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MATH 555: Section H01
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MATH 555 Homework 8

Problem 1 For the following series $\sum_{n=1}^{\infty} a_n$ say if they converge or diverge and why.

(a) $\displaystyle\sum_{n=1}^{\infty} \frac{2^n+1}{3^n+5}$.

(b) $\displaystyle\sum_{n=1}^{\infty} \frac{(2n)!}{(n!)^2}$.

(c) $\displaystyle\sum_{n=1}^{\infty} \frac{n!}{n^n}$.

(d) $\displaystyle\sum_{n=1}^{\infty} \frac{\sqrt{n+1}}{n^2+1}$.

(a) We can write

$$\begin{aligned}
\lim_{n\to\infty} \frac{a_{n+1}}{a_n} &= \lim_{n\to\infty} \frac{\frac{2^{n+1}+1}{3^{n+1}+5}}{\frac{2^n+1}{3^n+1}} \\
&= \lim_{n\to\infty} \frac{(2^{n+1}+1)(3^n+5)}{(2^n+1)(3^{n+1}+5)} \\
&= \lim_{n\to\infty} \frac{(2^{n+1})(3^n)+5(2^{n+1})+3^n+5}{(2^n)(3^{n+1})+5(2^n)+3^{n+1}+5} \\
&= \lim_{n\to\infty} \frac{2(6^n)+10(2^n)+3^n+5}{3(6^n)+5(2^n)+3(3^n)+5} \\
&= \lim_{n\to\infty} \frac{2(6^n)+\frac{10(6^n)}{3^n}+\frac{6^n}{2^n}+\frac{5(6^n)}{6^n}}{3(6^n)+\frac{5(6^n)}{3^n}+\frac{3(6^n)}{2^n}+\frac{5(6^n)}{6^n}} \\
&= \lim_{n\to\infty} \frac{2+\frac{10}{3^n}+\frac{1}{2^n}+\frac{5}{6^n}}{3+\frac{5}{3^n}+\frac{3}{2^n}+\frac{5}{6^n}} \\
&= \frac{\lim_{n\to\infty}\left(2+\frac{10}{3^n}+\frac{1}{2^n}+\frac{5}{6^n}\right)}{\lim_{n\to\infty}\left(3+\frac{5}{3^n}+\frac{3}{2^n}+\frac{5}{6^n}\right)} && \text{(limit property)} \\
&= \frac{2+0+0+0}{3+0+0+0} && \text{(splitting limit and evaluating)} \\
&= \frac{2}{3} < 1.
\end{aligned}$$

So by the ratio test, the series converges. □

(b) We can write

$$\lim_{n\to\infty} \frac{a_{n+1}}{a_n} = \lim_{n\to\infty} \frac{\frac{(2(n+1))!}{((n+1)!)^2}}{\frac{(2n)!}{(n!)^2}}$$

$$
\begin{aligned}
&= \lim_{n\to\infty} \frac{(2n+2)!(n!)^2}{((n+1)!)^2(2n)!} \\
&= \lim_{n\to\infty} \frac{(2n+2)(2n+1)}{(n+1)^2} \\
&= \lim_{n\to\infty} \frac{4n^2+6n+2}{n^2+2n+1} \\
&= \lim_{n\to\infty} \frac{8n+6}{2n+2} && \text{(L'Hôpital's Rule)} \\
&= \lim_{n\to\infty} \frac{8}{2} && \text{(L'Hôpital's Rule)} \\
&= 4 > 1.
\end{aligned}
$$

So by the ratio test, the series diverges.

(c) We can write

$$
\begin{aligned}
\lim_{n\to\infty} \frac{a_{n+1}}{a_n} &= \lim_{n\to\infty} \frac{\frac{(n+1)!}{(n+1)^{n+1}}}{\frac{n!}{n^n}} \\
&= \lim_{n\to\infty} \frac{(n+1)!(n^n)}{(n+1)^{n+1}(n!)} \\
&= \lim_{n\to\infty} \frac{(n+1)(n^n)}{(n+1)^{n+1}} \\
&= \lim_{n\to\infty} \left(\frac{n}{n+1}\right)^n \\
&= \lim_{n\to\infty} \left(1-\frac{1}{n+1}\right)^n \\
&= \frac{\lim_{n\to\infty} \left(1-\frac{1}{n+1}\right)^{n+1}}{\lim_{n\to\infty} \left(1-\frac{1}{n+1}\right)} && \text{(limit property)} \\
&= \frac{e^{-1}}{1} \\
&= \frac{1}{e} < 1.
\end{aligned}
$$

So by the ratio test, the series converges.

