A Synthesis of Binomial Option Pricing Models for Lognormally Distributed Assets

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The finance literature has revealed no fewer than 11 alternative versions of the binomial option pricing model for options on lognormally distributed assets. These models are derived under a variety of assumptions and in some cases require information that is ordinarily unnecessary to value options. This paper provides a review and synthesis of these models, showing their commonalities and differences and demonstrating how 11 diverse models all produce the same result in the limit. Some of the models admit arbitrage with a finite number of time steps and some fail to capture the correct volatility. This paper also examines the convergence properties of each model and finds that none exhibit consistently superior performance over the others. Finally, it demonstrates how a general model that accepts any arbitrage-free risk neutral probability will reproduce the Black-Scholes-Merton model in the limit.

■Option pricing theory has become one of the most powerful tools in economics and finance. The celebrated Black-Scholes-Merton model not only led to a Nobel Prize but completely redefined the financial industry. Its sister model, the binomial or two-state model, has also attracted much attention and acclaim, both for its ability to illustrate the essential ideas behind option pricing theory with a minimum of mathematics and to value many complex options.

The origins of the binomial model are somewhat unclear. Options folklore has it that around 1975 William Sharpe, later to win a Nobel Prize for his seminal work on the Capital Asset Pricing Model, suggested to Mark Rubinstein that option valuation should be feasible under the assumption that the underlying stock price can change to one of only two possible outcomes.[1] Sharpe subsequently developed the idea in the first edition of his textbook.[2] Perhaps the best-known and most widely cited original paper on the model is Cox, Ross, and Rubinstein (1979), but almost simultaneously, Rendleman and Bartter (1979) presented the same model in a slightly different manner.

Over the years, there has been an extensive body of research designed to improve the model.[3] In the literature the model has appeared in a variety of forms. Anyone attempting to understand the model can become bewildered by the array of formulas that all purport to accomplish the desired result of showing how to value an option and hedge an option position. These formulas have many similarities but notable

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[1]Not surprisingly, this story does not appear formally in the options literature but is related by Mark Rubinstein in RiskBooks (2003, p. 581).

[2]See Sharpe, Alexander, and Bailey (1998) for the current edition of this book.

[3]See, for example, Boyle (1988), Omberg (1988), Tian (1993), Figlewski and Gao (1999), Baule and Wilkens (2004) (for trinomials), He (1990) (for multiple state variables), and Rogers and Stapleton (1998), Breen (1991), Broadie and Detemple (1997), and Joshi (2007) (for American option pricing). See also Widdicks, Andricopoulos, Newton, and Duck (2002), Walsh (2003), Johnson, Pawlukiewicz, and Mehta (1997) for various other modifications, and Leisen and Reimer (1996) for a study of the model's convergence.

differences. Another source of confusion is that some presentations use opposite notation.[4] But more fundamentally, the obvious question is how so many different candidates for the inputs of the binomial model can exist and how each can technically be correct.

The objective of this paper is to synthesize the different approaches within a body of uniform notation and provide a coherent treatment of each model. Each model is presented with its distinct assumptions. Detailed derivations are omitted but are available in a supplemental document on the journal website or from the author.

Some would contend that it is wasteful to study a model that, for European options, in the limit equals the Black-Scholes-Merton model. Use of the binomial model, they would argue, serves only a pedagogical purpose. But it is difficult to consider the binomial model as a method for deriving the values of more complex options without knowing how well it works for the one scenario in which the true continuous limit is known. An unequivocal benchmark is rare in finance.

For options on lognormally distributed assets, the literature contains no less than 11 distinct versions of the binomial model. Some of the models are improperly specified and can lead to arbitrage profits for a finite number of time steps, while some do not capture the exogenous volatility. Several models focus first on fitting the binomial model to the physical process, rather than the risk neutral process, thereby requiring that the expected return on the stock be known, an unnecessary requirement in arbitrage-free pricing. I show that the translation from the physical to the risk neutral process has produced some misleading results. The paper also provides an examination of the convergence properties of each model and concludes with a demonstration that *any* risk neutral probability (other than one or zero) will correctly price an option in the limit.

My focus is exclusively on models for pricing options on lognormally distributed assets and not on interest rates. Hence, these models can be used for options on stocks, indices, currencies, and possibly commodities. Cash flows are ignored on the underlying, but these can be easily added. The paper begins with a brief overview of the model that serves to establish the notation and terminology.

[4]In some versions, the mean arithmetic return is α while the mean log return is μ. In others the opposite notation is used. Although there is no notational standard in the options literature, the inconsistent use of these symbols is a significant cost to comparing the models.

I. Basic Review of the Binomial Model

The continuously compounded risk-free rate per annum is r. Consider a risky asset priced at S that can move up to state "+" for a value of uS or down to state "–" for a value of dS. Let there be a call option expiring in one period with exercise price X. The value of the option in one period is c_u if the "+" state occurs and c_d if the "–" state occurs.

A. Deriving the Binomial Model

Now construct a portfolio consisting of Δ units of the asset and B dollars invested in the risk-free asset. This portfolio replicates the call option if its outcomes are the same in both states, that is,

$$\Delta S_u + B = c_u$$
$$\Delta S_d + B = c_d .$$

The unknowns are B and Δ. Rearranging to isolate B, setting the results equal to each other, and solving for B gives

$$\Delta = \frac{c_u - c_d}{S(u-d)} .$$

Since both values, c_u and c_d, are known, I substitute for Δ in either equation and solve for B. Then, given knowledge of Δ, S, and B, I obtain

$$c = \frac{\pi c_u + (1-\pi)c_d}{\exp(rh)} \tag{1}$$

as the value of the option, and

$$\pi = \frac{\exp(rh) - d}{u-d} \tag{2}$$

as the risk-neutral probability, sometimes referred to as the pseudo-probability or equivalent martingale probability, with h as the period length or time to expiration, T, divided by the number of binomial time periods, N. Extension to the multiperiod case follows and leads to the same result that the option value at a node, given the option values at the next possible two nodes, is given by Equation (1).

B. Specification of the Binomial Parameters

At times I will need to work with raw or discrete returns and at others times, I will work with continuous or log returns. Let the concept of return refer to the future price divided by the current price, or technically one plus the rate of return. Let the expected price one period later be $E(S_1)$ and the expected raw return be $E(S_1)/S$. The true probability of an up move is q. Thus, the per-period expected raw return is

$$E\left(\frac{S_1}{S}\right) = qu + (1-q)d. \tag{3}$$

The per-period expected log return is

$$E\left(\ln\left[\frac{S_1}{S}\right]\right) = q\ln u + (1-q)\ln d. \tag{4}$$

The variance of the raw return is

$$\begin{aligned} E\left(\frac{S_1}{S}\right)^2 - \left[E\left(\frac{S_1}{S}\right)\right]^2 &= (u-\alpha)^2 q + (d-\alpha)^2(1-q) \\ &= (u-d)^2 q(1-q). \end{aligned} \tag{5}$$

The variance of the log return is

$$\begin{aligned} E\left[\left(\ln\left(\frac{S_1}{S}\right)\right)^2\right] - \left[E\left(\ln\left(\frac{S_1}{S}\right)\right)\right]^2 &= (\ln u - \mu)^2 q + (\ln d - \mu)^2(1-q) \\ &= (\ln(u/d))^2 q(1-q). \end{aligned} \tag{6}$$

These parameters describe the actual probability distribution of the stock return, or the physical process. Option valuation requires transformation of the physical process to the risk neutral process. Typically, the user knows the volatility of the log return as given by the physical process, a value that may have been estimated using historical data or obtained as an implied volatility. In any case, I assume that volatility is exogenous and constant, as is usually assumed in continuous-time option pricing.

II. Fitting the Binomial Model

In early research on the binomial model, several papers examined fitting a binomial model to a continuous-time process, and each provided different prescriptions on how to do so. Before examining these models, I will review the basic concepts from the continuous-time models that are needed to fit the binomial model.

A. Basic Continuous-Time Concepts for the Binomial Model

The results in this section are from the Black-Scholes-Merton model. It starts by proposing that the log return is normally distributed with mean μ and variance σ^2. Given that $\ln(S_{t+dt}/S_t) = \ln(S_{t+dt}) - \ln(S_t)$, the stochastic process is proposed as

$$d\ln(S_t) = \mu dt + \sigma dW_t, \tag{7}$$

where μ and σ^2 are the annualized expected return and variance, respectively, as given by $E[d\ln(S_t)] = \mu dt$ and $Var[d\ln(S_t)] = \sigma^2 dt$, and dW_t is a Weiner process.

