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MATH 576: Section H01
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MATH 576 Homework 10

Problem 1 Let $G = \{10 \mid \{10 \mid -7\}\}$. Compute $m(G)$ and $t(G)$.

We can write

$$G = \{10 \mid \{10 \mid -7\}\} = \{10 \mid 10 + \{0 \mid -17\}\} = 10 + \{0 \mid \{0 \mid -17\}\} = 10 + +_{17}.$$

Then, we have

$$m(G) = m(10 + +_{17}) = m(10) + m(+_{17}) = 10 + 0 = 10,$$

and since $t(10) = -1 \neq 0 = t(+_{17})$ we can write

$$t(G) = \max\{t(10), t(+_{17})\} = \max\{-1, 0\} = 0.$$

Problem 2 Find the mean value and the temperature of the game

$$n + a\cdot\uparrow + *m \pm x_1 \ldots \pm x_r,$$

where n is a number, a is an integer, m is a nonnegative integer, and $x_1 > \ldots > x_r > 0$ are numbers.

We have

$$\begin{aligned} m(n + a\cdot\uparrow + *m \pm x_1 \ldots \pm x_r) &= m(n) + m(a\cdot\uparrow) + m(*m) + m(\pm x_1) + \ldots + m(\pm x_r) \\ &= n + 0 + 0 + 0 + \ldots + 0 \\ &= n. \end{aligned}$$

Also, we have $t(a\cdot\uparrow + *m) = 0$ since $a\cdot\uparrow + *m$ is an infinitesimal, so we have

$$\begin{aligned} t(n + a\cdot\uparrow + *m \pm x_1 \ldots \pm x_r) &= \max\{t(n), t(a\cdot\uparrow + *m), t(\pm x_1), \ldots, t(\pm x_r)\} \\ &= \max\{-1, 0, x_1, \ldots, x_r\} \\ &= x_1. \end{aligned}$$

Problem 3 Let x and y be two numbers with $x > y \geq 0$. Find examples of games G and H such that $t(G) = t(H) = x$ and $t(G + H) = y$.

Consider $G = \pm x$ and $H = \pm x \pm y$. Then $t(G) = x$, and since $t(\pm x) = x \neq y = t(\pm y)$ we have

$$t(H) = t(\pm x \pm y) = \max\{t(\pm x), t(\pm y)\} = \max\{x, y\} = x.$$

Also, we have

$$t(G + H) = t(\pm x \pm y \pm x) = t(\pm y) = y.$$

$\square$

Problem 4 Let G_n be the position in Grundy's game with n tokens in a single heap (recall that in Grundy's game, the only legal move is to split the tokens in a heap into two nonempty heaps of unequal size). Prove that in misère play, G_3, G_6 and G_9 are all $\mathcal{P}$-positions. (Hint: Proceed iteratively as you would in normal play, using the misère mex and misère nim addition rules.)

We can use the mex rule to write:

- $G_1 = G_2 = \{\} = *0$
- $G_3 = \{G_1 + G_2\} = \{*0 + *0\} = \{*0\} = *$
- $G_4 = \{G_1 + G_3\} = \{*0 + *\} = \{*\} = *0$
- $G_5 = \{G_1 + G_4, G_2 + G_3\} = \{*0 + *0, *0 + *\} = \{*0, *\} = *2$
- $G_6 = \{G_1 + G_5, G_2 + G_4\} = \{*0 + *2, *0 + *0\} = \{*0, *2\} = *$
- $G_7 = \{G_1 + G_6, G_2 + G_5, G_3 + G_4\} = \{*0 + *, *0 + *2, * + *0\} = \{*, *2, *\} = *0$
- $G_8 = \{G_1 + G_7, G_2 + G_6, G_3 + G_5\} = \{*0 + *0, *0 + *, * + *2\} = \{*0, *, *3\} = *2$
- $G_9 = \{G_1 + G_8, G_2 + G_7, G_3 + G_6, G_4 + G_5\} = \{*0 + *2, *0 + *0, * + *, *0 + *2\} = \{*2, *0, *0, *2\} = *$.

Since $* = \{*0\}$ has all its options in $\mathcal{N}$, $* \in \mathcal{P}$. Therefore, we have $G_3 = G_6 = G_9 = * \in \mathcal{P}$. □

Problem 5 Let S_n be the position in Subtraction-$\{1, 4, 9\}$ with n tokens in a single heap. Compute the values in misère play of S_i for $0 \leq i \leq 10$.

We can use the mex rule to write:

- $S_0 = \{\} = *0$
- $S_1 = \{S_0\} = \{*0\} = *$
- $S_2 = \{S_1\} = \{*\} = *0$
- $S_3 = \{S_2\} = \{*0\} = *$
- $S_4 = \{S_0, S_3\} = \{*0, *\} = *2$
- $S_5 = \{S_1, S_4\} = \{*, *2\} = *0$
- $S_6 = \{S_2, S_5\} = \{*0, *0\} = \{*0\} = *$
- $S_7 = \{S_3, S_6\} = \{*, *0\} = *0$
- $S_8 = \{S_4, S_7\} = \{*2, *0\} = *$
- $S_9 = \{S_0, S_5, S_8\} = \{*0, *0, *\} = \{*0, *\} = *2$
- $S_{10} = \{S_1, S_6, S_9\} = \{*, *, *2\} = \{*, *2\} = *0$.