Econ 675 Assignment 1

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October 9, 2018

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1 Kernal Density Estimation

1.1 Part 1

Start by noting that

$$\hat{f}^{(s)}(x) = \frac{(-1)^s}{nh^{1+s}} \sum_{i=1}^{n} k^{(s)} \left(\frac{x_i - x}{h} \right)$$

Now taking the expectation of $\hat{f}^{(s)}(x)$ that we can apply the linearity of expectations to move the expectation inside the sum. Then we can use the i.i.d. assumption to show the sum is just n times the expectation. This leaves us with

$$\mathrm{E}[\hat{f}^{(s)}(x)] = \mathrm{E}\left[\frac{(-1)^s}{h^{1+s}} k^{(s)} \left(\frac{x_i - x}{h} \right) \right] = \int_{-\infty}^{\infty} \frac{(-1)^s}{h^{1+s}} k^{(s)} \left(\frac{z - x}{h} \right) f(z) dz$$

Where the second equality is just by the definition of the expectation. Next we use integration by parts. Note that

$$\int_{-\infty}^{\infty} \frac{(-1)^s}{h^{1+s}} k^{(s)} \left(\frac{z - x}{h} \right) f(z) dz = - \int_{-\infty}^{\infty} \frac{(-1)^s}{h^{s}} k^{(s-1)} \left(\frac{z - x}{h} \right) f^{(1)}(z) dz$$

Iterating this s times gives us

$$\int_{-\infty}^{\infty} \frac{(-1)^s}{h^{1+s}} k^{(s)} \left(\frac{z - x}{h} \right) f(z) dz = (-1)^s \int_{-\infty}^{\infty} \frac{(-1)^s}{h} k \left(\frac{z - x}{h} \right) f^{(s)}(z) dz = \int_{-\infty}^{\infty} \frac{1}{h} k \left(\frac{z - x}{h} \right) f^{(s)}(z) dz$$

Next we apply change of variables. let $u = \frac{z-x}{h}$ Note that $du = \frac{1}{h} dz$ so we get

$$\int_{-\infty}^{\infty} k(u) f^{(s)}(x + hu) du$$

Next we Taylor expand $f^{(s)}(x+hu)$ to the P^{th} order about x. Recall from properties of the kernal estimator that $\int_{-\infty}^{\infty} k(u)du = 1$ and that $\int_{-\infty}^{\infty} k(u)u^j du = 0$ for all $j \neq p$ This gives us

$$f^{(s)}(x) + \frac{1}{P!} f^{(s+P)}(x) h^P \int_{-\infty}^{\infty} k(u)u^p du + o(h^P) = f^{(s)}(x) + \frac{1}{P!} f^{(s+P)}(x) h^p \mu_P(k) + o(h^P)$$

which is the desired result.

Now for the variance recall again that

$$\hat{f}^{(s)}(x) = \frac{(-1)^s}{nh^{1+s}} \sum_{i=1}^{n} k^{(s)}\left(\frac{x_i - x}{h}\right)$$

So by the i.i.d. assumption we can get that

$$\mathrm{V}\left(\hat{f}^{(s)}(x)\right) = \frac{1}{nh^{2+2s}} \mathrm{V}\left(k^{(s)}\left(\frac{x_i - x}{h}\right)\right)$$

$$\begin{aligned}
\mathrm{V}\left(\hat{f}^{(s)}(x)\right) &= \frac{1}{nh^{2+2s}} \mathrm{V}\left(k^{(s)}\left(\frac{x_i - x}{h}\right)\right) && (1)\\
&= \frac{1}{n2h^{2+2s}} \mathrm{E}\left[\left(k^{(s)}\left(\frac{x_i - x}{h}\right)\right)^2\right] - \frac{1}{nh^{2+2s}} \mathrm{E}\left[\left(k^{(s)}\left(\frac{x_i - x}{h}\right)\right)^2\right]^2 && (2)\\
&= \frac{1}{nh^{2+2s}} \mathrm{E}\left[\left(k^{(s)}\left(\frac{x_i - x}{h}\right)\right)^2\right] - \frac{1}{n}\left(\frac{1}{h^{1+s}} \mathrm{E}\left[\left(k^{(s)}\left(\frac{x_i - x}{h}\right)\right)^2\right]\right)^2 && (3)\\
&= \frac{1}{nh^{2+2s}} \int_{-\infty}^{\infty} k^{(s)}\left(\frac{x_i - x}{h}\right)^2 f(z)dz + \frac{1}{nh^{2+2s}} f^{(n)}(X)^2 && (4)\\
&= \frac{1}{nh^{1+2s}} \int_{-\infty}^{\infty} k^{(s)}(u)^2 f(x+hu)du + o\left(\frac{1}{nh^{2+2s}}\right) && (5)\\
&= \frac{1}{nh^{1+2s}} \cdot \vartheta_s(K) + o\left(\frac{1}{nh^{2+2s}}\right) && (6)
\end{aligned}$$