(d) For all $n \in \mathbb{N}$, we have

$$
\begin{aligned}
n+1 &\le n + 2\sqrt{n} + 1 \\
&= (\sqrt{n}+1)^2 \\
\implies \sqrt{n+1} &\le \sqrt{n} + 1
\end{aligned}
$$

and

$$
\begin{aligned}
n^2 + 1 &\ge n^2 \\
\implies \frac{1}{n^2+1} &\le \frac{1}{n^2},
\end{aligned}
$$

so we can write

$$
\frac{\sqrt{n+1}}{n^2+1} \le \frac{\sqrt{n}+1}{n^2}.
$$

We will use this for the comparison test. Let $\langle b_n\rangle_{n=1}^{\infty}$ defined by

$$b_n = \frac{\sqrt{n}+1}{n^2} = \frac{1}{n^{3/2}} + \frac{1}{n^2}.$$

Then, we have

$$\sum_{n=1}^{\infty} b_n = \sum_{n=1}^{\infty} \frac{1}{n^{3/2}} + \sum_{n=1}^{\infty} \frac{1}{n^2}.$$

Both of these sums are p-series with $p > 1$, so both converge and thus the sum converges. We can use the comparison test to conclude that $\sum_{k=1}^{\infty} a_k$ converges. $\square$

Problem 2 (Alternate L'Hôpital's Rule) Let $f, g : [0,\infty) \to \mathbb{R}$ be differentiable functions with

$$\lim_{x\to\infty} f(x) = \infty, \quad \lim_{x\to\infty} g(x) = \infty.$$

Assume

$$\lim_{x\to\infty} \frac{f'(x)}{g'(x)} = L.$$

Prove that

$$\lim_{x\to\infty} \frac{f(x)}{g(x)} = L.$$

Let $\varepsilon > 0$. By definition of a limit, there exists some $a \in \mathbb{R}$ such that for all $x \in \mathbb{R}$,

$$x \geq a \implies \left|\frac{f'(x)}{g'(x)} - L\right| < \varepsilon. \tag{1}$$

Let $x > a$. Then by the Cauchy Mean Value Theorem, there exists some $\xi \in (a,x)$ such that

$$f'(\xi)(g(x) - g(a)) = g'(\xi)(f(x) - f(a)),$$

and since we know that $\frac{f'}{g'}$ is defined we have

$$\frac{f'(\xi)}{g'(\xi)} = \frac{f(x) - f(a)}{g(x) - g(a)}.$$

Since $\xi \geq a$, we can use equation 1 to write

$$\left|\frac{f(x) - f(a)}{g(x) - g(a)} - L\right| < \varepsilon.$$

Thus, by definition

$$\lim_{x\to\infty} \frac{f(x) - f(a)}{g(x) - g(a)} = L. \tag{2}$$

We now show that

$$\lim_{x\to\infty} \frac{f(x)}{f(x) - f(a)} = 1.$$

We note that we have

$$\begin{aligned}
\left|\frac{f(x)}{f(x) - f(a)} - 1\right| &= \left|\frac{f(a) + f(x) - f(a)}{f(x) - f(a)} - 1\right| \\
&= \left|\frac{f(a)}{f(x) - f(a)} + 1 - 1\right|
\end{aligned}$$

$$= \left| \frac{f(a)}{f(x) - f(a)} \right|. \tag{3}$$

Let $\varepsilon' > 0$. We have from the assumption that $\lim_{x\to\infty} f(x) = \infty$, so by definition there exists some $a' \in \mathbb{R}$ such that for all $x \in \mathbb{R}$,

$$x \geq a' \implies f(x) > \frac{|f(a)|}{\varepsilon'} + f(a).$$

Let $x \geq a'$. Then, we have

$$\begin{aligned} f(x) &> \frac{|f(a)|}{\varepsilon'} + f(a) \\ \implies \frac{\big|f(a)\big|}{\varepsilon'} &< f(x) - f(a) \\ &\leq \big|f(x) - f(a)\big| \\ \implies \frac{1}{\big|f(x) - f(a)\big|} &< \frac{\varepsilon'}{\big|f(a)\big|} \\ \implies \left| \frac{f(a)}{f(x) - f(a)} \right| &< \varepsilon' \\ \implies \left| \frac{f(x)}{f(x) - f(a)} - 1 \right| &< \varepsilon'. && \text{(from equation 3)} \end{aligned}$$

