CLASSIFICATION OF CONVEX ANCIENT SOLUTIONS TO FREE BOUNDARY CURVE SHORTENING FLOW IN CONVEX DOMAINS.

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Abstract. We classify convex ancient curve shortening flows with free boundary on general bounded convex domains.

Contents

1. Introduction

A smooth one-parameter family of immersions $X : M_1 \times [0,T) \to \mathbb{R}^2$ is said to evolve via curve shortening flow if it satisfies

$$\partial_t X(u,t) = \vec{\kappa} \tag{1}$$

where $\vec{\kappa}(u,t)$ is the curvature vector of $\Gamma_t := X(M_1,t)$. It was shown by Gage and Hamilton [GH86] that, given any compact, convex, embedded initial condition Γ_0, there exists a solution to (1), which exists on a maximal time interval, and further, Grayson extended this result for an arbitrary embedded closed curve. A solution to curve shortening flow is called *ancient* if it exists on some time interval which contains an interval of the form $(-\infty,a]$ for some $a<\infty$, which by time translation we can take to be 0. Examples of ancient solutions include the stationary line, the shrinking circle, the Angenent oval and the grim reaper. The free boundary curve shortening flow is the following Neumann problem:

$$\begin{cases} \partial_t X(u,t) = \vec{\kappa} \quad \text{on } \overset{\circ}{M}_1 \\ X_t(\partial M_1) \subset \partial\Omega \\ \langle \nu, \nu^\Omega \circ X \rangle = 0 \ \text{ on } \partial M_1, \end{cases} \tag{2}$$

where $\Omega \subset \mathbb{R}^2$ is some closed and connected domain with boundary $\partial\Omega$. That is, the endpoints of the family Γ_t remain orthogonal to a fixed supporting curve $\partial\Omega$. It has been shown that convex curves which lie inside a convex domain, with free boundary with respect to $\partial\Omega$, remain convex with respect to the free boundary curve shortening flow and moreover, shrink to a point on the boundary curve [Sta96a, Sta96b]. Very recently, Langford and Zhu [LZ23] proved a Grayson-type theorem in the free boundary case, by proving that embedded curves converge in infinite time to a (unique) "critical chord", or contract in finite time to a "round half-point" on $\partial\Omega$. The classification of convex ancient solutions to the free boundary curve shortening flow was initiated by the first named author and Langford [BL], who proved that, up to rotation, there is a unique solution to this problem on the disk. In this paper we will expand those methods in order to produce the following analogous result on strictly convex domains of $\mathbb{R}^2$. In this more general setting, we can no longer rely on the existence of certain symmetries of our free boundary curve, and so the barrier method used in [BL] needs to be treated with more care. Similarly, in order to achieve boundary height estimates, we use a Taylor expansion argument on the free boundary curve, contrast to what was done in the case of a a disc, in which the relation $|\kappa_s| = \kappa$ is used at the boundary of the solution. With these adapted techniques, we are able to prove the following theorem:

Theorem 1.1. *Let Ω be a convex domain in $\mathbb{R}^2$ with smooth boundary. Then, modulo time translation, for each diameter of Ω, there exist precisely two convex, locally uniformly convex, ancient solutions to the free boundary curve shortening flow in Ω, one lying on each side of the diameter. By a diameter, we mean a line segment intersecting $\partial\Omega$ orthogonally.*

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2. Existence

In this section we will provide an explicit construction of a non-trivial ancient solution to the free boundary curve shortening flow emanating from a diameter of a convex domain. In the ensuing discussion, we shall let Ω be a strictly convex domain in $\mathbb{R}^2$ with smooth boundary and consider a diameter of Ω, D, that is, a line segment which intersects $\partial\Omega$ orthogonally. Note that such a diameter always exists; consider the segment which maximises the Euclidean distance amongst pairs of points on $\partial\Omega$. By scaling, translating and rotating, we may and henceforward will assume that D lies on the x-axis with endpoints $\pm e_1$.

2.1. **Barriers.**

We parameterise $\partial\Omega$ via the turning angle

$$\Phi : [-\tfrac{\pi}{2}, \tfrac{3\pi}{2}] \to \mathbb{R}^2 \tag{3}$$

so that the tangent and outward unit normal to $\partial\Omega$ at $\Phi(\omega)$ are given by $\tau^\Omega(\omega) = (\cos\omega, \sin\omega)$ and $\nu^\Omega(\omega) = (\sin\omega, -\cos\omega)$. Notice that with this parameterisation and under our assumption on the diameter D, $\Phi(\pm\frac{\pi}{2}) = \pm e_1$. Define $C_r^\pm$ to be two circles of radius r that lie inside Ω and are tangent to $\partial\Omega$ at the points $\pm e_1$ respectively, which we can always do since the boundary is smooth (see Figure 2). Moreover, we choose r to satisfy

$$4r \leq \kappa^\Omega(\omega) \leq \frac{1}{4r} \tag{4}$$

for all ω, the reason for which will be made apparent later.

Definition 2.1. *Let Γ be a convex curve intersecting a circle C at a point p, and consider the radial segment R passing through p and the centre of C. We shall say that Γ intersects C at* an acute angle *at p, if the tangent to Γ at p locally separates Γ and R inside C.*

Next we show that convex curves in Ω intersecting the boundary orthogonally near D with $y > 0$ intersect $C_r^\pm$ in acute angles. This will enable us to create upper barriers for a solution of (2).

Lemma 2.2. *Let Γ be a convex curve inside Ω which intersects the boundary orthogonally at two points in $\{r > y > 0\}$. Then Γ intersects C_r^- (resp. C_r^+) transversally at two points, and at the intersection point p with the smallest (resp. larger) x-coordinate, Γ intersects C_r^- (resp. C_r^+) at an acute angle.*

Proof. We prove the lemma for C_r^-, as the proof for C_r^+ is similar. Consider a parametrisation of C_r^- by turning angle

$$C : [-\tfrac{\pi}{2}, \tfrac{3\pi}{2}] \to \mathbb{R}^2 \,.$$

For any $\theta_0 \in (\pi, \frac{3\pi}{2})$, using the fact that $C(\frac{3\pi}{2}) = \Phi(\frac{3\pi}{2}) = -e_1$, we have

$$\begin{aligned} \langle e_1, \Phi(\theta_0)\rangle - \langle e_1, C(\theta_0)\rangle &= \int_{\frac{3\pi}{2}}^{\theta_0} \langle e_1, \Phi'(\theta) - C'(\theta)\rangle d\theta \\ &= \int_{\frac{3\pi}{2}}^{\theta_0} \cos\theta \left(\frac{1}{\kappa(\theta)} - r\right) d\theta \geq 0 \,, \end{aligned} \tag{5}$$

with the last inequality being true because of (4).

Since Γ is convex, we can parametrise it by turning angle, $\gamma := \gamma(\theta)$, and since $\Gamma \cap \partial\Omega \subset \{y > 0\}$, we can assume that there exists a point $x \in \Gamma$ so that $x = \gamma(0)$. Let $p_0 = \gamma(\omega_0)$ and $p_1 = \gamma(\omega_1)$ be the points of intersection of Γ with $\partial\Omega$ and C_r^- respectively with the smaller x-coordinates. Denoting by $\theta_1 \in (\frac{\pi}{2}, \frac{3\pi}{2})$ the angle for which $p_1 = C(\theta_1)$, the lemma is equivalent to showing that

$$(\cos\theta_1, \sin\theta_1) \cdot (\cos\omega_1, \sin\omega_1) < 0 \,, \tag{6}$$

that is, the tangent vectors of Γ and C_r^- at p_1 form an angle that is bigger than $\frac{\pi}{2}$.

Let $\theta_0 \in (\pi, \frac{3\pi}{2})$ be such that $\Phi(\theta_0) = p_0$. Then

$$(\cos\omega_0, \sin\omega_0) = (\sin\theta_0, -\cos\theta_0) \,,$$

and therefore, $\theta_0 = \omega_0 + \frac{3\pi}{2}$. Consider the tangent to Γ at p_0 and let p_2 be the point of intersection of this tangent and C_r^- with the smaller x-coordinate. Let $\theta_2 \in (\frac{\pi}{2}, \frac{3\pi}{2})$ be such that $p_2 = C(\theta_2)$. Then, using the convexity of Γ we have that

$$\langle p_0, e_1\rangle < \langle p_2, e_1\rangle < \langle p_1, e_1\rangle$$

and thus, using (5), we obtain $\theta_0 > \theta_2 > \theta_1$. Now, since $\theta_1 > \theta_0$, by convexity of Γ, we have

$$\omega_1 + \tfrac{3\pi}{2} > \theta_1 \,.$$

Recalling the domains of definition for ω_1 and θ_1, this implies (6). $\square$

Remark 2.3. It should be noted here (as it will be useful later) that the proof of Lemma 2.2 does not require that the curvature of C_r^- is constant but merely that the minimum curvature of C_r^- is bigger than the maximum of $\partial\Omega$ around the point of contact.

