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CONTINUOUS DATA ASSIMILATION ALGORITHM FOR SIMPLIFIED BARDINA MODEL

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Abstract. We present a continuous data assimilation algorithm for three-dimensional viscous simplified Bardina turbulence model, based on the fact that dissipative dynamical systems possess finite degrees of freedom. We construct an approximating solution of simplified Barbina model through an interpolant operator which is obtained using observational data of the system. This interpolant is inserted to theoric model coupled to a relaxation parameter, and our main result provides conditions on the finite-dimensional spatial resolution of collected measurements sufficient to ensure that the approximating solution converges to the theoric solution of the model. Global well-posedness of approximating solutions and related results with degrees of freedom are also presented.

1. **Introduction.** Data analysis is a wide range of techniques that conciliates mathematical models and physical observations with the goal of optimizing forecasts in evolutionary phenomenons. Due to the high degree of freedom of many models, a hard obstacle is to predict suitable initialization points, which in general are deduced starting from discrete and heterogeneous grids of collected observations. Approaches to this problem have been topic of countless theoretical and numerical studies. See [15] for an extensive overview about these issues in weather prediction. In this work, we deal with a technique denominated *continuous data assimilation*, which consists basically in inserting a forcing term (nudging) directly into the prognostic equations, as the latter is being integrated in time. In general, nudging processes are related with the difference between the solution and the observations. The target is to drive the system towards physical considerations and there are many different ways to perform this. In [3], the authors proposed a continuous data

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assimilation algorithm where interpolant observables are used, introducing a feedback control term that forces the model towards the theoric solution corresponding to the observations.

Following the ideas in [3], the algorithm can be described as proceeding. Suppose that the dynamical system is governed by

$$\frac{dv}{dt} = F(v), \tag{1}$$

where initial data $v(0)$ is unknown. Assume that is possible to obtain a linear interpolation operator $I_h(v(t))$ through observational measurements, over the time $[0, T]$. Here $h > 0$ is a coarse spatial resolution, related to accurency of operator. Then, we consider the system

$$\begin{aligned} &\frac{dw}{dt} = F(w) - \eta(I_h(w) - I_h(v)),\\ &w(0) = w_0, \end{aligned} \tag{2}$$

where $\eta > 0$ is a Newtonian relaxation parameter and the initial condition w_0 is taken arbitrary. The linear operator I_h must be constructed via methods that fit with the sort of collected data. Some examples are determining modes ([26],[32]), volume elements ([31],[32]) and nodal values ([25],[27], [32]).

The conjecture is to determine suitable conditions, in terms of physical parameters, for values of η and h, large and small enough respectively, to ensure w is directed to the reference solution v, when time goes to infinity. Then, we can choose an appropriated initial condition to (1), for example $w(0) = v(T)$, for some $T > 0$.

The above algorithm was rigorously proved to be true (with exponential rate of convergence) to the two-dimensional Navier-Stokes equations in [3] (see also [20]). Extensions to the Bérnard convection problem ([19] and [21]) and Brinkman-Forchheimer-extended Darcy model ([38]) were also proved. See [29] and [37] for computational experiments.

In [1], authors has proved the algorithm is valid to the three-dimensional Navier-Stokes-α equations, (also called Camassa-Holm equations,[22])

$$\left\{ \begin{array}{l} \dfrac{\partial v}{\partial t} - u \times (\nabla \times v) - \nu \Delta v + \nabla p = f,\\ v = (1 - \alpha^2 \Delta) u,\\ \nabla \cdot u = 0, \end{array} \right. \tag{3}$$

with data assimilation algorithm of the form

$$\left\{ \begin{array}{l} \dfrac{\partial z}{\partial t} - w \times (\nabla \times z) - \nu \Delta z + \nabla p\\ \qquad\qquad\qquad = f - \eta(I_h(w) - I_h(u)) + \eta \alpha^2 \Delta (I_h(w) - I_h(u)),\\ z = (1 - \alpha^2 \Delta) w,\\ \nabla \cdot w = 0. \end{array} \right. \tag{4}$$

System (3) can be seen as a Kelvin-filtered Navier-Stokes equations with the filter being the inverse of the Helmholtz operator $(I - \alpha^2 \Delta)^{-1}$ and α representing the width of the filter ([23] and [17]). Numerical results pointed the connection between Camassa-Holm equations and turbulence (see [11],[9],[10] and references therein). Similar equations are also related with second grade fluids and Euler-Poincaré models as can be found in [30], [6], [33] and [13].

Note that for appliance of continuous data assimilation algorithm (2), the diffusion relaxation $\Delta(I_h(w) - I_h(u))$ was added (see (4)). From the analytical point of view, it is present due to the derivatives of higher order in the non-linearity. In [16], authors performed numerical studies with the shallow water equations comparing methods involving direct insertion, data assimilation having only Newtonian relaxation and data assimilation including diffusion relaxation. For distinct experiments, they obtained different results and concluded that further inspections are required to determine the practical efficacy of each approach (see also [39]). In the context of Navier-Stokes-α system, it seems to be unknown if the algorithm is convergent without diffusion in the nudging. In this paper, we prove that it is true for a related system called three-dimensional viscous simplified Bardina turbulence model

$$\begin{cases} \dfrac{\partial v}{\partial t} - \nu\Delta v + (u\cdot\nabla)u + \nabla p = f, \\ v = (1-\alpha^2\Delta)u, \\ \nabla\cdot u = 0. \end{cases} \tag{5}$$

Vector $u = (u_1(x,t), u_2(x,t), u_3(x,t))$ is the spatial (filtered) velocity field, $p = p(x,t)$ is a modified scalar pressure field, $f = f(x,t)$ is a given external force; $\nu > 0$ is kinematic viscosity and $\alpha > 0$ is a fixed length-scale parameter. This model was considered by Layton and Lewandowski [34] being a simpler approximation of the Reynold stress tensor proposed by Bardina et al [5], which is called Bardina model. The main difference of simplified Bardina model from other alpha subgrid scale turbulence models (see e.g.,[7],[35],[41]) is that the nonlinear term can be distinguished easily. Moreover, notice that the simplified Bardina system is consistent with other alpha models in the sense that if $\alpha = 0$, we have $u = v$ and we formally recover the classical three-dimensional Navier-Stokes equations (3D NSE).

The following continuous data assimilation is considered

$$\begin{cases} \dfrac{\partial z}{\partial t} - \nu\Delta z + (w\cdot\nabla)w + \nabla p = f - \eta(I_h(w) - I_h(u)), \\ z = (1-\alpha^2\Delta)w, \\ \nabla\cdot w = 0, \end{cases} \tag{6}$$

subject to periodic boundary condition $\Omega = [0,L]^3$, namely,

$$u(x,t) = u(x + Le_i, t),\ \forall\,(x,t)\in\mathbb{R}^3\times[0,T), \tag{7}$$

where e_1, e_2 and e_3 are the canonical basis of $\mathbb{R}^3$ and $L > 0$ the fixed period.

We prove that, for suitable values of the relaxation parameter $\eta > 0$ and the coarse spatial resolution $h > 0$, solution w of (6) converges, at exponential rate, to solution u of reference system (5), independently of the initial condition $w(0)$. The restrictions on η and h will depend on physical constants (see Theorem 3.3 for details), such as kinematic viscosity and the size of domain.

