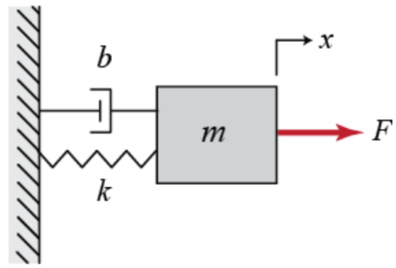
Exercises Sheet

1. Calculate the transfer Function $G(s) = \frac{X(s)}{F(s)}$



- We depend on Force balance equations

$$\sum F = 0$$

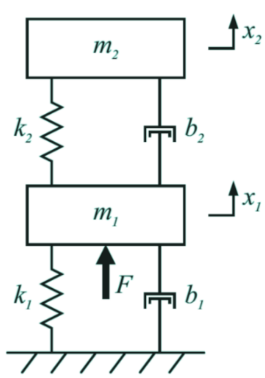
$$m\ddot{x} + b\dot{x} + kx = f(t)$$

- Convert to Laplace transform

$$[ms^2 + bs + k]X(s) = F(s)$$

$$\frac{X(s)}{F(s)} = \frac{1}{ms^2 + bs + k}$$

2. Calculate the transfer Function $H_1(s) = \frac{X_1(s)}{F(s)}$ and $H_2(s) = \frac{X_2(s)}{F(s)}$

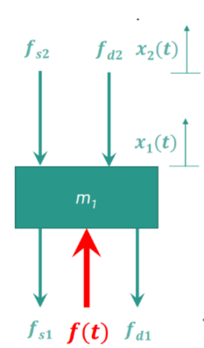


- Force balance equations

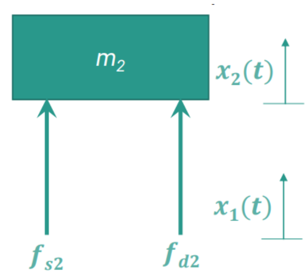
$$\sum F = 0$$

$$m_1\ddot{x}_1 + b_1\dot{x}_1 + k_1x_1 + b_2(\dot{x}_1 - \dot{x}_2) + k_2(x_1 - x_2) = f(t)$$

$$m_1\ddot{x}_1 + (b_1 + b_2)\dot{x}_1 + (k_1 + k_2)x_1 = b_2\dot{x}_2 + k_2x_2 + f(t) \rightarrow \{1\}$$



- Force balance equations at m_2:



$$\sum F = 0$$
$$m_2\ddot{x}_2 + b_2(\dot{x}_2 - \dot{x}_1) + k_2(x_2 - x_1) = 0$$
$$m_2\ddot{x}_2 + b_2\dot{x}_2 + k_2x_2 = b_2\dot{x}_1 + k_2x_1 \rightarrow \{2\}$$

- Laplace Transform

$$m_1\ddot{x}_1 + (b_1 + b_2)\dot{x}_1 + (k_1 + k_2)x_1 = b_2\dot{x}_2 + k_2x_2 + f(t) \rightarrow \{1\}$$

$$m_1s^2X_1(s) + (b_1 + b_2)sX_1(s) + (k_1 + k_2)X(s) = b_2sX_2(s) + k_2X_2(s) + F(s)$$

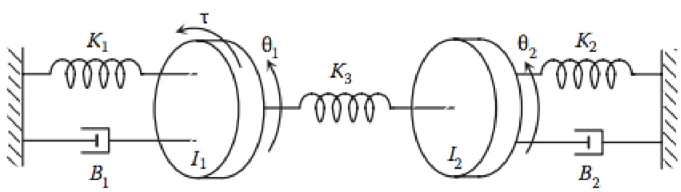
$$\boldsymbol{X_1(s)}[m_1s^2 + (b_1 + b_2)s + (k_1 + k_2)] = \boldsymbol{X_2(s)}[b_2s + k_2] + \boldsymbol{F(s)}$$

$$m_2\ddot{x}_2 + b_2\dot{x}_2 + k_2x_2 = b_2\dot{x}_1 + k_2x_1 \rightarrow \{2\}$$
$$m_2s^2X_2(s) + b_2sX_2(s) + k_2X_2(s) = b_2sX_1(s) + k_2X_1(s)$$
$$X_2(s)[m_2s^2 + b_2s + k_2] = X_1(s)[b_2s + k_2]$$

- Transfer Function

$$\frac{X_1(s)}{F(s)} = \frac{m_2s^2 + b_2s + k_2}{m_1m_2s^4 + (m_2(b_1 + b_2) + m_1b_2)s^3 + (m_1k_2 + m_2(k_1 + k_2) + b_1b_2)s^2 + (b_1k_2 + b_2k_1)s + k_1k_2}$$
$$\frac{X_2(s)}{F(s)} = \frac{b_2s + k_2}{m_1m_2s^4 + (m_2(b_1 + b_2) + m_1b_2)s^3 + (m_1k_2 + m_2(k_1 + k_2) + b_1b_2)s^2 + (b_1k_2 + b_2k_1)s + k_1k_2}$$