Examine now the raw return, dS_t/S_t. Letting $G_t = \ln(S_t)$, then $S_t = e^{G_t}$. Needed are the partial derivatives, $\partial S_t/\partial G_t = e^{G_t}$, and $\partial^2 S_t/\partial G_t^2 = e^{G_t}$. Applying Itô's Lemma to S_t obtains

$$dS_t = \frac{\partial S_t}{\partial G_t} dG_t + \frac{1}{2}\frac{\partial^2 S_t}{\partial G_t^2} dG_t^2.$$

Noting that $dG_t = \mu dt + \sigma dW_t$, then $dG_t^2 = \sigma^2 dt$. Substituting these results and the partial derivatives obtains

$$\frac{dS_t}{S_t} = (\mu + \sigma^2/2)dt + \sigma dW_t.$$

Define α as the expected value of the raw return so that

$$\frac{dS_t}{S_t} = \alpha dt + \sigma dW_t, \tag{8}$$

and $\alpha = \mu + \sigma^2/2$. The expectation of dS_t/S_t is $E[dS_t/S_t] = \alpha dt$ and $\text{Var}[dS_t/S_t] = \sigma^2 dt$. It is evident that the model assumes no difference in the volatilities of the raw and logarithmic processes in continuous time. This result is the standard assumption and derives from the fact that Itô's Lemma is used to transform the log process to the raw process. Technically, the variance of the raw process is

$$Var\left(\frac{dS_t}{S_t}\right) = \left(e^{\sigma^2 dt} - 1\right)e^{2\alpha dt}, \tag{9}$$

which is adapted to continuous time from Aitchison and Brown (1957). The difference in the variance defined as $\sigma^2 dt$ and in Equation (9) lies in the fact that the stochastic process for dS_t/S_t is an approximation. This subtle discrepancy is the source of some of the differences, however small, in the various binomial models.[5]

One final result is needed. The expected value of S

[5]Equation (9) is derived in Aitchison and Brown (1957) by taking the moment-generating function for the lognormal distribution. The difference in using Equation (9) in comparison to σ is quite small over most reasonable ranges of volatility and expected return. For example, with a 10% expected return, 30% volatility, and one day as a rough approximation of dt, the difference in instantaneous volatility is only 0.000005.

at the horizon T is given as[6]

$$E\left[S_T\right]=S\exp\left[\left(\mu+\sigma^2/2\right)T\right]=S\exp\left[\alpha T\right]. \quad (10)$$

B. Fitting the Binomial Model to a Continuous-Time Process

Several of the papers on the binomial model proceed to fit the model to the continuous-time process by finding the binomial parameters u, d, and q that force the binomial model mean and variance to equal the continuous-time model mean and variance. Thus, in this approach the binomial model is fit to the physical process. These parameters are then used as though they apply to the risk neutral process when valuing the option. As shown shortly, this is a dangerous step.

The binomial equations for the physical process are

$$q\ln u+\left(1-q\right)\ln d=\mu h \quad (11)$$

and

$$q\left(1-q\right)\left(\ln\left(u/d\right)\right)^2=\sigma^2 h, \quad (12)$$

or

$$\left(u-d\right)^2 q\left(1-q\right)=\left(e^{\sigma^2 h}-1\right)e^{2\alpha h}, \quad (13)$$

depending on whether one is fitting the log variance or raw variance. The volatility defined in the Black-Scholes-Merton model is the log volatility so the log volatility specification would seem more appropriate. But because the variance of the raw return is deterministically related to the variance of the log return, fitting the model to the variance of the raw return will still give the appropriate values of u and d.[7]

To convert the physical process to the risk-neutral process, a small transformation is needed. The mean raw return α is set to the risk-free rate r. Alternatively, the mean log return μ is set to $r-\sigma^2/2$.[8] But fitting the model to the equations for the physical process is at best unnecessary and at worst, misleading. Recall that the Black-Scholes-Merton model requires knowledge of the stock price, exercise price, risk-free rate, time to expiration, and log volatility but not the expected return. Fitting the binomial model to the physical process is unnecessary and adds the requirement that the expected return be known, thereby eliminating the main advantage of arbitrage-free option pricing over preference-based pricing.

As is evident from basic option pricing theory, the arbitrage-free and correct price of the option is derived from knowledge of the volatility with the condition that the expected return equals the risk-free rate. It follows that correct specification of the binomial model should require only that these two conditions be met. Let π be the risk neutral probability. The correct mean specification is

$$\pi u+\left(1-\pi\right)d=e^{rh}. \quad (14)$$

This expression is then turned around to isolate π:

$$\pi=\frac{e^{rh}-d}{u-d}. \quad (15)$$

Either Equation (14) or (15) is a necessary condition to guarantee the absence of arbitrage.[9] Surprisingly, not all binomial option pricing models satisfy Equation (14). Note that this condition is equivalent to, under risk neutrality, forcing the binomial expected raw return, *not the expected log return*, to equal the continuous risk-free rate. In other words, the correct value of π should come by specifying Equation (14), not

$$\pi\ln u+\left(1-\pi\right)\ln d=\left(r-\sigma^2/2\right)h, \quad (16)$$

which comes from adapting Equation (11) to the risk neutral measure and setting the log expected return μ to its risk neutral analog, $r-\sigma^2/2$. Surprisingly, many of the binomial models in the literature use this improper specification.[10]

The no-arbitrage condition is a necessary but not sufficient condition for the binomial model to yield the

[6]For proof, see the appendix in Jarrow and Turnbull (2000, p. 112).

[7]One would, however, need to exercise some care. Assume that the user knows the log variance. Then the raw variance can be derived from the right-hand side of Equation (9), which then becomes the right-hand side of Equation (13). If the user knows the log variance, then it becomes the right-hand side of Equation (12). If the user has empirically estimated the raw and log variances, the former can be used as the right-hand side of Equation (13) and the latter can be used as the right-hand side of Equation (12). But then Equations (12) and (13) might lead to different values of u and d, because the empirical rand log stochastic processes are unlikely to conform precisely to the forms specified by theory.

[8]See Jarrow and Turnbull (2000) for an explanation of this transformation.

[9]The proof is widely known. One simply relaxes this constraint whereupon a self-financed portfolio of long stock-short bond (or vice versa as appropriate) always generates positive value even in the worst outcome. It will follow that there exists a measure, like π, such that $\pi u + (1-\pi)d = e^{rh}$, which is Equation (14).

[10]A correct logarithmic specification of the no-arbitrage condition would involve taking the log of Equation (14). If this modified version of Equation (14) were solved, the model would be correct. I found no instances in the literature in which this alternative approach is used.

correct option price. The model must also be calibrated to the correct volatility. This constraint is met by using the risk-neutral analog of Equation (5),

$$(u-d)^2 \pi (1-\pi) = \left(e^{\sigma^2 h} - 1\right) e^{2rh} \tag{17}$$

or Equation (6),

$$\left(\ln(u/d)\right)^2 \pi (1-\pi) = \sigma^2 h. \tag{18}$$

Either condition will suffice because both provide the correct raw or log volatility.

C. Convergence of the Binomial Model to the Black-Scholes-Merton Model

Three of the most widely cited versions of the binomial model, Cox, Ross, and Rubinstein (1979); Rendleman and Bartter (1979); and Jarrow and Rudd (1983), provide proofs that their models converge to the BSM model when $N \to \infty$. Recall that each model is characterized by formulas for u, d, and the probability. Hsia (1983) has provided a proof that demonstrates that convergence can be shown under less restrictive assumptions. For risk neutral probability π, Hsia's (1983) proof shows that the binomial model converges to the BSM model if $N\pi \to \infty$ as $N \to \infty$. To meet this requirement, $0 < \pi < 1$ is all that is necessary.[11] This result may seem surprising for it suggests that the risk neutral probability can be set at any arbitrary value such as 0.1 or 0.8. In the literature some versions of the binomial model constrain the risk neutral probability to ½ and as shown later, all versions of the model have risk neutral probabilities that converge to ½. But Hsia's (1983) proof shows that *any* probability other than zero or one will lead to convergence. This interesting result is examined later.

D. Alternative Binomial Models

I now examine the 11 binomial models that have appeared in the literature.

1. Cox-Ross-Rubinstein

Cox, Ross, and Rubinstein (1979), henceforth CRR, is arguably the seminal article on the model. Their Equations (2) and (3) (p. 234) show the option value as given by my Equation (1) with the risk neutral probability specified as my Equation (2). CRR then proceed to examine how their formula behaves when $N \to \infty$ (their p. 246-251). They do this by choosing u, d, and q so that their model converges in the limit to the expected value and variance of the physical process. Thus, they solve for u, d, and q using the physical process, my Equations (11) and (12). Note that Equations (11) and (12) constitute a system of two equations and three unknowns. CRR propose a solution while implicitly imposing a necessary third condition, $ud = 1$, an assumption frequently found in the literature. Upon obtaining their solution, they then assume the limiting condition that $h^2 = 0$. This condition is necessary so that the correct variance is recovered, though the mean is recovered for any N. Their solutions are:

$$u = e^{\sigma\sqrt{h}}, \quad d = e^{-\sigma\sqrt{h}}, \tag{19}$$

with physical probability

$$q = \frac{1}{2} + \frac{1}{2}\frac{\mu}{\sigma}\sqrt{h}.$$

As previously discussed, for risk-neutral option valuation, q and μ are unnecessary. CRR recognize that the only condition required to prevent arbitrage is the equivalent of my Equation (15). To guarantee the absence of arbitrage, they discard their solution for q, accept Equation (15) as the solution for π, and retain their formulas for u and d. But their formulas for u and d are the solutions based on the log mean in Equation (11), not the raw, arbitrage-prohibitive mean as in Equation (15). Thus, their formulas for u and d are technically incorrect for finite N and do not recover the correct volatility.