In this paper, we consider a linear interpolant operator $I_h : \dot{H}^2(\Omega) \longmapsto L^2(\Omega)$ satisfying the approximation property

$$\|I_h g - g\|_{L^2} \le \mathcal{C}[h]\|g\|_{H^2} \tag{8}$$

for every $g \in \dot{H}^2(\Omega)$, where $\mathcal{C} : [0,\infty) \longmapsto [0,\infty)$ is a continuous real function, with unit of length square and $\mathcal{C}[h] \overset{h\to 0^+}{\longrightarrow} 0$. We present here three examples of such interpolants. The first is given by the projector onto low Fourier modes: considering $\phi_k(x) = L^{-3}e^{\frac{2\pi i}{L^3}k\cdot x}$, with $|k| \le \lfloor\frac{1}{2\pi h}\rfloor$, Then define $I_h : \dot{H}^2(\Omega) \to \dot{L}^2(\Omega)$ as

$$I_h(\varphi) = P_k\varphi = \sum_{|k|\le\lfloor\frac{L}{2\pi h}\rfloor} \widehat{\varphi}_k\phi_k(x), \tag{9}$$

where $\varphi(x) = \sum_{k\in\mathbb{Z}^3} \widehat{\varphi}_k \phi_k(x)$. It satisfies inequality (8), with $\mathcal{C}[h] = \frac{C}{2\pi} hL$. The use of Fourier modes projection, as well the concept of continuous data assimilation used along this paper, is motivated by the notion of stabilization via proportional-type feedback controllers for parabolic systems. An approach of exponential stabilization by finite-dimensional feedback controllers for the Navier-Stokes equations is found in [4].

The second example of interpolant operator is the volume element operator. Dividing $\Omega = [0, L]^3$ in N cubes Ω_k, of same edge, then $I_h : \dot{H}^2(\Omega) \to \dot{L}^2(\Omega)$ given by

$$I_h(\varphi) = \sum_{k=1}^{N} \frac{\chi_{\Omega_k}(x)N}{L^3} \int_{\Omega_k} \varphi(y)\, dy, \tag{10}$$

with $h = LN^{-\frac{1}{3}}$,i.e., the edge of each Ω_k. This interpolant operator satisfies (8) with $\mathcal{C}[h] = \frac{C}{2\pi} hL$.

Finally, the third example is obtained by observational measurements of velocity on discrete points x_k of cube Ω, that can be divided in N cubes Ω_k, as previous example, with $x_k \in \Omega_k$. Then $I_h : \dot{H}^2(\Omega) \to L^2(\Omega)$ defined as

$$I_h(\varphi) = \sum_{k=1}^{N} \varphi(x_k) u \chi_{\Omega_k}(x). \tag{11}$$

Then (11) satisfies (8), with $\mathcal{C}[h] = \left(\frac{Ch^2L^2}{(2\pi)^2} + Ch^4\right)^{\frac{1}{2}}$. See [3] and [1], for more details in two and three dimensions, respectively.

For complementing the theory, we present a result involving the degree of freedom of the system obtained via interpolant operator $I_h(\cdot)$. In addition to strengthening the theoretical approach, the goal is to provide suitable enough estimates to the size of the grid of observations, to ensure the convergence $u(t) - \tilde{u}(t) \to 0$, when time goes to infinity, of two solutions u and $\tilde{u}$ of system (5) and their respective forces, f and $\tilde{f}$. The result is given in Theorem 3.2, where we prove that the system (5) is asymptotically determined by $I_h(\cdot)$ if h is less than a constant that depends on some physical parameters. Furthermore, Fourier modes, nodal points and volume elements can be written in a similar abstract form

$$I_h(g) = \sum_{i=1}^{N_h} s^i(g)\omega_i, \tag{12}$$

where ω_i are functions and s^i linear functionals, as presented in (9), (10) and (11). Then, estimates to the degree of freedom given by $\{s^i\}$, $i = 1, ..., N_h$ can be easily obtained (see Corollary 1).

In [18], a similar result for a α-regularization model of the 3D Navier-Stokes equations was obtained. The authors applied the data assimilation algorithm for the case of 3D Leray-α model (see [12]), in periodic domain $\Omega = [0, L]^3$:

$$\begin{cases} \dfrac{dv}{dt} - \nu\Delta v + (u\cdot\nabla)v = -\nabla p + f \\ v = u - \alpha^2\Delta u \\ \nabla\cdot v = \nabla\cdot u = 0, \end{cases}$$

which differs from the model proposed here by the nonlinearity, composed of the filtered (smoother) and non-filtered velocities field. Also, for data assimilation algorithm application for 3D Leray-α model, the pressure term is considered by authors, while in this work we are making use of Leray Projetor (see section 2.1). In [18], the estimates are obtained using only coarse mesh observations of any two components of the three-dimensional velocity field. However, the range we obtained for the Newtonian relaxation parameter η is greater than in [18] (see 28). Besides, our condition to the interpolant 8 is more general.

This paper is organized as follows: In Section 2, we present the functional setting concerning Navier-Stokes and simplified Bardina equations, with inequalities and notations commonly used. Then, we state our results in Section 3. In Section 4 is presented some key estimates. Finally, the main results are proved in Section 5.

2. **Preliminaries.**

2.1. **Basic definitions and inequalities.**

Let $\Omega = [0, L]^3$ the periodic box for some period L. In order to formulate the problem, this section presents some basic definitions and functional settings. We denote by L^p ($\dot{L}^p$) and H^m ($\dot{H}^m$) the usual three-dimensional Lebesgue and Sobolev vector spaces (with mean zero), respectively. Due the fact that solutions of (5)-(7) satisfy

$$\frac{d}{dt}\int_\Omega u(x,t)dx = \int_\Omega f(x,t)dx,$$

the restriction of the analysis to functions with means zero is justified. Consider the set

$$\mathcal{V} = \left\{ \phi \in (C^\infty(\Omega))^3;\ \phi \text{ satisfies } \nabla\cdot\phi = 0 \text{ and } \int_\Omega \phi(x)\,dx = 0 \right\}.$$

We define H and V the closures of $\mathcal{V}$ in L^2 and H^1, respectively. They are Hilbert spaces, with $V \subset H \equiv H' \subset V'$ with dense inclusions and continuous injections (see [40]), endowed with inner products

$$(u,v) = \sum_{i=1}^{3}\int_\Omega u^i(x)v^i(x)\,dx \quad \text{and} \quad ((u,v)) = \sum_{i,j=1}^{3}\int_\Omega \partial_j u^i(x)\partial_j v^i(x)\,dx$$

respectively. The norms in H and V are denoted by

$$|u| := \|u\|_{L^2} = \sqrt{(u,u)} \qquad \text{and} \qquad \|u\| := \|u\|_{H^1} = \sqrt{((u,u))}.$$

We denote by $\mathcal{P} : L^2(\Omega) \to H$ the classical Helmholtz-Leray orthogonal projection and $A = -\mathcal{P}\Delta$ the Stokes operator under periodic boundary condition, i.e., $A = -\Delta$, with domain $D(A) = (H^2(\Omega))^3 \cap V$. The set $D(A)$ is a Banach space with norm $\|u\|_{D(A)} := |Au|$. By spectral theory, there exists a sequence of eigenfunctions $(u_n)_{n\in\mathbb{N}}$ such that

$$(u_n)_{n\in\mathbb{N}} \text{ is an orthonormal basis of } H \quad \text{and} \tag{13}$$

and

$$(\lambda_n^{-\frac{1}{2}}u_n)_{n\in\mathbb{N}} \text{ is an orthonormal basis of } V, \tag{14}$$

where $(\lambda_n)_{n\in\mathbb{N}}$ is the set of eigenvalues $Au_i = \lambda_i u_i$ and $0 < \lambda_i \le \lambda_{i+1}$ for $i \in \mathbb{N}$. Denoting $\lambda_1 = \left(\frac{2\pi}{L}\right)^2$ the smallest eigenvalue of A, we have the Poincaré inequalities

$$|u|^2 \le \lambda_1^{-1}\|u\|^2 \text{ for all } u \in V \text{ and } \|u\|^2 \le \lambda_1^{-1}|Au|^2 \text{ for all } u \in D(A). \tag{15}$$

Let V' and D' be the topological dual of V and $D(A)$, respectively. For each $\alpha > 0$, we consider the linear homeomorphism

$$(I + \alpha^2 A)^{-1} : D' \longmapsto H$$

with the estimates

$$\|(I + \alpha^2 A)^{-1}\|_{\mathcal{L}(D';H)} < 1.$$

We recall some particular three-dimensional cases of the Gagliardo-Nirenberg inequality (see [28]):

$$\begin{cases} \|g\|_{L^6(\Omega)} \leq C\|g\|, & \text{for all } g \in V; \\ \|g\|_{L^3(\Omega)} \leq C|g|^{\frac{1}{2}}\|g\|^{\frac{1}{2}}, & \text{for all } g \in V; \\ \|g\|_{L^\infty(\Omega)} \leq C\|g\|^{\frac{1}{2}}\,|Ag|^{\frac{1}{2}}, & \text{for all } g \in D(A), \end{cases} \tag{16}$$

where C is a dimensionless constant.