So

$$\lim_{x\to\infty} \frac{f(x)}{f(x) - f(a)} = 1, \tag{4}$$

and by similar reasoning we have

$$\lim_{x\to\infty} \frac{g(x) - g(a)}{g(x)} = 1. \tag{5}$$

We can use these results to write

$$\begin{aligned} \lim_{x\to\infty} \frac{f(x)}{g(x)} &= \lim_{x\to\infty} \left[\left(\frac{f(x)}{g(x)} \right) \left(\frac{f(x) - f(a)}{f(x) - f(a)} \right) \left(\frac{g(x) - g(a)}{g(x) - g(a)} \right) \right] \\ &= \lim_{x\to\infty} \left[\left(\frac{f(x) - f(a)}{g(x) - g(a)} \right) \left(\frac{f(x)}{f(x) - f(a)} \right) \left(\frac{g(x) - g(a)}{g(a)} \right) \right] \\ &= \lim_{x\to\infty} \left(\frac{f(x) - f(a)}{g(x) - g(a)} \right) \lim_{x\to\infty} \left(\frac{f(x)}{f(x) - f(a)} \right) \lim_{x\to\infty} \left(\frac{g(x) - g(a)}{g(a)} \right) && \text{(limit property)} \\ &= (L)(1)(1) && \text{(from equations 2, 4, 5)} \\ &= L. \end{aligned}$$

$\square$

Problem 3 Let k be an integer and let

$$P_0(x) = 1$$
$$P_1(x) = x - \left(k + \frac{1}{2}\right)$$
$$P_2(x) = \frac{(x-k)(x-k-1)}{2}.$$

Let $f(x)$ be a function on $[k, k+1]$ that is twice continuously differentiable.

(a) Show

$$P_1'(x) = 1$$
$$P_2'(x) = P_1(x).$$

(b) Integrate

$$\int_k^{k+1} f(x)\,dx = \int_k^{k+1} P_1'(x)f(x)\,dx$$

by part twice to get

$$\int_k^{k+1} f(x)\,dx = P_1(x)f(x)\Big|_k^{k+1} - P_2(x)f'(x)\Big|_k^{k+1} + \int_k^{k+1} P_2(x)f''(x)\,dx.$$

(c) Show this simplifies down to

$$\int_k^{k+1} f(x)\,dx = \frac{f(x)+f(k+1)}{2} + \int_k^{k+1} P_2(x)f''(x)\,dx.$$

(d) Now define a function $B : \mathbb{R} \to \mathbb{R}$ by

$$B(x) = \frac{(x-k)(k+1-x)}{2} = -P_2(x) \text{ when } k \le x \le k+1.$$

This function is periodic with period 1, that is $B(x+1) = B(x)$. Show that with this notation we have

$$\frac{f(k)+f(k+1)}{2} = \int_k^{k+1} f(x)\,dx + \int_k^{k+1} B(x)f''(x)\,dx.$$

Also show

$$0 \le B(x) \le \frac{1}{8}.$$

(e) Now sum the equality for $(f(k)+f(k+1))/2$ from $k=1$ to $n-1$ and rearrange a bit to get

$$\sum_{k=1}^{n} f(k) = \int_1^n f(x)\,dx + \frac{f(1)+f(n)}{2} + \int_1^n B(x)f''(x)\,dx. \tag{1}$$

This gives a precise relation between sums and integrals of the same function and is a special case of the Euler-Maclaurin Summation Formula.

(a) Since $k + \frac{1}{2}$ is a constant, $P_1'(x) = 1 = P_0(x)$ is clear. We can then use the product rule to write

$$P_2'(x) = \frac{(x-k)(1)+(x-k-1)(1)}{2} = x - k - \frac{1}{2} = x - \left(k + \frac{1}{2}\right) = P_1(x).$$

(b) Let $k \in \mathbb{N}$. Then, we can use integration by parts to write

$$\begin{aligned}
\int_k^{k+1} f(x)\,dx &= \int_k^{k+1} P_1'(x)f(x)\,dx && (P_1'(x) = 1 \text{ from (a)}) \\
&= P_1(x)f(x)\Big|_k^{k+1} - \int_k^{k+1} P_1(x)f'(x)\,dx && (u = f(x),\ dv = P_1'(x)\,dx) \\
&= P_1(x)f(x)\Big|_k^{k+1} - \int_k^{k+1} P_2'(x)f'(x)\,dx && (P_2'(x) = P_1(x) \text{ from (a)}) \\
&= P_1(x)f(x)\Big|_k^{k+1} - \left[P_2(x)f'(x)\Big|_k^{k+1} - \int_k^{k+1} P_2(x)f''(x)\,dx\right] \\
& && (u = f(x),\ dv = P_2'(x)\,dx) \\
&= P_1(x)f(x)\Big|_k^{k+1} - P_2(x)f'(x)\Big|_k^{k+1} + \int_k^{k+1} P_2(x)f''(x)\,dx.
\end{aligned}$$