As it turns out, however, their solutions for u and d are correct in the limit, because in that case u and d approach one and are infinitesimally different from their log values. Also, the risk neutral probability, Equation (15), converges to ½ using CRR's expressions for u and d. CRR acknowledge, and it has been noted in the literature, that their solution recovers the volatility only in the limit, but the reason has not always been clear. Their reference to the volatility has always referred to the volatility obtained using the physical measure.[12] It is now apparent that the volatility computed using the risk neutral probabilities is also incorrect except in the limit. The problem arises from the simple fact that CRR

[11]The other requirements not noted by Hsia (1983) are that the choice of u, d, and π must force the binomial model volatility to equal the true volatility and the mean must guarantee no arbitrage.

[12]See Cox, Ross, and Rubinstein (1979, p. 248-249).

fit the binomial model to the physical process, simultaneously deriving the physical probability q, and then substitute the arbitrage-free formula for π as q. Had they imposed the arbitrage-free condition directly into the solution, they would have obtained different formulas, as we will see in another model.

The so-called binomial model is really a family of models that, under surprisingly mild conditions, all converge in the limit to the Black-Schules-Merton model.

2. Rendleman-Bartter and Jarrow-Rudd-Turnbull

Because of their similarities, discussion of the Rendleman-Bartter (RB) approach is combined with discussion of the Jarrow-Rudd (JR) approach and later appended with the Jarrow-Turnbull (JT) approach. These approaches also fit the binomial model to the physical process. The RB approach specifies the log mean and log variance Equations (11) and (12), and solves these two equations to obtain:

$$u = e^{\mu h + \sqrt{\frac{1-q}{q}}\sigma\sqrt{h}}, \quad d = e^{\mu h - \sqrt{\frac{q}{1-q}}\sigma\sqrt{h}}.$$

Because these formulas do not specify the value of q, they are too general to be of use. In risk neutralizing the model, RB assume that $\mu = r - \sigma^2/2$ and a probability of ½. It is important to note that this probability does not guarantee the absence of arbitrage because it is arbitrarily established and not derived by conditioning on the arbitrage-free specification, Equation (14). Clear distinctions between the arbitrage-free risk-neutral probability need to be made, which results from Equation (14), and the risk neutral probability obtained by solving whatever condition is imposed, such as specifying the log mean, Equation (11). This latter probability is denoted as π^* and called the risk neutral proxy probability. Hence, for RB $\pi^* = ½$. The JR approach basically starts with these conditions. There is a great deal of confusion over this version of the binomial model. I will examine it carefully and attempt to bring some clarity.

JR solve the same two equations, the log mean and log variance, and assume that q, the physical probability, is ½, which leads to the following formulas for u and d:

$$u = e^{\mu h + \sigma\sqrt{h}}, \quad d = e^{\mu h - \sigma\sqrt{h}}$$

(Jarrow and Rudd, 1983, p. 188).[13] They then proceed to show that these formulas result in convergence of their model to the BSM model. Note that q is the physical probability. Normally the binomial model would be shown to converge to the BSM model using the risk neutral probability. These values for u and d are not consistent with the absence of arbitrage because they are derived by constraining the log mean, not the raw mean. That is, they are consistent with the risk neutral proxy probability π^* but not with the risk neutral probability π. In the limit, it can be shown that π converges to ½ when JR's formulas for u and d are used, and with $\pi^* = ½$ all is well and the JR model is arbitrage-free in the limit. Combined with Hsia's (1983) proof, the JR model clearly converges to BSM.

Thus, for finite N, JR's formulas do not prohibit arbitrage, but there is yet another problem. JR make the interesting comment (p. 188) that their formulas result in recovery of the volatility for any N, while Cox, Ross, and Rubinstein's (1979) parameters recover the volatility only in the limit. As I will show, the first portion of this statement is misleading. Their choice of ½ as a probability does indeed recover the correct volatility for finite N, but this result is obtained only by using the physical probability. A risk neutral probability of ½ is obtained only in the limit. Hence, the volatility is not recovered for finite N when the risk neutral probability is used. For option pricing, of course, it is the risk neutral probability that counts.

JR risk neutralize their formulas by specifying (p. 190) that $\mu = r - \sigma^2/2$, thereby leading to their solutions:

$$u = e^{(r-\sigma^2/2)h + \sigma\sqrt{h}}, \quad d = e^{(r-\sigma^2/2)h - \sigma\sqrt{h}}, \tag{20}$$

but again, these formulas are consistent only with a probability of ½ and risk neutrality as specified by $\mu = r - \sigma^2/2$. Simply converting the mean is not sufficient to ensure risk neutrality for a finite number of time steps.[14]

[13]These are JR's equations (13-18). In their models the up and down factors are denoted as u and v, respectively, with $S^+ = Se^u$ and $S^- = Se^v$.

[14]A close look at JR shows that q is clearly the physical probability. On page 187, they constrain q to equal the risk neutral probability, with their symbol for the latter being ϕ. But this constraint is not upheld in subsequent pages whereupon they rely on convergence in the limit to guarantee the desired result that arbitrage is prevented and BSM is obtained. This point has been recognized in a slightly different manner by Nawalkha and Chambers (1995, p. 608).

A number of years later, JT derive the same model but make a much clearer distinction between the physical and risk neutral processes. They fix π^* at its arbitrage-free value and show for their up and down parameters that

$$\pi^* = \pi = \frac{e^{rh} - d}{u - d} = \frac{e^{\sigma^2 h/2} - e^{-\sigma\sqrt{h}}}{e^{\sigma\sqrt{h}} - e^{-\sigma\sqrt{h}}}. \quad (21)$$

Like CRR, the correct specification of π ensures that their model does not admit arbitrage. But, because their solutions for u and d were obtained by specifying the log mean, these solutions are not technically correct for finite N. The mean constraint is met, so there must be an error somewhere, which has to be in the variance. Thus, their model does not recover the variance for finite N using the risk neutral probabilities. It returns the correct variance either when the physical probability is used or in the limit with the risk neutral probability converging to ½.

For future reference, this model will be called the RBJRT model and referred only to the last version of the model in which the no-arbitrage constraint is applied to obtain π. I have shown that it does not recover the correct volatility for finite N. Now I will consider a model that fits a binomial tree to the physical process but does prevent arbitrage and recovers the correct volatility.

3. Chriss

Chriss's model (1996) specifies the raw mean and log variance of the physical process. The former is given by

$$qu + (1-q)d = e^{\alpha h}, \quad (22)$$

and the latter by Equation (12). He then assumes that q = ½. The solutions are

$$u = \frac{2e^{\alpha h + 2\sigma\sqrt{h}}}{e^{2\sigma\sqrt{h}} + 1}, \quad d = \frac{2e^{\alpha h}}{e^{2\sigma\sqrt{h}} + 1}.$$

The risk-neutralized analogs are found by substituting r for α:

$$u = \frac{2e^{rh + 2\sigma\sqrt{h}}}{e^{2\sigma\sqrt{h}} + 1}, \quad d = \frac{2e^{rh}}{e^{2\sigma\sqrt{h}} + 1}. \quad (23)$$

Note that because Chriss' mean specification is the raw mean, transformation to risk neutrality by $\alpha = r$ correctly returns the no-arbitrage condition, Equation (15). Thus, for the Chriss model, $\pi = \pi^*$ = ½ for all N, and the model correctly preserves the no-arbitrage condition and recovers the volatility for any number of time steps.

4. Trigeorgis

The Trigeorgis (1991) model transforms the original process into a log process. That is, let $X = \ln S$ and specify the binomial process as the change in X, or ΔX. The solutions for the physical process are

$$u = e^{\sqrt{\sigma^2 h + \mu^2 h^2}}, \quad d = e^{-\sqrt{\sigma^2 h + \mu^2 h^2}}, \quad q = \frac{1}{2} + \frac{1}{2}\frac{\mu h}{\sqrt{\sigma^2 h + \mu^2 h^2}}.$$

Note that if $h^2 = 0$, the Trigeorgis model is the same as the CRR model. Trigeorgis then risk neutralizes the model by assuming that $\mu = r - \sigma^2/2$. The results are

$$u = e^{\sqrt{\sigma^2 h + \left(r - \sigma^2/2\right)^2 h^2}}, \quad d = e^{-\sqrt{\sigma^2 h + \left(r - \sigma^2/2\right)^2 h^2}}. \quad (24)$$

Trigeorgis' risk neutral probability comes simply from substitution of $r - \sigma^2/2$ for μ in the formula for q, thereby obtaining

$$\pi^* = \frac{1}{2} + \frac{1}{2}\frac{\left(r - \sigma^2/2\right)h}{\sqrt{\sigma^2 h + \left(r - \sigma^2/2\right)h^2}}.$$

Of course this is the risk neutral proxy probability and is not given by the no-arbitrage condition. Therefore, it is not arbitrage-free for finite N, though it does recover the correct volatility. In the limit, Trigeorgis's risk neutral proxy probability, π^*, converges to ½ and the arbitrage-free risk neutral probability, π, converges to ½ , so the Trigeorgis model is arbitrage-free in the limit.