We define the bilinear form $B : D(A) \times D(A) \to H$ as the continuous operator

$$B(u, v) = \mathcal{P}[(u \cdot \nabla)v].$$

For $u, v, w \in D(A)$, the bilinear term B has the property

$$(B(u, v), w) = -(B(u, w), v)$$

and hence

$$(B(u, w), w) = 0. \tag{17}$$

Besides,

$$|(B(u, v), w)| \leq C|u|^{\frac{1}{2}}\|u\|^{\frac{1}{2}}\|v\|\,\|w\|, \tag{18}$$

For every $u, v \in D(A)$ and $w \in H$, we have

$$|(B(u, v), w)| \leq C\|u\|^{\frac{1}{2}}|Au|^{\frac{1}{2}}\,\|v\||w|. \tag{19}$$

We write the incompressible three-dimensional simplified Bardina equations (5) using functional settings as

$$\begin{aligned} &\frac{dv}{dt} + \nu Av + B(u, u) = f, \\ &v = u + \alpha^2 Au, \end{aligned} \tag{20}$$

with initial condition $u(0) = u_0$ and thereby $v(0) = u_0 + \alpha^2 Au_0$.

2.2. **Regular solutions.** We recall the global well-posedness results for three-dimensional viscous simplified Bardina model (5), found in [34],[8].

Theorem 2.1 (Existence and Uniqueness of Weak Solutions)**.** *Let $f \in L^2([0, T]; H)$ and $u(0) = u_0 \in V$. Then the system (5) has a unique weak solution $u \in C([0, T); V)$ $\cap$ $L^2([0, T]; D(A))$, with $\frac{du}{dt} \in L^2([0, T]; H)$.*

Next, note that, using functional setting again, the continuous data assimilation equations (6) is equivalent to

$$\begin{cases} \dfrac{dz}{dt} + \nu Az + B(w, w) = f - \eta\mathcal{P}(I_h(w) - I_h(u)), \\ z = w + \alpha^2 Aw, \\ \nabla \cdot w = 0, \end{cases} \tag{21}$$

on the interval $[0, T]$. Moreover, Leray projector implies that (8) becomes

$$|\mathcal{P}(I_h g - g)| \leq |I_h g - g| \leq \mathcal{C}[h]|Ag| \tag{22}$$

for every $g \in D(A)$, where $\mathcal{C}[h]$ is the same as (8).

We enunciate now the global well-posedness of data assimilation equations (21), which will be proved in section 5.1.

Theorem 2.2 (Global well-posedness)**.** *Let $f \in L^2([0,T];H), u_0 \in V$ and u be the solution of simplified Bardina equations (5), with initial data u_0 (see Theorem 2.1). Let $w_0 \in V$ and $\eta > 0$ given. Suppose that I_h is linear and satisfies (22) and $\mathcal{C}[h]$ satisfies*

$$\mathcal{C}[h] \leq \sqrt{\frac{\nu\alpha^2}{\eta}}. \tag{23}$$

Then the continuous data assimilation system (21) with interpolant I_h possess a unique solution with regularity

$$w \in C([0,T);V) \cap L^2([0,T];D(A)) \quad \text{and} \quad \frac{dw}{dt} \in L^2([0,T];H).$$

Finally, there exists a continuous dependence with respect to initial data w_0 in V-norm.

3. Main results.

3.1. Vanishing limits and degree of freedom to the system (5).

We start stating a result concerning the vanishing limit of the length-scale parameter α. We prove that as $\alpha \to 0$, solutions u_α of the model (20) converge to a solution u of the NSE. Hence, the system (20) can be seen as a regularized approximation to 3D NSE.

We observe that in [34], the authors have proved a similar result. There, they considered the initial data for (20) of the form $u_0 = (I - \alpha^2\Delta)^{-1}v_0$, with $v_0 \in L^2$, and therefore $u_0 \in H^2 \cap V$. Here, we consider $u_0 \in V$. In this sense, our result is different from theirs and we present a detailed proof in section (5).

Also, see [40] and [14] for definitions of Leray-weak and strong solutions to the Navier-Stokes equations.

Theorem 3.1. *Let $u_0 \in V$ and $f \in L^2([0,T];H)$. For each fixed $\alpha > 0$, consider the solution u_α of (20) in $(0,T)$, with initial data $u_\alpha(0) = u_0$. Then, there exists a sequence $(\alpha_j)_{j\in\mathbb{N}}$ and a function $u \in L^\infty([0,T];H) \cap L^2([0,T];V)$ such that:*

- *P1.* $\alpha_j \overset{j\to\infty}{\longrightarrow} 0^+$;
- *P2.* $u_{\alpha_j} \overset{j\to\infty}{\longrightarrow} u$ *strongly in $L^2([0,T];H)$, weakly in $L^2([0,T];V)$ and weakly-* in $L^\infty([0,T];H)$;*
- *P3. u is a Leray weak solution to the Navier-Stokes equations in $(0,T)$, with initial data $u(0) = u_0$.*

Furthermore, there is a time $0 < T^* \leq T$ such that:

- *P4.* $u_{\alpha_j} \overset{j\to\infty}{\longrightarrow} u$ *strongly in $L^2([0,T^*];V)$, weakly in $L^2([0,T^*];D(A))$ and weakly-* in $L^\infty([0,T^*];V)$;*
- *P5. u is a strong solution to the Navier-Stokes equations in $[0,T^*)$.*

Finally, if $\|u_0\|$ and $\|f\|_{L^2([0,T];H)}$ are small enough, we can choose $T^* = T$.

The next result we show that, under suitable assumptions, the system (20) is asymptotically determined by the observational measurements. For that, we make use of the dimensionless generalized Grashof number in three dimensions

$$Gr(f) := \frac{1}{\nu^2 \lambda_1^{\frac{3}{4}}} \limsup_{t\to\infty} |f(t)|, \tag{24}$$

where λ_1 is given in (15). For this result, we consider the time-dependent external force $f \in L^\infty([0,\infty); H)$.

Theorem 3.2. *Let u and $\tilde{u}$ be global solutions of simplified Bardina equations (5), with respective external forces f and $\tilde{f}$ in $L^\infty([0,\infty); H)$. Suppose that I_h is an interpolant linear operator such that (22) holds and the solutions u and $\tilde{u}$ satisfies*

$$\lim_{t\to\infty} |\mathcal{P}(I_h(u(t) - \tilde{u}(t)))| = 0. \tag{25}$$

and

$$\lim_{t\to\infty} |f(t) - \tilde{f}(t)| = 0. \tag{26}$$

Additionally, suppose that

$$\mathcal{C}[h] < C\frac{\lambda_1 \alpha^4}{(Gr(f))^2} \tag{27}$$

is valid, where C is a dimensionless positive constant, depending only on constant presented in (16). Then $\|u(t) - \tilde{u}(t)\| \to 0$ in V-norm.

Remark 1. As mentioned in introduction, a wide range of operators are of the form (12), such as (9),(10) and (11). In these cases, $\{s^i\}, i = 1, ..., N_h$ can be used to measure the degree of freedom of the infinity-dimensional system. Due to the fact that $\mathcal{C}[h]$ and N_h are dependent functions ($\mathcal{C}[h]$ decreases when N_h increases), the degree of freedom is determined by (27).

Stationary solutions are completely determined by functional related with the degree of freedom.

Corollary 1. *Let u and $\tilde{u}$ be time-independent solutions of (5) with the same time-independent external forces $f = \tilde{f} \in H$. Suppose that the interpolant operator is given by the form (12) and satisfies (27). If*

$$s^i[u] = s^i[\tilde{u}], \text{ for all } i = 1, ..., N_h,$$

then $u = \tilde{u}$.