(c) We have

$$\begin{aligned}
\int_k^{k+1} f(x)\,dx &= P_1(x)f(x)\Big|_k^{k+1} - P_2(x)f'(x)\Big|_k^{k+1} + \int_k^{k+1} P_2(x)f''(x)\,dx \\
&= P_1(k+1)f(k+1) - P_1(k)f(k) - P_2(k+1)f'(k+1) \\
&\quad + P_2(k)f'(k) + \int_k^{k+1} P_2(x)f''(x)\,dx \\
&= \left((k+1) - \left(k+\frac{1}{2}\right)\right) f(k+1) - \left(k - \left(k+\frac{1}{2}\right)\right) f(k) \\
&\quad - \left(\frac{(k+1-k)(k+1-k-1)}{2}\right) f'(k+1) + \left(\frac{(k-k)(k-k-1)}{2}\right) f'(k) \\
&\quad + \int_k^{k+1} P_2(x)f''(x)\,dx \\
&= \frac{1}{2}f(k+1) + - \left(-\frac{1}{2}\right) f(k) + 0 + 0 + \int_k^{k+1} P_2(x)f''(x)\,dx \\
&= \frac{f(x)+f(k+1)}{2} + \int_k^{k+1} P_2(x)f''(x)\,dx.
\end{aligned}$$

(d) The first follows quickly:

$$\begin{aligned}
\int_k^{k+1} f(x)\,dx &= \frac{f(x)+f(k+1)}{2} + \int_k^{k+1} P_2(x)f''(x)\,dx \\
\implies \frac{f(x)+f(k+1)}{2} &= \int_k^{k+1} f(x)\,dx - \int_k^{k+1} P_2(x)f''(x)\,dx \\
\implies \frac{f(x)+f(k+1)}{2} &= \int_k^{k+1} f(x)\,dx + \int_k^{k+1} B(x)f''(x)\,dx \qquad (B(x) = -P_2(x))
\end{aligned}$$

Also, since $-B(x) = P_2(x)$, and from part (a) $P_2'(x) = P_1(x)$, we have

$$B'(x) = -P_1(x) = -x + k + \frac{1}{2}.$$

We have $B'(x) = 0$ when $x = k + \frac{1}{2}$, and as $B'(x)$ switches from positive to negative at this point $x = k + \frac{1}{2}$ is a local maximum. We have

$$B\left(k+\frac{1}{2}\right) = \frac{\left(k+\frac{1}{2}-k\right)\left(k+1-\left(k+\frac{1}{2}\right)\right)}{2} = \frac{\left(\frac{1}{2}\right)\left(1-\frac{1}{2}\right)}{2} = \frac{1}{8},$$

so the maximum of $B(x)$ is $\frac{1}{8}$. Since $k \le x \implies x - k \ge 0$ and $x \le k+1 \implies k+1-x \ge 0$, clearly

$$\frac{(x-k)(k+1-x)}{2} \ge 0.$$

So $0 \le B(x) \le \frac{1}{8}$.

(e) Using these results, we have

$$\begin{aligned}
\sum_{k=1}^{n} f(k) &= \frac{f(1)}{2} + \frac{f(1)}{2} + \frac{f(2)}{2} + \frac{f(2)}{2} + \cdots + \frac{f(n-1)}{2} + \frac{f(n-1)}{2} + \frac{f(n)}{2} + \frac{f(n)}{2} \\
&= \frac{f(1)}{2} + \frac{f(1)+f(2)}{2} + \frac{f(2)+f(3)}{2} + \cdots + \frac{f(n-1)+f(n)}{2} + \frac{f(n)}{2}
\end{aligned}$$

$$= \frac{f(1)}{2} + \sum_{k=1}^{n-1} \left(\frac{f(k) + f(k+1)}{2} \right) + \frac{f(n)}{2}$$

$$= \frac{f(1)}{2} + \sum_{k=1}^{n-1} \left(\int_k^{k+1} f(x)\,dx + \int_k^{k+1} B(x) f''(x)\,dx \right) + \frac{f(n)}{2} \qquad \text{(from (d))}$$

$$= \int_1^n f(x)\,dx + \frac{f(1) + f(n)}{2} + \int_1^n B(x) f''(x)\,dx. \qquad \text{(rearranging)}$$

Problem 4 Let $f(x) = \ln(x)$. We will derive Stirling's Formula.