5. Wilmott1 and Wilmott2

Wilmott (1998) derives two binomial models. He specifies the raw mean and raw variance of the physical process, Equations (22) and (17). His first model, referred to here as Wil1, assumes $ud = 1$. The solutions for the physical process are

$$u = \frac{1}{2}\left(e^{-\alpha h} + e^{(\alpha+\sigma^2)h}\right) + \frac{1}{2}\sqrt{\left(e^{-\alpha h} + e^{(\alpha+\sigma^2)h}\right)^2 - 4}$$

$$d = \frac{1}{2}\left(e^{-\alpha h} + e^{(\alpha+\sigma^2)h}\right) - \frac{1}{2}\sqrt{\left(e^{-\alpha h} + e^{(\alpha+\sigma^2)h}\right)^2 - 4}.$$

The physical probability q is found easily from the mean condition,

$$q = \frac{e^{\alpha h} - d}{u - d}.$$

Risk neutralizing the model is done by simply substituting r for α:

$$u=\frac{1}{2}\left(e^{-rh}+e^{\left(r+\sigma^2\right)h}\right)+\frac{1}{2}\sqrt{\left(e^{-rh}+e^{\left(r+\sigma^2\right)h}\right)^2-4}$$
$$u=\frac{1}{2}\left(e^{-rh}+e^{\left(r+\sigma^2\right)h}\right)-\frac{1}{2}\sqrt{\left(e^{-rh}+e^{\left(r+\sigma^2\right)h}\right)^2-4} \qquad (25)$$

and the risk neutral probability π is correctly given by Equation (14). Because the raw mean constraint is upheld, this model prohibits arbitrage, and it also recovers the volatility. In addition, π converges to ½ in the limit.

The second version of the model, which I will call Wil2, assumes that q = ½. The solutions for the physical process are

$$u=e^{\alpha h}\left(1+\sqrt{e^{\sigma^2 h}-1}\right), \quad d=e^{\alpha h}\left(1-\sqrt{e^{\sigma^2 h}-1}\right).$$

Risk neutralizing the model gives the solutions

$$u=e^{rh}\left(1+\sqrt{e^{\sigma^2 h}-1}\right), \quad d=e^{rh}\left(1-\sqrt{e^{\sigma^2 h}-1}\right). \qquad (26)$$

Here π is forced to a value of ½ and this specification correctly prevents arbitrage because Equation (14) is upheld. In addition, the volatility is recovered.

6. Jabbour-Kramin-Young

Jabbour, Kramin, and Young (2001), henceforth JKY, provide a review of several well-known binomial models and introduce some new ones. They classify their models into three families: 1) the Rendleman-Bartter approach (referred to here as JKYRB models); 2) alternative binomial models or ABMC models (referred to here as JKYABMC models); and 3) discrete-time Geometric Brownian motion or ABMD models (referred to here as JKYABMD models). Each family is identified by its specification of the mean and variance and a conditioning constraint.

For the physical process, JKYRB models specify the mean and variance of the log process, Equations (11) and (18). JKYABMC models specify the mean of the raw process, Equation (22), and an approximation of the volatility of the raw process,

$$q(1-q)(u-d)^2 \cong \sigma^2 h. \qquad (27)$$

Recall that the volatility of the raw process should technically have $e^{2\alpha h}(e^{\sigma^2 h}-1)$ on the right-hand side.[15] JKYABMD models specify an approximation of the raw mean as

$$qu+(1-q)d \cong 1+\alpha h, \qquad (28)$$

where $1 + \alpha h$ is an approximation of $e^{\alpha h}$.

Risk neutralizing these models requires changing either α to r or μ to $r - \sigma^2/2$. Because the JKYRB and JKYABMD models specify the mean of the log process instead of the mean of the raw process, they admit arbitrage for finite N. Because they use an approximation of the raw volatility, they do not precisely recover the volatility.

As previously noted, the mean and volatility specifications establish two equations, but three unknowns. A third assumption, the conditioning constraint, is required to obtain solutions for u, d, and either q or π. Each model class is appended with the number "1," "2," or "3" to indicate the third assumption. Models appended with a "1" assume $ud = 1$. Models appended with a "2" assume $ud = e^{2\alpha h}$. Models appended with a "3" simply assume that q = ½ or π = ½.

Under these assumptions, model JKYRB1 is equivalent to Trigeorgis. Model JKYRB2 is unique and has solutions for the physical process of

$$u=e^{\mu h+\frac{1-q}{\sqrt{q(1-q)}}\sigma\sqrt{h}}, \quad d=e^{\mu h-\frac{q}{\sqrt{q(1-q)}}\sigma\sqrt{h}}.$$

Many of the JKY models have the same solution for the probability, q, which is

$$q=\frac{1}{2}\left[1-\frac{m}{\sqrt{4+m^2}}\right]. \qquad (29)$$

For JKYRB2,

$$m=\sigma\sqrt{h}. \qquad (30)$$

Risk neutralizing by setting $\mu = r - \sigma^2/2$ gives

$$u=e^{\left(r-\sigma^2/2\right)h+\frac{1-\pi^*}{\sqrt{\pi^*\left(1-\pi^*\right)}}\sigma\sqrt{h}},$$
$$d=e^{\left(r-\sigma^2/2\right)h-\frac{\pi^*}{\sqrt{\pi^*\left(1-\pi^*\right)}}\sigma\sqrt{h}}, \qquad (31)$$

$$\pi^*=\frac{1}{2}\left[1-\frac{m}{\sqrt{4+m^2}}\right], \qquad (32)$$

with m given in Equation (30). Because π^* is the probability obtained by the log mean constraint, it is not the arbitrage-free risk neutral probability, π. Both, however, converge to ½ in the limit, so arbitrage is

[15] As shown earlier, this approximation is obtained from derivation of the stochastic process of the raw return by applying Itôs Lemma to the log process. It can also be shown to arise from application of the expression $e^x \cong 1+x+x^2/2+\ldots$

prohibited in the limit.

Model JKYRB3 is equivalent to RBJRT.

JKYAMBC1 is equivalent to Wil2. JKYABMC2 is unique and has the following solutions for the physical process:

$$u = e^{\alpha h}\left(1+\frac{1-q}{\sqrt{q(1-q)}}\sqrt{e^{\sigma^2 h}-1}\right),$$
$$d = e^{\alpha h}\left(1-\frac{q}{\sqrt{q(1-q)}}\sqrt{e^{\sigma^2 h}-1}\right),$$

with q given by Equation (29)and m given by

$$m=\sqrt{e^{\sigma^2 h}-1}\,. \tag{33}$$

Risk neutralizing by setting α to r gives

$$\begin{aligned} u &= e^{rh}\left(1+\frac{1-\pi^*}{\sqrt{\pi^*(1-\pi^*)}}\sqrt{e^{\sigma^2 h}-1}\right), \\ d &= e^{rh}\left(1-\frac{\pi^*}{\sqrt{\pi^*(1-\pi^*)}}\sqrt{e^{\sigma^2 h}-1}\right), \end{aligned} \tag{34}$$

with π^* the same as in Equation (32) and with m given in Equation (33). Because π^* is not obtained by meeting the arbitrage-free constraint, it does not equal the arbitrage-free risk neutral probability, π. Both, however, converge to ½ in the limit, so arbitrage is prohibited in the limit.

JKYABMC3 is equivalent to Wil2.

The solutions for JKYABMD1 are:

$$u = 1+\alpha h+\frac{1-q}{\sqrt{q(1-q)}}\sigma\sqrt{h},$$
$$d = 1+\alpha h-\frac{q}{\sqrt{q(1-q)}}\sigma\sqrt{h},$$

with q given by Equation (29) and m as

$$m=\frac{1+\sigma^2 h-(1+\alpha h)^2}{(1+\alpha h)\sigma\sqrt{h}}\,.$$

Risk neutralizing leads to the solution,

$$\begin{aligned} u &= 1+rh+\frac{1-\pi^*}{\sqrt{\pi^*(1-\pi^*)}}\sigma\sqrt{h}, \\ d &= 1+rh-\frac{\pi^*}{\sqrt{\pi^*(1-\pi^*)}}\sigma\sqrt{h}, \end{aligned} \tag{35}$$

with π^* as given in Equation (32) and[16]

$$m=\frac{1+\sigma^2 h-(1+\alpha h)^2}{(1+rh)\sigma\sqrt{h}}\,. \tag{36}$$

Because π^* is not obtained by meeting the arbitrage-free constraint, it does not equal the arbitrage-free risk neutral probability, π. Both, however, converge to ½ in the limit, so arbitrage is prohibited in the limit.