3.2. **The convergence result.** We now state the main result, which ensures that

when time goes to infinity, the solution of the continuous data assimilation system (21), converges exponentially to the solution of classical simplified Bardina system (20), independently of the initial condition imposed to (21).

Theorem 3.3. *Let u be a global solution of the incompressible three-dimensional simplified Bardina equations (20), with external force $f \in L^\infty([0,\infty); H)$. Suppose that I_h is a linear interpolant operator which satisfies (22). Additionally, let w be a global solution of continuous data assimilation system (20), for η large enough such that*

$$\eta > C_2 \frac{\nu (Gr(f))^2}{\alpha^4 \lambda_1} - \frac{\nu \lambda_1}{2}, \tag{28}$$

where C_2 is a dimensionless positive constant (depending only of constant given in (16)). Suppose also that η and h are related in the sense that $\mathcal{C}[h]$ satisfies (23). Then $\|u(t) - w(t)\| \to 0$, in V-norm, as $t \to \infty$, at an exponential rate.

4. **Key estimates.**

4.1. **Estimates to the system (20).** Next, we state some lemmas which contain useful estimates.

Lemma 4.1. *Let u be a weak solution of (20) in $(0, T)$, with initial data $u(0) = u_0 \in V$. Then*

$$\frac{1}{2}\frac{d}{dt}\left(|u(t)|^2 + \alpha^2\|u(t)\|^2\right) + \nu\left(\|u(t)\|^2 + \alpha^2|Au(t)|^2\right) = (f(t), u(t)) \tag{29}$$

for every $0 \leq t < T$. Furthermore, there are dimensionless positive constants C_3, C_4 and C_5 (depending only of constants given in (16)) such that

$$\left\langle \frac{d}{dt}(u + \alpha^2 Au), g \right\rangle_{D'} \leq C_3\left(\nu|u| + \nu\alpha^2|Au| + \frac{\|u\|^2}{\lambda_1^{3/4}} + \frac{|f|}{\lambda_1}\right)|Ag| \tag{30}$$

for every $g \in D(A)$. Also,

$$(\frac{du}{dt}, g) \leq C_4\left(\nu\|u\| + |u|^{\frac{1}{2}}\|u\|^{\frac{3}{2}} + \frac{|f|}{\lambda_1^{1/2}}\right)\|g\| \tag{31}$$

for every $g \in V$. Furthermore,

$$\frac{d}{dt}\|u\|^2 + \nu|Au|^2 \leq C_5\left(\frac{1}{\nu^3}\|u\|^6 + \frac{1}{\nu}|f|^2\right). \tag{32}$$

Proof. The equality (29) is immediately obtained by taking the duality $\langle .,. \rangle_{D'}$ of the first expression of (20) with u and using (17). Similarly, for all $g \in D(A)$,

$$\left\langle \frac{d}{dt}(u + \alpha^2 Au), g \right\rangle_{D'} + \overbrace{\nu(Au, g)}^{L_1} + \overbrace{\nu\alpha^2(Au, Ag)}^{L_2} + \overbrace{(B(u, u), g)}^{L_3} = \overbrace{(f, g)}^{L_4}.$$

Using Cauchy-Schwarz Inequality, (15) and (18), we estimate the above terms as follows:

$$\begin{gathered} L_1 = \nu(Au, g) = \nu(u, Ag) \leq \nu|u|\,|Ag|, \\ L_2 = \nu\alpha^2(Au, Ag) \leq \nu\alpha^2|Au|\,|Ag|, \\ L_3 = (B(u, u), g) \leq C|u|^{\frac{1}{2}}\|u\|^{\frac{3}{2}}\|g\| \leq \frac{C}{\lambda_1^{3/4}}\|u\|^2|Ag|, \\ L_4 = (f, g) \leq \frac{1}{\lambda_1}|f|\,|Ag|, \end{gathered}$$

which proves (30). Besides, applying $(I + \alpha^2 A)^{-1}$ in (20), we have

$$\frac{du}{dt} + \nu Au + (I + \alpha^2 A)^{-1}B(u, u) = (I + \alpha^2 A)^{-1}f. \tag{33}$$

Taking the L^2-action of (33) with g, we have

$$(\frac{du}{dt}, g) + \overbrace{\nu(Au, g)}^{L_6} + \overbrace{((I + \alpha^2 A)^{-1}B(u, u), g)}^{L_7} = \overbrace{((I + \alpha^2 A)^{-1}f, g)}^{L_8}. \tag{34}$$

We have

$$L_6 = \nu(Au, g) \le \nu\|u\|\,\|g\|,$$

$$L_7 = (B(u,u), (I+\alpha^2 A)^{-1}g) \le C|u|^{\frac{1}{2}}\|u\|^{\frac{3}{2}}\|g\|,$$

$$L_8 = (f, (I+\alpha^2 A)^{-1}g) \le |f|\,|g| \le \frac{1}{\lambda_1^{1/2}}|f|\,\|g\|.$$

Therefore (31) holds true. Finally, choosing $g = Au \in H$ in (34), we get

$$\frac{1}{2}\frac{d}{dt}\|u\|^2 + \nu|Au|^2 + \overbrace{(B(u,u), (I+\alpha^2 A)^{-1}Au)}^{L_9} = \overbrace{(f, (I+\alpha^2 A)^{-1}Au)}^{L_{10}}.$$

Finally, to obtain (32), we use Young's Inequality and (19):

$$L_9 \le C\|u\|^{\frac{3}{2}}|Au|^{\frac{3}{2}} \le \frac{\nu}{4}|Au|^2 + \frac{27C}{4\nu^3}\|u\|^6,$$

$$L_{10} \le \frac{\nu}{4}|Au|^2 + \frac{1}{\nu}|f|^2.$$

□

Lemma 4.2. *For $T > 0$ fixed and u the solution of simplified Bardina equations (20) (given by Theorem 2.1), there exists a time $t_0 > 0$ depending on initial data $u(0) = u_0 \in V$, such that for $t \ge t_0$ we have*

$$\|u(t)\|^2 \le \frac{2\nu^2 (Gr(f))^2}{\lambda_1^{1/2}\alpha^2} \tag{35}$$

and

$$\int_t^{t+T} \|u(s)\|^2 + \alpha^2|Au(s)|^2 ds \le (2 + \nu\lambda_1 T)\frac{\nu(Gr(f))^2}{\lambda_1^{1/2}}, \tag{36}$$

where $Gr(f)$ is given in (24).

Proof. By (15) and Young inequality, we obtain

$$(f,u) \le |f||u| \le \frac{1}{2\lambda_1\nu}|f|^2 + \frac{\nu}{2}\|u\|^2.$$

Using equality (29), we have that

$$\frac{d}{dt}\left(|u(t)|^2 + \alpha^2\|u(t)\|^2\right) + \nu\left(\|u(t)\|^2 + \alpha^2|Au(t)|^2\right) \le \frac{1}{\lambda_1\nu}|f(t)|^2, \tag{37}$$

and applying (15) again,

$$\frac{d}{dt}\left(|u(t)|^2 + \alpha^2\|u(t)\|^2\right) + \lambda_1\nu\left(|u(t)|^2 + \alpha^2\|u(t)\|_2\right) \le \frac{1}{\lambda_1\nu}|f(t)|^2.$$

By the Gronwall inequality, we conclude that ,$\forall\, 0 \le s \le t$,

$$\begin{aligned} |u(t)|^2 + \alpha^2\|u(t)\|^2 \le{}& e^{-\lambda_1\nu(t-s)}(|u(s)|^2 + \alpha^2\|u(s)\|^2) \\ &+ \frac{1}{\lambda_1\nu}\int_s^t e^{-\lambda_1\nu(t-r)}|f(r)|^2 dr. \end{aligned} \tag{38}$$

Therefore, inequality (35) follows from (38) and (36) follows from (37). □

4.2. **Estimates to the data assimilation system (21).** In this subsection, we present estimates to the data assimilation system (21), which are similar to those previously presented.