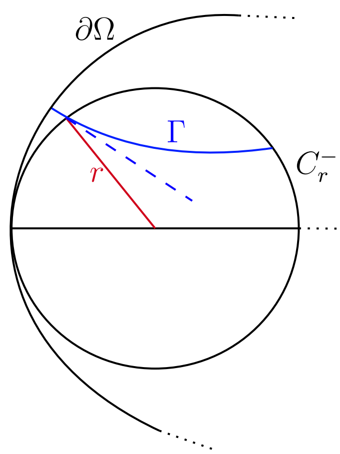

FIGURE 1. Γ intersects C_r^- at an Acute Angle.

With all this in mind, we are ready to construct upper barriers for solutions to the free boundary curve shortening flow in Ω. Consider the arcs

$$K_\omega := \{(x,y) \in B^1 : x^2 + (\csc\omega - y)^2 = \cot^2\omega\}, \omega \in (0, \tfrac{\pi}{2})$$

which intersect S^1 orthogonally at the points $(\pm\cos\omega, \sin\omega)$. In [BL], it was observed that if we set $\omega(t) = \arcsin e^{2t}$, $t \in (-\infty, 0)$, then the inward normal speed of $K_{\omega(t)}$ is no less than its curvature. By scaling and translating we see that the same is true for

$$K_t^\pm := rK_{\omega(r^{-2}t)} \pm (1-r)e_1,$$

which are now families of curves that intersect $C_r^\pm$ orthogonally. Define the curves

$$K_t := K_t^+ \cup L_t \cup K_t^- \tag{7}$$

where L_t is the line joining the point $K_t^- \cap C_r^-$ with the larger x-coordinate to the point of $K_t^+ \cap C_r^+$ with the smaller x-coordinate (see Figure 2). The maximum principle along with Lemma 2.2 gives us the following:

Proposition 2.4. *A convex solution to the free boundary curve shortening flow in Ω which lies below $K_{\hat t}$ and a time t_0 for some $\hat t \in (-\infty, 0)$ must lie below $K_{\hat t + t - t_0}$ for all $t > t_0$ such that $t - t_0 + \hat t < 0$.*

2.2. **Old-but-not-ancient solutions.** For each $\rho > 0$, choose a smooth curve Γ^ρ in Ω with the following properties:

(a) $\Gamma^\rho \subset \Omega \cap \{(x,y) \in \mathbb{R}^2 \mid y > 0\}$.

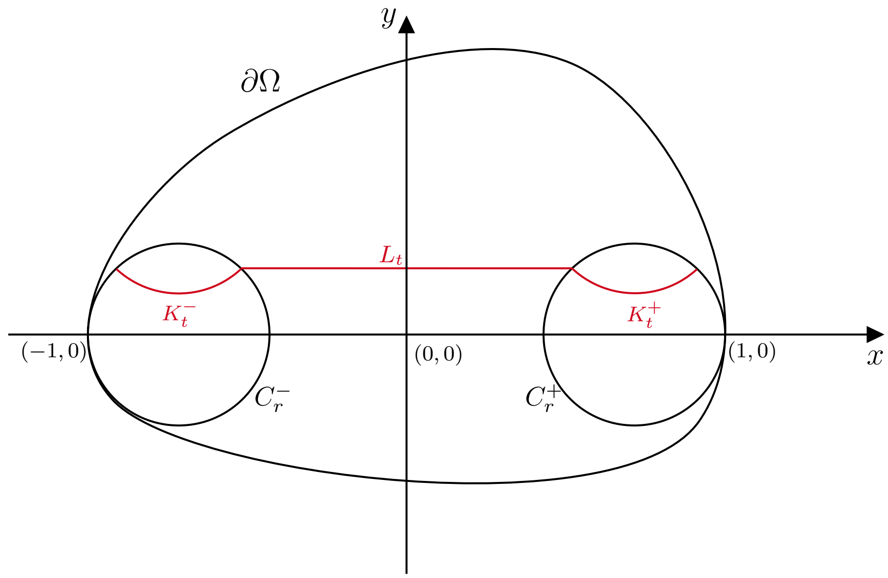

FIGURE 2. The Curves $K_t = K_t^+ \cup L_t \cup K_t^-$ in Ω.

(b) Γ^ρ meets $\partial\Omega$ orthogonally and lies below the line $y = \rho$.
(c) $\Gamma^\rho \cap \Omega$ is the relative boundary of a convex region $\Omega^\rho \subset \Omega \cap \{y > 0\}$.
(d) The curvature κ^ρ of Γ^ρ has a unique critical point at which κ^ρ is minimised.

Such a curve always exists. In particular we have the following.

Lemma 2.5. *Define the scaled and horizontally shifted Angenent oval*

$$A_{t_\rho}^{\lambda_\rho,\xi_\rho} := \{(x,y) \in \Omega \mid \sin(\lambda_\rho y) = e^{\lambda_\rho^2 t_\rho} \cosh(\lambda_\rho(x - \xi_\rho))\}.$$

Then, for all $\rho < r$, there exist λ_ρ, ξ_ρ and t_ρ such that the curve $\Gamma^\rho := A_{t_\rho}^{\lambda_\rho,\xi_\rho}$ satisfies (a)-(d). *Moreover, as $\rho \to 0$, $\lambda_\rho \to \lambda_0$, where λ_0 is defined to be the solution of*

$$\lambda^2 - \lambda(\kappa^\Omega(e_1) + \kappa^\Omega(-e_1)) \coth(2\lambda) + \kappa(e_1)\kappa^\Omega(-e_1) = 0,$$

which satisfies $\lambda > \max\{\kappa^\Omega(e_1), \kappa^\Omega(-e_1)\}$. Additionally, $\xi_\rho \to \xi_0$ where

$$\xi_0 = 1 - \frac{1}{\lambda_0} \cosh^{-1}\left(\frac{1}{\sqrt{1 - \frac{\kappa^\Omega(e_1)^2}{\lambda_0^2}}} \right).$$

The proof of the existence of such $A_{t_\rho}^{\lambda_\rho,\xi_\rho}$ and properties thereof can be found in the appendix.

The idea to construct an ancient solution is to take subsequential limits of the flows coming out of the curves Γ^ρ, as defined in Lemma 2.5, as $\rho \to 0$.

Lemma 2.6. *For each Γ^ρ, with ρ sufficiently small, there exists a K_{t_ρ} as defined by (7), such that K_{t_ρ} lies above Γ^ρ and $t_\rho \to -\infty$ as $\rho \to 0$.*

Proof. Given $\rho < r$, we can pick K_{t_ρ} so that it is tangent to the line $y = \rho$ at two points. It follows that K_{t_ρ} lies above Γ^ρ. As $\rho \to 0$, $\omega(r^{-2}t_\rho) = \arcsin(e^{2(r^{-2}t_\rho)}) \to 0$ which implies $t_\rho \to -\infty$. $\square$

The work of Stahl [Sta96a, Sta96b], now yields the following *old-but-not-ancient solutions.*

Lemma 2.7. *For each $\rho > 0$, there exists a solution to the free boundary curve shortening flow $\{\Gamma^\rho_t\}_{t\in[\alpha_\rho,0)}$ with $\Gamma^\rho_{\alpha_\rho} = \Gamma^\rho$. Furthermore, this solution satisfies the following:*

1. *Γ^ρ_t is convex and locally uniformly convex for each $t \in (\alpha_\rho, 0)$.*
2. *The curvature κ^ρ of Γ^ρ_t has only one critical point at which κ^ρ has a minimum.*
3. *$\alpha_\rho \to -\infty$ as $\rho \to 0$.*

Proof. Existence of a maximal solution to curve shortening flow out of Γ^ρ which meets $\partial\Omega$ orthogonally was proven by Stahl [Sta96a, Sta96b], similarly it was shown there that the solution remains convex, locally uniformly convex and shrinks to a point on the boundary at the final time. We obtain our solution $\{\Gamma^\rho_t\}_{t\in[\alpha_\rho,0)}$ via time translation. By [Sta96a], we know that at a boundary point $|\kappa^\rho_s| = \kappa^\rho\kappa^\Omega$ and so an application of Sturm's theorem [Ang88] to κ^ρ_s proves (2). Finally property (3) is an immediate consequence to Lemma 2.6 and Proposition 2.4. $\square$

Next, we wish to show that we can obtain estimates for the curvature and its derivatives for the the flows $\{\Gamma^\rho_t\}_{t\in[\alpha_\rho,0)}$ which are uniform in ρ. In the following, we fix $\rho > 0$, drop the super/sub-script ρ and fix the following notation;

$$\underline{\kappa}(t) := \min_{\Gamma_t} \kappa = \kappa(p(t)) \tag{8}$$

and

$$\{q^\pm(t)\} = \partial\Omega \cap \Gamma_t \tag{9}$$

with $x(q^-) < x(q^+)$. We define $\theta_\pm \in (0, \pi)$ to be such that if we parameterise Γ_t via the turning angle, $\gamma_t = \gamma_t(\theta)$, as described for Φ

in the beginning of subsection 2.1, then the domain of γ_t is $[-\theta_-, \theta_+]$. Finally, we will write

$$\overline{\kappa}_+(t) := \kappa(q^+(t)), \ \ \overline{\kappa}_-(t) := \kappa(q^-(t)).$$

Lemma 2.8. *For any old-but-not-ancient solution, with ρ sufficiently small, there exists a constant C such that for all $t < -\frac{2|\Omega|}{\pi}$,*

$$\sin\left(\frac{\theta_+(t)+\theta_-(t)}{2}\right) \leq Ce^{rt}$$

where r is defined as in (4).