3. Calculate Transfer function $H_1(s) = \dfrac{\theta_1(s)}{\tau(s)}$ and $H_2(s) = \dfrac{\theta_2(s)}{\tau(s)}$



- Torque balance equation for first inertia

$$\sum \tau = 0$$
$$j_1\ddot{\theta}_1 + b_1\dot{\theta}_1 + k_1\theta_1 + k_3(\theta_1 - \theta_2) = \tau(t) \rightarrow \{1\}$$
$$j_1\ddot{\theta}_1 + b_1\dot{\theta}_1 + (k_1 + k_3)\theta_1 = k_3\theta_2 + \tau(t)$$

- Laplace Transform

$$\boldsymbol{\theta_1(s)}[j_1s^2 + b_1s + (k_1 + k_3)] = \boldsymbol{\theta_2(s)}k_3 + \boldsymbol{\tau(s)}$$

- Torque balance equation for second inertia

  $\sum \tau = 0$

  $j_2\ddot{\theta}_2 + b_2\dot{\theta}_2 + k_2\theta_2 + k_3(\theta_2 - \theta_1) = 0 \rightarrow \{2\}$

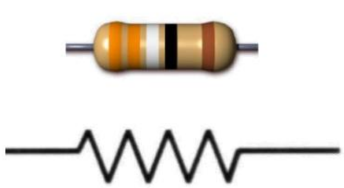
  $j_2\ddot{\theta}_2 + b_2\dot{\theta}_2 + (k_2 + k_3)\theta_2 = k_3\theta_1$
- Laplace transform

  $\boldsymbol{\theta_2(s)}[j_2s^2 + b_2s + (k_2 + k_3)] = \boldsymbol{\theta_1}(s)k_3$
- Transfer function $H_1(s) = \dfrac{\theta_1(s)}{\tau(s)}$

$$H_1(s) = \frac{\theta_1(s)}{\theta_2(s)} \cdot \frac{\theta_2(s)}{\tau(s)} = \frac{j_2s^2 + b_2s + (k_2 + k_3)}{(j_1j_2s^4 + (j_1b_2 + j_2b_1)s^3 + [j_1(k_2 + k_3) + b_1b_2 + j_2(k_1 + k_3)]s^2 + [b_1(k_2 + k_3) + b_2(k_1 + k_3)]s + (k_1k_2 + k_1k_3 + k_2k_3))}$$

- Find Transfer function $H_2(s) = \dfrac{\theta_2(s)}{\tau(s)}$ by yourself

Resistance



V-I in time domain

$\nu_R(t) = i_R(t)R$

V-I in s domain

$V_R(s) = I_R(s)R$

Inductance

V-I in time domain

$\nu_L(t) = L\dfrac{di_L(t)}{dt}$

V-I in s domain

$V_L(s) = sLI_L(s)$

Capacitance

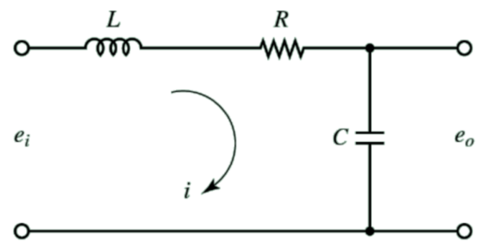
V-I in time domain

$\nu_c(t) = \dfrac{1}{C}\displaystyle\int i_c(t)dt$

V-I in s domain

$V_c(s) = \dfrac{1}{Cs}I_c(s)$

4. Calculate Transfer Function $G(s) = \frac{E_o(s)}{E_i(s)}$



- Kirchhoff's Voltage Law KVL

$$\sum V = 0$$

$$v_L(t) + v_R(t) + v_c(t) = e_i(t)$$

$$L\frac{di}{dt} + Ri(t) + \frac{1}{c}\int i\,dt = e_i(t) \rightarrow \{1\}$$

$$LsI(s) + RI(s) + \frac{1}{Cs}I(s) = E_i(s)$$

$$I(s)\left[Ls + R + \frac{1}{Cs}\right] = E_i(s)$$

$$\frac{I(s)}{E_i(s)} = Ls + R + \frac{1}{Cs}$$

- KVL

$$\sum V = 0$$

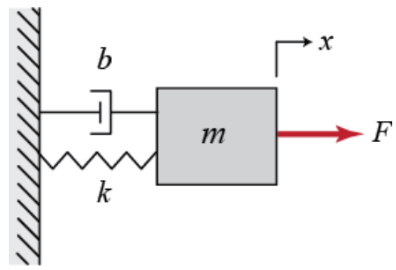
$$e_o(t) = v_c(t) = \frac{1}{C}\int i\,dt \rightarrow \{2\}$$

