Econ 675 Assignment 1

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Contents

1   Kernal Density Estimation

1.1   Q1 Part 1

Start by noting that

$$\hat{f}^{(s)}(x) = \frac{(-1)^s}{nh^{1+s}} \sum_{i=1}^{n} k^{(s)} \left(\frac{x_i - x}{h} \right)$$

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*Shouts out to Ani, Paul, Tyler, Erin, Caitlin and others for all the help with this

Now taking the expectation of $\hat{f}^{(s)}(x)$ that we can apply the linearity of expectations to move the expectation inside the sum. Then we can use the i.i.d. assumption to show the sum is just n times the expectation. This leaves us with

$$\mathrm{E}[\hat{f}^{(s)}(x)] = \mathrm{E}\left[\frac{(-1)^s}{h^{1+s}}k^{(s)}\left(\frac{x_i - x}{h}\right)\right] = \int_{-\infty}^{\infty}\frac{(-1)^s}{h^{1+s}}k^{(s)}\left(\frac{z-x}{h}\right)f(z)dz$$

Where the second equality is just by the definition of the expectation. Next we use integration by parts. Note that

$$\int_{-\infty}^{\infty}\frac{(-1)^s}{h^{1+s}}k^{(s)}\left(\frac{z-x}{h}\right)f(z)dz = -\int_{-\infty}^{\infty}\frac{(-1)^s}{h^s}k^{(s-1)}\left(\frac{z-x}{h}\right)f^{(1)}(z)dz$$

Iterating this s times gives us

$$\int_{-\infty}^{\infty}\frac{(-1)^s}{h^{1+s}}k^{(s)}\left(\frac{z-x}{h}\right)f(z)dz = (-1)^s\int_{-\infty}^{\infty}\frac{(-1)^s}{h}k\left(\frac{z-x}{h}\right)f^{(s)}(z)dz = \int_{-\infty}^{\infty}\frac{1}{h}k\left(\frac{z-x}{h}\right)f^{(s)}(z)dz$$

Next we apply change of variables. let $u = \frac{z-x}{h}$ Note that $du = \frac{1}{h}dz$ so we get

$$\int_{-\infty}^{\infty}k(u)f^{(s)}(x+hu)du$$

Next we Taylor expand $f^{(s)}(x+hu)$ to the Pth order about x. Recall from properties of the kernal estimator that $\int_{-\infty}^{\infty}k(u)du = 1$ and that $\int_{-\infty}^{\infty}k(u)u^j du = 0$ for all $j \neq p$ This gives us

$$f^{(s)}(x) + \frac{1}{P!}f^{(s+P)}(x)h^P\int_{-\infty}^{\infty}k(u)u^p du + o(h^P) = f^{(s)}(x) + \frac{1}{P!}f^{(s+P)}(x)h^p\mu_P(k) + o(h^P)$$

which is the desired result.

Now for the variance recall again that

$$\hat{f}^{(s)}(x) = \frac{(-1)^s}{nh^{1+s}}\sum_{i=1}^{n}k^{(s)}\left(\frac{x_i - x}{h}\right)$$

So by the i.i.d. assumption we can get that

$$\mathrm{V}\left(\hat{f}^{(s)}(x)\right) = \frac{1}{nh^{2+2s}}\mathrm{V}\left(k^{(s)}\left(\frac{x_i - x}{h}\right)\right)$$

$$V\left(\hat{f}^{(s)}(x)\right) = \frac{1}{nh^{2+2s}} V\left(k^{(s)}\left(\frac{x_i - x}{h}\right)\right) \tag{1}$$

$$= \frac{1}{n2h^{2+2s}} E\left[\left(k^{(s)}\left(\frac{x_i - x}{h}\right)\right)^2\right] - \frac{1}{nh^{2+2s}} E\left[\left(k^{(s)}\left(\frac{x_i - x}{h}\right)\right)^2\right]^2 \tag{2}$$

$$= \frac{1}{nh^{2+2s}} E\left[\left(k^{(s)}\left(\frac{x_i - x}{h}\right)\right)^2\right] - \frac{1}{n}\left(\frac{1}{h^{1+s}} E\left[\left(k^{(s)}\left(\frac{x_i - x}{h}\right)\right)^2\right]\right)^2 \tag{3}$$

$$= \frac{1}{nh^{2+2s}} \int_{-\infty}^{\infty} k^{(s)}\left(\frac{x_i - x}{h}\right)^2 f(z)dz + \frac{1}{nh^{2+2s}} f^{(n)}(X)^2 \tag{4}$$

$$= \frac{1}{nh^{1+2s}} \int_{-\infty}^{\infty} k^{(s)}(u)^2 f(x + hu)du + o\left(\frac{1}{nh^{2+2s}}\right) \tag{5}$$

$$= \frac{1}{nh^{1+2s}} \cdot \vartheta_s(K) + o\left(\frac{1}{nh^{2+2s}}\right) \tag{6}$$

1.2 Q1 part 2

We start with the following AMISE

$$AIMSE[h] = \int \left[\left(h_n^P \cdot \mu_P(K) \cdot \frac{f^{(P+s)}(x)}{P!}\right)^2 + \frac{1}{nh_n^{1+2s}} \cdot \vartheta_s(K) \cdot f(x)\right] dx$$

Using the ϑ notation so $\vartheta_{P+s}(f) = \int (f^{(P+s)}(x))^2$ and recalling that f(x) integrates to 1 we can rewrite this as

$$= h_n^{2P}\left(\frac{\mu_P(K)}{P!}\right)^2 \vartheta_{P+s}(f) + \frac{\vartheta_s(K)}{nh_n^{1+2s}}$$

Now taking first order conditions and solving for h

$$\frac{d}{dh}AIMSE[h] = 2Ph_n^{2p-1}\left(\frac{\mu_P(K)}{P!}\right)^2 \vartheta_{P+s}(f) - (1+2s)\frac{\vartheta_s(K)}{nh_n^{2+2s}} = 0$$

$$\implies 2Ph^{1+2P+2s}\left(\frac{\mu_P(K)}{P!}\right)^2 \vartheta_{P+s}(f) = (1+2s)\frac{\vartheta_s(K)}{n}$$

Thus, we get the AIMSE-optimal bandwidth choice.

$$h_{AIMSE_s} = \left[\frac{(2s+1)(P!)^2}{2P} \frac{\vartheta_s(K)}{\vartheta_{s+P}(f) \cdot \mu_P(K)^2} \frac{1}{n}\right]^{\frac{1}{1+2P+2s}}$$

Least squares cross-validation is a fully automatic data-driven method of selecting the smoothing parameter h. THis is based on the principle of selecting bandwidth that minimizes the integrated squared error of the resulting estimate. The estimate used is

$$\hat{h}_{CV} = \arg\min_h \frac{1}{n^2h} \sum_{i=1}^{n}\sum_{j=1}^{n} \bar{k}\left(\frac{X_i - X_j}{h}\right) - \frac{2}{n(n-1)h}\sum_{i=1}^{n}\sum_{j=1,i\neq j}^{n} k\left(\frac{X_i - X_j}{h}\right)$$

1.3 Monte Carlo experiment

1.3.1 Q1 Part 3 a

First, we want to compute the theoretically optimal bandwidth for $s = 0$, $n = 1000$, using the Epanechnikov kernel ($P = 2$), with the following Gaussian DGP:

$$x_i \sim 0.5\mathcal{N}(-1.5, -1.5) + 0.5\mathcal{N}(1, 1)$$

Filling in what we know so far we have :

$$h_{AIMSE_s} = \left[\frac{\vartheta_0(K)}{\vartheta_2(f) \cdot \mu_2(K)^2} \frac{1}{1000} \right]^{\frac{1}{5}}$$

So we need the second moment of K and the first moment of the second derivative of k squared. We can get two of these values from the table in Bruce Hanson's nonparametric notes. Giving us.

$$h_{AIMSE_s} = \left[\frac{\frac{3}{5}}{\vartheta_2(f) \cdot \frac{1}{5}^2} \frac{1}{1000} \right]^{\frac{1}{5}}$$

The second derivative of the normal density φ with mean μ variance σ^2 is

$$\varphi''_{\mu,\sigma^2}(x) = \frac{1}{\sqrt{2\pi\sigma^2}} e^{\frac{-(x-\mu)^2}{2\sigma^2}} \left[\left(\frac{(x-\mu)}{\sigma^2} \right)^2 - \frac{1}{\sigma^2} \right]$$

now useing the linearity of integrals we can find $\vartheta_2(f)$

$$\vartheta_2(f) = \int_{-\infty}^{\infty} [0.5\varphi''_{1,1}(x) + 0.5\varphi''_{-1.5,1.5}(x)]^2 dx \approx 0.03883397$$