Lemma 4.3. *Let $f \in H, w(0) = w_0 \in V$, and w the weak solution of data assimilation system (21), with initial data $w(0) = w_0$. We have that*

$$\frac{1}{2}\frac{d}{dt}\left(|w(t)|^2 + \alpha^2\|w(t)\|^2\right) + \nu\left(\|w(t)\|^2 + \alpha^2|Aw(t)|^2\right) \leq (f + \eta\mathcal{P}(I_h(u(t))), w(t)) + \frac{\eta}{2}(\mathcal{C}[h])^2|Aw(t)|^2 - \frac{\eta}{2}|w(t)|^2, \tag{39}$$

for all $0 \leq t < T$. Furthermore, there is a dimensionless positive constant C_9 (depending only of constants given in (16))such that

$$\left\langle \frac{d}{dt}(w + \alpha^2 Aw), g\right\rangle_{D'} \leq C_9\psi|Ag|, \tag{40}$$

for all $g \in D(A)$, where

$$\psi := \left(\nu\alpha^2 + \frac{\eta\mathcal{C}[h]}{\lambda_1}\right)|Aw| + \left(\nu + \frac{\eta}{\lambda_1}\right)|w| + \frac{\|w\|^2}{\lambda_1^{3/4}} + \frac{|f|}{\lambda_1} + \frac{\eta}{\lambda_1}|\mathcal{P}I_h w|.$$

Proof. To obtain inequality (39), we take the duality $\langle .,.\rangle_{D'}$ in (21) with w, use (22) and Young's Inequality for the following estimate:

$$\begin{aligned} -\eta(\mathcal{P}I_h(w-u), w) &= -\eta(\mathcal{P}I_h(w) - w, w) - \eta|w|^2 + \eta(\mathcal{P}I_h(u), w) \\ &\leq \eta\mathcal{C}[h]|Aw|\,|w| - \eta|w|^2 + \eta(\mathcal{P}I_h(u), w) \\ &\leq \frac{\eta}{2}(\mathcal{C}[h])^2|Aw|^2 - \frac{\eta}{2}|w|^2 + \eta(\mathcal{P}I_h(u), w). \end{aligned}$$

Using Poincaré's Inequality, estimate (40) is obtained similarly as (30), only including the following estimate

$$\begin{aligned} -\eta(\mathcal{P}I_h(w-u), g) &= -\eta(\mathcal{P}I_h(w) - w, g) - \eta(w, g) + \eta(\mathcal{P}I_h(u), g) \\ &\leq \eta\mathcal{C}[h]|Aw|\,|g| + \eta|w|\,|g| + \eta|\mathcal{P}I_h(u)|\,|g| \\ &\leq \frac{\eta\mathcal{C}[h]}{\lambda_1}|Aw|\,|Ag| + \frac{\eta}{\lambda_1}|w|\,|Ag| + \frac{\eta}{\lambda_1}|\mathcal{P}I_h(u)|\,|Ag|. \end{aligned}$$

□

Lemma 4.4. *Let w and $\tilde{w}$ be weak solutions of data assimilation system (21) in $(0,T)$, according to Theorem 2.2, with initial values $w(0) = w_0 \in V$ and $\tilde{w}(0) = w_0 \in V$ and external force f and $\tilde{f}$, respectively. For $W = w - \tilde{w}$, there exists a dimensionless positive constant C_{10} (depending only of constant given in (16)) such that*

$$\frac{d}{dt}\left(|W|^2 + \alpha^2\|W\|^2\right) + \nu\left(\|W\|^2 + \alpha^2|AW|^2\right) \leq \frac{C_{10}}{\nu^3}\|\tilde{w}\|^4|W|^2 + \frac{\eta}{2}(\mathcal{C}[h])^2|AW|^2 + \frac{|f-\tilde{f}|^2}{2\lambda_1\nu}. \tag{41}$$

Proof. Subtracting data assimilation equations (21) for w and $\tilde{w}$ yields

$$\frac{d}{dt}(W + \alpha^2 AW) + \nu A(W + \alpha^2 AW) + B(w,w) - B(\tilde{w},\tilde{w}) = -\eta\mathcal{P}I_h W + \mathcal{P}(f - \tilde{f}). \tag{42}$$

Since $B(w,w) - B(\tilde{w},\tilde{w}) = B(W,W) + B(\tilde{w},W) + B(W,\tilde{w})$, taking the duality $\langle .,.\rangle_{D'}$ in (42) with W and using (17), we have

$$\frac{1}{2}\frac{d}{dt}(|W|^2+\alpha^2\|W\|^2)+\nu(\|W\|^2+\alpha^2|AW|^2)+(B(W,\tilde{w}),W)$$
$$=-\eta(\mathcal{P}I_hW,W)+(\mathcal{P}(f-\tilde{f}),W). \quad (43)$$

From (18) we obtain

$$(B(W,\tilde{w}),W)\le\|W\|^{\frac{3}{2}}|W|^{\frac{1}{2}}\|\tilde{w}\|\le\frac{\nu}{4}\|W\|^2+\frac{C_{10}}{\nu^3}\|\tilde{w}\|^4|W|^2$$

Also, using (22) we get

$$\begin{aligned}-\eta(\mathcal{P}I_hW,W) &= -\eta(\mathcal{P}I_h[W]-W,W)-\eta|W|^2\le\eta\mathcal{C}[h]|AW|\,|W|-\eta|W|^2\\ &\le \frac{\eta}{4}(\mathcal{C}[h])^2|AW|^2.\end{aligned}$$

Finally, we have

$$(\mathcal{P}(f-\tilde{f}),W)\le|f-\tilde{f}||W|\le\frac{1}{\lambda_1^{\frac{1}{2}}}|f-\tilde{f}|\|W\|\le\frac{1}{4\lambda_1\nu}|f-\tilde{f}|^2+\frac{\nu}{4}\|W\|^2.$$

Therefore by (43) and inequalities proves (41). □

Finally, we recall the generalized Gronwall inequality, which proof can be found in [31] and [24].

Lemma 4.5 (Uniform Gronwall Inequality)**.** *Let $T>0$ fixed and $\gamma,\beta:\mathbb{R}_+^*\longrightarrow\mathbb{R}$ locally integrable real valued functions. Suppose that the conditions below are valid:*

$$\lim_{t\to\infty}\beta(t)=0,$$

$$\liminf_{t\to\infty}\int_t^{t+T}\gamma(s)\,ds>0 \qquad \text{and} \qquad \limsup_{t\to\infty}\int_t^{t+T}\gamma^-(s)\,ds=\Gamma<\infty,$$

where $\gamma^-=\max\{-\gamma,0\}$. Moreover, suppose that $Y:(0,\infty)\longmapsto[0,\infty)$ is an absolutely continuous function such that

$$\frac{dY}{dt}+\gamma(t)Y\le\beta(t),$$

almost everywhere in $(0,\infty)$. Then $Y(t)\to0$, as $t\to\infty$. Furthermore, if $\beta(t)\equiv0$, then $Y(t)\to0$, as $t\to\infty$, exponentially.

