpha = \nu = 1$
$\beta = \mu = 2$
$\gamma = \gamma = 3$

Reemplazando $\nu = 2$, $\mu = 3$ y $\gamma = 4$.

$$\frac{\partial F_{32}}{\partial x^4} + \frac{\partial F_{24}}{\partial x^3} + \frac{\partial F_{43}}{\partial x^2} = 0$$

F_{24}

F_{32}

F_{43}

$$\left[\frac{\partial cB^z}{\partial x^4} + \frac{\partial cB^y}{\partial x^3} + \frac{\partial cB^x}{\partial x^2}\right] = 0$$

Reorganizando:

$$\left(\frac{\partial}{\partial x^2}, \frac{\partial}{\partial x^3}, \frac{\partial}{\partial x^4}\right) \cdot (cB^x, cB^y, cB^z) = 0$$

$$\nabla \cdot \mathbb{B}_i = 0$$

Ahora, con la misma identidad de Bianchi pero los indices con otros valores seria:

$\nu = 1 \quad \mu = 2 \quad \gamma = 3$

$$\rightarrow \frac{\partial F_{21}}{\partial x^3} + \frac{\partial F_{13}}{\partial x^2} + \frac{\partial F_{32}}{\partial x^1} = 0$$

$$-\frac{\partial E_x}{\partial x^3} + \frac{\partial E_y}{\partial x^2} + \frac{\partial c B_z}{\partial x^1} = 0 \qquad X = (t, x^2, x^3, x^4)$$

$$\rightarrow c\frac{\partial B_z}{\partial t} - \frac{\partial E_z}{\partial x^3} + \frac{\partial E_y}{\partial x^2} \quad (1)$$ componentes en $\hat{k}$.

$\nu = 1, \quad \mu = 3, \quad \gamma = 4.$

$$\rightarrow \frac{\partial F_{31}}{\partial x^4} + \frac{\partial F_{14}}{\partial x^3} + \frac{\partial F_{43}}{\partial x^1} = 0$$

$$\rightarrow -\frac{\partial E_y}{\partial x^4} + \frac{\partial E_z}{\partial x^3} + \frac{\partial c B_x}{\partial x^1} = 0$$

$$\rightarrow c\frac{\partial B_x}{\partial t} + \frac{\partial E_z}{\partial x^3} - \frac{\partial E_y}{\partial x^4} = 0 \quad (2)$$ componentes en $\hat{\imath}$.

$\nu = 1, \mu = 2, \gamma = 4.$

$$\rightarrow \frac{\partial F_{21}}{\partial x^4} + \frac{\partial F_{14}}{\partial x^2} + \frac{\partial F_{42}}{\partial x^1} = 0$$

$$\rightarrow \cdot -1 \quad -\frac{\partial E_x}{\partial x^4} + \frac{\partial E_z}{\partial x^2} + \left(-\frac{c\partial B_y}{\partial x^1}\right) = 0 \quad \cdot -1$$

$$c\frac{\partial B_y}{\partial t} + \frac{\partial E_x}{\partial x^4} - \frac{\partial E_z}{\partial x^2} = 0 \quad (3)$$ componente en y

Asociando 1, 2, 3:

$$\left(-\frac{\partial E_z}{\partial x^3} + \frac{\partial E_y}{\partial x^2}\right)\hat{k} + \left(\frac{\partial E_z}{\partial x^3} - \frac{\partial E_y}{\partial x^4}\right)\hat{\imath} + \left(\frac{\partial E_x}{\partial x^4} - \frac{\partial E_z}{\partial x^2}\right)$$

$$+ \frac{\partial}{\partial t}(B_x, B_y, B_z) = 0$$

$$= \nabla \times \mathbb{E} + \frac{\partial}{\partial t}\mathbb{B} = 0$$