Where the approximation comes from R
Finally, pluging this in gives the theoretically optimal bandwidth is:

$$h* = 0.8267532$$

1.3.2 Q1 Part 3 b

Below Is the table of $\widehat{IMSE}^{lI}$ results and $\widehat{IMSE}^{LO}$ results by bandwidth h. My stata code was significantly slower and so I only ran 10 repetitions. Even with that the plots give us generally the same idea.
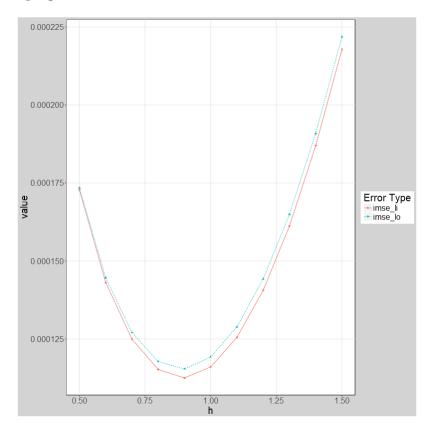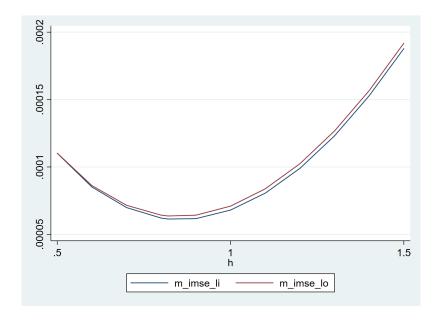
R TABLE

h	imse_li	imse_lo	d_h_hat
0.5	0.000173	0.000173	0.988
0.6	0.000143	0.000145	0.988
0.7	0.000125	0.000127	0.988
0.8	0.000115	0.000118	0.988
0.9	0.000112	0.000115	0.988
1	0.000116	0.000119	0.988
1.1	0.000126	0.000129	0.988
1.2	0.000141	0.000144	0.988
1.3	0.000161	0.000165	0.988
1.4	0.000187	0.000191	0.988
1.5	0.000218	0.000222	0.988

STATA TABLE

h	m_imse_li	m_imse_lo
0.500	0.000110	0.000110
0.600	8.51e-05	8.62e-05
0.700	6.99e-05	7.16e-05
0.800	6.21e-05	6.42e-05
0.820	6.14e-05	6.36e-05
0.900	6.17e-05	6.42e-05
1	6.80e-05	7.09e-05
1.100	8.06e-05	8.38e-05
1.200	9.89e-05	0.000102
1.300	0.000123	0.000127
1.400	0.000153	0.000157
1.500	0.000188	0.000192

My graphs from both programs are below

1.3.3 Q1 Part 3 c

Intuitively the difference between the two estimators, LI and LO, is that the LI includes the extra zero term in the sum since we include $x_i - x_i$. As the size of the sample increases this contribution to the overal average will go to zero. Meaning that the LI IMSE will also converge to the correct estimate. s

1.3.4 Q1 Part 3 d

The "d_h_hat" column of the graph in part c is my calculation of this over the 1000 iterations. The value it came up with was 1.04. This is somewhat close but, as expected, not exactly correct.

2 Question 2: Linear Smoothers, Cross-validation and Series

2.1 Q2 Part 1

For local polynomial regression we want to estimate $e(x) = \mathrm{E}[y_i|x_i = x]$. The idea of a local polynomial regression is to estimate e(x) around the point x using a polynomial of degree p. We estimate this polynomial with weighted least squares. For a given x, we want to solve.

$$\hat{\boldsymbol{\beta}}_{LP}(x) = \arg\min_{\beta \in \mathbb{R}^{p+1}} \sum_{i=1}^{n} [y_i - \boldsymbol{r}_p(x_i - x)'\boldsymbol{\beta}]^2 K(\frac{x_i - x}{h})$$

where $\boldsymbol{r}_p(x) = (1, x, x^2, ..., x^p)'$ and $\hat{e}(x) = \hat{\beta}_0$
from the lecture notes we can get that

$$\hat{\boldsymbol{\beta}}_{LP}(x) = (\boldsymbol{R}'_{\boldsymbol{p}}\boldsymbol{W}\boldsymbol{R}'_{\boldsymbol{p}})^{-1}\boldsymbol{R}'_{\boldsymbol{p}}\boldsymbol{W}\boldsymbol{y}$$

I think This simplifies to the following

$$\hat{e}(x) = e'_1 \left(\sum_{i=1}^{n} \boldsymbol{r}_p(x_i - x)\boldsymbol{r}_p(x_i - x)'w_i \right)^{-1} \left(\sum_{i=1}^{n} \boldsymbol{r}_p(x_i - x)w_iy_i \right)$$

where $wi = K(\frac{x_i - x}{h})$

Now for the series estimation.

$$\hat{\boldsymbol{\beta}}_{\boldsymbol{series}} = \arg \min_{\beta \in \mathbb{R}^{p+1}} \sum_{i=1}^{n} (y_i - \boldsymbol{r}_p(x_i)' \boldsymbol{\beta})^2$$

where $\boldsymbol{r}_p(x_i) = (1, x_x, x_i^2, ..., x_i^p)$ and

$$\hat{e}(x) = \boldsymbol{r}_p(x)' \hat{\boldsymbol{B}}_{series}$$

Together we get

$$\hat{\boldsymbol{B}}_{series} = (\boldsymbol{R}'_p \boldsymbol{R}_p)^{-1} \boldsymbol{R}_p \boldsymbol{y}$$

so

$$\hat{e}(x) = \boldsymbol{r}_p(x)' (\boldsymbol{R}'_p \boldsymbol{R}_p)^{-1} \boldsymbol{R}_p \boldsymbol{y}$$

Writing this in linear summation form I believe we get

$$\hat{e}(x) = \boldsymbol{r}_p(x)' \left(\sum_{i=1}^{n} \boldsymbol{r}_p(x_i) \boldsymbol{r}_p(x_i)' \right)^{-1} \left(\sum_{i=1}^{n} \boldsymbol{r}_p(x_i) y_i \right)$$

2.2 Q2 Part 2

We want to choose the tuning parameter to minimize the mean squared leave one out error which is

$$\hat{c} = \arg \min_{c} \frac{1}{n} \sum_{i=1}^{n} (y_i - \hat{e}_{(i)}(x_i; c))^2$$

where $\hat{e}_{(i)}(x_i)$ is the estimator of the regression function that leaves out x_i

We can write the local polynomial series estimator as

$$\hat{e}(x) = \boldsymbol{S} \boldsymbol{y}$$

Where S is the smoothing matrix. Note that the rows of S sum to one so $\boldsymbol{S1} = \boldsymbol{1}$. For the leave one out estimator we want to use $\boldsymbol{S}$ but with the i_{th} row and column removed. If we let the elements of $\boldsymbol{s}$ be denoted by w_{ij} than deleting the i_{th} column means that the i_{th} row will now sum to $1 - w_{ij}$. So, we divide by $1 - w_{ij}$ to renormalize and get the the leave-one-out estimator is

$$\hat{e}_{(i)}(x_i) = \frac{1}{1 - w_{ij}} \sum_{j=1 j \neq i}^{n} w_{ij} y_i$$

The full sample estimator is

$$\hat{e}(x_i) = \sum_{j=1}^{n} w_{ij} y_i$$

Together we can get that

$$\hat{e}_{(i)}(x_i)(1 - w_{ij}) = \sum_{j=1 j \neq i}^{n} w_{ij} y_i$$

$$\implies \hat{e}_{(i)}(x_i) = \sum_{j=1 j \neq i}^{n} w_{ij} y_i + w_{ij} \hat{e}_{(i)}(x_i) = \sum_{j=1}^{n} w_{ij} y_i + w_{ij} \hat{e}_{(i)}(x_i) - w_{ij} y_i = \hat{e}_{(i)}(x_i) + w_{ij} \hat{e}_{(i)}(x_i) - w_{ij} y_i$$

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$$\implies y - \hat{e}_{(i)}(x_i) = y - \hat{e}_{(i)}(x_i) - w_{ij}\hat{e}_{(i)}(x_i) + w_{ij}y_i$$
$$= y - \hat{e}_{(i)}(x_i) + w_{ij}(y_i - \hat{e}_{(i)}))$$

$$\implies y_i - \hat{e}_{(i)}(x_i) = \frac{1}{1 - w_{ij}}(y_i - \hat{e}(x_i))$$

Which is what we wanted

2.3 Q2 part 3

Note that we have iid data and the $\sum_{i=1}^{n} w_{n,i}(x_i) = 1$ first we want to find

$$\mathrm{E}[\hat{e}(x)|\boldsymbol{x}] = \mathrm{E}\left[\sum_{i=1}^{n} w_{n,i}(x_i)y_i|\boldsymbol{x}\right] = \sum_{i=1}^{n} \mathrm{E}\left[w_{n,i}(x_i)y_i|\boldsymbol{x}\right] = \sum_{i=1}^{n} w_{n,i}(x_i)\mathrm{E}\left[y_i|\boldsymbol{x}\right] = \mathrm{E}[y_i|\boldsymbol{x}]$$

Now as long as we have a bounded second moment we can use CLT to get asymptotic normality. Now to calculate the variance:

$$\mathrm{V}[\hat{e}(x)|\boldsymbol{x}] = \mathrm{V}\left[\sum_{i=1}^{n} w_{n,i}(x_i)y_i|\boldsymbol{x}\right] = \sum_{i=1}^{n} \mathrm{V}\left[w_{n,i}(x_i)y_i|\boldsymbol{x}\right] = \sum_{i=1}^{n} w_{n,i}(x_i)^2\mathrm{V}\left[y_i|\boldsymbol{x}\right]$$

