Ejercicios

I) $(P^{\dagger})^{-1} = C \Rightarrow C\,C^{-1} = \mathbb{I} \Rightarrow C^{-1} = P^{\dagger}$

$\Rightarrow \left[(P^{\dagger})^{-1} P^{\dagger}\right]^{-\dagger} = \mathbb{I}^{-\dagger} \Rightarrow P\left[(P^{\dagger})^{-1}\right]^{\dagger} = \mathbb{I}$

$\Updownarrow$

$(P^{\dagger})^{-1} = (P^{-1})^{\dagger}$

II) $(\mathbb{P}\mathbb{Q})^{-1} \Rightarrow \mathbb{P}\mathbb{Q} = C \Rightarrow C\,C^{-1} = \mathbb{I}$

$\Rightarrow (\mathbb{P}\mathbb{Q})(\mathbb{P}\mathbb{Q})^{-1} \Rightarrow$ Si $(\mathbb{P}\cdot\mathbb{Q})^{-1} = \mathbb{Q}^{-1}\mathbb{P}^{-1} \Rightarrow \mathbb{P}\mathbb{Q}\mathbb{Q}^{-1}\mathbb{P}^{-1} = \dots$

$\dots = \mathbb{P}\underbrace{\mathbb{I}\,\mathbb{P}^{-1}}_{\mathbb{P}^{-1}} = \mathbb{P}\,\mathbb{P}^{-1} = \mathbb{I} \Leftrightarrow (\mathbb{P}\mathbb{Q})^{-1} = \mathbb{Q}^{-1}\mathbb{P}^{-1}$

III) Si $[\mathbb{P},\mathbb{Q}] = 0 \Rightarrow \mathbb{P}\mathbb{Q} = \mathbb{Q}\mathbb{P}$ ($\mathbb{Q}^{-1}$ a izquierdas)

$\Rightarrow \mathbb{Q}^{-1}\mathbb{P}\mathbb{Q} = \underbrace{\mathbb{Q}^{-1}\mathbb{Q}}_{\mathbb{I}}\mathbb{P} \Rightarrow \mathbb{P} = \mathbb{Q}^{-1}\mathbb{P}\mathbb{Q}$

($\mathbb{Q}^{-1}$ a derechas) $\Rightarrow \mathbb{P}\mathbb{Q}^{-1} = \mathbb{Q}^{-1}\mathbb{P}\underbrace{\mathbb{Q}\mathbb{Q}^{-1}}_{\mathbb{I}}$

$\Rightarrow \mathbb{P}\mathbb{Q}^{-1} = \mathbb{Q}^{-1}\mathbb{P}$

IV) $(e^{\mathbb{P}})^{\dagger} \equiv \left[\sum_{n=0}^{\infty} \frac{\mathbb{P}^n}{n!}\right]^{\dagger} = \left[\mathbb{I} + \mathbb{P} + \frac{\mathbb{P}^2}{2!} + \dots + \frac{\mathbb{P}^n}{n!}\right]^{\dagger}$

Por propiedad

$(\mathbb{A}+\mathbb{B})^{\dagger} = \mathbb{A}^{\dagger} + \mathbb{B}^{\dagger} \Rightarrow \left[\mathbb{I}^{\dagger} + \mathbb{P}^{\dagger} + \left(\frac{\mathbb{P}^2}{2!}\right)^{\dagger} + \dots + \left(\frac{\mathbb{P}^n}{n!}\right)^{\dagger}\right]$

$= \left[\sum_{n=0}^{\infty} \left(\frac{\mathbb{P}^n}{n!}\right)^{\dagger}\right] = e^{\mathbb{P}^{\dagger}}$

V) $\mathbb{P} e^{\mathbb{Q}} \mathbb{P}^{-1} \equiv \mathbb{P}\left[\sum_{n=0}^{\infty} \frac{\mathbb{Q}^n}{n!}\right]\mathbb{P}^{-1} = \mathbb{P}\left[\mathbb{I} + \mathbb{Q} + \frac{\mathbb{Q}^2}{2!} \dots\right]\mathbb{P}^{-1}$