Proof. We use the parametrisation of Γ_t by turning angle

$$\gamma_t : [-\theta_-, \theta_+] \to \Omega,$$

so that the unit tangent vector to the curve at $\gamma_t(\theta)$ is given by $\tau_t(\theta) = (\cos\theta, \sin\theta)$. Let $\underline{\theta} := \underline{\theta}(t)$ be such that $\gamma(\underline{\theta}(t)) = p(t)$. Since Γ_t is convex, we have

$$2 \geq \langle q^+ - p, e_1\rangle = \int_{\underline{\theta}}^{\theta_+} \frac{\cos u}{\kappa(u)} du \geq \frac{1}{\overline{\kappa}_+}\int_{\underline{\theta}}^{\theta_+} (\cos u) du = \frac{\sin\theta_+ - \sin\underline{\theta}}{\overline{\kappa}_+}$$

and similarly

$$2 \geq \langle p - q^-, e_1\rangle = \int_{-\theta_-}^{\underline{\theta}} \frac{\cos u}{\kappa(u)} du \geq \frac{1}{\overline{\kappa}_-}\int_{-\theta_-}^{\underline{\theta}} (\cos u) du = \frac{\sin\underline{\theta} + \sin\theta_-}{\overline{\kappa}_-},$$

which yields

$$\overline{\kappa}_+ + \overline{\kappa}_- \geq \frac{\sin\theta_+ + \sin\theta_-}{2}.$$

At the boundary $|\kappa_s| = \kappa\kappa^\Omega$, and so

$$\frac{d\theta_+}{dt} = \overline{\kappa}_+\kappa^\Omega, \ \ \frac{d\theta_-}{dt} = \overline{\kappa}_-\kappa^\Omega,$$

therefore,

$$\begin{aligned}\frac{d(\theta_+ + \theta_-)}{dt} &\geq 4r(\overline{\kappa}_+ + \overline{\kappa}_-) \geq 2r(\sin\theta_+ + \sin\theta_-)\\ &= 4r\tan\left(\tfrac{\theta_+ + \theta_-}{2}\right)\cos\left(\tfrac{\theta_+ + \theta_-}{2}\right)\cos\left(\tfrac{\theta_+ - \theta_-}{2}\right).\end{aligned}$$

Consider the time t_0 for which

$$\theta_+(t) + \theta_-(t) = \frac{\pi}{2}. \tag{10}$$

Note that by considering ρ sufficiently small, we can ensure that such a $t_0 \geq \alpha_\rho$ exists. Thus for any $t < t_0$ we have

$$\frac{d(\theta_+ + \theta_-)}{dt} \geq 2r\tan\left(\frac{\theta_+ + \theta_-}{2}\right),$$

which, after integrating from t to t_0, yields

$$\sin\left(\frac{\theta_+ + \theta_-}{2}\right) \leq Ce^{rt}.$$

Finally, to bound t_0, by monotonicity of $\theta_\pm(t)$, we have

$$\theta_+(t) + \theta_-(t) \geq \frac{\pi}{2}$$

for all $t \geq t_0$. Define $A(t)$ to be the area of the convex region enclosed by Γ_t and $\partial\Omega$. Since

$$-\frac{dA}{dt} = \theta_+(t) + \theta_-(t),$$

integrating from t_0 to 0 gives us

$$A(t_0) \geq -\frac{\pi}{2}t_0,$$

and since $A(t_0) \leq |\Omega|$, we obtain $-t_0 \leq \frac{2|\Omega|}{\pi}$. □

Remark 2.9. We remark here, as it will be used later, that we can also find an upper bound for t_0, the first time satisfying (10). Indeed, consider a continuous function $h : [0, \frac{\pi}{2}] \to \mathbb{R}_{\geq 0}$ sending θ to the area of the convex region enclosed by $\partial\Omega$ and the line segment joining points $p_\theta = \Phi(\frac{\pi}{2} + \theta), q_\theta = \Phi(\pi + \theta) \in \partial\Omega$, where $\Phi(\theta)$ is the angle parametrisation defined in (3). By smoothness of the boundary, $h(\theta)$ is continuous, and $h(\theta) > 0$. By compactness of $\overline{\Omega}$, $\inf_{\theta\in[0,\frac{\pi}{2}]} h(\theta) > 0$. Convexity of $\{\Gamma_t\}_{t\in[\alpha_\rho,0)}$ implies that $A(t_0) \geq \inf_{\theta\in[0,\frac{\pi}{2}]} h(\theta)$, independent of α_ρ. Thus, by using the fact that

$$-\frac{dA}{dt} = \theta_+(t) + \theta_-(t) \leq \pi$$

for all $t < 0$, integrating from t_0 to 0 and using the fact that $A(t_0) \geq \inf_{\theta\in[0,\frac{\pi}{2}]} h(\theta)$, gives the upper bound for t_0.

With Lemma 2.8, we can obtain uniform estimates which yield the following.

Proposition 2.10. *For any diameter D of Ω, there exist two convex, locally uniformly convex, ancient solutions to the free boundary curve shortening flow in Ω, one lying on each side of the diameter. As $t \to -\infty$ both of these solutions converge to D.*

Proof. For each ρ sufficiently small, consider the old-but-not-ancient solution $\{\Gamma_t^\rho\}_{t\in[\alpha_\rho,0)}$ as constructed in Lemma 2.7. If we represent Γ_t^ρ as a graph over the x-axis; $x \mapsto y^\rho(x,t)$, then convexity and Lemma 2.8 implies that $|y_x^\rho| = |\tan\theta|$ can be bounded uniformly in ρ, which in turn implies the gradient of this graph representation is uniformly bounded in ρ. Therefore, Stahl's (global in space, interior in time) Ecker-Huisken type estimates [Sta96a] imply uniform-in-ρ bounds for the curvature and its derivatives. Therefore, the limit

$$\{\Gamma_t^\rho\}_{t\in[\alpha_\rho,0)} \to \{\Gamma_t\}_{t\in(-\infty,0)}$$

exists in the smooth topology (globally in space on compact subsets of time), and the limit satisfies the curve shortening flow with free boundary in Ω. By our uniform bounds on t_0 in Remark 2.9, it is easily verified that this limit is not the trivial solution. Since each Γ_t is the limit of convex boundaries, we conclude that the limit is also convex at each time-slice, and, by [Sta96b, Corollary 4.5], Γ_t is also locally uniformly convex for all t. The second solution is achieved by repeating the construction in $\Omega \cap \{y < 0\}$. $\square$

3. Asymptotics for the Height

In this section, we fix the ancient solution, $\{\Gamma_t\}_{t\in(-\infty,0)}$, that we have constructed in Lemma 2.10 as a limit of old-but-not-ancient solutions $\{\Gamma_t^\rho\}_{t\in[-\alpha_\rho,0)}$ with initial condition $\Gamma_{\alpha_\rho}^\rho = \Gamma^\rho$ as defined in Lemma 2.5. We shall aim to show that as a graph, $\lim_{t\to-\infty} e^{-\lambda_0^2 t} y(x,t)$ exists in $(0,\infty)$ for all $x \in (-1,1)$. This asymptotic behaviour will be used to show uniqueness in the next section. The argument presented in this section follows the general idea of that in [BL, Section 2.4]. However, in this case we have to account for the lack of symmetry. For instance, the point for which the minimum curvature occurs for some time-slice of our constructed solution need not occur on the y-axis. Moreover, the boundary maximum principles become more complicated as the relation $\kappa_s = \kappa$ is no longer valid. In order to deal with the fact that $\partial\Omega$ doesn't have constant curvature, we use a Taylor expansion argument around D.

Lemma 3.1. *For all sufficiently negative time,*

$$|\langle \gamma, \nu\rangle| \le \Lambda\kappa$$

for some positive constant $\Lambda > 0$.

Proof. Note that $\lim\limits_{t\to-\infty} \langle\gamma,\nu\rangle - \Lambda\kappa = 0$. Assume there is a first space-time for which this is equal to some positive number ϵ. If this point occurs on the interior, then we would have

$$\begin{aligned} 0 \leq (\partial_t - \Delta)(\langle \gamma, \nu \rangle - \Lambda\kappa) &= \kappa^2 \langle \gamma, \nu \rangle - 2\kappa - \Lambda\kappa^3 \\ &= -2\kappa + \kappa^2(\langle \gamma, \nu \rangle - \Lambda\kappa) \\ &= \kappa(-2 + \epsilon\kappa) < 0 \end{aligned}$$

where we have estimated $\kappa, \epsilon < 1$, which is absurd. On the (right) boundary

$$(\langle \gamma, \nu \rangle - \Lambda\kappa)_s = \kappa(\langle \gamma, \tau \rangle - \Lambda\kappa^\Omega) < 0,$$

where we adjust Λ if needed (the computation at the left boundary point is analogous). The Hopf-boundary point lemma then implies that such a point cannot occur on the boundary. Taking $\epsilon \to 0$ proves the result. $\square$

Next we will prove curvature estimates, which will allow us to obtain estimates of the height function. As introduced in Section 2, we will use the notation $\overline{\kappa}_\pm$, $\theta_\pm$, but now for the ancient solution $\{\Gamma_t\}_{t \in (-\infty, 0)}$ instead of the old-but-not-ancient solutions $\{\Gamma_t^\rho\}_{t \in [\alpha_\rho, 0)}$. Similarly, we define $\overline{\kappa} = \max\{\overline{\kappa}_+, \overline{\kappa_-}\}$.