Then if we assume homoroskedasticity we get the estimator

$$\hat{\mathrm{V}}(x) = \hat{\sigma}^2 \sum_{i=1}^{n} w_{n,i}(x)^2$$

Where $\hat{\sigma}^2 = \frac{1}{n-1}\sum_{i=1}^{n}(y_i - \hat{e}(x_i))^2$

2.4 Q2 part 4

The pointwise asymptotically valid 95% convidence interval for $e(x)$ is

$$CI_{95}(x) = [\hat{e}(x) - 1.96\sqrt{\hat{\mathrm{V}}(x)}, \hat{e}(x) + 1.96\sqrt{\hat{\mathrm{V}}(x)}]$$

This is just a confidence interval for a given point. applying this to a grid of points across the line and interpreting that as a band for the function is incorrect. For uniformly valid inference we need that the estimate is less that the cutoff for all values of x, not just one specific x.
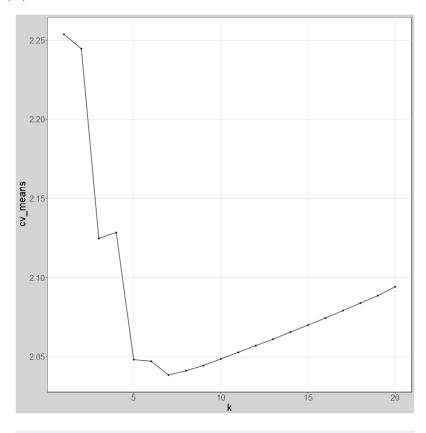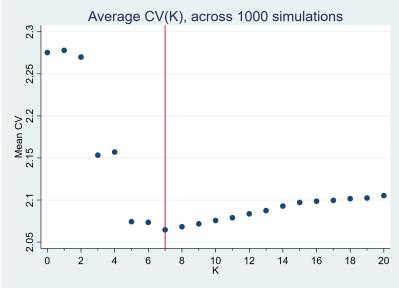
2.5 Q2 part 5

2.5.1 Q2 part 5 a

See the code in appendix
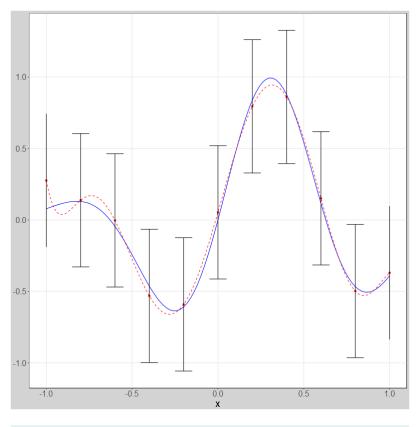
2.5.2   Q2 part 5 B

The plot of the CV(K) simulations is below

2.6   Q2 part 5 C

My plot is below. I used homoscedastic standard errors. The dotted line is my estimate
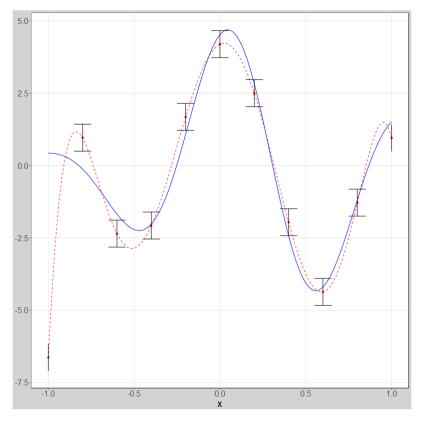
Mu_hat(x) across 1000 simulations

2.7 Q2 part 5 D

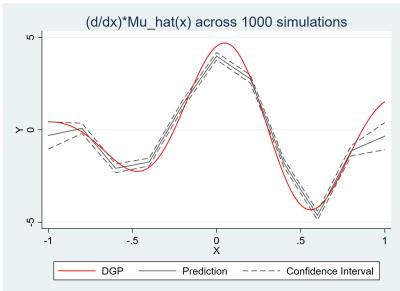Calculating the derivative of $u(x)$ we get

$$e^{(0.1\cdot(4x-1)^2)}[5\cdot cos(5x) - 0.8\cdot(4x-1)sin(5x)]$$

Taking the derivative of the estimated euqation we get

$$\hat{u}'(x) = \beta_1 + 2\beta_2 x + 3\beta_3 x^2 + 4beta_4 x^3 + 5\beta_5 x^4 + 6\beta_6 x^5 + 7\beta_7 x^6$$

I plot the corresponding curves below. The dotted line is my estimate

(d/dx)*Mu_hat(x) across 1000 simulations

3 Question 3: Semiparametric Semi-Linear Model

3.1 Q3 part 1

first note that θ_0 cannot contain a constant. since $\alpha + g(z) = [\alpha + c] + [g(z) - c] \equiv \alpha_{new} + g_{new}(z)$ the sum of the new g and intercept are observationally equivalent to the old ones so they cannot be

identified. From Li and Racine page 223 we can see that the requirements for identifiability are that $E[(t_i - h_0(x_i))(t_i - h_0(x_i))']$ is positive definite. 3

To prove the moment condition let's start with the expectation of interest and apply the law of iterated expectations.

$$E[(t_i - h_o(x_i))(y_i - t_i\theta_0)] = E[E[(t_i - h_o(x_i))(y_i - t_i\theta_0)|x_i]] = E[E[(t_i - h_o(x_i))(g_0(x_i) + \epsilon_i)|x_i]]$$

$$= E[E[(t_i - h_o(x_i))g_o(x_i)|x_i]] + E[E[(t_i - h_o(x_i))\epsilon_i|x_i]]$$
$$= E[g_o(x_i)E[(t_i - h_o(x_i))|x_i]] + E[(t_i - h_o(x_i))E[\epsilon_i|x_i, i_i]] = 0$$

Now to find a closed form solution for θ_0.

$$E[(t_i - h_o(x_i)y_i)] - E[(t_i - h_o(x_i)t_i)]\theta_0 = 0$$
$$\implies \theta_0 = \frac{E[(t_i - h_o(x_i)y_i)]}{E[(t_i - h_o(x_i)t_i)]}$$

The IV interpretation can be given as follows. Let $yi = t_i\theta_0 + g_0(x_i) + \epsilon_i = t_i\theta_0 + \mu_i$. Now μ_i is uncorrelated with t_i so we can define and instrument $z_i = t_i - h_0(x_i)$ which has the property of $E[z_i\mu_i] = 0$ and $E[t_i z_i] \neq 0$ so it is a valid instrument.

3.2 Q3 part 2

(a) As the question asks we will consider the power series approximation.

$$E[y_i|x_i] \approx t_i\theta_0 + p^K(x_i)'\gamma_k$$

Next, as the question instructs, we can use the partition regression formula and get the OLS estimator

$$\hat{\theta}(K) = (t'M_xt)^{-1}t'M_xY$$

Where here $t = (t_1, ..., t_n)'$ and $M_p = I - P_{r_p(x)}$

and $P_{r_p(x)} = R_p(R_p'R_p)^{(-1)}R_p'$

$$R_p = \begin{bmatrix} 1 & (x_1) & (x_1)^2 & \cdots & (x_1)^p \\ 1 & (x_2) & (x_2)^2 & \cdots & (x_2)^p \\ \vdots & \vdots & \vdots & \ddots & \vdots \\ 1 & (x_n) & (x_n)^2 & \cdots & (x_n)^p \end{bmatrix}$$

(b)
We used the moment condition when discussing the IV estimate interpretation in part 1 to find

$$\theta_0 = \frac{E[(t_i - h_o(x_i)y_i)]}{E[(t_i - h_o(x_i)t_i)]}$$

So we can estimate this with

$$\left(\frac{1}{n}\sum_{i=1}^{n} t_i - h_o(x_i)t_i\right)^{-1}\left(\frac{1}{n}\sum_{i=1}^{n} t_i - h_o(x_i)y_i\right)$$

3.3 Q3 part 3

(a)

WE can just use the partialing out method above to get

$$\hat{\theta}(K) = (t'M_pt)^{-1}t'M_p(t\theta_0 + R_p\gamma_k + e) = \theta + (t'M_pt)^{-1}t'M_pe$$

Now with iid data and conditional l heteroskedasticity, we can use the WLLN and CLT as usual to get normality and the sandwich form variance matrix

(b)

The asymptotically valid 95% confidence interval is just the same as usual then

$$CI_{95} = [\hat{\theta}(K) - 1.96\sqrt{V_{HCO}}, \hat{\theta}(K) + 1.96\sqrt{V_{HCO}}]$$