$$E_o(s) = \frac{1}{Cs}I(s)$$

$$\frac{E_o(s)}{E_i(s)} = \frac{1}{Cs}$$

$$\frac{E_o(s)}{E_i(s)} = \frac{E_o(s)}{I(s)} \cdot \frac{I(s)}{E_i(s)} = \frac{\frac{1}{Cs}}{Ls + R + \frac{1}{Cs}} = \frac{1}{LCs^2 + RCs + 1}$$

5. State-Space modelling
    - It's different for modelling-like transfer function and it's designed to simplify the systems with multiple inputs and multiple outputs and find the relations between them
    - State-space model is widely used at advanced control
    - In our course we will use it for system identification and parameter estimation.
6. Find the state-space model for the following system



- Force balance equation

$$\sum F = 0$$

$$m\ddot{x} + b\dot{x} + kx = f(t)$$

- Decomposed to first order states

$$x_1 = x$$
$$x_2 = \dot{x} = \dot{x}_1$$
$$x_3 = \ddot{x} = \dot{x}_2$$

- Separate the highest order on LHS

$$\ddot{x} = \frac{1}{m}[f(t) - b\dot{x} - kx]$$

- Replace state variable with state vector

$$\dot{x}_1 = (0)x_1 + (1)x_2 + (0)f(t)$$
$$\dot{x}_2 = -\frac{k}{m}x_1 - \frac{b}{m}x_2 + \frac{1}{m}f(t)$$

- Formulate state-space function

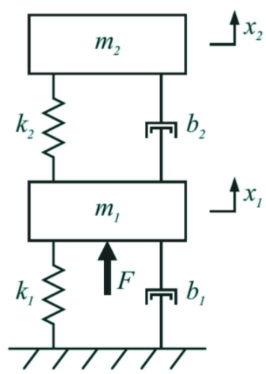
$$\dot{x} = Ax + Bu$$

$$\begin{bmatrix} \dot{x}_1 \\ \dot{x}_2 \end{bmatrix} = \begin{bmatrix} 0 & 1 \\ -\dfrac{k}{m} & -\dfrac{b}{m} \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \end{bmatrix} + \begin{bmatrix} 0 \\ \dfrac{1}{m} \end{bmatrix} f(t)$$

$$y = Cx + Du$$

$$y = \begin{bmatrix} 1 & 0 \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \end{bmatrix}$$

7. Find the state-space model for the following system



- Force balance equation

$$\sum F = 0$$

$$m_1\ddot{x}_1 + (b_1 + b_2)\dot{x}_1 + (k_1 + k_2)x_1 = b_2\dot{x}_2 + k_2x_2 + f(t) \rightarrow \{1\}$$

$$m_2\ddot{x}_2 + b_2\dot{x}_2 + k_2x_2 = b_2\dot{x}_1 + k_2x_1 \rightarrow \{2\}$$

- Decomposed to first order states

$$z_1 = x_1$$
$$z_2 = \dot{x}_1 = \dot{z}_1$$
$$z_3 = \ddot{x}_1 = \dot{z}_2$$
$$z_4 = x_2$$
$$z_5 = \dot{x}_2 = \dot{z}_4$$
$$z_6 = \ddot{x}_2 = \dot{z}_5$$

- Replace state variable with state vector

$$\dot{z}_1 = (0)z_1 + (1)z_2 + (0)z_4 + (0)z_5 + (0)f(t)$$

$$\dot{z}_2 = -\frac{(k_1+k_2)}{m_1}z_1 - \frac{(b_1+b_2)}{m_1}z_2 + \frac{k_2}{m_1}z_4 + \frac{b_2}{m_1}z_5 + \frac{1}{m_1}f(t) \rightarrow \{1\}$$

$$\dot{z}_4 = (0)z_1 + (0)z_2 + (0)z_4 + (1)z_5 + (0)f(t)$$

$$\dot{z}_5 = \frac{k_2}{m_2}z_1 + \frac{b_2}{m_2}z_2 - \frac{k_2}{m_2}z_4 - \frac{b_2}{m_2}z_5 + (0)f(t\,)$$

