This Assignment is compulsory, and contributes 7.5% towards your final grade. It should be submitted by 2pm on Thursday 23 May 2024. In the absence of a medical certificate or other valid documented excuse, assignments submitted after the due date will attract a penalty as outlined in the course profile. Prepare your assignment as a **pdf file**, either by typing it, writing on a tablet or by scanning/photographing your handwritten work. Ensure that your name, student number and tutorial group number appear on the first page of your submission. Check that your pdf file is legible and that the file size is not excessive. Files that are poorly scanned and/or illegible may not be marked. Upload your submission using the Gradescope submission link in Blackboard. Remember that you must allocate pages of your assignment to each question after you have uploaded the file!

1. **(7 marks)** Recall that S_4 is the group consisting of the set of all bijections from $\{1,2,3,4\}$ to $\{1,2,3,4\}$ together with the binary operation $\circ$ denoting composition of functions. You may use arrow diagrams to represent bijections of S_4 and perform compositions of bijections.

- **(a)** Is S_4 abelian? Justify your answer.
- **(b)** Determine the cyclic subgroup $\langle f \rangle$ of S_4, where $f \in S_4$ is the bijection defined by

$$f(1) = 3, \quad f(2) = 4, \quad f(3) = 1, \quad f(4) = 2.$$

- **(c)** Determine a cyclic subgroup of S_4 that is isomorphic to $(\mathbb{Z}_4, +)$, and give an isomorphism between these two groups.

2. **(7 marks)** Define addition $+$ and multiplication $\cdot$ on $\mathbb{R} \times \mathbb{R}$ by

$$(a, b) + (c, d) = (a + c, b + d), \quad (a, b) \cdot (c, d) = (ac - bd, ad + bc).$$

for all $(a, b), (c, d) \in \mathbb{R} \times \mathbb{R}$. You can assume that $(\mathbb{R} \times \mathbb{R}, +)$ forms an abelian group with the identity element $(0, 0)$. Additionally, you can assume that the set $((\mathbb{R} \times \mathbb{R}) - \{(0, 0)\}, \cdot)$ is closed under the multiplication $\cdot$, and this multiplication is associative.

- **(a)** Prove that $(\mathbb{R} \times \mathbb{R}, +, \cdot)$ is a field.
- **(b)** Solve the equation

$$(1, 2) \cdot x + (2, 4) = (1, 1)$$

  in the field $(\mathbb{R} \times \mathbb{R}, +, \cdot)$.

3. (**6 marks**) You have a bag containing 20 distinct small balls. In this question your answers may be expressed in terms of binomial coefficients or factorials.

- **(a)** In how many different ways can you choose a set of 5 balls from the bag? No justification needed.
- **(b)** Suppose that of the 20 balls in the bag, 8 balls are red and 12 balls are blue. A set of 5 balls is selected at random from the bag. What is the probability that the set of 5 balls contains at most 2 red balls? Justify your answer.
- **(c)** There are 4 players in a game, and you need to distribute all balls from the bag into 4 sets of 5 balls each, such that each player receives exactly one set of 5 balls. The order in which the specific sets are assigned to the players matters, while the order of balls within each set does not matter. Given these conditions, in how many different ways can you distribute these balls to the 4 players? Justify your answer.

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4. (5 marks) Three pairs of best friends have made reservations to dine together at a long table. In how many different ways can these six people be seated in a row, such that at least one pair of best friends is seated side by side? Justify your answer. (Hint: Consider using the inclusion/exclusion principle.)