3.4 Q3 part 4

3.4.1 Q3 part 4 a

see code in appendix

3.4.2 Q3 part 4 b

My results from R are in the table below

K	Theta	Bias	S.D	V_HCO	Rejection rate
6.000	1.841	0.467	3.607	0.462	0.980
11.000	-0.167	0.231	0.082	0.218	0.093
21.000	-0.161	0.232	0.080	0.214	0.093
26.000	-0.162	0.235	0.081	0.214	0.102
56.000	-0.140	0.227	0.071	0.199	0.128
61.000	-0.124	0.221	0.064	0.193	0.122
126.000	0.008	0.113	0.013	0.100	0.094
131.000	0.008	0.114	0.013	0.100	0.093
252.000	0.008	0.128	0.017	0.094	0.149
257.000	0.008	0.130	0.017	0.094	0.153
262.000	0.008	0.132	0.017	0.094	0.154
267.000	0.009	0.132	0.018	0.094	0.162
272.000	0.009	0.133	0.018	0.094	0.162
277.000	0.008	0.135	0.018	0.094	0.170

theta_hat	se_hat	bias	cov	svar	K
3.002	0.114	2.002	0	0.688	1
0.734	0.140	-0.266	0.00500	0.249	2
0.734	0.140	-0.266	0.00500	0.249	3
0.766	0.139	-0.234	0.00600	0.223	4
0.766	0.139	-0.234	0.00600	0.223	5
0.796	0.139	-0.204	0.00500	0.256	6
0.796	0.139	-0.204	0.00500	0.256	7
0.790	0.139	-0.210	0.00600	0.259	8
0.790	0.139	-0.210	0.00600	0.259	9
0.793	0.139	-0.207	0.00500	0.250	10
0.791	0.139	-0.209	0.00600	0.230	11
0.779	0.139	-0.221	0.00600	0.242	12
0.771	0.139	-0.229	0.00700	0.227	13
0.795	0.138	-0.205	0.00700	0.220	14

3.4.3   Q3 part 4 c

Using cross-validation, I get $\hat{K}_{cv} = 126$. We can see from Table 1, across the simulations, $\hat{K}_{cv}$ gives a low rejection rate, but other estimators have lower bias and variance.

4   Appendix

4.1   R Code

pset 2 Labor

```r
#=======================================#



   #==========================#
# ==== Metrics 675 ps 2 ====
#==========================#




#=======================================#
# ==== load packages and clear data ====
#=======================================#

library(data.table)
library(doParallel)
library(foreach)
library(ggplot2)
library(Matrix)

# clear data and consol
rm(list = ls(pos = ".GlobalEnv"), pos = ".GlobalEnv")
options(scipen = 999)
cat("\f")

# set options
opt_test_run <- TRUE

# set attributes for plot to default ea theme
plot_attributes <- theme( plot.background = element_rect(fill = "lightgrey"),
                          panel.grid.major.x = element_line(color = "gray90"),
                          panel.grid.minor  = element_blank(),
                          panel.background = element_rect(fill = "white", colour = "black") ,
                          panel.grid.major.y = element_line(color = "gray90"),
                          text = element_text(size= 20),
                          plot.title = element_text(vjust=0, colour = "#0B6357",face = "bold", size = 40

#=================================================#
# ==== Question 1: Kernel Density Estimation ====
#=================================================#

#================#
# ==== Part a ====
#================#


# now to find the theoretically optimal H I need to calculate integral of second derivative.
# second dericative of normal function is
phi_2 <- function(x, mean, v){

  dnorm(x=x,mean=mean,sd=sqrt(v))*(((x - mean)/v)^2-(1/v))
```

```r
}

# now create the function to integrate
f_int <- function(x){

  f_out <- (.5*phi_2(x=x, -1.5,1.5) + .5*phi_2(x=x, 1,1))^2
  return(f_out)
}

# and the integral is
v2k <- integrate(f_int, lower = -Inf, upper = Inf)$val

# so optimal bandwith is
h_opt <- (15/(v2k*1000))^(1/5)

#=================#
# ==== part b/d ====
#=================#

# set parms
n      <- 1000
M <- ifelse(opt_test_run, 10, 1000)


# kernal function
K0 <- function(u){

  out <- .75 * (1-u^2) * (abs(u) <= 1)
  return(out)
}

# define the true f(x) function
f_x <- function(x){

  .5*dnorm(x, -1.5,sqrt(1.5))+.5*dnorm(x,1,1)
}


#==============================#
# ==== Make imse function ====
#==============================#

# define variables for debug
# in_data <- r_dt
# x_v <- "rdraw"

# generate data for debugging functions
# start data.table for random data, take a random draw for weighted normals
# r_dt <- data.table( r1 = sample(1:2,prob=c(.5,.5),size=n,replace=T) )
#
# # draw a random number from appropriate normal dist according to r1
# r_dt[r1 == 1, rdraw := rnorm(.N,-1.5,1.5)]
# r_dt[r1 == 2, rdraw := rnorm(.N,1,1)]
```

```r
# r_dt[, r1 := NULL]
# in_data <- r_dt
# h_v <- c(.5,.6)
# x_v <- "rdraw"
# i <- 1

imse_f <- function(in_data, x_v, h_v = NULL, f_x = f_x){

  # copy the data to aviod editing it in global enviorment
  data <- copy(in_data)

  # add a constant for the merge
  in_data[, const := 1]

  # cartesian merge to get all pairs
  paired_dt <- merge(in_data, in_data, by = "const", allow.cartesian = TRUE)

  # get new variable names after the merge. This kind of annoyingly general for a HW assingment. I regr
  x_vx <- paste0(x_v, ".x")
  x_vxi <- paste0(x_v, ".y")

  # initialize a list for output from each h
  ouput_list <- vector("list", length= length(h_v))

  # now do the imse calculations for each h in h_v
  for(i in 1:length(h_v)){

    h <- h_v[[i]]

    # get the kernal thing for each pair
    paired_dt[, k_x := K0((get(x_vxi) - get(x_vx))/h)]

    # now mean the kernal by rdraw.x and devide by h
    f_hats <- paired_dt[, list(f_hat_x = mean(k_x)/h), by = x_vx]

    # now get the f_hats for the leave one out by deleating the observation where x= xi. This will be r
    # 1, M+2, 2M+3, 3M+4 ... so eq(1, M*M, M+1) should take care of those
    paired_dt_lo <- paired_dt[-c(seq(1, n*n, n+1)), ]

    # now get the mean of the f_hats leacing out the x
    f_hats_lo <- paired_dt_lo[, list(f_hat_x = mean(k_x)/h), by = x_vx]

    # now add in f_x for each
    f_hats[, f_x := f_x(get(x_vx))]
    f_hats_lo[, f_x := f_x(get(x_vx))]

    # now do squared error
    f_hats[, sq_er := (f_hat_x - f_x)^2]
    f_hats_lo[, sq_er := (f_hat_x - f_x)^2]

    # now get imse
    imse_li <- f_hats[, mean(sq_er)]
    imse_lo <- f_hats_lo[, mean(sq_er)]
```

```r
    # now put into a data.table and put in list
    ouput_list[[i]] <- data.table(imse_li = imse_li, imse_lo= imse_lo, h = h)
  }

  output <- rbindlist(ouput_list)

  return(output[])


}



#=========================#
# ==== run simulations ====
#=========================#

# note: pretty sure it would be faster yet to just include the simulations in the by group of the data
# operations in the IMSE function. Probably marginally faster but kind of hard to wrap my head around.
# update: I tried this an it exceeded R's vector length limit. Might be a workaround, unsure.

# define squared phi_2 function for part d
phi_2_sq <- function(x , mean, v){

  phi_2(x =x , mean = mean, v = v)^2
}


# now set up function to run simulations, make sure to pass in user defined functons/vars or foreach ca
sim_function <- function(i, n, f_x, phi_2, h_v){

  # generate data
  # start data.table for random data, take a random draw for weighted normals
  r_dt <- data.table( r1 = sample(1:2,prob=c(.5,.5),size=n,replace=T) )

  # draw a random number from appropriate normal dist according to r1
  r_dt[r1 == 1, rdraw := rnorm(.N,-1.5,1.5)]
  r_dt[r1 == 2, rdraw := rnorm(.N,1,1)]
  r_dt[, r1 := NULL]


  # get IMSE
  results_i <- imse_f(in_data = r_dt, x_v = "rdraw" ,f_x = f_x ,h_v =h_v)
  results_i[, sim := i]

  # now get mean and SE or part d
  mean_i <- r_dt[, mean(rdraw)]
  var_i <- r_dt[, var(rdraw)]

  # calculate "optimal bandwidth" under the procdure from part D
  vok <- 3/5
  u2k2 <- (1/5)^2

  # and the integral is
  v2phi <- integrate(phi_2_sq, mean = mean_i, v = var_i, lower = -Inf, upper = Inf)$val
```