5. **Proof of main results.**

Proof of Theorem 2.2. Existence of solutions can be obtained via standard Galerkin procedure, using a basis of eigenfunctions with properties (13) and (14). Denoting

$$F(t)=f(t)+\eta\mathcal{P}I_hu(t),$$

we have that $F\in L^2([0,T];H)$, since $u\in L^2([0,T];D(A))$ and (22). Consider the linear spanned space $H_m=[u_1,...,u_m]$, the projections $P_m:H\longrightarrow H_m$ and the approximated problem

$$\begin{cases}\dfrac{dz_m}{dt}+P_mB(w_m,w_m)+\nu Az_m=P_mF-\eta P_mI_h(w_m)\\ w_m(0)=P_m(w_0),\end{cases} \quad (44)$$

with $z_m = w_m + \alpha^2 A w_m$, where $w_m(x,t) = \sum_{j=1}^{m} g_{j,m}(t) u_j(x)$. Existence and uniqueness of short time ($[0, T_m)$) solutions to the above system is obtained by classical ODE theory. Subsequently, we prove now uniform bounds for w_m independently of m, which guarantees existence in time $[0,T)$ of each w_m. It is easy to check that the inequality (39) remains true to the system (44). Therefore, we have the estimates

$$\begin{aligned}\frac{1}{2}\frac{d}{dt}\left(|w_m|^2 + \alpha^2\|w_m\|^2\right) \quad &+ \quad \nu\left(\|w_m\|^2 + \alpha^2|Aw_m|^2\right)\\ &\leq \quad (F(t), w_m) + \frac{\eta}{2}(\mathcal{C}[h])^2|Aw_m|^2 - \frac{\eta}{2}|w_m|^2\\ &\leq \quad \frac{1}{2\eta}|F|^2 + \frac{\eta}{2}(\mathcal{C}[h])^2|Aw_m|^2.\end{aligned}$$

Since by hypothesis $\mathcal{C}[h]$ is small enough such that $\mathcal{C}[h] \leq \sqrt{\frac{\nu\alpha^2}{\eta}}$, we obtain

$$\frac{d}{dt}\left(|w_m|^2 + \alpha^2\|w_m\|^2\right) + \nu\left(\|w_m\|^2 + \alpha^2|Aw_m|^2\right) \leq \frac{1}{\eta}|F|^2.$$

We get for all $t \in [0, T_m)$,

$$\begin{aligned}|w_m(t)|^2 + \alpha^2\|w_m(t)\|^2 + \nu\int_0^t(\|w_m(s)\|^2 + \alpha^2|Aw_m(s)|^2)ds\\ \leq (|u_m(0)|^2 + \alpha^2\|u_m(0)\|^2) + \frac{1}{\eta}\int_0^T |F(s)|^2 ds := K_1(T).\end{aligned}$$

Since the estimate above is uniform in m and t, we have existence on the interval $[0,T)$, for all m. Moreover, we have the following estimates

$$\|w_m\|^2_{L^2([0,T];D(A))} \leq \frac{K_1(T)}{\nu\alpha^2} \quad \text{and} \quad \|w_m\|^2_{L^\infty([0,T];V)} \leq \frac{K_1(T)}{\alpha^2}. \tag{45}$$

Now, in order to apply Aubin-Lion Theorem (see,e.g.,[36]), we estabilish uniform estimates in m for $\frac{dw_m}{dt}$. We can proceed as in (40) to obtain

$$\left\langle \frac{d}{dt}(w_m + \alpha^2 A w_m), g\right\rangle_{D'} \quad \leq \quad C_9\psi_m|Ag|, \forall g \in D(A),$$

where

$$\psi_m := \left(\nu\alpha^2 + \frac{\eta\mathcal{C}[h]}{\lambda_1}\right)|Aw_m| + \left(\nu + \frac{\eta}{\lambda_1}\right)|w_m| + \frac{\|w_m\|^2}{\lambda_1^{3/4}} + \frac{|f|}{\lambda_1} + \frac{\eta}{\lambda_1}|\mathcal{P}I_h w_m|.$$

Since $\|\psi_m\|_{L^2([0,T];H)}$ is bounded uniformly in m by (45), we can conclude that $\frac{d}{dt}z_m = \frac{d}{dt}(w_m + \alpha^2 A w_m)$ is bounded uniformly in $L^2([0,T];D')$. Equivalently, $\frac{d}{dt}w_m$ is bounded uniformly in $L^2([0,T];H)$. By Aubin-Lions compactness theorem (see e.g., [2],[36]) and Banach-Alaoglu Theorem, we obtain a subsequence of approximated solutions, which we also denote by $\{w_m\}_{m\in\mathbb{N}}$, such that

$$\begin{aligned}&w_m \to w \text{ weakly in } L^2([0,T], D(A)),\\ &w_m \to w \text{ strongly in } L^2([0,T];V),\\ &\frac{dw_m}{dt} \to \frac{dw}{dt} \text{ weakly in } L^2([0,T],H),\end{aligned}$$

and, for non-filtered velocity,

$$
\begin{aligned}
&z_m \to z \ \text{ weakly in } \ L^2([0,T],H),\\
&z_m \to z \ \text{ strongly in } \ L^2([0,T];V'),\\
&\frac{dz_m}{dt} \to \frac{dz}{dt} \ \text{ weakly in } \ L^2([0,T],D').
\end{aligned}
$$

Now, it is straightforward to pass the weak limit in (44) to obtain that w is a solution of (21).

Let us prove the continuous dependence on initial data of the solutions and as a result, the uniqueness of solutions. Let w and $\tilde{w}$ be two solutions and denote $W = w - \tilde{w}$. By (41) and $\mathcal{C}[h] \leq \sqrt{\frac{\nu\alpha^2}{\eta}}$, we obtain

$$
\frac{d}{dt}\left(|W|^2 + \alpha^2\|W\|^2\right) + \frac{\nu}{2}\left(\|W\|^2 + \alpha^2|AW|^2\right) \leq \frac{C_{10}}{\nu^3}\|\tilde{w}\|^4|W|^2 + \frac{|f-\tilde{f}|^2}{2\lambda_1\nu}.
$$

Using Gronwall inequality, we obtain

$$
|W(t)|^2 + \alpha^2\|W(t)\|^2 \leq e^{\frac{C_{10}}{\nu^3}\int_0^t \|\tilde{w}(s)\|^4 ds}\left(\xi_0 + \frac{1}{2\lambda_1\nu}\int_0^t |f(s)-\tilde{f}(s)|^2 ds\right),
$$

where $\xi_0 = |W_0|^2 + \alpha^2\|W_0\|^2$. It implies continuous dependence of the regular solution. □

Proof of Theorem 3.1. To obtain the convergence of the solutions u_{α_j} of the system (20), as $\alpha_j \to 0^+$, and relate the limit to the Navier-Stokes equations, we use estimates (29) and (31) and get

$$
\begin{aligned}
&(u_\alpha)_{\alpha>0} \ \text{ is bounded in } \ L^\infty([0,T],H)\cap L^2([0,T],V),\\
&\left(\frac{du_\alpha}{dt}\right)_{\alpha>0} \ \text{ is bounded in } \ L^{\frac{4}{3}}([0,T],V'),\\
&(\alpha^2 A u_\alpha)_{\alpha>0} \ \text{ is bounded in } \ L^2([0,T],H).
\end{aligned}
$$

Aubin-Lions compactness theorem and the Banach-Alaoglu theorem ensure the existence of a function $u \in L^\infty([0,T],H)\cap L^2([0,T],V)$, and a sequence $(\alpha_j)_{j\in\mathbb{N}}$ such that $\alpha_j \overset{j\to\infty}{\to} 0^+$ and

$$
\begin{gathered}
u_{\alpha_j} \overset{j\to\infty}{\to} u \ \text{ weakly in } \ L^2([0,T],V) \ \text{ and } \ \text{ weakly-* in } \ L^\infty([0,T],H),\\
u_{\alpha_j} \overset{j\to\infty}{\to} u \ \text{ strongly in } \ L^2([0,T],H),\\
\alpha_j^2 A u_{\alpha_j} \overset{j\to\infty}{\to} 0 \ \text{ weakly in } \ L^2([0,T],H).
\end{gathered}
$$

We can apply the limit in the weak formulation: for all $g \in D(A)$, and $\phi \in C_0^1[0,T)$,

$$
\begin{aligned}
\int_0^T (u_{\alpha_j}(t),(I+\alpha^2 A)g)\phi_t(t)\,dt \quad &= \int_0^T (B(u_{\alpha_j}(t),u_{\alpha_j}(t)),g)\phi(t)\,dt\\
&+ \nu\int_0^T ((u_{\alpha_j}(t),g))\phi(t)\,dt\\
&+ \nu\alpha_j^2\int_0^T (Au_{\alpha_j}(t),Ag)\phi(t)\,dt\\
&- \phi(0)(u(0),(I+\alpha^2 A)g) - \int_0^T (f,g)\phi(t)\,dt,
\end{aligned}
$$

and check that u is a weak solution of Navier-Stokes equations. Applying $\liminf$ as $j \to \infty$ in the inequality below

$$\begin{aligned}\frac{1}{2}\left(|u_{\alpha_j}(t)|^2 + \alpha_j^2\|u_{\alpha_j}(t)\|^2\right) + \quad & \nu\int_0^T \left(\|u_{\alpha_j}(s)\|^2 + \alpha_j^2|Au_{\alpha_j}(s)|^2\right) ds \\ \leq \quad & \int_0^T (f, u_{\alpha_j}(s))\, ds + \frac{1}{2}\left(|u_0|^2 + \alpha_j^2\|u_0\|^2\right),\end{aligned}$$

we obtain that u satisfies the energy inequality, and therefore u is a Leray-weak solution.

