Econ 675 Assignment 1

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1 Kernal Density Estimation

1.1 Part 1

Start by noting that

$$\hat{f}^{(s)}(x) = \frac{(-1)^s}{nh^{1+s}} \sum_{i=1}^{n} k^{(s)} \left(\frac{x_i - x}{h} \right)$$

Now taking the expectation of $\hat{f}^{(s)}(x)$ that we can apply the linearity of expectations to move the expectation inside the sum. Then we can use the i.i.d. assumption to show the sum is just n times the expectation. This leaves us with

$$\mathrm{E}[\hat{f}^{(s)}(x)] = \mathrm{E}\left[\frac{(-1)^s}{h^{1+s}} k^{(s)} \left(\frac{x_i - x}{h} \right) \right] = \int_{-\infty}^{\infty} \frac{(-1)^s}{h^{1+s}} k^{(s)} \left(\frac{z - x}{h} \right) f(z) dz$$

Where the second equality is just by the definition of the expectation. Next we use integration by parts. Note that

$$\int_{-\infty}^{\infty} \frac{(-1)^s}{h^{1+s}} k^{(s)} \left(\frac{z - x}{h} \right) f(z) dz = - \int_{-\infty}^{\infty} \frac{(-1)^s}{h^{s}} k^{(s-1)} \left(\frac{z - x}{h} \right) f^{(1)}(z) dz$$

Iterating this s times gives us

$$\int_{-\infty}^{\infty} \frac{(-1)^s}{h^{1+s}} k^{(s)} \left(\frac{z - x}{h} \right) f(z) dz = (-1)^s \int_{-\infty}^{\infty} \frac{(-1)^s}{h} k \left(\frac{z - x}{h} \right) f^{(s)}(z) dz = \int_{-\infty}^{\infty} \frac{1}{h} k \left(\frac{z - x}{h} \right) f^{(s)}(z) dz$$

Next we apply change of variables. let $u = \frac{z-x}{h}$ Note that $du = \frac{1}{h} dz$ so we get

$$\int_{-\infty}^{\infty} k(u) f^{(s)}(x + hu) du$$

Next we Taylor expand $f^{(s)}(x + hu)$ to the P^{th} order about x. Recall from properties of the kernal estimator that $\int_{-\infty}^{\infty} k(u) du = 1$ and that $\int_{-\infty}^{\infty} k(u) u^j du = 0$ for all $j \neq p$ This gives us

$$f^{(s)}(x) + \frac{1}{P!} f^{(s+P)}(x) h^P \int_{-\infty}^{\infty} k(u) u^p du + o(h^P) = f^{(s)}(x) + \frac{1}{P!} f^{(s+P)}(x) h^p \mu_P(k) + o(h^P)$$

which is the desired result.

Now for the variance recall again that

$$\hat{f}^{(s)}(x) = \frac{(-1)^s}{nh^{1+s}} \sum_{i=1}^{n} k^{(s)} \left(\frac{x_i - x}{h} \right)$$

So by the i.i.d. assumption we can get that

$$\mathrm{V}\left(\hat{f}^{(s)}(x)\right) = \frac{1}{nh^{2+2s}} \mathrm{V}\left(k^{(s)}\left(\frac{x_i - x}{h}\right)\right)$$

$$\begin{aligned}
\mathrm{V}\left(\hat{f}^{(s)}(x)\right) &= \frac{1}{nh^{2+2s}} \mathrm{V}\left(k^{(s)}\left(\frac{x_i - x}{h}\right)\right) && (1)\\
&= \frac{1}{n2h^{2+2s}} \mathrm{E}\left[\left(k^{(s)}\left(\frac{x_i - x}{h}\right)\right)^2\right] - \frac{1}{nh^{2+2s}} \mathrm{E}\left[\left(k^{(s)}\left(\frac{x_i - x}{h}\right)\right)^2\right]^2 && (2)\\
&= \frac{1}{nh^{2+2s}} \mathrm{E}\left[\left(k^{(s)}\left(\frac{x_i - x}{h}\right)\right)^2\right] - \frac{1}{n}\left(\frac{1}{h^{1+s}} \mathrm{E}\left[\left(k^{(s)}\left(\frac{x_i - x}{h}\right)\right)^2\right]\right)^2 && (3)\\
&= \frac{1}{nh^{2+2s}} \int_{-\infty}^{\infty} k^{(s)}\left(\frac{x_i - x}{h}\right)^2 f(z) dz + \frac{1}{nh^{2+2s}} f^{(n)}(X)^2 && (4)\\
&= \frac{1}{nh^{1+2s}} \int_{-\infty}^{\infty} k^{(s)}(u)^2 f(x + hu) du + o\left(\frac{1}{nh^{2+2s}}\right) && (5)\\
&= \frac{1}{nh^{1+2s}} \cdot \vartheta_s(K) + o\left(\frac{1}{nh^{2+2s}}\right) && (6)
\end{aligned}$$

1.2 part 2

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Thus, we get the AIMSE-optimal bandwidth choice.

$$h_{AIMSE_s} = \left[\frac{(2s+1)(P!)^2}{2P} \frac{\vartheta_s(K)}{\vartheta_{s+P}(f) \cdot \mu_P(K)^2} \frac{1}{n}\right]$$