JKYABMD2 has solutions for the physical process of

$$u = 1+\alpha h+\frac{1-q}{\sqrt{q(1-q)}}\sigma\sqrt{h},$$
$$d = 1+\alpha h-\frac{q}{\sqrt{q(1-q)}}\sigma\sqrt{h},$$

with q given by Equation (29) and m as

$$m=\frac{e^{2\alpha h}+\sigma^2 h-(1+\alpha h)^2}{(1+\alpha h)\sigma\sqrt{h}}\,.$$

These formulas are not exactly as reported in JKY. They make a further approximation using $e^x \cong 1+x$. I report the solution without this approximation. As a result of this adjustment, I refer to this model as JKYABMD2c.

Risk neutralizing gives the solutions

$$\begin{aligned} u &= 1+rh+\frac{1-\pi^*}{\sqrt{\pi^*(1-\pi^*)}}\sigma\sqrt{h}, \\ d &= 1+rh-\frac{\pi^*}{\sqrt{\pi^*(1-\pi^*)}}\sigma\sqrt{h}, \end{aligned} \tag{37}$$

with π^* given by Equation (32), and

$$m=\frac{e^{2rh}+\sigma^2 h-(1+rh)^2}{(1+rh)\sigma\sqrt{h}}\,. \tag{38}$$

Because π^* is not obtained by meeting the arbitrage-free constraint, it does not equal the arbitrage-free risk neutral probability, π. Both, however, converge to ½ in the limit, so arbitrage is prohibited in the limit.

JKYABMD3 has solutions for the physical process

[16]The formula for π in Equation (32) comes from solving a quadratic equation. As is well-known, such equations have two roots. For some of the JKY models, it can be shown that one sign is the correct one. For the JKYABMD1 model, both signs are acceptable solutions. JKY report the formula with the minus sign as the correct one, and we shall use it from this point on, but we should be aware that yet another solution exists.

of

$$u = 1 + \alpha h + \sigma\sqrt{h}, \quad d = 1 + \alpha h - \sigma\sqrt{h}, \quad q = \frac{1}{2}.$$

Risk neutralizing gives the solutions

$$u = 1 + rh + \sigma\sqrt{h}, \quad d = 1 + rh - \sigma\sqrt{h}, \quad \pi^* = \frac{1}{2}. \quad (39)$$

Because π^* is not obtained by meeting the arbitrage-free constraint, it does not equal the arbitrage-free risk neutral probability, π. Both, however, converge to ½ in the limit, so arbitrage is prohibited in the limit.

7. Avellaneda and Laurence

I will take a look at one additional model that appears unique but is not. Avellaneda and Laurence (1999) (AL) take a notably different approach to solving for u, d, and q. For the risk neutral-process, the expected return is specified for the raw return as in Equation (14). For the volatility, they specify the ratio of u to d in terms of a constant ω,

$$u / d = e^{2\omega\sqrt{h}}.$$

They specify the log volatility as in Equation (12). Their solutions are:

$$u = \frac{e^{rh+\omega\sqrt{h}}}{\pi e^{\omega\sqrt{h}} + (1-\pi)e^{-\omega\sqrt{h}}},$$

$$d = \frac{e^{rh-\omega\sqrt{h}}}{\pi^* e^{\omega\sqrt{h}} + (1-\pi^*)e^{-\omega\sqrt{h}}},$$

$$\pi^* = \frac{1}{2}\left(1 \pm \sqrt{1-\sigma^2/\omega^2}\right).$$

Of course, these solutions contain an unknown ω. AL note that if ω is set to σ, then π^* = ½ and

$$u = \frac{e^{(r-\sigma^2/2)h+\sigma\sqrt{h}}}{\cosh\left(\sigma\sqrt{h}\right)}, \quad d = \frac{e^{(r-\sigma^2/2)h-\sigma\sqrt{h}}}{\cosh\left(\sigma\sqrt{h}\right)}.$$

These formulas have occasionally appeared in the literature.[17] Nonetheless, the model is not unique as algebraic rearrangement shows that it is equivalent to Chriss.

E. Anomalies in the Models

Consider some desirable conditions. The risk neutral probability, π or π^*, should range between zero and one, u exceed one, and d be less than one. Thus, I examine four conditions: (a) π (or π^*) < 1, (b) π (or π^*) > 0, (c) u > 1, and (d) d < 1. Conditions (a) and (b) establish that π (or π^*) is a probability measure. If u < 1, the asset value decreases when intuition suggests that it should increase. If d > 1, the asset value increases when intuition suggests that it should decrease.

For π, condition (a) is equivalent to the following:

$$\pi < 1 \Rightarrow \frac{\exp(rh) - d}{u - d} < 1$$
$$\Rightarrow \exp(rh) < u.$$

With u always greater than d, which is true for each model, condition (b) is met as:

$$\pi > 0 \Rightarrow \frac{\exp(rh) - d}{u - d} > 0$$
$$\Rightarrow \exp(rh) > d.$$

When the modeler proposes π^* as the risk neutral probability, we examine it directly in relation to zero and one.

Detailed proofs are contained in the supporting document. I find that all of the models pass the test that the risk neutral probability or its proxy exceeds zero, but the CRR and RBJRT models can have $\pi > 1$. This result for CRR is well-known.[18] It arises when $h > (\sigma/r)^2$, which is likely to occur with low volatility, high interest rates, and a long time to expiration. Sufficiently low volatility is unlikely to apply when modeling stock prices, but exchange rate volatility is often less than 0.1. Thus, long-term foreign exchange options where the interest rate is high can have a risk neutral probability greater than one.[19] For RBJRT, the risk neutral probability can exceed one if $h < 4/\sigma^2$. Although the volatility of some commodities has been known to exceed 100%, the volatility of most stocks is less than 100%. Therefore, for most stocks, $4/1^2 = 4$, so the problem exists only if $h > 4$. For very low volatility, as in the foreign exchange market, the time step would have to be extremely large. Thus, it would take exceptionally large volatility and a very small number of time steps relative to the option maturity for the risk neutral probability to exceed one for the RBJRT model.

Of course, if the risk neutral probability exceeds one, a model could still correctly value the option. But, as

[17]See, for example, Carpenter (1998).

[18]See, for example, Chriss (1996, p. 239).

[19]For example, if r = .1, σ = .05, and T = 5. In that case, we would require N > 20.

previously noted, the CRR and RBJRT models use the u and d formulas from the physical process, which is derived by constraining the log mean, not the raw mean. It is the raw mean that guarantees no arbitrage.

For the other two desirable conditions that the up factor exceeds one and the down factor is less than one, only the RBJRT methodology permits an up factor that can be less than one. Interestingly, seven of the eleven models permit a down factor greater than one. Only the models of Trigeorgis, Wil1, and the JKYABMD1 model have no anomalies.

If the correct mean and volatility are captured, any binomial probability other than zero or one will produce the Black-Schules-Merton price in the limit.

These anomalies are interesting but usually occur only with extreme values of the inputs and/or a small number of time steps relative to the option maturity. They can usually be avoided when actually pricing an option. The greatest risk they pose is probably when the model is used for illustrative purposes.

III. Model Comparisons

Table I illustrates an example for valuing a call option in which the asset is priced at 100, the exercise price is 100, and the volatility is 30%.[20] The continuous risk-free rate is 5% and the option expires in one year. In all cases, I use the probability π or π^* as specified by the authors of the respective models. I show the values for 1, 5, 10, 20, 30, 50, 75 and 100 time steps. The correct value, as given by the Black-Scholes-Merton formula, is 14.23. At 50 times steps all of the prices are within 0.06. At 100 time steps, all of the prices are within 0.03.

To further investigate the question of which models perform best, I vary the inputs by letting the volatility be 0.10, 0.30, and 0.50; the time to expiration be 0.25, 1.0, and 4.0; and the moneyness be 10% out-of-the-money, at-the-money, and 10% in-the-money. These inputs comprise 27 unique combinations. I examine several characteristics of the convergence of these models to the Black-Scholes-Merton value.

A. An Initial Look at Convergence

Let $b(N)$ be the value computed for a given binomial model with N time steps and BSM be the true Black-Scholes-Merton value. Binomial models are commonly described as converging in a pattern referred to as "odd-even." That is, when the number of time steps is odd (even), the binomial price tends to be above (below) the true price. We will call this phenomenon the "odd-even" property. Interestingly, my numerical analyses show that the odd-even phenomenon never occurs for any model with out-of-the-money options. For at-the-money options, the odd-even phenomenon always occurs for the JKYABMD1 model and occasionally for other models. Odd-even convergence never occurs for any inputs for JKYRB2, JKYABMC2, and JKYABMD2c. Thus, the odd-even property is not a consistent phenomenon across models.

Next, I examine whether a model exhibits monotonic convergence, defined as

$$\left|e(N)\right| < \left|e(N-1)\right| \forall N > 1,$$

where $|e(N)| = |b(N) - \text{BSM}|$. That is, successive errors are smaller. Only the Trigeorgis model exhibits monotonic convergence and it does so for only one of the 27 combinations of inputs examined. Because monotonic convergence is virtually non-existent, we examine a slight variation. Suppose each alternate error is smaller than the previous one, a phenomenon we call *alternating monotonic convergence*, defined as

$$\left|e(N)\right| < \left|e(N-2)\right| \forall N > 2.$$

As it happens, however, alternating monotonic convergence never occurs.