Next, we prove stronger convergence for small interval of time. Using inequality (32) and dividing the both sides by $(\nu^2\lambda_1^{1/2} + \|u_\alpha(t)\|^2)^3$, we have that

$$\frac{\frac{d}{dt}\|u_\alpha(t)\|^2}{(\nu^2\lambda_1^{1/2} + \|u_\alpha(t)\|^2)^3} \leq C_5\left(\frac{C}{\nu^3} + \frac{|f|^2}{\nu^7\lambda_1^{3/2}}\right).$$

Integrating over the interval $(0,t)$, it follows that

$$(\nu^2\lambda_1^{1/2} + \|u_\alpha(t)\|^2)^2 \leq \frac{(\nu^2\lambda_1^{1/2} + \|u_0\|)^2}{1 - 2C_5\left(\nu^2\lambda_1^{1/2} + \|u_0\|^2\right)^2\left(\int_0^t \frac{C}{\nu^3} + \frac{|f(s)|^2}{\nu^7\lambda_1^{3/2}}ds\right)}.$$

Using (32) again, we ensure the existence of a small time $0 < T^* \leq T$ such that

$$(u_\alpha)_{\alpha>0} \text{ is bounded in } L^\infty([0,T^*],V) \cap L^2([0,T^*],D(A)). \tag{46}$$

Thus, holds true the convergences

$$u_{\alpha_j} \overset{j\to\infty}{\rightarrow} u \text{ weakly in } L^2([0,T^*],D(A)),$$

$$u_{\alpha_j} \overset{j\to\infty}{\rightarrow} u \text{ weakly-}* \text{ in } L^\infty([0,T^*],V),$$

and u is a strong solution to the Navier-Stokes equations in $(0,T^*)$.

Finally, suppose that $\|u_0\|$ and $|f|$ are small enough to satisfy

$$\|u_0\|_V^2 + \frac{C_5}{\nu}\sup_{0\leq t\leq T}\left\{\exp\left(-\frac{\lambda_1\nu t}{2}\right)\int_0^t \exp\left(\frac{\lambda_1\nu r}{2}\right)|f(r)|^2 dr\right\} \leq \frac{1}{4}\sqrt{\frac{\lambda_1\nu^4}{C_5}},$$

where C_5 the constant given in (32). Let T^* the supremum over all $0 < \tilde{T} < T$ such that

$$\|u_\alpha(t)\|^2 \leq \sqrt{\frac{\lambda_1\nu^4}{2C_5}},$$

for all $0 \leq t \leq \tilde{T}$. Assume, by contradiction, that $T^* < T$. By (32), we have

$$\frac{d}{dt}\|u_\alpha(t)\|^2 + \frac{\nu\lambda_1}{2}\|u_\alpha(t)\|^2 \leq \frac{C_5}{\nu}|f(t)|^2,$$

for all $0 < t < T^* < T$. By Grönwall inequality, we obtain

$$\begin{aligned}\|u_\alpha(t)\|^2 \quad &\leq \quad \|u(0)\|^2 + C_5\exp\left(-\frac{\nu\lambda_1 t}{2}\right)\int_0^t \exp\left(\frac{\nu\lambda_1 r}{2}\right)\frac{|f(r)|^2}{\nu}dr \\ &\leq \quad \frac{1}{4}\sqrt{\frac{\lambda_1\nu^4}{C_5}} < \sqrt{\frac{\lambda_1\nu^4}{2C_5}},\end{aligned}$$

for all $0 < t < T^*$, which contradicts the maximality of T^*, provided that solutions is time-continuous. As a result, we can choose $T^* = T$ in (46). □

Proof of Theorem 3.2. Set $v = u + \alpha^2 Au, \tilde{v} = \tilde{u} + \alpha^2 A\tilde{u}$, and the differences $U := \tilde{u} - u$ and $V = \tilde{v} - v$. Then U solves the equation

$$\frac{dV}{dt} + \nu AV + B(U,U) + B(u,U) + B(U,u) = \tilde{f} - f,$$

provided that $B(\tilde{u},\tilde{u}) - B(u,u) = B(U,U) + B(u,U) + B(U,u)$. Taking the duality $\langle \, , \, \rangle_{D'}$ with U and using (17), gives

$$\frac{1}{2}\frac{d}{dt}(|U|^2 + \alpha^2\|U\|^2) + \nu(\|U\|^2 + \alpha^2|AU|^2) = -(B(U,u),U) + (\tilde{f} - f, U)$$

Using the following estimates, which are obtained from (18), Young and Poincaré inequalities

$$(f - \tilde{f}, U) \leq |(\tilde{f} - f)|\,|U| \leq \frac{1}{\lambda^{\frac{1}{2}}}|\tilde{f} - f|\|U\| \leq \frac{1}{2\nu\lambda_1}|\tilde{f} - f|^2 + \frac{\nu}{2}\|U\|^2,$$

$$|(B(U,u),U)| \leq C\|u\|\,|U|^{\frac{1}{2}}\|U\|^{\frac{3}{2}} \leq \frac{C}{\lambda_1^{\frac{1}{4}}}\|u\|\,\|U\|^2,$$

we obtain

$$\frac{d}{dt}\left(|U|^2 + \alpha^2\|U\|^2\right) + \nu\left(\|U\|^2 + \alpha^2|AU|^2\right) \leq \frac{C}{\lambda_1^{\frac{1}{4}}}\|u\|\,\|U\|^2 + \frac{1}{\nu\lambda_1}|\tilde{f} - f|^2. \quad (47)$$

Additionally, by (22) and Young's inequality, we have that

$$\begin{aligned}\|U\|^2 &= (U, AU) \leq |U|\,|AU| \leq (|U - \mathcal{P}(I_h(U))| + |\mathcal{P}(I_h(U))|)\,|AU| \\ &\leq (\mathcal{C}[h]|AU| + |\mathcal{P}(I_h(U))|)\,|AU| \leq 2\mathcal{C}[h]|AU|^2 + \frac{1}{4\mathcal{C}[h]}|\mathcal{P}(I_h(U))|^2.\end{aligned}$$

Therefore, using the inequality above, the first term on the right side of (47) is estimated as follows:

$$\begin{aligned}\frac{C}{\lambda_1^{\frac{1}{4}}}\|u\|\,\|U\|^2 &\leq \frac{\nu\alpha^2}{4\mathcal{C}[h]}\left(2\mathcal{C}[h]|AU|^2 + \frac{1}{4\mathcal{C}[h]}|\mathcal{P}(I_h(U))|^2\right) \\ &+ \frac{C^2\alpha^2\mathcal{C}[h]}{\nu\lambda_1^{\frac{1}{2}}\alpha^4}\|u\|^2\|U\|^2. \end{aligned} \quad (48)$$

By (47), (48) and Poincaré inequalities (15), we obtain

$$\begin{aligned}\frac{d}{dt}\left(|U|^2 + \alpha^2\|U\|^2\right) + \left(\frac{\lambda_1\nu}{2} - \frac{C^2\mathcal{C}[h]}{\nu\lambda^{\frac{1}{2}}\alpha^4}\|u\|^2\right)\left(|U|^2 + \alpha^2\|U\|^2\right) \\ \leq \frac{1}{\lambda_1\nu}|f - \tilde{f}|^2 + \frac{\nu\alpha^2}{4^2(\mathcal{C}[h])^2}|\mathcal{P}(I_h(U))|^2.\end{aligned}$$