Por composición de operadores (distributividad)

$: \mathbb{P}\mathbb{I}\mathbb{P}^{-1} + \mathbb{P}\mathbb{Q}\mathbb{P}^{-1} + \mathbb{P}\frac{\mathbb{Q}^2}{2!}\mathbb{P}^{-1} + \dots + \mathbb{P}\frac{\mathbb{Q}^n}{n!}\mathbb{P}^{-1}$

$= \sum_{n=0}^{\infty} \mathbb{P}\frac{\mathbb{Q}^n}{n!}\mathbb{P}^{-1} = e^{\mathbb{P}\mathbb{Q}\mathbb{P}^{-1}}$

b) $\mathbb{A} = \mathbb{A}^\dagger \Rightarrow \widetilde{\mathbb{A}} = \mathbb{U}^{-1}\mathbb{A}\mathbb{U} = \mathbb{U}^{-1}\mathbb{A}^\dagger\mathbb{U}$

$\Rightarrow \widetilde{\mathbb{A}}^\dagger = (\mathbb{U}^{-1}\mathbb{A}\mathbb{U})^\dagger = \mathbb{U}^\dagger\mathbb{A}^\dagger(\mathbb{U}^{-1})^\dagger$

Pero $\mathbb{U}^\dagger = \mathbb{U}^{-1} \Rightarrow \mathbb{U}^{-1}\mathbb{A}^\dagger(\mathbb{U}^{-1})^{-1} = \mathbb{U}^{-1}\mathbb{A}\mathbb{U} = \widetilde{\mathbb{A}}$

c) $\mathbb{A} = \mathbb{A}^\dagger \Rightarrow e^{i\mathbb{A}} = \left[\sum_{n=0}^{\infty} \frac{(i\mathbb{A})^n}{n!}\right] = \left[\sum_{n=0}^{\infty} \frac{(i\mathbb{A}^\dagger)^n}{n!}\right]$

$= \left[\sum_{n=0}^{\infty} \frac{(-i\mathbb{A})^n}{n!}\right] = \left[\mathbb{I} - i\mathbb{A} + \frac{\mathbb{A}^2}{2!} + \frac{i\mathbb{A}^3}{3!} - \frac{\mathbb{A}^4}{4!} \dots \pm \frac{i\mathbb{A}^n}{n!}\right]$

d) $\mathbb{K}^\dagger = -\mathbb{K} \Rightarrow \widetilde{\mathbb{K}} = \mathbb{U}^{-1}\mathbb{K}^\dagger\mathbb{U} = -\mathbb{U}^{-1}\mathbb{K}\mathbb{U}$

$\Rightarrow \widetilde{\mathbb{K}}^\dagger = (\mathbb{U}^{-1}\mathbb{K}\mathbb{U})^\dagger = \mathbb{U}^\dagger\mathbb{K}^\dagger(\mathbb{U}^{-1})^\dagger$

$= -\mathbb{U}^{-1}\mathbb{K}\mathbb{U} = \widetilde{\mathbb{K}}$

$\widetilde{\mathbb{K}} = i\mathbb{A}$ y si $i\mathbb{A}$ es hermítico, entonces

$i\mathbb{A} = \mathbb{U}^{-1} i\mathbb{A}\mathbb{U}$

e) $(AB)^\dagger = B^\dagger A^\dagger$ y si $A^\dagger = A$ y $B = B^\dagger$

$\Rightarrow (AB)^\dagger = BA$ y si conmutan

$(AB)^\dagger = AB$

f) I) $S = -S^\dagger \Rightarrow [\mathbb{I}-S, \mathbb{I}+S] = \ldots$

$\ldots = (\mathbb{I}-S)(\mathbb{I}+S) - (\mathbb{I}+S)(\mathbb{I}-S) = \ldots$

$\ldots = \mathbb{I} + S - S - SS - (\mathbb{I} - S + S - SS) = \ldots$

$\ldots = \cancel{\mathbb{I}} \cancel{+S} \cancel{-S} - \cancel{SS} \cancel{-\mathbb{I}} + \cancel{S} - \cancel{S} + \cancel{SS} = 0$

