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MATH 554: Section H01
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December 5, 2023

MATH 554 Homework 17

Problem 1 Let $f : A \to B$ be a bijection between sets. Prove that f has an inverse.

Let $f^{-1} \subseteq B \times A$ be a relation defined by

$$f^{-1} = \{(b, a) \in B \times A : f(a) = b\}.$$

We first claim that this is a function from B to A, which is true if for all $b \in B$, there exists precisely one $a \in A$ such that $(b, a) \in f^{-1}$. Let $b \in B$. Since f is surjective, there exists $a \in A$ with $f(a) = b$. Then by definition $(b, a) \in f^{-1}$, so existence holds. To show uniqueness, suppose there exist $a_1, a_2 \in A$ with $(b, a_1), (b, a_2) \in f^{-1}$. Then we have $f(a_1) = b$ and $f(a_2) = b$, so since f is injective and $f(a_1) = b = f(a_2)$, we have $a_1 = a_2$ as desired. So f^{-1} is a well-defined function.

We next check that f^{-1} has the property that for all $a \in A$ we have $f^{-1}(f(a)) = a$ and for all $b \in B$ we have $f(f^{-1}(b)) = b$. First, let $a \in A$. Then, $(f(a), a) \in f^{-1}$ because $f(a) = f(a)$, so $f^{-1}(f(a)) = a$. Next, let $b \in B$. Then, $(b, f^{-1}(b)) \in f^{-1}$ by definition of being a function, so by our definition of f^{-1} we have $f(f^{-1}(b)) = b$. Therefore, f^{-1} is a well-defined inverse. $\square$

Problem 2 Let $f : A \to B$ be a bijection.

(a) Show that the inverse $f^{-1} : B \to A$ is also a bijection.

(b) Show that $(f^{-1})^{-1} = f$.

(a) To see that f^{-1} is injective, let $b_1, b_2 \in B$ such that $f^{-1}(b_1) = f^{-1}(b_2)$. Then we have

$$b_1 = f(f^{-1}(b_1)) = f(f^{-1}(b_2)) = b_2.$$

To see that f^{-1} is surjective, let $a \in A$. Then $f(a) \in B$ and $f^{-1}(f(a)) = a$, so a is in the range of f^{-1}. Therefore, f^{-1} is bijective because it is both injective and surjective.

(b) Two functions $\phi : D_\phi \to C_\phi$, $\psi : D_\psi \to C_\psi$ are equal if $D_\phi = D_\psi$, $C_\phi = C_\psi$, and for all $x \in D_\phi = D_\psi$, we have $\phi(x) = \psi(x)$. Since the codomain of f^{-1} is A, the domain of $(f^{-1})^{-1}$ is A, and since the domain of f^{-1} is B, the codomain of $(f^{-1})^{-1}$ is B. Thus f and $(f^{-1})^{-1}$ have the same domain and codomain.

Next, let $a \in A$. Since $(f^{-1})^{-1}$ is an inverse of f^{-1}, we have that for all $a \in A$,

$$f^{-1}\left((f^{-1})^{-1}(a)\right) = a.$$

So we have

$$f\left(f^{-1}\left((f^{-1})^{-1}(a)\right)\right) = f(a),$$

and thus $(f^{-1})^{-1}(a) = f(a)$. Therefore, we have $(f^{-1})^{-1}(a) = f(a)$. $\square$

Problem 3 Let $(E,d),(E',d')$ be metric spaces with E compact, and let $f : E \to E'$ be a continuous bijection between metric spaces. Prove that $f^{-1} : E' \to E$ is continuous.

To show that f^{-1} is closed, it suffices to show that the pre-images of closed sets in E are closed. Let $K \subseteq E$ be closed. Since E is compact, K is compact. Then, the preimage $(f^{-1})^{-1}[K] = f[K]$ from Problem 2, and since the continuous image of a compact set is compact, $f[K]$ is compact. This implies that preimage of K under f^{-1} is closed, and therefore f^{-1} is continuous. $\square$

Problem 4 Let $f : [a,b] \to [\alpha,\beta]$ be an increasing continuous function with $f(a) = \alpha$ and $f(b) = \beta$. Prove that f is bijective and that the inverse f^{-1} is continuous.

First, since f is increasing, we have that $x_1 < x_2$ implies $f(x_1) < f(x_2)$. Let $x_1, x_2 \in [a,b]$ such that $x_1 \neq x_2$. Then either $x_1 < x_2$ in which case $f(x_1) < f(x_2)$, or $x_2 < x_1$ in which $f(x_2) < f(x_1)$. Either we do not have $f(x_1) = f(x_2)$, and thus f is injective.

Next, let $y \in [\alpha,\beta]$. If $y = \alpha$ then $f(a) = y$ and if $y = \beta$ then $f(b) = y$, but otherwise we have $\alpha < y < \beta$. Since f is continous, we have $f(x)-y$ is continuous with $f(a)-y < 0$ and $f(b)-y > 0$. So by the intermediate value theorem, there exists $\xi \in [a,b]$ with $f(\xi) - y = 0$ and thus $f(\xi) = y$. Thus, f is surjective. Therefore f is bijective. Since $[a,b]$ is compact, then, from problem 3 we have that f^{-1} is continuous. $\square$

Problem 5 Let K be a closed bounded subset of $\mathbb{R}^2$. Show that there exist $\mathbf{x}_*, \mathbf{y}_*, \mathbf{z}_* \in K$ so that the triangle $\triangle\mathbf{x}_*\mathbf{y}_*\mathbf{z}_*$ has maximum area of all triangles with vertices in K.

Since the continuous image of a compact set is compact, and we have $A(\mathbf{x}_*, \mathbf{y}_*, \mathbf{z}_*)$ is continuous and $K \times K \times K$ is compact, the image of $A(\mathbf{x}_*, \mathbf{y}_*, \mathbf{z}_*)$ is compact in $\mathbb{R}$. Thus, it is closed and bounded, and so the image contains its supremum A_*. Therefore, there exist $\mathbf{x}_*, \mathbf{y}_*, \mathbf{z}_* \in K$ with $A(\mathbf{x}_*, \mathbf{y}_*, \mathbf{z}_*) = A_*$. $\square$