Lemma 3.2. *There exists a constant C, so that for sufficiently negative time,*

$$\underline{\kappa} \leq Ce^{rt},$$

where r is defined in (4).

Proof. Let $t < 0$, and let q_+, q_- and p be defined by (8) and (9). Then,

$$\langle q_+ - p, e_1 \rangle = \int_{\underline{\theta}}^{\theta_+} \frac{\cos(u)}{\kappa} du \leq \frac{\sin\theta_+ - \sin\underline{\theta}}{\underline{\kappa}}$$

and

$$\langle p - q_-, e_1 \rangle = \int_{-\theta_-}^{\underline{\theta}} \frac{\cos(u)}{\kappa} du \leq \frac{\sin\underline{\theta} + \sin\theta_-}{\underline{\kappa}}.$$

After adding these inequalities, we obtain

$$\underline{\kappa} \leq \frac{\sin\theta_+ + \sin\theta_-}{\langle q_+ - q_-, e_1 \rangle} \leq C \sin\left(\frac{\theta_+ + \theta_-}{2}\right)$$

for sufficiently negative time. The result then follows from Lemma 2.8. $\square$

Lemma 3.2 allows us to obtain the following sharper estimates.

Lemma 3.3. *There exist constants C_1, C_2 so that for sufficiently negative time*

$$\overline{\kappa} \leq C_1 e^{rt}$$

and

$$|\kappa_s| \leq C_2 \kappa,$$

where r is as defined in (4).

Proof. First note that the first inequality follows from the second; if $|\kappa_s| \leq C_2\kappa$, then $(\log \kappa)_s \leq C_2$. By integrating from the point of minimum curvature to the point of maximum curvature and noting that $\mathrm{Length}(\Gamma_t) \leq 2$, we have for small enough t

$$2C_2 \geq \log \frac{\overline{\kappa}}{\underline{\kappa}},$$

which implies $\overline{\kappa} \leq e^{2C_2}\underline{\kappa}$. The first inequality then follows from Lemma 3.2. For the second inequality, by Lemma 3.1, it suffices to show that $|\kappa_s| - C\kappa + \langle \gamma, \nu \rangle \leq 0$ for some constant C. For each $0 < \varepsilon \leq 1$, define a function

$$f_\varepsilon = |\kappa_s| - C\kappa + \langle \gamma, \nu \rangle - \varepsilon(e^t + 1).$$

Clearly $\lim_{t\to-\infty} f_\varepsilon = -\varepsilon < 0$. We will show that for sufficiently negative times, independent of ε, f_ε remains negative. First of all, we can ensure that $f_\varepsilon < 0$ at the boundary, since, by using Lemma 3.1,

$$\begin{aligned} f_\varepsilon &\leq \kappa\kappa^\Omega - C\kappa + \langle \gamma, \nu \rangle \\ &\leq \kappa(\kappa^\Omega - C + \Lambda), \end{aligned}$$

and so we can pick C to ensure that f_ε is always negative on the boundary, and we also ensure that $3C + \Lambda > 1$. Now assume there exists a first time at which $f_\varepsilon = 0$ at some point. Such a point must be in the interior and moreover, since $C > \Lambda$, at that point we must have $\kappa_s \neq 0$ and without loss of generality we assume $\kappa_s > 0$. If we denote by T_C the time for which $\kappa < \frac{1}{2(3C+\Lambda)}$ for all $t < T_C$, then for all $t < T_C$, at the first point for which $f_\varepsilon = 0$, we have

$$\begin{aligned} 0 &\leq (\partial_t - \Delta) f_\varepsilon \\ &\leq 4\kappa^2\kappa_s - C\kappa^3 + \kappa^2\langle \gamma, \nu \rangle - 2\kappa - \varepsilon e^t \\ &\leq 4\kappa^2(C\kappa - \langle \gamma, \nu \rangle + \varepsilon(e^t + 1)) - C\kappa^3 + \kappa^2\langle \gamma, \nu \rangle - 2\kappa - \varepsilon e^t \\ &= 3C\kappa^3 + 4\kappa^2\varepsilon + (4\kappa^2 - 1)\varepsilon e^t - 3\kappa^2\langle \gamma, \nu \rangle - 2\kappa \\ &\leq (3C + \Lambda)\kappa^3 + 4\kappa^2 + (4\kappa^2 - 1)\varepsilon e^t - 2\kappa \\ &< 0 \end{aligned}$$

for sufficiently negative time, where we have used used the Lemma 3.1 in the penultimate inequality above. This is a contradiction which completes the proof. $\square$

Remark 3.4. Notice that proof of Lemmas 3.1, 3.2 and 3.3 did not depend on the solution constructed in Lemma 2.10 and are therefore true for *any* ancient solution to the free boundary curve shortening flow in Ω.

With these curvature estimates in mind, we are ready to prove the following height estimates. These height estimates do depend on the particular ancient solution constructed in Lemma 2.10.

Lemma 3.5. *There are positive constants n and $\hat{n}$, depending only on the boundary curve Ω such that the ancient solution $\{\Gamma_t\}_{t\in(-\infty,0)}$ satisfies*

$$\frac{\kappa}{y}e^{ny} \geq \lambda_0^2 - \hat{n}e^{rt}$$

for sufficiently negative time.

Proof. We will prove that $\frac{\kappa}{y}e^{ny} \geq \lambda^2 - \hat{n}e^{rt}$ on each old-but-not-ancient solution $\{\Gamma_t^\lambda\}_{t\in(\alpha_t^\lambda,0)}$ for λ sufficiently close to λ_0. Indeed, on the initial curve we have

$$\frac{\kappa}{y}e^{ny} \geq \lambda^2\cos\theta \geq \lambda^2(1-\sin^2\theta) \geq \lambda^2 - Ce^{2rt},$$

where we have used Lemma 2.8 in the last inequality. Thus, the bound is always true at the initial curve, for sufficiently large $-\alpha_t^\lambda$, provided $\hat{n} \geq C$. We will now show that the bound remains true during the flow for at least a small period of time independent of λ. Presume there is a first time in which $\frac{\kappa}{y}e^{ny} + ne^{rt} = \lambda^2$. If such a point were to occur in the interior, then

$$\left(\frac{\kappa}{y}e^{ny}\right)_s = 0,$$

which means

$$\frac{(\kappa e^{ny})y_s^2}{y^3} = \frac{(\kappa e^{ny})_s y_s}{y^2}$$

at such a point. Therefore

$$\begin{aligned}
0 &\geq (\partial_t - \Delta)\left(\frac{\kappa}{y}e^{ny} + \hat{n}e^{rt}\right)\\
&= \frac{1}{y}(\partial_t - \Delta)\left(\kappa e^{ny}\right) + \hat{n}re^{rt}\\
&\geq \lambda^2 n\left(-n\sin^2\theta - 2\frac{\kappa_s}{\kappa}\sin\theta\right) + \hat{n}re^{rt}\\
&\geq \lambda^2 n(-n\sin^2\theta - 2C_2\sin\theta) + \hat{n}re^{rt}
\end{aligned}$$

for sufficiently negative time, where we have used Lemma 3.3 in the last inequality. Using Lemma 2.8, we can estimate $\sin\theta \leq Ce^{rt}$ and so

$$\begin{aligned} 0 &\geq (\partial_t - \Delta)\left(\frac{\kappa}{y}e^{ny} + \hat{n}e^{rt}\right) \\ &\geq \lambda^2 n e^{rt}\left(\frac{\hat{n}r}{\lambda^2 n} - 2C_2C - nC^2e^{rt}\right), \end{aligned}$$

which can be made positive for sufficiently negative time, provided we adjust $\hat{n}$ so that $\hat{n}r > 2C_2C\lambda^2 n$. This is a contradiction and so the first time in which $\frac{\kappa}{y}e^{ny} + \hat{n}e^{rt} = \lambda^2$ cannot happen in the interior. Similarly, if we presume that this minimum occurs at the right-most boundary point, then the Hopf boundary point lemma implies

$$0 > \left(\frac{\kappa}{y}e^{ny} + \hat{n}e^{rt}\right)_s = \frac{\kappa}{y}e^{ny}\left(\kappa^\Omega - \frac{\sin\theta}{y} + n\sin\theta\right). \tag{11}$$

However, we claim the right-hand side can be made positive by choosing n sufficiently large. Indeed, by considering angle parametrisation $\Phi(\theta)$ as defined by (3), and the Taylor expansion of the height function on the boundary curve, $\langle\Phi(\frac{\pi}{2}+\theta), e_2\rangle$, around $\theta = 0$, with respect to θ, we obtain

$$y(\theta) = \frac{\theta}{\kappa_+^\Omega} + O(\theta^2)$$

from which, it follows

$$\kappa^\Omega - \frac{\sin\theta}{y} \geq -C\sin\theta$$

for some constant C. Therefore,

$$\begin{aligned} 0 &> \left(\frac{\kappa}{y}e^{ny} + ne^{ct}\right)_s \\ &= \frac{\kappa}{y}e^{ny}\left(\kappa^\Omega - \frac{\sin\theta}{y} + n\sin\theta\right) \\ &\geq 0, \end{aligned}$$

so by choosing $n > C$, and the claim follows. The same argument after reflecting about the y-axis demonstrates that this minimum cannot occur at the left-most boundary point either. This completes the proof. □

Lemma 3.6. *There exists a positive constant m depending only on the boundary curve Ω such that the ancient solution $\{\Gamma_t\}_{t\in(-\infty,0)}$ satisfies*

$$\frac{\kappa}{y}e^{-ny} \leq \lambda_0^2 + me^{rt}$$

for sufficiently negative time.