I then attempt to identify at which step a model is deemed to have acceptably converged. For a given time step, I compute the average of the current computed price and the previous computed price. I then identify the time step at which this average price is within 0.01 of the BSM price with the added criterion that the difference must remain less than 0.01 through step 100. The results are presented in Tables II, III, and IV. One consistent result in all three tables is that the RBJRT and Chriss models produce the same results. Further examination shows that the values of u and d are not precisely equal for both models for all values

[20] The choices of stock price and exercise price are not particularly important as long as moneyness is consistent. Standard European options, indeed most options, have values that are linearly homogeneous with respect to the stock price and exercise price. Thus, after choosing a stock price and exercise price, we could use a scale factor to change to any other stock price and exercise price, and we would obtain an option price that differs only by the scale factor.

Table I. Some Numerical Examples

The table contains the binomial option value for various time steps (N) for a call option with stock price of 100, exercise price of 100, volatility of 0.30, risk-free rate of 0.05, and time to expiration of one year for each of the 11 binomial models. The risk neutral probability is p or p^* as specified by the authors of the models. The Black-Scholes-Merton option value is 14.23.

N	CRR	RBJRT	Chriss	Trigeorgis	Wil1	Wil2	JKYABMD1	JKYRB2	JKYABMC2	JKYABMD2c	JKYABMD3
1	16.96	17.00	17.00	16.97	17.79	17.78	16.69	17.17	17.24	16.15	16.65
5	14.79	14.79	14.79	14.79	14.93	14.92	14.74	14.69	14.70	14.51	14.73
10	13.94	14.00	14.00	13.94	14.00	14.05	13.92	14.39	14.40	14.31	13.97
20	14.08	14.13	14.13	14.09	14.12	14.15	14.07	14.36	14.36	14.32	14.11
30	14.13	14.17	14.17	14.13	14.16	14.19	14.13	14.33	14.33	14.30	14.16
50	14.17	14.20	14.20	14.17	14.19	14.21	14.17	14.29	14.29	14.27	14.19
75	14.27	14.27	14.27	14.27	14.28	14.27	14.26	14.25	14.25	14.24	14.26
100	14.20	14.22	14.22	14.20	14.21	14.23	14.20	14.24	14.24	14.23	14.22

Table II. Convergence Time Step for Binomial Models by Moneyness

The table shows the average time step N at which convergence is achieved where the error is defined as $|(b(N) + b(N\text{-}1))/2 - \text{BSM}|$ where $b(N)$ is the value computed by the given binomial model for time step N, BSM is the correct value of the option as computed by the Black-Scholes-Merton model, and convergence is defined as an error of less than 0.01 for all remaining time steps through 100. The exercise price is 100, the risk-free rate 0.05, the volatilities are is 0.10, 0.30, and 0.50, and the times to expiration are 0.25, 1.0, and 4.0. Out-of-the-money options have a stock price 10% lower than the exercise price, and in-the-money options have a stock price 10% higher than the exercise price. These parameters combine to create nine options for each moneyness class. A maximum of 100 time steps is used. For models that did not converge by the 100[th] time step, a value of 100 is inserted.

Model	Moneyness (S/100)		
	Out-of-the-Money	At-the-Money	In-the-Money
CRR	66.86	40.25	54.29
RBJRT	55.57	54.30	55.02
Chriss	55.57	54.30	55.02
Trigeorgis	62.57	31.91	63.66
Wil1	66.93	48.08	69.08
Wil2	66.64	59.85	55.06
JKYABMD1	59.20	50.51	63.24
JKYRB2	61.67	60.39	57.34
JKYABMC2	61.92	60.47	58.58
JKYABMD2c	61.92	63.55	63.17
JKYABMD3	63.47	56.88	63.93

Table III. Convergence Time Step for Binomial Models by Time to Expiration

The table shows the average time step N at which convergence is achieved where the error is defined as $|(b(N) + b(N\text{-}1))/2 - \text{BSM}|$ where $b(N)$ is the value computed by the given binomial model for time step N, BSM is the correct value of the option as computed by the Black-Scholes-Merton model, and convergence is defined as an error of less than 0.01 for all remaining time steps through 100. The exercise price is 100, the risk-free rate 0.05, the volatilities are is 0.10, 0.30, and 0.50, and the moneyness is 10% out-of-the-money, at-the-money, and 10% in-the-money. The times to expiration are shown in the columns. These parameters combine to create nine options for each time to expiration. A maximum of 100 time steps is used. For models that did not converge by the 100th time step, a value of 100 is inserted.

Model	Time to Expiration (T)		
	0.25	1.00	4.00
CRR	33.33	54.33	87.89
RBJRT	43.33	67.78	79.00
Chriss	43.33	67.78	79.00
Trigeorgis	33.78	53.33	84.67
Wil1	38.33	72.33	93.33
Wil2	48.44	76.67	82.56
JKYABMD1	33.67	53.78	99.22
JKYRB2	51.44	74.44	81.44
JKYABMC2	53.22	74.78	81.44
JKYABMD2c	46.78	61.22	100.00
JKYABMD3	43.44	61.44	97.44

Table IV. Convergence Time Step for Binomial Models by Volatility

The table shows the average time step N at which convergence is achieved where the error is defined as $|(b(N) + b(N\text{-}1))/2 - \text{BSM}|$ where $b(N)$ is the value computed by the given binomial model for time step N, BSM is the correct value of the option as computed by the Black-Scholes-Merton model, and convergence is defined as an error of less than 0.01 for all remaining time steps through 100. The exercise price is 100, the risk-free rate 0.05, the times to expiration are 0.25, 1.00, and 4.00, and the moneyness is 10% out-of-the-money, at-the-money, and 10% in-the-money. The volatilities are shown in the columns. These parameters combine to create nine options for each time to expiration. A maximum of 100 time steps is used. For models that did not converge by the 100th time step, a value of 100 is inserted.

Model	Volatility (σ)		
	0.10	0.30	0.50
CRR	40.22	61.89	73.44
RBJRT	25.78	68.89	95.44
Chriss	25.78	68.89	95.44
Trigeorgis	38.00	62.33	71.44
Wil1	40.11	74.67	89.22
Wil2	29.78	80.11	97.78
JKYABMD1	42.78	63.22	80.67
JKYRB2	27.33	82.22	97.78
JKYABMC2	27.33	82.44	99.67
JKYABMD2c	45.33	67.78	94.78
JKYABMD3	48.33	60.22	93.78

of N, but they are very close and become essentially equal for fairly small values of N.

Table II illustrates that for at-the-money options, the Trigeorgis model performs best followed by CRR and Wil1. The worst model is JKYABMD2c, followed by JKYABMC2 and JKYRB2. For in-the-money options, the best model is CRR, followed by Chriss-RBJRT with Wil2 very close behind. The worst is Wil1, followed by JKYABMD3 and Trigeorgis. For out-of-the-money options, the best are RBJRT-Chriss, followed by JKYABMD1. The worst is Wil1, followed by CRR and Wil2.

Table III shows that convergence is always faster with a shorter time to expiration. This result should not be surprising. With a fixed number of time steps, a shorter time to expiration means that the time step is smaller. For the medium maturity, the fastest convergence is achieved by the Trigeorgis model, followed by JKYABMD1 and CRR. The worst performance is by Wil2, followed by JKYABMC2 and JKYRB2. For the shortest maturity, the best performance is by CRR, followed by JKYABMD1 and Trigeorgis; and the worst performance is by JKYABMC2, followed by JKYRB2 and Wil2. For the longest maturity, the best performance is by RBJRT-Chriss, followed by JKYRB2 and JKYABMC2 (tied). The worst performance is by JKYABMD2c, followed by JKYABMD1 and JKYABMD3.

Table IV shows that convergence is always slower with higher volatility. For the lowest volatility, the fastest models are RBJRT-Chriss (tied), followed by JKYRB2 and JKYABMC2 (tied). The slowest model is JKYABMD3, followed by JKYABMD2c and JKYABMD1. For medium volatility, the fastest model is JKYABMD3, followed by CRR and Trigeorgis; and the slowest is JKYABMC2, followed by JKYRB2 and Wil2. For the highest volatility, the fastest models are Trigeorgis, followed by CRR and JKYABMD1; while the slowest is JKYABMC2, followed by JKYRB2 and Wil2 (tied).

It is difficult to draw conclusions about which are the fastest and slowest models. Each model finishes in the top or bottom four at least once. Although the tests are not independent, we can gain some insight by assigning a simple ranking (1 = best, 11 = worst) and tally the performance across all nine groupings. CRR has the best performance with the lowest overall score of 36, while Trigeorgis is at 37, and RBJRT and Chriss are at 38. The highest scores and, thus, worst performance are JKYABMC2 at 71.5, followed by Wil2 at 69.5 and JKYABMD2c at 67.5. These rankings are useful and could suggest that CRR, Trigeorgis, RBJRT, and Chriss might be the best set of models, but they are not sufficient to declare a definitive winner.

Whether a model converges acceptably can be defined by whether the error is within a tolerance for a given time step. I calculate the error for the 100th time step. These results also reveal no consistent winner among the models. Most model values are within four cents of the true value on the 100th time step, and the differences are largest with long maturity and/or high volatility, consistent with my previous finding that short maturity and low volatility options are the fastest to price.