4

```r
  # now calculate h optimal
  h_opt <- (vok/ (u2k2 *v2phi*n))^(1/5)

  # put that bad boy in the table
  results_i[, d_h_hat := h_opt]

  # return the rsults for all of q2
  return(results_i[])

}


# make a vector of h's
h_v <- seq(.5, 1.5,.1)

# lets time this sucker
start_t <- Sys.time()


# parallel setup
cl <- makeCluster(4, type = "PSOCK")
registerDoParallel(cl)

# run simulations in parallel
output_list <- foreach(sim = 1 : M,
                       .inorder = FALSE,
                       .packages = "data.table",
                       .options.multicore = list(preschedule = FALSE, cleanup = 9)) %dopar% sim_functio

# stop clusters
stopCluster(cl)


# AND TIME
run_time1 <- Sys.time() - start_t

  # bind list
  output_dt <- rbindlist(output_list)

  # now take the mean of imse
  part_b_res <- output_dt[, list(imse_li = mean(imse_li), imse_lo = mean(imse_lo), d_h_hat = mean(d_h_ha

  # make them pretty
  part_b_res_pretty <- signif(part_b_res, 3)
  part_b_res_pretty[, colnames(part_b_res_pretty)] <- lapply(part_b_res_pretty[,colnames(part_b_res_pret

  # make the graph
  # melt the data to work better with ggplot
  part_b_res[, d_h_hat := NULL ]
  plot_data <-  melt.data.table(part_b_res, id.vars = "h", variable.name = "Error Type")
  plot_1_3_b <- ggplot(data = plot_data, aes(x = h, y = value, color = `Error Type`, shape = `Error Typ

  plot_1_3_b <- plot_1_3_b + geom_point() + geom_line() + plot_attributes
```

```r
  plot_1_3_b



#===================#
# ==== save data ====
#===================#

  # only save data if this isn't a test run
  if(!opt_test_run){
    # save IMSE by h results
    print(xtable(part_b_res_pretty, type = "latex",
                 digits = 3),
          file = "C:/Users/Nmath_000/Documents/Code/courses/econ 675/PS_2_tex/Q1_p3_b.tex",
          include.rownames = FALSE,
          floating = FALSE)

    # save the plot
    png("c:/Users/Nmath_000/Documents/Code/courses/econ 675/PS_2_tex/plot_1_3_b.png", height = 800, wid
    print(plot_1_3_b)
    dev.off()

  }



#=====================#
# ==== Question 2 ====
#=====================#



#==========================#
# ==== A: generate data ====
#==========================#

  gen_data_2.5.a <- function(){

    # start data.table with random x's. get a chi squared too cause i need that for the epsilon
    r_dt <- data.table(x = runif(n,-1,1), chi_sq = rchisq(n,5))

    # create a noise clumn epsilon,
    r_dt[, eps := x^2*(chi_sq-5)]

    # now calcualte y
    r_dt[, y := exp(-0.1*(4*x-1)^2)*sin(5*x) + eps]

    # drop the chi_sq column
    r_dt[, chi_sq := NULL]

    # return the random data
    return(r_dt[])
```

```r
    }


#=========================#
# ==== B do experiment ====
#=========================#

# generate some random data
r_dt <- gen_data_2.5.a()


# write a function to apply accross simulations
power_s_fun <- function(sim = NULL){

  r_dt <- gen_data_2.5.a()
  r_dt[, const :=1]

  # store results in a list
  results <- vector("list", length = 20)
  # make the 20 squared variables
  #note: im makeing an extra column. Ill fix this if I have time but this is easy for now
  for(i in 1:20){

  r_dt[, temp := x^i]
  setnames(r_dt, "temp", paste0("x_exp_", i))


  # conver things to matrices to get the y hats
  x_mat <- as.matrix(r_dt[, c(grep("x_exp", colnames(r_dt), value = TRUE), "const"), with = FALSE])
  y_mat <- as.matrix(r_dt[, y])

  # get the projection matrix
  X.Q <- qr.Q(qr(x_mat))
  XX <- tcrossprod(X.Q)
  Y.hat <- XX %*% y_mat

  # now put this crap in a data.table to calculate cv
  res <- data.table(y_hat = Y.hat, w = diag(XX), y = r_dt[, y])

  # now calculate cv
  res[, cv_n := ((y - y_hat.V1)/(1-w))^2]

  # now get the mean of cv_i to get cv
  res <-  data.table(cv = res[, mean(cv_n)], k = i)
  setnames(res, "cv", paste0("cv_", sim))
  results[[i]] <- res


  }

  print(sim)
  # bind results
return(rbindlist(results))
```

```r
}

start_t <- Sys.time()

# parallel setup
cl <- makeCluster(4, type = "PSOCK")
registerDoParallel(cl)

# run simulations in parallel
all_out <- foreach(sim_i = 1 : M,
                        .inorder = FALSE,
                        .packages = "data.table",
                        .options.multicore = list(preschedule = FALSE, cleanup = 9)) %dopar% power_s_fu


# now merge all results
all_out_dt <-Reduce(function(x, y) merge(x, y, by = "k"), all_out)

# stop clusters
stopCluster(cl)

# check time
run_time2 <- Sys.time() - start_t

# row sum my data to get the averaage cv for each k
all_out_dt[, k := NULL]
mean_cv <- data.table( cv_means = rowMeans(all_out_dt), k = 1:20)

# now plot that bad boy

# initialize base data mapping for plot
plot_2_5_b <- ggplot(data = mean_cv, aes(x = k, y = cv_means))

plot_2_5_b <- plot_2_5_b + geom_point(size = 1) + geom_line() + plot_attributes
plot_2_5_b

#=================#
# ==== part c ====
#=================#


    # write a function to apply accross simulations
    B_fun <- function(sim = NULL){

      r_dt <- gen_data_2.5.a()
      r_dt[, const :=1]


      # make the y vars
      for(i in 1:7){

        r_dt[, temp := x^i]
        setnames(r_dt, "temp", paste0("x_exp_", i))
```

```r
    }

    # conver things to matrices to get the y hats
    x_mat <- as.matrix(r_dt[, c(grep("x_exp", colnames(r_dt), value = TRUE), "const"), with = FALS
    y_mat <- as.matrix(r_dt[, y])

    # get betas
    B <- Matrix::solve(Matrix::crossprod(x_mat, x_mat))%*%(Matrix::crossprod(x_mat, y_mat))

    # get weights
    X.Q <- qr.Q(qr(x_mat))
    XX <- tcrossprod(X.Q)
    weights <- diag(XX)
    Y.hat <- XX %*% y_mat

    # now square the weights
    weights_sq <- weights^2

    # now get se
    se <- sqrt(sum(weights_sq) * var(y_mat - Y.hat))



    # put the stuff in a list
    output <- list()
    output[["B"]] <- B
    output[["se"]] <- se

  # return the betas
  return(output)

}

start_t <- Sys.time()

# okay now run this shit 1000 times
bw_stuff <- lapply(c(1:M), B_fun)

run_time3 <- Sys.time() - start_t

# now do some dumb stuff because its late
b_list <- list()
se_list <- list()
for(i in 1:M){

  b_list[[i]] <- bw_stuff[[i]][["B"]]
  se_list[[i]] <- bw_stuff[[i]][["se"]]
}

b_mat <- do.call(cbind, b_list)
se_mat <- do.call(cbind, se_list)

# sum the rows
```

```r
    betas <- rowMeans(b_mat)
    se <-    rowMeans(se_mat)

    # now write a function to plot the u hat funciton
    u_hat_fun <- function(x){

      betas[[8]] + betas[[1]]*x + betas[[2]]*x^2 + betas[[3]]*x^3 + betas[[4]]*x^4 + betas[[5]]*x^5 +
    }


    # write out true function
    true_fun <- function(x){

      exp(-0.1*(4*x-1)^2)*sin(5*x)

    }


#======================#
# ==== part c plot ====
#======================#


    # plot the true functin
    plot_2_5_c <- ggplot(data = data.frame(x = 0), mapping = aes(x = x))
    plot_2_5_c <- plot_2_5_c + stat_function(fun = true_fun,
                                            color = "blue")
    plot_2_5_c <- plot_2_5_c + plot_attributes + xlim(-1,1)


    # now add u hat function
    plot_2_5_c <- plot_2_5_c + stat_function(fun = u_hat_fun,
                                            color = "red", linetype = 2)

    plot_2_5_c <- plot_2_5_c + scale_colour_identity("Function", guide="legend",
                                            labels = c("U hat", "True U"),
                                            breaks = c("red", "blue")) + theme(axis.title.y=element_
    # create some data to plot with the standard errors
    plot_data <- data.table(x = seq(-1,1,.2))
    plot_data[, y_hat := u_hat_fun(x)]
    plot_data[, se := se]

    plot_2_5_c <- plot_2_5_c + geom_point(data = plot_data, mapping = aes(x = x, y = y_hat),
                                            color = "red")

    plot_2_5_c <- plot_2_5_c + geom_errorbar(data = plot_data, aes(ymin=y_hat-se, ymax=y_hat+se), wid