Proof. We will prove the result on each old-but-not-ancient solution $\{\Gamma_t^\lambda\}_{t\in(\alpha_t^\lambda,0)}$ where λ is sufficiently close to λ_0. For sufficiently negative time we have

$$\begin{aligned}(\partial_t-\Delta)\left(\frac{\kappa}{y}e^{-ny}\right) &\le \frac{1}{y}\left(\kappa^3e^{-ny}+2\kappa_s n\sin\theta e^{-ny}\right)+2\left\langle\nabla\left(\frac{\kappa}{y}e^{-ny}\right),\frac{\nabla y}{y}\right\rangle\\ &\le C(\kappa^2+\sin\theta)\left(\frac{\kappa}{y}e^{-ny}\right)+2\left\langle\nabla\left(\frac{\kappa}{y}e^{-ny}\right),\frac{\nabla y}{y}\right\rangle\\ &\le Ce^{rt}\left(\frac{\kappa}{y}e^{-ny}\right)+2\left\langle\nabla\left(\frac{\kappa}{y}e^{-ny}\right),\frac{\nabla y}{y}\right\rangle,\end{aligned}$$

where we have used Lemma 3.3 in the last inequality. At the boundary, we have

$$\left(\frac{\kappa}{y}e^{-ny}\right)_s\le\frac{\kappa}{y}e^{-ny}\left(\kappa^\Omega-\frac{\sin\theta}{y}-n\sin\theta\right)$$

which is negative by our choice of n in the Lemma 3.5. Hence the Hopf boundary point lemma and the ODE comparison principle imply

$$\max_{\Gamma_t}\frac{\kappa}{y}e^{-ny}\le C\max_{\Gamma_{\alpha_t}}\frac{\kappa}{y}e^{-ny}.$$

But now,

$$(\partial_t-\Delta)\left(\frac{\kappa}{y}e^{-ny}\right)\le Ce^{rt}\max_{\Gamma_{\alpha_t}}\frac{\kappa}{y}e^{-ny}+2\left\langle\nabla\left(\frac{\kappa}{y}e^{-ny}\right),\frac{\nabla y}{y}\right\rangle,$$

which by the ODE comparison principle again, implies

$$\max_{\Gamma_t}\frac{\kappa}{y}e^{-ny}\le(1+Ce^{rt})\max_{\Gamma_{\alpha_t}}\frac{\kappa}{y}e^{-ny}.$$

Since on the initial time-slice $\Gamma_{a_t}=A_t^{\lambda,\xi}$,

$$\frac{\kappa}{y}e^{-ny}=\frac{\lambda\tan(\lambda y)}{y}\cos\theta e^{-ny},$$

the claim follows by letting $\lambda\to\lambda_0$. □

We are now able to prove the major result of this section.

Proposition 3.7. *If we parameterise Γ_t as a graph over the x-axis, then the limit*

$$A(x):=\lim_{t\to-\infty}e^{-\lambda_0^2t}y(x,t)$$

exists in $(0,\infty)$ for all $x\in(-1,1)$ on the constructed ancient solution.

Proof. First we show that

$$e^{-\lambda_0^2 t} y(x,t) < C \tag{12}$$

for some constant C and for all t small enough. By Lemma 3.5, for sufficiently negative time,

$$\begin{aligned}(\log y(t) - \lambda_0^2 t)_t &= \frac{\kappa}{y\cos\theta} - \lambda_0^2 \\ &\geq (e^{-ny} - 1)\lambda_0^2 - \hat{n} e^{rt}.\end{aligned}$$

We claim that there exists a positive constant a such that

$$f(y,t) := 1 - ae^{rt} - e^{-ny} \leq 0$$

for all t sufficiently negative. Indeed $\lim_{t\to-\infty} f = 0$, and for sufficiently negative time,

$$\begin{aligned}f_t &= -are^{rt} + \frac{n\kappa}{\cos\theta} e^{-ny} \\ &\leq -are^{rt} + 2nCe^{rt},\end{aligned}$$

where we have used Lemma 3.3 and have estimated $\sec\theta < 2$. Thus, we can pick a so that there exists a $T < 0$ so that for any $t < T$, we have $f_t \leq 0$. This proves the claim, and so

$$(\log y(t) - \lambda_0^2 t)_t \geq -(a\lambda_0^2 + \hat{n})e^{rt}.$$

Integrating from t to T proves that $\log y(t) - \lambda_0^2 t$ is uniformly bounded from above for all $t < T$, hence (12) is true for all t sufficiently small. Now we will prove

$$e^{-\lambda_0^2 t} y(x,t) > \tilde{C} \tag{13}$$

for some constant $\tilde{C} > 0$ and for all t small enough. By Lemma 3.6, for sufficiently negative time,

$$\begin{aligned}(\log y(t) - \lambda_0^2 t)_t &= \frac{\kappa}{y\cos\theta} - \lambda_0^2 \\ &\leq (\frac{e^{ny}}{\cos\theta} - 1)\lambda_0^2 + 4me^{rt}.\end{aligned}$$

Similar to before, we claim that there is a positive constant b such that

$$g := 1 + be^{rt} - \frac{e^{ny}}{\cos\theta} \geq 0$$

for all t sufficiently negative. Indeed, $\lim_{t\to-\infty} g = 0$ and, for sufficiently negative time,

$$\begin{aligned}g_t &= -\frac{e^{ny} n\kappa}{\cos^2\theta} - \frac{e^{ny}\theta_t \sin\theta}{\cos^2\theta} + bre^{rt} \\ &\geq -8nC_1 e^{rt} - 16Ce^{rt} + bre^{rt},\end{aligned}$$

where we have used Lemma 2.8, Lemma 3.3 and have used the estimates $-\frac{1}{\cos\theta} > -2$, $-e^{ny} > -2$ and $-\theta_t \geq -2$. Thus, we can pick $b > 0$ so that there exists $T < 0$ so that for any $t < T$, $g_t \geq 0$. This proves the claim, and so

$$(\log y(t) - \lambda_0^2 t)_t \leq (b\lambda_0^2 + 4m)e^{rt}.$$

Integrating from t to T proves that $\log y(t) - \lambda_0^2 t$ is bounded from below for all $t < T$, hence (13) is true for all t sufficiently small. Now, we show that the limit exists, from which, our uniform bounds above imply the result. By a direct calculation

$$\begin{aligned} \frac{d}{dt}\left(e^{-\lambda_0^2 t} y(x,t)\right) &= \left(\frac{\kappa}{y\cos\theta} - \lambda_0^2\right) y e^{-\lambda_0^2 t} \\ &\geq \tilde{C}\left((e^{-ny} - 1)\lambda_0^2 - \hat{n}e^{rt}\right) \\ &\geq \tilde{C}\left(-a\lambda_0^2 - \hat{n}\right) e^{rt}, \end{aligned}$$

where we have used (13) and Lemma 3.5. Therefore, we conclude that

$$\lim_{t\to-\infty} e^{-\lambda_0^2 t} y(x,t)$$

exists in $(0,\infty)$. □

4. Uniqueness

Let $\{\Gamma_t\}_{t\in(-\infty,0)}$ by *any* convex, locally uniformly convex ancient solution to the free boundary curve shortening flow in Ω. We may assume by Stahl's theorem [Sta96b] that Γ_t contracts to a point on the boundary $\partial\Omega$ as $t \to 0$.

Lemma 4.1. *Γ_t converges in C^∞ to a diameter as $t \to -\infty$.*

Proof. The proof of this is the same as that shown in [BL, Lemma 3.1], *mutatis mutandis.* □

Therefore, by scaling and translating, we may assume without loss of generality that the backwards limit is $D = [-1, 1]$ as assumed thus far.

Lemma 4.2. *There exists a constant C such that for sufficiently negative time*

$$\overline{\kappa} \leq C\underline{\kappa}$$

on the ancient solution Γ_t.