II) $(\mathbb{I}-S)(\mathbb{I}+S) = \mathbb{I} \cancel{+S} \cancel{-S} - SS = \mathbb{I} - SS$

$= \mathbb{I} + SS^\dagger \Rightarrow \left[\mathbb{I} + SS^\dagger\right]^\dagger = \mathbb{I} + (SS^\dagger)^\dagger$

$= \mathbb{I} + (S^\dagger)^\dagger S^\dagger = \mathbb{I} + SS^\dagger$

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$(\mathbb{I}-S)(\mathbb{I}+S)^{-1} \Rightarrow \left[(\mathbb{I}-S)(\mathbb{I}+S)^{-1}\right]^{-1} = \ldots$

$\ldots = \left[(\mathbb{I}+S)^{-1}\right]^{-1} (\mathbb{I}-S)^{-1} = (\mathbb{I}+S)(\mathbb{I}-S)^{-1}$

$\Rightarrow$

$\gamma \left[(\mathbb{I}-S)(\mathbb{I}+S)^{-1}\right]^T = \left[(\mathbb{I}+S)^{-1}\right]^T (\mathbb{I}-S)^T$

Invertible

$\dagger = T$ en transformaciones $\in \mathbb{R}$

$$= \left[(\mathbb{I}+S)^T\right]^{-1}(\mathbb{I}+S) = (\mathbb{I}-S)^{-1}(\mathbb{I}+S)$$

$$\uparrow$$

$$S = -S^T$$

$$y$$

$$\mathbb{I} = \mathbb{I}^T$$

2) $\mathbb{R} = \begin{pmatrix} \cos\theta & \operatorname{sen}\theta \\ -\operatorname{sen}\theta & \cos\theta \end{pmatrix}$

$$\Rightarrow (\mathbb{I}-S)(\mathbb{I}+S)^{-1} = \left[\begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix} - \begin{pmatrix} a & b \\ c & d \end{pmatrix}\right]\left[\begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix} + \begin{pmatrix} a & b \\ c & d \end{pmatrix}\right]^{-1}$$

$$= \begin{pmatrix} 1-a & -b \\ -c & 1-d \end{pmatrix}\begin{pmatrix} 1+a & b \\ c & 1+d \end{pmatrix}^{-1} = \frac{1}{\det S}\begin{pmatrix} 1-a & -b \\ -c & 1-d \end{pmatrix}\begin{pmatrix} d+1 & -b \\ -c & a+1 \end{pmatrix}$$

$$= \frac{1}{\det S}\begin{pmatrix} -a+bc-ad+d+1 & -2b \\ -2c & a+bc-ad-d+1 \end{pmatrix} = \begin{pmatrix} \cos\theta & \operatorname{sen}\theta \\ -\operatorname{sen}\theta & \cos\theta \end{pmatrix}$$

$$\begin{cases} -2b = \operatorname{sen}\theta \wedge -2c = \operatorname{sen}\theta \Rightarrow b = -c \\ -a+bc-ad+d+1 = \cos\theta \Rightarrow -a-b^2-ad+d+1 = \cos\theta \\ a+bc-ad-d+1 \Rightarrow a-b^2-ad-d+1 = \cos\theta \end{cases}$$

$$\Rightarrow -a \cancel{-b^2} \cancel{-ad} + d \cancel{+1} = a \cancel{-b^2} \cancel{-ad} - d \cancel{+1} \Rightarrow a = d$$