    # print it out to see if it looks alright
    plot_2_5_c

#======================#
# ==== part d pot ====
#======================#
```

```r
    # create derivative funciton
    # write out true function
    true_fun_d <- function(x){

      exp(-0.1*(4*x-1)^2)*(5*cos(5*x) - 0.8*(4*x-1)*sin(5*x))

    }

    # write out estimated polynomial
    est_fun_d <- function(x){
     betas[[1]] + 2*betas[[2]]*x + 3*betas[[3]]*x^2 + 4*betas[[4]]*x^3 + 5*betas[[5]]*x^4 + 6*betas[[
    }

    # plot the true functin
    plot_2_5_d <- ggplot(data = data.frame(x = 0), mapping = aes(x = x))
    plot_2_5_d <- plot_2_5_d + stat_function(fun = true_fun_d,
                                             color = "blue")
    plot_2_5_d <- plot_2_5_d + plot_attributes + xlim(-1,1)


    # now add u hat function
    plot_2_5_d <- plot_2_5_d + stat_function(fun = est_fun_d,
                                             color = "red", linetype = 2)

    plot_2_5_d <- plot_2_5_d + scale_colour_identity("Function", guide="legend",
                                                     labels = c("U hat", "True U"),
                                                     breaks = c("red", "blue")) + theme(axis.title.y=
    # create some data to plot with the standard errors
    plot_data <- data.table(x = seq(-1,1,.2))
    plot_data[, y_hat := est_fun_d(x)]
    plot_data[, se := se]

    plot_2_5_d <- plot_2_5_d + geom_point(data = plot_data, mapping = aes(x = x, y = y_hat),
                                          color = "red")

    plot_2_5_d <- plot_2_5_d + geom_errorbar(data = plot_data, aes(ymin=y_hat-se, ymax=y_hat+se), wid

    # print it out to see if it looks alright
    plot_2_5_d

#====================#
# ==== save plots ====
#====================#

    # only save data if this isn't a test run
    if(!opt_test_run){

      # save the plot
      png("c:/Users/Nmath_000/Documents/Code/courses/econ 675/PS_2_tex/plot_2_5_b.png", height = 800,
      print(plot_2_5_b)
      dev.off()
```

```r
        # save the plot
        png("c:/Users/Nmath_000/Documents/Code/courses/econ 675/PS_2_tex/plot_2_5_c.png", height = 800,
        print(plot_2_5_c)
        dev.off()

        # save the plot
        png("c:/Users/Nmath_000/Documents/Code/courses/econ 675/PS_2_tex/plot_2_5_d.png", height = 800,
        print(plot_2_5_d)
        dev.off()



    }



#===================#
# ==== question 3 ====
#===================#

  #================#
  # ==== Part a ====
  #================#

    d    = 5
    theta_n = 1


    data_gen <- function(n) {
      X <- matrix(runif(n*d,-1,1), n, d)
      V <- rnorm(n)
      x.norm = sapply(1:n,function(i) t(X[i,])%*%X[i,])
      E      = 0.3637899*(1+x.norm)*V
      g0.x   =exp(x.norm)

      U <- rnorm(n)
      tt <- as.numeric((sqrt(x.norm)+U)>1)
      Y <- tt + g0.x + E
      return(list(Y=Y, X=X, tt=tt))
    }


    # generate the polynomial basis
    gen.P = function(Z,K) {
      if (K==0)   out = NULL;
      if (K==1)   out = poly(Z,degree=1,raw=TRUE);
      if (K==2)  {out = poly(Z,degree=1,raw=TRUE); for (j in 1:ncol(Z)) out = cbind(out,Z[,j]^2);}
      if (K==2.5) out = poly(Z,degree=2,raw=TRUE);
      if (K==3)  {out = poly(Z,degree=2,raw=TRUE); for (j in 1:ncol(Z)) out = cbind(out,Z[,j]^3);}
      if (K==3.5) out = poly(Z,degree=3,raw=TRUE);
      if (K==4)  {out = poly(Z,degree=3,raw=TRUE); for (j in 1:ncol(Z)) out = cbind(out,Z[,j]^4);}
      if (K==4.5) out = poly(Z,degree=4,raw=TRUE);
      if (K==5)  {out = poly(Z,degree=4,raw=TRUE); for (j in 1:ncol(Z)) out = cbind(out,Z[,j]^5);}
```

```r
      if (K==5.5) out = poly(Z,degree=5,raw=TRUE);
      if (K>=6)  {out = poly(Z,degree=5,raw=TRUE); for (k in 6:K) for (j in 1:ncol(Z)) out = cbind(out,
      ## RETURN POLYNOMIAL BASIS
      return(out)
   }

  #===============#
  # ==== part b ====
  #===============#

  n    <- 500
  K    <- c(1, 2, 2.5, 3, 3.5, 4, 4.5, 5, 5.5, 6, 7, 8, 9, 10)
  K.r <- c(6, 11, 21, 26, 56, 61, 126, 131, 252, 257, 262, 267, 272, 277)
  nK   <- length(K)
  M <- ifelse(opt_test_run, 10, 1000)
  theta.hat <- matrix(NaN, ncol=nK, nrow=M)
  se.hat     <- theta.hat

  set.seed(123)
  ptm <- proc.time()
  for (m in 1:M) {
    data <- data_gen(n)
    X <- data$X
    Y <- data$Y
    tt <- data$tt
    for (k in 1:nK) {
      X.pol <- cbind(1, gen.P(X, K[k]))
      X.Q   <- qr.Q(qr(X.pol))
      MP      <- diag(rep(1,n)) - X.Q %*% t(X.Q)
      Y.M <- MP %*% Y
      tt.M <- MP %*% tt
      theta.hat[m, k] <- (t(tt.M) %*% Y.M) / (t(tt.M) %*% tt.M)
      Sigma <- diag((as.numeric((Y.M - tt.M*theta.hat[m, k])))^2)
      se.hat[m, k] <- sqrt(t(tt.M) %*% Sigma %*% tt.M) / (t(tt.M) %*% tt.M)
    }
  }
  proc.time() - ptm

  table <- matrix(NaN, ncol=6, nrow=nK)
  for (k in 1:nK) {
    table[k, 1] <- K.r[k]
    table[k, 2] <- mean(theta.hat[, k]) - 1                       # bias
    table[k, 3] <- sd(theta.hat[, k])                             # standard deviation
    table[k, 4] <- table[k, 2]^2 + table[k, 3]^2                  # mse
    table[k, 5] <- mean(se.hat[, k])                              # mean standard error
    table[k, 6] <- mean((theta.hat[, k] - 1.96 * se.hat[, k] > 1) |
                        (theta.hat[, k] + 1.96 * se.hat[, k] < 1))  # rejection rate
  }

  table <- data.table(table)
  setnames(table, colnames(table), c("K", "Theta", "Bias", "S.D", "V_HCO", "Rejection rate"))

#====================#
```

```r
# ==== save table ====
#====================#

    # save IMSE by h results
    print(xtable(table, type = "latex",
                 digits = 3),
          file = "C:/Users/Nmath_000/Documents/Code/courses/econ 675/PS_2_tex/Q3_4_b.tex",
          include.rownames = FALSE,
          floating = FALSE)



#====================#
# ==== Q3. 4. (c) ====
#====================#



    # cross validation function
    K.CV <- function(tt, X, Y) {
      temp <- rep(NaN, nK)
      for (k in 1:nK) {
        X.pol <- cbind(1, tt, gen.P(X, K[k]))
        X.Q   <- qr.Q(qr(X.pol))
        XX <- X.Q %*% t(X.Q)
        Y.hat <- XX %*% Y
        W <- diag(XX)
        temp[k] <- mean(((Y-Y.hat) / (1-W))^2)
      }
      return(which.min(temp))
    }

    theta.hat2 <- rep(NaN, M)
    se.hat2    <- theta.hat2
    K.hat2     <- theta.hat2

    set.seed(123)
    ptm <- proc.time()
    for (m in 1:M) {
      data <- data_gen(n)
      X <- data$X; Y <- data$Y; tt <- data$tt
      k.opt <- K.CV(tt, X, Y)
      X.pol <- cbind(1, gen.P(X, K[k.opt]))
      X.Q   <- qr.Q(qr(X.pol))
      MP    <- diag(rep(1,n)) - X.Q %*% t(X.Q)
      Y.M   <- MP %*% Y
      tt.M  <- MP %*% tt
      theta.hat2[m] <- (t(tt.M) %*% Y.M) / (t(tt.M) %*% tt.M)
      Sigma         <- diag((as.numeric((Y.M - tt.M*theta.hat[m, k])))^2)
      se.hat2[m]    <- sqrt(t(tt.M) %*% Sigma %*% tt.M) / (t(tt.M) %*% tt.M)
      K.hat2[m]     <- K.r[k.opt]
    }
    time4 <- proc.time() - ptm
```

```r
# summary of the cross validation
table(K.hat2)

# estimator
summary(theta.hat2)
sd(theta.hat2)
summary(se.hat2)
sd(se.hat2)

par(mfrow=c(1,2))
hist(theta.hat2, freq=FALSE, xlab="theta-hat", ylab="", main="")
lines(c(mean(theta.hat2), mean(theta.hat2)), c(-1, 20), col="red", lwd=3)
hist(se.hat2, freq=FALSE, xlab="s.e.", ylab="", main="")
lines(c(mean(se.hat2), mean(se.hat2)), c(-1, 80), col="red", lwd=3)


par(mfrow=c(1,2))
CI.l <- theta.hat2 - 1.96 * se.hat2
CI.r <- theta.hat2 + 1.96 * se.hat2

# rejection rate
mean(1 < CI.l | 1 > CI.r)
plot(1:M, CI.l, type="l", ylim=c(0,2), xlab="simulations", ylab="CI")
lines(1:M, CI.r)
abline(1, 0, col="red", lwd=2)

temp <- sort(CI.l, index.return=TRUE)
CI.l <- temp$x
CI.r <- CI.r[temp$ix]
plot(1:M, CI.l, type="l", ylim=c(0,2), xlab="simulations", ylab="CI")
lines(1:M, CI.r)
abline(1, 0, col="red", lwd=2)
```