B. A More Formal Look at Convergence

One problem with analyzing the convergence properties of a model is that it is difficult to definitively identify when convergence occurs. Visual observation and rules about differences being less than a specified value are useful but arbitrary. It is possible, however, to use a more mathematically precise definition of convergence. Leisen and Reimer (1996) (LR) provide a detailed analysis of the convergence of the CRR and RBJRT models using the notion of *order of convergence*. A model converges more rapidly the higher the order of convergence. Therefore, determining the order of convergence of these models avoids the subjectivity of the previous analysis.

Convergence of a binomial model is defined to occur with order ρ if there exists a constant k such that

$$\left|e(N)\right| \leq \frac{k}{N^{\rho}}.$$

Visual examination of the errors on a log-log graph can reveal the order of convergence. LR further show, however, that a better measure of convergence can be derived using the difference between the moments of the binomial and continuous-time distributions. These moments are defined as follows:

$$\bar{m}^2(N) := \pi u^2 + (1-\pi)d^2 - e^{(2r+\sigma^2)^h}$$
$$\bar{m}^3(N) := \pi u^3 + (1-\pi)d^3 - e^{3(2r+\sigma^2)h}$$
$$\wp(N) := \pi \ln u (u-1)^3 + (1-\pi)\ln d (d-1)^3.$$

The moments $\bar{m}^2(N)$ and $\bar{m}^3(N)$ are obviously related to the second and third moments. The third term is referred to as a pseudo-moment. Let $\rho(.)$ represent the order of convergence of the above moments and the pseudo-moment. LR show that the order of convergence of the binomial series is

$$\max\left\{1, \min\left(\rho\left(\overline{m}^2(N)\right), \rho\left(\overline{m}^3(N)\right), \rho\left(\wp(N)\right)\right) - 1\right\}.$$

In other words, the order of convergence is the minimum of the orders of convergence of the two moments and the pseudo-moment minus one with an overall minimum order of convergence of one. They show that the order of convergence can be derived mathematically and they do so for the three models they examine. These proofs, however, are quite detailed and cumbersome and, as they note, visual inspection of these moments with a graph is equally effective.

I examine the order of convergence using the moments and pseudo-moments of each of the eleven models. Because of the excessive space required, I present the results only for the Chriss model. Figures 1, 2, and 3 illustrate various characteristics of the convergence of the Chriss model for the previously used inputs. Because the LR error analysis uses common logs, I show only the time steps starting with 10.

Figure 1 is the option price graphed against the number of time steps, with the BSM value represented by the horizontal line. The convergence is oscillatory, exhibiting the odd-even pattern previously noted.[21] Figure 2 shows the absolute value of the error, which exhibits a wavy pattern. The solid line was created by proposing values for k and ρ such that the error bound always lies above the absolute value of the error. The value of k is not particularly important, but the value of ρ indicates the order of convergence. Here, $\rho = 1$. A value of $\rho = 2$ would force the bound below the wavy error line. Thus, the order of convergence is clearly one. Figure 3 shows the moments and pseudo-moments as defined by Liesen and Reimer (1996). The pseudo-moments $\wp(N)$ and $\overline{m}^2(N)$ are almost indistinguishable. The heavy solid line is the simple function $1/N^\rho$ where ρ is the order of convergence of the moments and pseudo-moments. In this case, $\rho = 2$ provides the best fit. Therefore, following Theorem 1 of Leisen and Reimer (1996), the order of convergence is one, confirming my direct examination of the error.

These graphs were generated for all of the models and all have an order of convergence of one. In the limit all models produce the correct option value, but of course limit analysis make N essentially infinite. As shown earlier, seven of the eleven models admit arbitrage with finite N, but these opportunities vanish in the limit. Also, the values of π and π^* converge to ½ in the limit. These results suggest that a model that correctly prevents arbitrage for all N and sets the risk neutral probability π at ½ for any N might be superior. That model is the Chriss model. And yet, there is no evidence that the Chriss model consistently performs best for finite N.

C. Why the Models Converge

I have shown that all the models converge, but it is not clear why. As Hsia's (1983) proof shows, the requirement for convergence is not particularly demanding, but clearly one cannot arbitrarily choose formulas for u, d, and π.

As noted, it is possible to prove that all of the formulas for either π or π^*converge to ½ in the limit. I will examine why this result occurs. Focusing on π, I divide the models into four categories: 1) models that assume π = ½ (Chriss, Wil2, JKYABMD3); 2) models that assume $ud = e^{2rh}$ (JKYRB2, JKYABMC2, JKYABMD2c); 3) models that assume $ud = e^{2(r-\sigma^2/2)h}$ (RBJRT); and 4) models that assume $ud = 1$ (CRR, Trigeorgis, Wil1, JKYABMD1).[22] For 1), there is no need to examine the limiting condition. For 2), 3) and 4), general convergence is shown in the aforementioned supporting document.

Thus, all of the models either have π or π^* converge to ½. The other requirements are that the models return the correct mean and volatility in the limit. I will look at how they achieve this result. Re-classify the models according to their assumptions about the mean. Group (a) includes all models that correctly use the arbitrage-free specification of the raw mean, Equation (14) (CRR, RBJRT, Chriss, Wil1, Wil2, JKYABMC2). Group (b) includes all models that correctly use the raw mean specification but use $1+rh$ instead of e^{rh} (JKYABMD1, JKYABMD2c, and JKYABMD3). Group (c) includes the models that specify the log mean, Equation (16) (Trigeorgis and JKYRB2). Obviously Group (a) will correctly converge to the proper mean. Group (b) will do so as well, because e^{rh} is well approximated by $1 + rh$ in the limit. Group (c) uses the specification [Equation (16)], $\pi^*\ln u + (1-\pi^*)\ln d = (r - \sigma^2/2)h$. Using the approximation $\ln u \cong u - 1$ and likewise for d, we have

[21]As previously noted, the Chriss model does not exhibit this property for every case.

[22]It is important to understand why RBJRT is classified in this manner and not in any other group. RB and JR obtain their solutions by setting the physical probability q to ½. Their solution derives from using the mean of the log process, and thus, is not arbitrage-free. JT then impose the arbitrage-free condition and, hence, correctly use π for the risk neutral probability, but this constraint cannot lead to their formulas for u and d. Their formulas can be obtained only by imposing a third condition, which can be inferred to be the one stated here.

Figure 1. Convergence of the Chriss Model to the Black-Scholes-Merton Model

This figure shows the option price obtained by the Chriss model against the Black-Scholes-Merton model (indicated by the solid line) for time steps 10 to 100. The stock price is 100, the exercise price is 100, the risk-free rate is 0.05, the time to expiration is one year, and the volatility is 0.30.

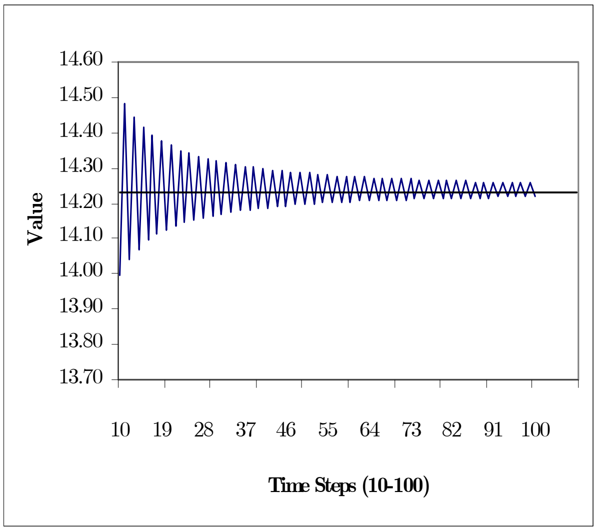


Figure 2. Absolute Value of the Convergence Error for the Chriss Model and its Order Bound Function

This figure shows the absolute value of the error for the option price obtained by the Chriss model against the Black-Scholes-Merton model for time steps 10 to 100. The stock price is 100, the exercise price is 100, the risk-free rate is 0.05, the time to expiration is one year, and the volatility is 0.30. Because the error bound is linear in logs, a log-log scale is used. The upper bound is the dark shaded line based on an order of convergence of one.