Proof. The proof of this is identical to that in Lemma 3.3 as per Remark 3.4. □

Lemma 4.3. *If we parameterise Γ_t as a graph over the x-axis, then there is a constant C such that*

$$\sup_{x\in(-1,1)}\left(\limsup_{t\to-\infty} e^{-\lambda_0^2 t}y(x,t)\right) < C,$$

Proof. Denote by $\{\hat{\Gamma}_t\}_{t\in(-\infty,0)}$ the solution constructed in Lemma 2.10. Then for all sufficiently negative time, $\Gamma_t \cap \hat{\Gamma}_t \neq \emptyset$. Indeed, if this were not true, then they would be disjoint for all times and therefore have to contract to the same point on the boundary at $t = 0$, which contradicts the avoidance principle. Therefore, for any t sufficiently negative, there exists an $x_0 \in (-1, 1)$ such that $y(x_0, t) < \hat{y}(x_0, t)$, and therefore, by Proposition 3.7

$$e^{-\lambda_0^2 t}y(x_0,t) \le e^{-\lambda_0^2 t}\hat{y}(x_0,t) < C,$$

Now let x be some other point in $(-1, 1)$. Then, by Lemma 4.2, there exists a constant $\tilde{C}$ such that for sufficiently negative time

$$y(x,t) = \int_{-\infty}^t y_t(x,t)dt = \int_{-\infty}^t \frac{\kappa(x,t)}{\cos\theta}dt \le \tilde{C}\int_{-\infty}^t \frac{\kappa(x_0,t)}{\cos\theta_0}dt = \tilde{C}y(x_0,t).$$

Hence, by once again using Proposition 3.7

$$e^{-\lambda_0^2 t}y(x,t) \le \tilde{C}e^{-\lambda_0^2 t}y(x_0,t) < C.$$

□

We will now examine the limiting behaviour of the height function on the general ancient solution.

Proposition 4.4. *For some constant A, we have*

$$e^{\lambda_0^2 t}y(x,t) \to A\left(\cosh(\lambda_0 x) + \frac{\kappa_1-\kappa_2}{2\lambda_0 - (\kappa_1+\kappa_2)\tanh\lambda_0}\sinh(\lambda_0 x)\right)$$

uniformly as $t \to -\infty$, where $\kappa_1 := \kappa^\Omega(1,0)$, $\kappa_2 := \kappa^\Omega(-1,0)$.

Proof. For each $\tau < 0$, let $y^\tau(x,t) := e^{-\lambda_0^2\tau}y(x,t+\tau)$ defined on the time-translated flow $\{\Gamma_t^\tau\}_{t\in(-\infty,-\tau)}$ where $\Gamma_t^\tau := \Gamma_{t+\tau}$. Lemma 4.3, implies a uniform bound for y^τ on $\{\Gamma_t^\tau\}_{t\in(-\infty,T]}$ for any $T \in \mathbb{R}$. Alaoglu's theorem therefore yields a sequence of times $\tau_j \to -\infty$ such that y^{τ_j} converges in the weak* topology as $j \to \infty$ to some $y^\infty \in L^2_{\mathrm{loc}}([-1,1]\times(-\infty,\infty))$. Since convexity and the boundary condition imply a uniform bound for $\nabla^\tau y^\tau$ on any time interval of the form $(-\infty, T]$, where ∇^τ is the gradient on Γ_t^τ, we may also arrange that the convergence in uniform in space at time zero, say. For any j, note that y^{τ_j} satisfies the following boundary value problem;

$$\begin{cases}(\partial_t - \Delta^{\tau_j})y^{\tau_j} = 0 \;\text{ in } \Gamma_t^\tau \\ \langle \nabla^{\tau_j} y^{\tau_j}, N\rangle = y \cdot f \;\text{ on } \partial\Gamma_t^\tau,\end{cases} \tag{14}$$

where $f = \frac{\sin\theta}{y}$, N is the outward unit normal to $\partial\Omega$ and Δ^τ is the Laplacian on Γ_t^τ. Since y^{τ_j} satisfies (14) then necessarily

$$\int_{-\infty}^{-\tau_j}\int_{\Gamma_t^{\tau_j}} y^{\tau_j}(\partial_t - \Delta^{\tau_j})^*\eta = 0$$

for all smooth η which are compactly supported in time and satisfy

$$\nabla^\tau \eta \cdot N = \eta \cdot f \;\text{ on } \partial\Gamma_t^{\tau_j},$$

where $(\partial_t - \Delta^{\tau_j})^* = -(\partial_t + \Delta^{\tau_j})$ is the formal L^2-adjoint of the heat operator. Since $\{\Gamma_t^{\tau_j}\}_{t\in(-\infty,-\tau_j)}$ converges uniformly in the smooth topology to the stationary interval $\{[-1,1]\times\{0\}\}_{t\in(-\infty,\infty)}$ as $j\to\infty$, we may parameterise each flow $\{\overline{\Gamma}_t^{\tau_j}\}_{t\in(-\infty,-\tau_j)}$ over $I := [-1,1]$ by a family of embeddings $\gamma_t^j : I\times(-\infty,-\tau_j)\to\Omega$ which converge in $C^\infty_{\mathrm{loc}}(I\times(-\infty,\infty))$ to the stationary embedding Γ^∞, which is characterised by $(x,t)\mapsto xe_1$. Given $\eta\in C_0^\infty(I\times(-\infty,\infty))$ satisfying $\eta_z(1) = \eta\kappa_1$, and $\eta_z(-1) = -\eta\kappa_2$ (recall that $f(e_1) = \kappa_1$ and $f(-e_1) = -\kappa_2$). Set $\eta^j = \phi^j\eta$, where $\phi^j : [-1,1]\times(-\infty,\tau_j)\to\mathbb{R}$ is defined by

$$\phi^j(z,t) = e^{s^j(z,t)},$$

where $s_z^j(z,t) = (|\gamma_z^j(z,t)|-1)f(\gamma^j(z))$. This ensures that $\nabla^{\tau_j}\eta^j\cdot N = \eta^j\cdot f$, and hence

$$\int_{-\infty}^{-\tau_j}\int_{\Gamma_t^{\tau_j}} y^{\tau_j}(\partial_t - \Delta^{\tau_j})^*\eta^j = 0.$$

Since $\phi^j\to 1$ in $C^\infty_{\mathrm{loc}}(I\times(-\infty,\infty))$, then weak$*$ convergence of y^{τ_j} to y^∞ as $j\to\infty$ implies

$$\int_{-\infty}^{\infty}\int_{\Gamma^\infty} y^\infty(\partial_t - \Delta^{\tau_j})^*\eta = 0.$$

Thus by the L^2 theory for the heat equation, y^∞ satisfies

$$\begin{cases} y_t^\infty = y_{xx}^\infty \;\text{ in } [-1,1] \\ y_x^\infty(\pm1) = \pm y\cdot f((\pm1,0)) \;\text{ on } \partial\Gamma_t^\tau.\end{cases}$$

Finally, we characterise the limit. Separation of variables leads us to consider the problem

$$\begin{cases} -\varphi_{xx} = \mu\varphi \ \text{ in } [-1,1] \\ \varphi_x(\pm 1) = \pm\varphi \cdot f((\pm 1, 0)) \ \text{ on } \partial\Gamma_t^\tau. \end{cases}$$

After a long calculation, one finds that the negative eigenspace restricted to convex functions is one dimensional. This eigenspace corresponds to the eigenvalue λ_0, as defined in Lemma 2.5 and the corresponding eigenfunction is given by

$$\varphi_{\lambda_0} := \cosh\lambda_0 x + \frac{\kappa_1 - \kappa_2}{2\lambda_0 - (\kappa_1+\kappa_2)\tanh\lambda_0}\sinh\lambda_0 x.$$

Note that in general, there might be a second negative eigenvalue, however, in that case the corresponding eigenfunction is not convex. Thus,

$$y^\infty(x,t) = Ae^{\lambda_0^2 t}\left(\cosh\lambda_0 x + \frac{\kappa_1 - \kappa_2}{2\lambda_0 - (\kappa_1+\kappa_2)\tanh\lambda_0}\sinh\lambda_0 x\right)$$

for some $A \geq 0$, and by the avoidance principle, such an A is unique. □

Uniqueness of the constructed ancient solution now follows directly from the avoidance principle.