4.2   STATA Code

```stata
1    * NOTES:
2    * My code fricking CRAAAWLS. It is extremely slow. I guess trying to jerry rig how
3    * I did this in R into the weird world of stata was not a great idea.
4
5
6    clear all
7    set more off, perm
8
9    global dir "c:\Users\Nmath_000\Documents\Code\courses\econ 675\PS_2_tex\"
10
11   cap log close
12   log using $pset2_stata_log.smcl, replace
13   ************************************
14   ************ Question 1 ************
15   ************************************
16
17    global hvalues .5 .6 .7 .8 0.8199 .9 1 1.1 1.2 1.3 1.4 1.5
18   * global hvalues .5
19   local h = .5
20   local i = 1
21   global n = 1000
22
23   * I need to only do 10 simulations because of how slow this thing is
24   global m = 10
25   set obs $n
26
27   * replace with for loop eventually
28   forvalues i = 1/10{
29   di `i'
30   * start loop
31   clear
32   set obs $n
33   * generate random data
34   gen z_o = uniform()
35   gen xi = rnormal(-1.5, sqrt(1.5)) if z_o < .5
36   replace xi = rnormal(1,1) if z_o >= .5
37
38   * drop zero one var
39   drop z_o
40
41   * gen constaant for merge
42   gen const = 1
43
44   * save as temp file for merge
45   tempfile rand_i
46    save "`rand_i'"
47
48
49    * rename variable
50    rename xi x
51
52   * try merging this with teacher level enr_staff file
53   joinby const using `rand_i'
54
55   * now loop over h values
56   foreach h in $hvalues {
57
58       di `h'
59       * make h for file names
60       local h_n: subinstr local h "." "", all
61
62       *preserve data before I mess with is
63       preserve
64
65       * gnerate u
66       gen u =   (xi-x )/`h'
67
68       * calculate kernal for pairs
69       gen kern = (.75*(1-u^2)*(abs(u)<=1))/`h'
70
```

```stata
71
72          * get means
73          replace x = round(x,.00001)
74          bys x: egen fhats = mean(kern)
75          egen tag = tag(x)
76          keep if tag == 1
77          drop xi const u kern tag
78
79          * add in f_x
80          gen f_x = .5*normalden(x, -1.5, sqrt(1.5)) + .5*normalden(x, 1, 1)
81
82          * find sq error
83          gen sq_er = (fhats-f_x)^2
84
85          * now get  imse_li
86          egen imse_li = mean(sq_er)
87          egen tag2 = tag(imse_li)
88          keep if tag2 == 1
89
90          keep imse_li
91
92          * fill in sum info
93          gen sim = `i'
94          gen h = `h'
95
96
97          * save temp data
98          tempfile imseli_`h_n'_`i'
99           save "imseli_`h_n'_`i'", replace
100
101         * restore data, preserve it for next thing
102         restore
103
104         preserve
105         * now do the leave on out, drop columns with the same x xi
106         * this is bad coding but STATA is terrible so this is what it deserves
107         keep if x != xi
108
109         * gnerate u
110         gen u =  (xi-x )/`h'
111
112         * calculate kernal for pairs
113         gen kern = (.75*(1-u^2)*(abs(u)<=1))/`h'
114
115         * collaps data to get means \* collapse data
116         replace x = round(x,.00001)
117         bys x: egen fhats = mean(kern)
118         egen tag = tag(x)
119         keep if tag == 1
120         drop xi const u kern tag
121
122         * add in f_x
123         gen f_x = .5*normalden(x, -1.5, sqrt(1.5)) + .5*normalden(x, 1, 1)
124
125         * find sq error
126         gen sq_er = (fhats-f_x)^2
127
128         * now get  imse_li
129         egen imse_lo = mean(sq_er)
130         egen tag2 = tag(imse_lo)
131         keep if tag2 == 1
132
133         keep imse_lo
134
135
136         * fill in sum info
137         gen sim = `i'
138         gen h = `h'
139
140
```

```
141         * save temp data
142         tempfile imselo `h_n' `i'
143         quietly save "imselo_`h_n'_`i'" , replace
144
145         * restore data for next h
146         restore
147
148     }
149     }
150     * now, because I dont think stata has lists we just load all that back in and stack it
151     * clear out data
152     clear
153
154     forvalues i = 1/$m{
155     foreach h in $hvalues {
156
157         * make h for file names
158         local h_n: subinstr local h "." "", all
159
160         append using "imseli_`h_n'_`i'.dta"
161         append using "imselo_`h_n'_`i'.dta"
162
163     }
164     }
165
166     * Now collapse data to get mean leave on in and out across iteratiosn by h
167         bys h: egen m_imse_li = mean(imse_li)
168         bys h: egen m_imse_lo = mean(imse_lo)
169         egen tag = tag(h)
170         keep if tag == 1
171         keep h m_imse_li m_imse_lo
172
173
174     * graph this stuff
175     line m_imse_li m_imse_lo h
176
177     graph export "$dir\stata_plot_1_3_b.png", replace
178
179     dataout, save($dir\stata_table_1_3_b.tex) tex replace
180
181
182     **************
183     **** Problem 2
184     **************
185     **************
186     **** Problem 2.5.b
187     **************
188     clear all
189     set obs 1000
190     * Define cross validation function: CV(list, i): vars=variable list, i = max polynomial
191     mata
192         void CV(vars, i) {
193             st_view(y=., ., "y")
194             st_view(X=., ., tokens(vars))
195             XpX   = cross(X, X)
196             XpXinv  = invsym(XpX)
197             b  = XpXinv*cross(X, y)
198             w = diagonal(X*XpXinv*X')
199             muhat = X*b
200             num = (y - muhat):*(y - muhat)
201             den= (J(1000,1,1) - w):*(J(1000,1,1) - w)
202             div = num:/den
203             CV = mean(div)
204             CV
205             st_numscalar("mCV"+strofreal(i), CV)
206         }
207     end
208     * Program which runs the monte-carlo experiment
209     program CVsim, rclass
210         drop _all
```

```stata
211         set obs 1000
212         forvalues i = 0/20 {
213             gen CV`i' = 0
214         }
215         gen x = runiform(-1,1)
216         gen e = x^2*(rchi2(5)-5)
217         gen y = exp(-0.1*(4*x-1)^2)*sin(5*x)+e
218         forvalues i = 0/20 {
219             gen x`i' = x^`i'
220         }
221         forvalues i = 0/20 {
222             global xlist = "x0-x`i'"
223             di "$xlist"
224             mata CV("$xlist", `i')
225             replace CV`i' = mCV`i'
226         }
227     end
228     * Run the experiment
229     set seed 12345
230     simulate CV0=CV0 CV1=CV1 CV2=CV2 CV3=CV3 CV4=CV4 CV5=CV5 CV6=CV6 CV7=CV7 CV8=CV8 ///
231         CV9=CV9 CV10=CV10 CV11=CV11 CV12=CV12 CV13=CV13 CV14=CV14 CV15=CV15 ///
232         CV16=CV16 CV17=CV17 CV18=CV18 CV19=CV19 CV20=CV20, reps(100) nodots: CVsim
233     collapse *
234     gen i = 1
235     reshape long CV, i(i) j(k)
236     sort CV
237     local min = k[1]
238     twoway scatter CV k, ytitle("Mean CV") xtitle("K") xlabel(0(2)20) xmtick(0(1)20) xline(`min'
        ) title("Average CV(K), across 1000 simulations")
239     graph export "$dir\stata_plot_2_5_b.png", replace
240
241     ****************
242     ***Problem 2.5.c
243     ****************
244
245     * Program which runs the monte-carlo experiment for mu_0
246     program muhatsim, rclass
247         drop _all
248         set obs 1000
249         gen x = runiform(-1,1)
250         gen e = x^2*(rchi2(5)-5)