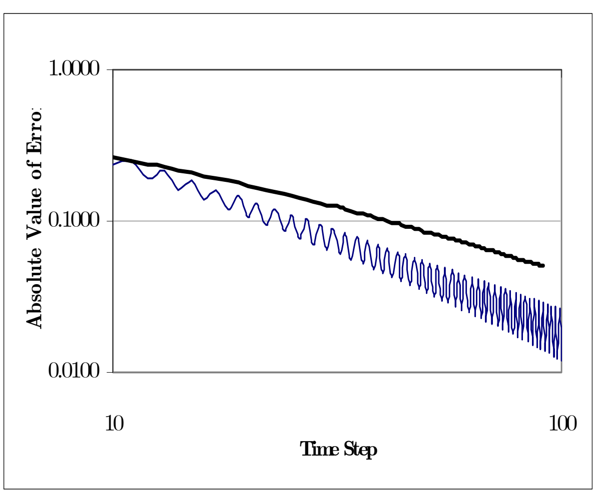
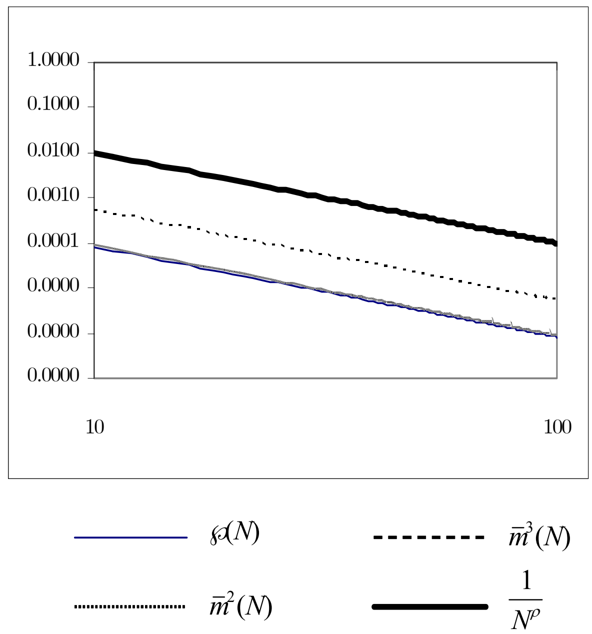


Figure 3. Absolute Value of the Moments and Pseudo-moments for the Chriss Model and its Order Bound Function

This figure shows the absolute value of the error for the second and third moments and the pseudo-moment as defined by Leisen and Reimer for the option price obtained by the Chriss model against the Black-Scholes-Merton model for time steps 10 to 100. The stock price is 100, the exercise price is 100, the risk-free rate is 0.05, the time to expiration is one year, and the volatility is 0.30. Because the error bound is linear in logs, a log-log scale is used. The upper bound is the dark shaded line based on an order of convergence of 2, which is consistent with order of convergence of the model of one.



$$\pi^*(u-1)+(1-\pi^*)(d-1)=(r-\sigma^2/2)h$$

$$\Rightarrow \pi^* u+1(1-\pi^*)d=1+rh-\sigma^2 h/2.$$

This specification is extremely close to that of Group (b), differing only by the variance term on the RHS, which goes to zero in the limit.

The next step is to consider the volatility. Let Group (a) consist of models that correctly specify the log volatility (CRR, RBJRT, Trigeorgis, JKYRB2, Chriss); (b) consist of models that correctly specify the raw volatility (Wil1, Wil2, and JKYABMC2); and (c) consist of models that use an approximation of the raw volatility, $\sigma^2 h \simeq e^{2rh}(e^{\sigma^2 h}-1)$ (JKYABMD1, JKYABMD2c, and JKYABMD3). Group (a) will obviously return the correct log volatility, and Group (b) will return the correct raw volatility. Either specification suffices because constraining the one volatility automatically constrains the other. Group (c) can be shown to be based on an acceptable approximation by using the Taylor series for the exponential function and assuming $h^k = 0$ for all k of power 2 or more.

Hence, all of the models work because in the limit they all have a binomial probability of ½, and they all return the risk-free rate as the mean and the correct volatility in the limit. Thus, any model with these characteristics will work. As shown in the next section, however, the constraints are not nearly that severe.

IV. A General Binomial Formula

As previously noted, Hsia's (1983) proof of the convergence of the binomial model to the Black-Scholes-Merton model shows that any probability is acceptable provided that u and d return the correct mean and volatility. This result suggests that any value of the risk neutral probability would lead to convergence if the correct mean and volatility are upheld. We now propose a general binomial model with arbitrary π that prohibits arbitrage and recovers the correct volatility for all N. Let the mean and variance be specified as follows:

Of course, these are Equations (14) and (18). The mean equation guarantees no arbitrage profits for all N. Now assume that π is known but its value is left unspecified. Solving for u and d gives

$$u = \frac{e^{rh+\sigma\sqrt{h}/\sqrt{\pi(1-\pi)}}}{\pi e^{\sigma\sqrt{h}/\sqrt{\pi(1-\pi)}} + (1-\pi)},$$

$$d = \frac{e^{rh}}{\pi e^{\sigma\sqrt{h}/\sqrt{\pi(1-\pi)}} + (1-\pi)},$$

and, of course,

$$\pi = \frac{e^{rh} - d}{u - d}.$$

For the special case where $\pi = ½$, the equations are equivalent to those of Chriss.

These equations tell us that we can arbitrarily set π to *any* value between 0 and 1 and be assured that the model will converge to the BSM value. This result is observed in Figure 4. Note that while convergence appears much smoother and faster with $\pi = ½$, the results are not much different for probabilities of ¼ and ¾. For $N = 100$, a probability of ¼ gives an option value of 14.27, while a probability of ¾ gives an option value of 14.15. The correct BSM value is 14.23.[23]

While yet one more binomial formula is not necessary, this model shows that binomial option pricing is a remarkably flexible procedure that makes only minimum demands on its user and the choice of probability is not one of them.

V. Conclusion

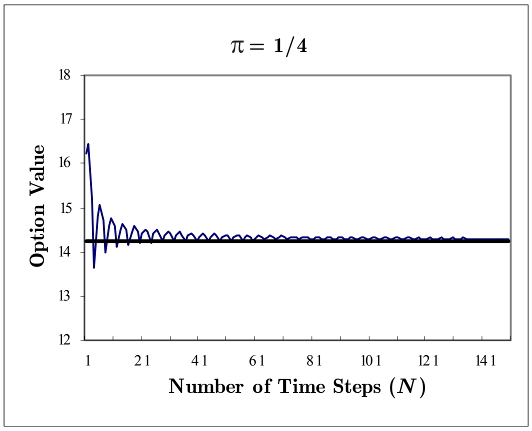
This paper synthesizes the research on binomial models for European options. It shows that the model is not a single model but a family of interpretations of a discrete-time process that converges to the continuous Brownian motion process in the limit and accurately prices options. That there are no less than 11 such members of this family may seem surprising. The fact that they all perform equally in the limit, even though some admit arbitrage for a finite number of time steps, is a testament to the extremely general nature of the Black-Scholes-Merton model and its modest requirements. It would seem that a preferable binomial model should prohibit arbitrage for a finite number of time steps and recover the correct volatility, and some models fail to meet these requirements. But perhaps most interesting of the results shown here is that given Hsia's elegant proof, the choice of the actual risk neutral probability is meaningless in the limit, though clearly a risk neutral probability of ½ assures the fastest convergence.■

[23] A general formula of this type even means that extreme probabilities, say 0.01 and 0.99, would also correctly price the option in the limit. I tested these extreme values, however, and the results are not impressive. For example, with a probability of 0.01 I obtain an option value of 13.93, while a probability of 0.99 gives an option value of 13.01 after 100 time steps. Convergence is extremely erratic and the order of convergence is difficult to determine. Nonetheless, in the limit, the correct option value is obtained.

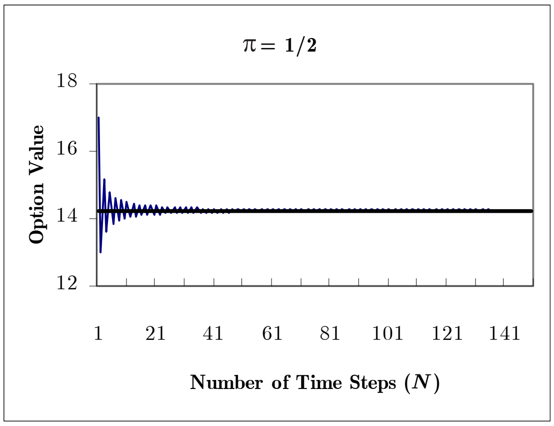
Figure 4. Convergence of a General Binomial Model that Prohibits Arbitrage and Allows any Probability between Zero and One

These figures show the value of the option computed from a general binomial model that assures the absence of arbitrage, recovery of the correct log volatility, and in which the probability can be arbitrarily chosen as indicated. The stock price is 100, the exercise price is 100, the risk-free rate is 0.05, the volatility is 0.30, and the option expires in one year. The horizontal line is the BSM value of 14.23.

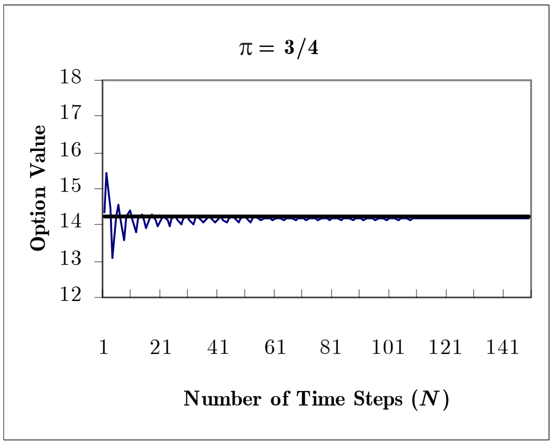
(a)



(b)



(c)



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