Theorem 4.5. *Modulo time translation, for each diameter of Ω, there exists precisely two convex, locally uniformly convex, ancient solution to the free boundary curve shortening flow in Ω, one lying on each side of the diameter.*

Proof. Consider two convex ancient solutions $\{\Gamma_t\}$ and $\{\Gamma'_t\}$ to (2) lying on one side of D. Given $\tau > 0$, consider the time-translated solution $\{\Gamma_t^\tau\}$ defined by $\Gamma_t^\tau = \Gamma'_{t+\tau}$. By the previous proposition;

$$e^{-\lambda_0^2 t}y^\tau(x,t) \to Ae^{\lambda_0^2\tau}\left(\cosh(\lambda_0 x) + \frac{\kappa_1 - \kappa_2}{2\lambda_0 - (\kappa_1+\kappa_2)\tanh\lambda_0}\sinh(\lambda_0 x)\right)$$

as $t \to -\infty$. Thus, Γ_t^τ lies above Γ_t for t sufficiently negative. The avoidance principle then ensures that Γ_t^τ lies above Γ_t for all $t \in (-\infty, -\tau)$. Taking $\tau \to 0$, we find that Γ'_t lies above Γ_t for all $t < 0$ by the avoidance principle. But both curves reach the same point at time zero, and so they must intersect for all $t < 0$. The strong maximum principle then implies the two solutions coincide for all t. □

Appendix A. Orthogonally Intersecting Angenent Ovals

Let $\Omega \subset \mathbb{R}^2$ be a compact, strictly convex domain and let D be any diameter of Ω. By shifting, rotating and dilating, we may assume that $\overline{D} = [-1, 1]$. We will show that for any $\rho > 0$, there exists a time-slice of an x-shifted Angenant oval of the form

$$A_t^{\lambda,\xi} := \{(x, y) \in \mathbb{R} \times (0, \tfrac{\pi}{2\lambda}) \mid \sin(\lambda y) = e^{\lambda^2 t} \cosh(\lambda(x - \xi))\}$$

such that $A_t^{\lambda,\xi}$ intersects $\partial\Omega$ orthogonally at two points that lie below the line $y = \rho$. Moreover we will show that the scale λ has a limit as $\rho \to 0$. We will do the construction in two parts, first by asserting that we have orthogonality at a single point on the boundary, and then demonstrating we have a large enough degree of freedom to achieve orthogonality at a second point on the boundary.

A.1. Orthogonality at a Single Point.

We adopt a graph parametrisation for $\partial\Omega \cap \{(x, y) \in \mathbb{R}^2 \mid y \geq 0\}$, $\phi(x)$ where $x \in [-1, 1]$. Pick a point on $\partial\Omega \cap \{(x, y) \in \mathbb{R}^2 \mid y \geq 0\}$, say $(x_0, \phi(x_0))$, such that $x_0 > 0$ and $\phi'(x_0) < 0$. Then it is easily verified that $A_t^{\lambda,\xi}$ passes through $(x_0, \phi(x_0))$ for any *'valid'* λ (the precise meaning of this will be established in the next lemma) and for any $t < 0$ where ξ is given by

$$\xi = x_0 - \frac{1}{\lambda} \cosh^{-1}\left(e^{-\lambda^2 t} \sin(\lambda\phi(x_0))\right). \tag{1}$$

Thus, we obtain a 2-parameter family of Angenant ovals which pass through the point $(x_0, \phi(x_0))$. We reduce this to a 1-parameter family, by enforcing orthogonality at that point.

Lemma A.1. *For each* $\lambda \in \left(\frac{1}{\phi(x_0)} \tan^{-1}(\frac{-1}{\phi'(x_0)}), \frac{\pi}{2\phi(x_0)}\right)$, *let* $A_t^{\lambda,\xi}$ *be the scaled Angenant oval where*

$$t = \frac{1}{2\lambda^2} \log\left(\sin^2(\lambda\phi(x_0)) - \frac{\cos^2(\lambda\phi(x_0))}{\phi'(x_0)^2}\right) \tag{2}$$

and

$$\xi = x_0 - \frac{1}{\lambda} \cosh^{-1}\left(\frac{\sin(\lambda\phi(x_0))}{\sqrt{\sin^2(\lambda\phi(x_0)) - \frac{\cos^2(\lambda\phi(x_0))}{\phi'(x_0)^2}}}\right). \tag{3}$$

Then $A_t^{\lambda,\xi}$ *intersects* $\partial\Omega$ *orthogonally at* $(x_0, \phi(x_0))$.

Proof. It is easy to verify that $(\tanh(\lambda(x_0 - \xi)), -\cot(\lambda\phi(x_0)))$ is normal to the Angenant oval at $(x_0, \phi(x_0))$. Similarly $(-\phi'(x_0), 1)$ is normal to $\partial\Omega$ at $(x_0, \phi(x_0))$. Hence, we require

$$\tanh(\lambda(x_0 - \xi))\phi'(x_0) + \cot(\lambda\phi(x_0)) = 0$$

which, after substituting into (1), can be solved for t. Substituting this t back into (1) yields (3).
In (2), we need the term inside the log to be strictly positive, i.e.,

$$0 < \sin^2(\lambda\phi(x_0)) - \frac{\cos^2(\lambda\phi(x_0))}{\phi'(x_0)^2}.$$

This tells us $\lambda > \frac{1}{\phi(x_0)} \tan^{-1}(\frac{-1}{\phi'(x_0)})$. Similarly, in the expression for ξ (3), we need the term in the parenthesis to be greater than one, which is always true provided $\lambda < \frac{\pi}{2\phi(x_0)}$. □

Definition A.2. *Given x_0 as in the beginning of the section, and λ, $A_t^{\lambda,\xi}$ as in Lemma A.1, define $A_{x_0}^{\lambda}$ to the be connected component of $A_t^{\lambda,\xi} \cap \Omega$ passing through $(x_0, \phi(x_0))$.*

A.2. **Orthogonality at the Second Point.** For each $(x_0, \phi(x_0)) \in \partial\Omega$ as in the beginning of section A.1, we have one parameter family of Angenant ovals $A_{x_0}^{\lambda}$ which intersect $\partial\Omega$ orthogonally at $(x_0, \phi(x_0))$. Now, let $\rho > 0$. We will now show that we can find x_0 and λ so that $A_{x_0}^{\lambda}$ intersects $\partial\Omega$ orthogonally at two points and lies in the strip $\{(x, y) \mid 0 < y < \rho\}$. We first show a preliminary lemma.

Lemma A.3. *Let ρ be such that the line $y = \rho$ intersects $\partial\Omega$ at two points. Then there exists a point x_0 with $\phi(x_0) < \rho$ and $\phi'(x_0) < 0$, and such that the following holds: $A_{x_0}^{\lambda}$ with $\lambda := \frac{\pi}{2\rho}$, as in A.2, intersects $\partial\Omega$ at $(x_0, \phi(x_0))$ orthogonally, and further, intersects $\partial\Omega$ at a second point $(\hat{x}, \rho)$, where $\phi'(\hat{x}) > 0$.*

Proof. By assumption, the line $y = \rho = \frac{\pi}{2\lambda}$ intersects $\partial\Omega$ at a point (x_1, ρ) where $\phi'(x_1) < 0$. Then for $x_0 > x_1$, $\phi(x_0) = \rho$ and so $\frac{\pi}{2\phi(x_0)} > \frac{\pi}{2\rho}$ from which the previous lemma implies $A_{x_0}^{\lambda}$ is well-defined. Recall that $A_{x_0}^{\lambda}$ is part of a scaled Angenent oval (as in Definition A.2) and the point on this Angenent oval with outward pointing unit normal equal to $-e_1$ has y-coordinate equal to ρ and x-coordinate given by

$$\hat{x} = \frac{-1}{\lambda} \cosh^{-1}\left(\frac{1}{\sin^2(\lambda\phi(x_0)) - \frac{\cos^2(\lambda\phi(x_0))}{\phi'(x_0)}}\right) + \xi.$$

As $x_0 \to 1$, $\hat{x} \to -\infty$. Additionally, one can check (by replacing ξ from Lemma A.1), as $x_0 \searrow x_1$, $\hat{x} \to x_0$. Therefore, by the intermediate value theorem, we can find an x_0 satisfying the consequent of the lemma. □

Lemma A.4. *For any $\rho > 0$, we can find x_0 and λ_ρ, as in Lemma A.1 so that $\phi(x_0) < \rho$ and $A_{x_0}^{\lambda_\rho}$ intersects $\partial\Omega$ orthogonally at two points and lies in the strip $\{(x, y) \mid 0 < y < \rho\}$.*

Proof. Given ρ, fix x_0 as given in Lemma A.3. Then, the angle between the tangent vectors of $A_{x_0}^{\frac{\pi}{2\rho}}$ and $\partial\Omega$ at $(\hat{x}, \phi(\hat{x}))$ is less than $\pi/2$. Keeping x_0 fixed, we consider now $A_{x_0}^{\lambda}$ as in Definition A.2 and we claim that there exists a λ_ρ such that $A_{x_0}^{\lambda_\rho}$ intersects $\partial\Omega$ orthogonally (at both points). By continuity, it suffices to show that there exists a λ such that the angle between the tangent vectors of $A_{x_0}^{\lambda}$ and $\partial\Omega$ at the *second* point of intersection is less than $\pi/2$. We do this by showing, in the following claim, that there exists λ for which the corresponding ξ (determined in the definition for $A_{x_0}^{\lambda}$ in Lemma A.1) is equal to -1.

Claim A.5. *There exists* $\lambda \in \left(\frac{1}{\phi(x_0)}\tan^{-1}(\frac{-1}{\phi'(x_0)}), \frac{\pi}{2\phi(x_0)}\right)$ *such that* $\xi = -1$.