- Formulate state-space function

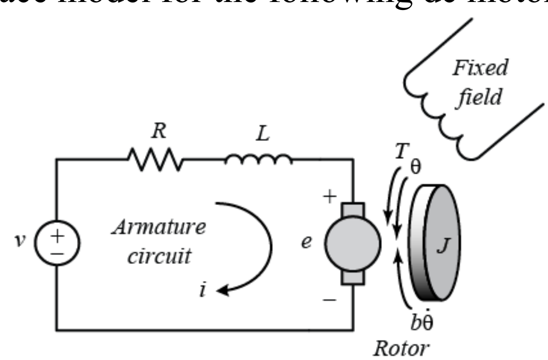
$$\dot{x} = Ax + Bu$$

$$\begin{bmatrix}\dot{z}_1\\ \dot{z}_2\\ \dot{z}_4\\ \dot{z}_5\end{bmatrix} = \begin{bmatrix} 0 & 1 & 0 & 0\\ -\frac{k_1+k_2}{m_1} & -\frac{b_1+b_2}{m_1} & \frac{k_2}{m_1} & \frac{b_2}{m_1}\\ 0 & 0 & 0 & 1\\ \frac{k_2}{m_2} & \frac{b_2}{m_2} & -\frac{k_2}{m_2} & -\frac{b_2}{m_2}\end{bmatrix}\begin{bmatrix}z_1\\ z_2\\ z_4\\ z_5\end{bmatrix} + \begin{bmatrix}0\\1\\0\\0\end{bmatrix}f(t)$$

$$y = Cx + Du$$

$$y = \begin{bmatrix}1 & 0 & 1 & 0\end{bmatrix}\begin{bmatrix}z_1\\ z_2\\ z_3\\ z_4\end{bmatrix}$$

8. Find the state-space model for the following dc motor design



- Torque balance equation for armature

$$\sum T = 0$$

$$T = K_t i \;\rightarrow \{1\}$$

Where:

- T: Torque of Rotor
- K_t: Motor torque constant
- i: Armature Current

- Kirchhoff's Voltage Law KVL

$$\sum V = 0$$

$$e = K_e\dot{\theta}$$

Where:

- e: Voltage source
- K_e: Back EMF constant
- $\dot{\theta}$: Angular Velocity

Note: for dc motor $\boldsymbol{K_t} = \boldsymbol{K_e} = \boldsymbol{K}$

- Derive the following governing equations based on Newton's 2nd law and Kirchhoff's voltage law

$$J\ddot{\theta} + b\dot{\theta} = Ki \ \rightarrow \{1\}$$

$$L\frac{di}{dt} + Ri = V - e$$

$$L\frac{di}{dt} + Ri = V \ - K\dot{\theta} \ \rightarrow \{3\}$$

- Decomposed to first order states

$$x_1 = \theta$$

$$x_2 = \dot{\theta} = \dot{x}_1$$

$$x_3 = \ddot{\theta} = \dot{x}_2$$

$$x_4 = i$$

$$x_5 = i = \dot{x}_4$$

- Replace state variable with state vector

$$\dot{x}_1 = (0)x_1 + (1)x_2 + (0)x_3 + (0)x_4 + (0)x_5 + (0)V(t)$$

$$\dot{x}_2 = (0)x_1 - \left(\frac{b}{J}\right)^{\cdot} x_2 + (0)x_3 + \left(\frac{K}{J}\right)x_4 + (0)x_5 + (0)V(t)$$

$$\dot{x}_4 = (0)x_1 - \left(\frac{K}{L}\right)x_2 + (0)x_3 - \left(\frac{R}{L}\right)x_4 + (0)x_5 + \left(\frac{1}{L}\right)V(t)$$

- Formulate state-space function

$$\dot{x} = Ax + Bu$$

$$\begin{bmatrix} \dot{x}_2 \\ \dot{x}_4 \end{bmatrix} = \begin{bmatrix} -\left(\frac{b}{J}\right) & \left(\frac{K}{J}\right) \\ -\left(\frac{K}{L}\right) & -\left(\frac{R}{L}\right) \end{bmatrix} \begin{bmatrix} x_2 \\ x_4 \end{bmatrix} + \begin{bmatrix} 0 \\ \frac{1}{L} \end{bmatrix} V(t)$$

$$\begin{bmatrix} \ddot{\theta} \\ i \end{bmatrix} = \begin{bmatrix} -\left(\frac{b}{J}\right) & \left(\frac{K}{J}\right) \\ -\left(\frac{K}{L}\right) & -\left(\frac{R}{L}\right) \end{bmatrix} \begin{bmatrix} \dot{\theta} \\ i \end{bmatrix} + \begin{bmatrix} 0 \\ \frac{1}{L} \end{bmatrix} V(t)$$

$$y = Cx$$

$$y = \begin{bmatrix} 1 & 0 \end{bmatrix} \begin{bmatrix} \dot{\theta} \\ i \end{bmatrix}$$