251         gen y = exp(-0.1*(4*x-1)^2)*sin(5*x)+e
252         forvalues p = 0/7 {
253             gen x`p' = x^`p'
254         }
255         reg y x0-x7, nocons
256         clear
257         set obs 11
258         gen n = _n
259         gen foo = 1
260         gen x = -1+(_n-1)/5
261         forvalues p = 0/7 {
262             gen x`p' = x^`p'
263         }
264         predict muhat
265         predict se, stdp
266         generate lb = muhat - invnormal(0.975)*se
267         generate ub = muhat + invnormal(0.975)*se
268
269
270         keep n muhat foo lb ub
271         reshape wide muhat lb ub, i(foo) j(n)
272     end
273     set seed 12345
274     simulate muhat1=muhat1 muhat2=muhat2 muhat3=muhat3 muhat4=muhat4 muhat5=muhat5 ///
275         muhat6=muhat6 muhat7=muhat7 muhat8=muhat8 muhat9=muhat9 muhat10=muhat10 muhat11=muhat11
        ///
276         ub1=ub1 ub2=ub2 ub3=ub3 ub4=ub4 ub5=ub5 ub6=ub6 ub7=ub7 ub8=ub8 ub9=ub9 ub10=ub10 ub11=
        ub11 ///
277         lb1=lb1 lb2=lb2 lb3=lb3 lb4=lb4 lb5=lb5 lb6=lb6 lb7=lb7 lb8=lb8 lb9=lb9 lb10=lb10 lb11=
```

```stata
        lb11, reps(1000) nodots: muhatsim
278     gen i =  n
279     reshape long muhat ub lb, i(i) j(grid)
280     collapse muhat ub lb, by(grid)
281     gen x = -1+ (grid-1)/5
282     twoway (function y = exp(-0.1*(4*x-1)^2)*sin(5*x), range(-1 1) lcolor(red)) ///
283         (line muhat x, lcolor(gs6)) (line lb x, lcolor(gs6) lpattern(dash)) (line ub x, lcolor(
        gs6) lpattern(dash)), ///
284         legend(order(1 "DGP" 2 "Prediction" 3 "Confidence Interval") rows(1)) ytitle(Y) xtitle(X
        ) title("Mu_hat(x) across 1000 simulations")
285     graph export "$dir\stata_plot_2_5_c.png", replace
286
287
288     ***************
289     * poblem 2.5.d
290     ***************
291
292
293     * Program which runs the monte-carlo experiment for mu_1
294     program dmuhatsim, rclass
295         drop _all
296         set obs 1000
297         gen x = runiform(-1,1)
298         gen e = x^2*(rchi2(5)-5)
299         gen y = exp(-0.1*(4*x-1)^2)*((0.8-3.2*x)*sin(5*x)+5*cos(5*x)) + e
300         forvalues p = 0/7 {
301             gen x`p' = x^`p'
302         }
303         reg y x0-x7, nocons
304         clear
305         set obs 11
306         gen n = _n
307         gen foo = 1
308         gen x = -1+(_n-1)/5
309         forvalues p = 0/7 {
310             gen x`p' = x^`p'
311         }
312         predict dmuhat
313         predict se, stdp
314         generate lb = dmuhat - invnormal(0.975)*se
315         generate ub = dmuhat + invnormal(0.975)*se
316
317
318         keep n dmuhat foo lb ub
319         reshape wide dmuhat lb ub, i(foo) j(n)
320     end
321     set seed 12345
322     simulate dmuhat1=dmuhat1 dmuhat2=dmuhat2 dmuhat3=dmuhat3 dmuhat4=dmuhat4 dmuhat5=dmuhat5 ///
323         dmuhat6=dmuhat6 dmuhat7=dmuhat7 dmuhat8=dmuhat8 dmuhat9=dmuhat9 dmuhat10=dmuhat10
        dmuhat11=dmuhat11 ///
324         ub1=ub1 ub2=ub2 ub3=ub3 ub4=ub4 ub5=ub5 ub6=ub6 ub7=ub7 ub8=ub8 ub9=ub9 ub10=ub10 ub11=
        ub11 ///
325         lb1=lb1 lb2=lb2 lb3=lb3 lb4=lb4 lb5=lb5 lb6=lb6 lb7=lb7 lb8=lb8 lb9=lb9 lb10=lb10 lb11=
        lb11, reps(1000) nodots: dmuhatsim
326     gen i = _n
327     reshape long dmuhat ub lb, i(i) j(grid)
328     collapse dmuhat ub lb, by(grid)
329     gen x = -1+ (grid-1)/5
330     twoway (function y = exp(-0.1*(4*x-1)^2)*((0.8-3.2*x)*sin(5*x)+5*cos(5*x)), range(-1 1)
        lcolor(red)) ///
331         (line dmuhat x, lcolor(gs6)) (line lb x, lcolor(gs6) lpattern(dash)) (line ub x, lcolor(
        gs6) lpattern(dash)), ///
332         legend(order(1 "DGP" 2 "Prediction" 3 "Confidence Interval") rows(1)) ytitle(Y) xtitle(X
        ) title("(d/dx)*Mu_hat(x) across 1000 simulations")
333     graph export "$dir\stata_plot_2_5_d.png", replace
334
335
336
337
338
```

```
339
340     ****************************
341     ******** Question 3 ********
342     ****************************
343     * DGP
344     clear all
345     drop _all
346     local theta = 1
347     local d = 5
348     local n = 500
349
350     set obs 1000
351
352     forvalues p = 1/14 {
353     gen se_hat`p' = .
354     gen theta_hat`p' = .
355
356     }
357     mata:
358     void polyloop(i) {
359
360     X   = uniform(`n',`d'):*2 :-1
361     ep  = invnormal(uniform(`n',1)):*0.3637899:*(1 :+ rowsum(X:^2))
362     gx  = exp(rowsum(X:^2))
363     T   = invnormal(uniform(`n',1)) + rowsum(X:^2):^.5 :>= 0
364     Y   = T + gx + ep
365     cons= J(500,1,1)
366
367     /*Raising to single powers */
368     X2  = X:^2
369     X3  = X:^3
370     X4  = X:^4
371     X5  = X:^5
372     X6  = X:^6
373     X7  = X:^7
374     X8  = X:^8
375     X9  = X:^9
376     X10 = X:^10
377
378     /*Kronekering, but this creates some duplicates*/
379     X1k = X#X
380     X2k = X2#X2
381     X3k = X3#X3
382     X4k = X4#X4
383
384     /* Manually removing duplicates...might be a better way to do this */
385     X1k = X1k[1::`n',2::5], X1k[1::`n', 8::10], X1k[1::`n',14::15], X1k[1::`n', 20]
386     X2k = X2k[1::`n',2::5], X2k[1::`n', 8::10], X2k[1::`n',14::15], X2k[1::`n', 20]
387     X3k = X3k[1::`n',2::5], X3k[1::`n', 8::10], X3k[1::`n',14::15], X3k[1::`n', 20]
388     X4k = X4k[1::`n',2::5], X4k[1::`n', 8::10], X4k[1::`n',14::15], X4k[1::`n', 20]
389
390     A = asarray_create("real",1)
391     asarray(A,1,X)
392     asarray(A,2,(asarray(A,1),X2))
393     asarray(A,3,(asarray(A,2),X1k))
394     asarray(A,4,(asarray(A,3),X3))
395     asarray(A,5,(asarray(A,4),X2k))
396     asarray(A,6,(asarray(A,5),X4))
397     asarray(A,7,(asarray(A,6),X3k))
398     asarray(A,8,(asarray(A,7),X5))
399     asarray(A,9,(asarray(A,8),X4k))
400     asarray(A,10,(asarray(A,9),X6))
401     asarray(A,11,(asarray(A,10),X7))
402     asarray(A,12,(asarray(A,11),X8))
403     asarray(A,13,(asarray(A,12),X9))
404     asarray(A,14,(asarray(A,13),X10))
405     theta_hat = I(1,14):*0
406     se_hat = I(1,14):*0
407     k_hat = I(1,14):*0
408
```

```
409   for (j=1; j<=14; j++) {
410   Z = qrsolve(cons,(T,asarray(A,j)))
411   ZZ  = Z*Z'
412   Yhat = ZZ*Y
413   W = diag(ZZ)
414   ZQ = (cons,asarray(A,j))*invsym((cons,asarray(A,j))'*(cons,asarray(A,j)))*(cons,asarray(A,j
      ))'
415   M = I(`n') - ZQ
416   YM = M*Y
417   TM = M*T
418   theta_hat[1,j] = (TM'*YM) / (TM'*TM)
419   sigma = diag(ZQ*(Y-T*theta_hat[1,j]))
420   se_hat[1,j] = sqrt(invsym(T'*ZQ*T)*(T'*ZQ*sigma*ZQ*T)*invsym(T'*ZQ*T))
421   st_store(i, "se_hat"+strofreal(j), se_hat[1,j])
422   st_store(i, "theta_hat"+strofreal(j), theta_hat[1,j])
423   }
424
425   }
426   end
427
428   forvalues i = 1/1000 {
429   mata polyloop(`i')
430   }
431
432   gen theta =  n
433
434   reshape long se_hat theta_hat, i(theta) j(K)
435
436   replace theta = 1
437
438   gen bias = theta_hat - theta
439   gen cov = ((theta_hat - invnormal(.975)*abs(se_hat) <= 1) & (theta_hat + invnormal(.975)*abs
      (se_hat) >= 1))
440
441   collapse se_hat theta_hat bias cov (sd) svar = theta_hat, by(K)
442
443
444   label var se_hat "SE"
445   label var theta_hat "Thetahat"
446   label var bias "Bias"
447   label var cov "Coverage"
448   label var svar "Sample Standard Dev."
449
450   order theta_hat se_hat bias cov svar
451   dataout, save($dir\stata_table_3_4_d.png) tex replace
452
```