Proof. $\xi = -1$ means we are trying to solve

$$x_0 - \frac{1}{\lambda}\cosh^{-1}\left(\frac{\sin(\lambda\phi(x_0))}{\sqrt{\sin^2(\lambda\phi(x_0)) - \frac{\cos^2(\lambda\phi(x_0))}{\phi'(x_0)^2}}}\right) = -1,$$

which after rearranging becomes

$$\phi'(x_0)^2\tanh^2\left(\lambda(x_0+1)\right) = \cot^2(\lambda\phi(x_0)).$$

Define a function

$$f(\lambda) := \phi'(x_0)^2\tanh^2\left(\lambda(x_0+1)\right) - \cot^2(\lambda\phi(x_0)).$$

If $\lambda = \frac{1}{\phi(x_0)}\tan^{-1}(\frac{-1}{\phi'(x_0)})$, then

$$f(\lambda) = \phi'(x_0)^2\left(\tanh^2(\lambda(x_0+1)) - 1\right) < 0.$$

On the other hand, if $\lambda = \frac{\pi}{2\phi(x_0)}$

$$f(\lambda) = \phi'(x_0)^2\tanh^2(\frac{\pi}{2\phi(x_0)}(x_0+1)) > 0,$$

this proves the claim. $\square$

$\square$

A.3. **Calculating the asymptotic behavior of λ_ρ (as in Lemma A.4) as $\rho \to 0$.** Now that we have justified the construction of the shifted and scaled Angenent oval which intersects $\partial\Omega$ orthogonally below the horizontal line $y = \rho$ (Lemma A.4), we will show that its scale factor λ_ρ, as $\rho \to 0$, does have a limit, which we will call λ_0. To do this, we consider a sequence ρ_i tending to 0, and for each ρ_i we considered the corresponding $A_{x_i}^{\lambda_i}$ as constructed in Lemma A.4. We first show that $\liminf_{i\to\infty}\lambda_i$ and $\limsup_{i\to\infty}\lambda_i$ are bounded below and above respectively.

Lemma A.6. *We have*

$$\max\{\kappa^\Omega(e_1), \kappa^\Omega(-e_1)\} \leq \liminf_{i\to\infty} \lambda_i \leq \limsup_{i\to\infty} \lambda_i \leq \sigma,$$

where σ solves the equation $\sigma \tanh \sigma = \max\{\kappa^\Omega(e_1), \kappa^\Omega(-e_1)\}$.

Proof. Consider a sequence $\rho_i \downarrow 0$ and the corresponding $A^{\lambda_i}_{x_i}$ as constructed in Lemma A.4. Recall that

$$\lambda_i > \frac{1}{\phi(x_i)} \tan^{-1}(\frac{-1}{\phi'(x_i)}).$$

Note that $x_i \to 1$, as $i \to \infty$, which gives $\liminf_{i\to\infty} \lambda_i > \kappa^\Omega(e_1)$, and since we could have done the entire construction in the previous section by considering x_i so that $\phi'(x_i) > 0$, the lower bound follows. We also note that, the above freedom to choose "side" for x_i, allows us to assume without loss of generality that $\kappa(e_1) \geq \kappa(-e_1)$ and, moreover, in the case of equality the following picture holds: If we denote by R the reflection about the y-axis, then $R(\partial\Omega \cap \{x \geq 0, 0 \leq y \leq \rho_i\})$ lies inside $\overline{\Omega}$ for all ρ_i sufficiently close to 0.

Consider now an Angenent oval $A^{\sigma_i}_{x_i}$ as in Lemma A.1 (see also Definition A.2) which is not shifted, that is

$$\xi_i = x_i - \frac{1}{\sigma_i} \cosh^{-1}\left(e^{-\sigma_i^2 t} \sin(\sigma_i \phi(x_i))\right) = 0. \tag{4}$$

Then, $A^{\sigma_i}_{x_i}$ intersects $R(\partial\Omega \cap \{x \geq 0, 0 \leq y \leq \rho_i\})$ orthogonally, and thus $\partial\Omega \cap \{x < 0\}$ at an acute angle (as per defintion 2.1, see figure 3 and Remark 2.3). Following the construction in Lemma A.4, we point out that as the shift ξ decreases, the scale λ also decreases, and so by decreasing ξ towards -1 (when the angle becomes obtuse), we can infer that $\lambda_i < \sigma_i$. Therefore $\limsup_{i\to\infty} \lambda_i \leq \sigma := \lim_{i\to\infty} \sigma_i$. To calculate σ, note that (4) can be written as

$$\cosh(\sigma_i x_i) = \frac{1}{\sqrt{1 - \frac{\cot^2(\sigma_i \phi(x_i))}{\phi'(x_i)^2}}}.$$

After taking the limit as $x_i \to 1$, this becomes

$$\cosh(\sigma) = \frac{1}{\sqrt{1 - \frac{\kappa^\Omega(e_1)^2}{\sigma^2}}},$$

which rearranges to

$$\sigma \tanh \sigma = \kappa^\Omega(e_1).$$

∎

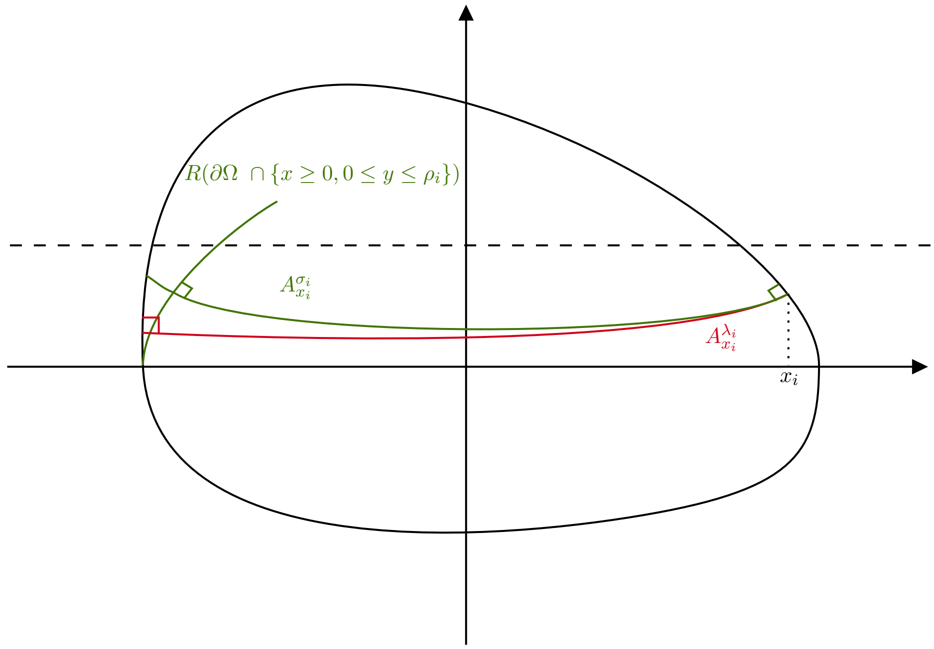

FIGURE 3. $A^{\sigma_i}_{x_i}$ intersects $\partial\Omega$ at an acute angle.

Lemma A.6 implies there exists a subsequence ρ_{i_j}, for which the corresponding λ_{i_j} converges, as $j \to \infty$, to some $\lambda_0 \in (0, \infty)$. We will show that λ_0 is independent of the sequence ρ_i. This then implies that the scale λ has a limit as $\rho \to 0$.

First note that the shifts ξ_{i_j} of $A^{\lambda_{i_j}}_{x_{i_j}}$, as in (3) in Lemma A.1, also have a limit, which we call ξ_0 and satisfies

$$\xi_0 := 1 - \frac{1}{\lambda_0} \cosh^{-1}\left(\frac{1}{\sqrt{1 - \frac{\kappa^\Omega(e_1)^2}{\lambda_0^2}}} \right).$$

Recall that if $A^{\lambda_{i_j}}_{x_{i_j}}$ intersects $\partial\Omega$ at a point $(\hat{x}, \phi(\hat{x})) \in \partial\Omega$ orthogonally (here $\hat{x}$ is either x_{i_j} or the corresponding x-coordinate of the point on $\partial\Omega \cap A^{\lambda_{i_j}}_{x_{i_j}}$, $(\hat{x}, \phi(\hat{x}))$ with $\phi'(\hat{x}) > 0$), then

$$\tanh(\lambda_{i_j}(\hat{x} - \xi_{i_j}))\phi'(\hat{x}) + \cot(\lambda_{i_j}\phi(\hat{x})) = 0,$$

which can be written as

$$\tanh(\lambda_{i_j}(\hat{x} - \xi_{i_j})) = -\frac{\cot(\lambda_{i_j}\phi(\hat{x}))}{\phi'(\hat{x})}.$$

Taking the limit as $j \to \infty$ and noting that $\hat{x} \to 1$ or $\hat{x} \to -1$ we obtain

$$\begin{cases} \tanh(\lambda_0(1-\xi_0)) & = \kappa^{\Omega}(e_1) \\ \tanh(\lambda_0(-1-\xi_0)) & = -\kappa^{\Omega}(-e_1). \end{cases} \tag{5}$$

The system (5) can be reduced (by using the addition formula for the hyperbolic tangent) to

$$\lambda_0^2 - \lambda_0(\kappa^{\Omega}(e_1) + \kappa^{\Omega}(-e_1)) \coth 2\lambda_0 + \kappa^{\Omega}(e_1)\kappa^{\Omega}(-e_1) = 0,$$

which, since $\lambda_0 \geq \kappa^{\Omega}(e_1), \kappa^{\Omega}(-e_1)$, determines λ_0 uniquely.

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