Now, we apply Lemma 4.5 with

$$\begin{cases} \gamma(t) := \dfrac{\lambda_1\nu}{2} - \dfrac{C^2\mathcal{C}[h]}{\nu\lambda^{\frac{1}{2}}\alpha^4}\|u(t)\|^2, \\[2ex] \beta(t) := \dfrac{1}{\lambda_1\nu}|f(t) - \tilde{f}(t)|^2 + \dfrac{\nu\alpha^2}{4^2(\mathcal{C}[h])^2}|\mathcal{P}I_h(U(t))|^2, \\[2ex] Y(t) := |U(t)|^2 + \alpha^2\|U(t)\|^2. \end{cases}$$

Using assumptions (25) and (26), we have that $\lim_{t\to\infty}\beta(t)=0$. Also, by (35) and hypothesis (27), we obtain

$$\begin{aligned}\limsup_{t\to\infty}\int_t^{t+(\lambda_1\nu)^{-1}}\gamma^-(s)ds &\leq \frac{C^2\mathcal{C}[h]}{\nu\lambda_1^{\frac{1}{2}}\alpha^4}\limsup_{t\to\infty}\int_t^{t+(\lambda_1\nu)^{-1}}\|u(s)\|^2ds\\ &\leq \frac{C^2\mathcal{C}[h]}{\nu\lambda_1^{\frac{1}{2}}\alpha^4}\cdot\frac{2\nu^2(Gr(f))^2}{\lambda_1^{\frac{1}{2}}\alpha^2}(\lambda_1\nu)^{-1}\\ &= \frac{2C^2(Gr(f))^2\mathcal{C}[h]}{\lambda_1^2\alpha^6}<\infty,\end{aligned}$$

and

$$\begin{aligned}\liminf_{t\to\infty}\int_t^{t+(\lambda_1\nu)^{-1}}\gamma(s)ds &\geq \frac{\lambda_1\nu}{2}(\lambda_1\nu)^{-1}-\frac{C^2\mathcal{C}[h]}{\nu\lambda_1^{\frac{1}{2}}\alpha^4}\limsup_{t\to\infty}\int_t^{t+(\lambda_1\nu)^{-1}}\|u(s)\|^2ds\\ &\geq \frac{\lambda_1\nu}{2}(\lambda_1\nu)^{-1}-\frac{C^2\nu\mathcal{C}[h](Gr(f))^2}{\lambda_1\alpha^4\nu}(2+\nu\lambda_1(\lambda_1\nu)^{-1})>0.\end{aligned}$$

Therefore

$$\lim_{t\to\infty}Y(t)=\lim_{t\to\infty}|U(t)|^2+\alpha^2\|U(t)\|^2=0.$$

□

Proof of Theorem 3.3. Define $\xi=w-u$. Then ξ satisfies the equations

$$\begin{aligned}&\frac{d}{dt}(\xi+\alpha^2A\xi)+\nu A(\xi+\alpha^2A\xi)+B(w,w)-B(u,u)=-\eta\mathcal{P}I_h(\xi),\\ &\nabla\cdot\xi=0.\end{aligned}\tag{49}$$

Taking the duality $\langle\ ,\ \rangle_{D'}$ in (49) with ξ, we have

$$\frac{1}{2}\frac{d}{dt}(|\xi|^2+\alpha^2\|\xi\|^2)+\nu(\|\xi\|^2+\alpha^2|A\xi|^2)+(B(w,w)-B(u,u),\xi)=-\eta(I_h\xi,\xi)$$

Using the fact that $B(w,w)-B(u,u)=B(\xi,\xi)+B(u,\xi)+B(\xi,u)$ and (17), we get

$$\frac{1}{2}\frac{d}{dt}(|\xi|^2+\alpha^2\|\xi\|^2)+\nu(\|\xi\|^2+\alpha^2|A\xi|^2)+(B(\xi,u),\xi)=-\eta(I_h\xi-\xi,\xi)-\eta|\xi|^2\tag{50}$$

The nonlinear term we estimate using (19) and Young's inequality:

$$|(B(\xi,u),\xi)|\leq\frac{C}{\lambda_1^{1/4}}|\xi|\,\|u\|\,|A\xi|\leq\frac{\nu\alpha^2}{4}|A\xi|^2+\frac{C}{\lambda_1^{1/2}\nu\alpha^2}\|u\|^2|\xi|^2,\tag{51}$$

and the right-hand side of (50) we estimate as follows:

$$\begin{aligned}-\eta(I_h\xi-\xi,\xi)-\eta|\xi|^2 &\leq \eta|I_h\xi-\xi|\,|\xi|-\eta|\xi|^2\\ &\leq \eta\mathcal{C}[h]|A\xi|\,|\xi|-\eta|\xi|^2\\ &\leq \frac{\eta}{2}(\mathcal{C}[h])^2|A\xi|^2-\frac{\eta}{2}|\xi|^2\\ &\leq \frac{\nu\alpha^2}{2}|A\xi|^2-\frac{\eta}{2}|\xi|^2,\end{aligned}\tag{52}$$

since (23) holds by hypothesis. From (50),(51) and (52), we obtain

$$\frac{1}{2}\frac{d}{dt}(|\xi|^2+\alpha^2\|\xi\|^2)+\nu(\|\xi\|^2+\alpha^2|A\xi|^2)\leq\frac{3\nu\alpha^2}{4}|A\xi|^2+\frac{C}{\lambda_1^{1/2}\nu\alpha^2}\|u\|^2|\xi|^2-\frac{\eta}{2}|\xi|^2,$$

and therefore

$$\frac{1}{2}\frac{d}{dt}(|\xi|^2+\alpha^2\|\xi\|^2)+\frac{\nu}{4}(\|\xi\|^2+\alpha^2|A\xi|^2)\le\left(\frac{C}{\lambda_1^{1/2}\nu\alpha^2}\|u\|^2-\frac{\eta}{2}\right)|\xi|^2.$$

By Poincaré Inequality,

$$\frac{d}{dt}(|\xi|^2+\alpha^2\|\xi\|^2)+\frac{\nu\lambda_1}{2}(|\xi|^2+\alpha^2\|\xi\|^2)\le\left(\frac{C}{\lambda_1^{1/2}\nu\alpha^2}\|u\|^2-\eta\right)|\xi|^2. \tag{53}$$

Defining

$$\delta(t)=\frac{\nu\lambda_1}{2}+\eta-\frac{C}{\lambda_1^{1/2}\nu\alpha^2}\|u(t)\|^2,$$

and denoting $\gamma(t)=\min\left\{\frac{\nu\lambda_1}{2},\delta(t)\right\}$, we can rewrite (53) as

$$\frac{d}{dt}(|\xi|^2+\alpha^2\|\xi\|^2)+\gamma(t)(|\xi|^2+\alpha^2\|\xi\|^2)\le 0.$$

From Lemma 4.2 and assumption (28), we get

$$\begin{aligned}\delta(t) &= \frac{\nu\lambda_1}{2}+\eta-\frac{C}{\lambda_1^{1/2}\nu\alpha^2}\|u(t)\|^2\\ &> \frac{\nu\lambda_1}{2}+\eta-\frac{C}{\lambda_1^{1/2}\nu\alpha^2}\frac{2\nu^2(Gr(f))^2}{\lambda_1^{1/2}\alpha^2}\\ &= \frac{\nu\lambda_1}{2}+\eta-C_2\frac{\nu(Gr(f))^2}{\lambda_1\alpha^4}>0.\end{aligned}$$

Therefore

$$\liminf_{t\to\infty}\int_t^{t+(\nu\lambda_1)^{-1}}\gamma(s)\,ds>0.$$

Besides that,

$$\limsup_{t\to\infty}\int_t^{t+(\nu\lambda_1)^{-1}}\gamma^-(s)\,ds=0.$$

Therefore, by Lemma 4.5, it follows that

$$|\xi(t)|^2+\|\xi(t)\|^2\to 0$$

i.e., $(w(t)-u(t))\to 0$, as $t\to\infty$, exponentially, in L^2 and H^1 norms. □

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