1.2 part 2

We start with the following AMISE

$$AIMSE[h] = \int \left[\left(h_n^P \cdot \mu_P(K) \cdot \frac{f^{(P+s)}(x)}{P!}\right)^2 + \frac{1}{nh_n^{1+2s}} \cdot \vartheta_s(K) \cdot f(x)\right] dx$$

Using the ϑ notation so $\vartheta_{P+s}(f) = \int (f^{(P+s)}(x))^2$ and recalling that f(x) integrates to 1 we can rewrite this as

$$= h_n^{2P} \left(\frac{\mu_P(K)}{P!}\right)^2 \vartheta_{P+s}(f) + \frac{\vartheta_s(K)}{nh_n^{1+2s}}$$

Now taking first order conditions and solving for h

$$\frac{d}{dh} AIMSE[h] = 2Ph_n^{2p-1} \left(\frac{\mu_P(K)}{P!}\right)^2 \vartheta_{P+s}(f) - (1+2s)\frac{\vartheta_s(K)}{nh_n^{2+2s}} = 0$$

$$\implies 2Ph^{1+2P+2s}\left(\frac{\mu_P(K)}{P!}\right)^2\vartheta_{P+s}(f)=(1+2s)\frac{\vartheta_s(K)}{n}$$

Thus, we get the AIMSE-optimal bandwidth choice.

$$h_{AIMSE_s}=\left[\frac{(2s+1)(P!)^2}{2P}\frac{\vartheta_s(K)}{\vartheta_{s+P}(f)\cdot\mu_P(K)^2}\frac{1}{n}\right]^{\frac{1}{1+2P+2s}}$$

Least squares cross-validation is a fully automatic data-driven method of selecting the smoothing parameter h. THis is based on the principle of selecting bandwidth that minimizes the integrated squared error of the resulting estimate.The estimate used is

$$\hat{h}_{CV}=\arg\min_h \frac{1}{n^2h}\sum_{i=1}^{n}\sum_{j=1}^{n}\bar{k}\left(\frac{X_i-X_j}{h}\right)-\frac{2}{n(n-1)h}\sum_{i=1}^{n}\sum_{j=1,i\neq j}^{n}k\left(\frac{X_i-X_j}{h}\right)$$

1.3 Monte Carlo experiment

1.3.1 a

First, we want to compute the theoretically optimal bandwidth for $s=0$, $n=1000$, using the Epanechnikov kernel ($P=2$), with the following Gaussian DGP:

$$x_i\sim 0.5\mathcal{N}(-1.5,-1.5)+0.5\mathcal{N}(1,1)$$

Filling in what we know so far we have :

$$h_{AIMSE_s}=\left[\frac{2*\vartheta_0(K)}{\vartheta_2(f)\cdot\mu_2(K)^2}\frac{1}{1000}\right]^{\frac{1}{5}}$$

So we need the second moment of K and the first moment of the second derivative of k squared. We can get two of these values from the table in Bruce Hanson's nonparametric notes. Giving us.

$$h_{AIMSE_s}=\left[\frac{\frac{3}{5}\cdot 2}{\vartheta_2(f)\cdot\frac{1}{5}^2}\frac{1}{1000}\right]^{\frac{1}{5}}$$

The second derivative of the normal density φ with mean μ variance σ^2 is

$$\varphi''_{\mu,\sigma^2}(x)=\frac{1}{\sqrt{2\pi\sigma^2}}e^{\frac{-(x-\mu)^2}{2\sigma^2}}\left[\left(\frac{(x-\mu)}{\sigma^2}\right)^2-\frac{1}{\sigma^2}\right]$$

now useing the linearity of integrals we can find $\vartheta_2(f)$

$$\vartheta_2(f)=\int_{-\infty}^{\infty}[0.5\varphi''_{1,1}(x)+0.5\varphi''_{-1.5,1.5}(x)]^2dx\approx 0.03883397$$

Where the approximation comes from R
Finally, pluging this in gives the theoretically optimal bandwidth is:

$$h*=0.8267532$$

1.3.2 b