(a) With this choice of $f(x)$ show that (1) becomes

$$\begin{aligned}
\ln(n!) &= \sum_{k=1}^{n} \ln(k) \\
&= \int_1^n \ln(x)\,dx + \frac{\ln(1) + \ln(n)}{2} + \int_1^n B(x) \ln''(x)\,dx \\
&= (x\ln(x) - x)\Big|_1^n + \frac{\ln(n)}{2} - \int_1^n \frac{B(x)}{x^2}\,dx \\
&= n\ln(n) - n + 1 + \frac{\ln(n)}{2} - \int_1^n \frac{B(x)}{x^2}\,dx.
\end{aligned}$$

(b) Note

$$0 < \int_1^n \frac{B(x)}{x^2}\,dx < \int_1^n \frac{1}{8x^2}\,dx = \frac{1}{8}\left(1 - \frac{1}{n}\right) < \frac{1}{8}$$

and use this to show

$$\int_0^\infty \frac{B(x)}{x^2}\,dx = \lim_{n\to\infty} \int_1^n \frac{B(x)}{x^2}\,dx$$

exists and

$$0 < \int_1^\infty \frac{B(x)}{x^2}\,dx \leq \frac{1}{8}.$$

(c) Rewrite the formula for $\ln(n!)$ as

$$\begin{aligned}
\ln(n!) &= \left(n + \frac{1}{2}\right)\ln(n) - n + 1 - \int_1^\infty \frac{B(x)}{x^2}\,dx + \int_n^\infty \frac{B(x)}{x^2}\,dx \\
&= \left(n + \frac{1}{2}\right)\ln(n) - n + C + R_n
\end{aligned}$$

where

$$C = 1 - \int_1^\infty \frac{B(x)}{x^2}$$

and

$$R_n = \int_1^n \frac{B(x)}{x^2}\,dx$$

satisfies $0 < R_n \leq \frac{1}{8n}$.

(d) Use this to conclude

$$n! = e^C n^{n+\frac{1}{2}} e^{-n} e^{R_n}.$$

(a) There isn't much to do here: the chain of equalities given come from formula (1) and then some algebra and calculus.

(b) By the calculation given, the sequence

$$\left\langle \int_1^n \frac{B(x)}{x^2} \right\rangle_{n=1}^{\infty}$$

is bounded above by $\frac{1}{8}$. Since the sequence is also monotone increasing ($B(x)$ and x^2 are non-negative, so the integrands of the integrals are non-negative), the sequence converges. Therefore,

$$\int_0^{\infty} \frac{B(x)}{x^2}\,dx = \lim_{n\to\infty} \int_1^n \frac{B(x)}{x^2}\,dx$$

exists and

$$0 < \int_1^{\infty} \frac{B(x)}{x^2}\,dx \leq \frac{1}{8}.$$

(c) There isn't much to do here either. This follows directly from (a).

(d) We have

$$\begin{aligned}
\ln(n!) &= \left(n + \frac{1}{2}\right)\ln(n) - n + C + R_n && \text{(from (c))} \\
\implies e^{\ln(n!)} &= e^{\left(n+\frac{1}{2}\right)\ln(n) - n + C + R_n} \\
\implies n! &= e^{\left(n+\frac{1}{2}\right)\ln(n)} e^{-n} e^{C} e^{R_n} && \text{(exponential property)} \\
\implies n! &= e^{C} n^{n+\frac{1}{2}} e^{-n} e^{R_n}. && \text{(rearranging)}
\end{aligned}$$

Problem 5 Use the asymptotic formulas

$$I_n \sim \frac{\sqrt{\pi}}{\sqrt{n}} \text{ and } n! \sim K n^{n+\frac{1}{2}} e^{-n}$$

to show

$$I_n \sim \frac{K}{\sqrt{2n}}$$

and thus conclude $K = \sqrt{2\pi}$.

We can write

$$\begin{aligned}
\lim_{n\to\infty} \frac{I_n}{K/\sqrt{2n}} &= \lim_{n\to\infty} \frac{I_n\sqrt{2n}}{K} \\
&= \lim_{n\to\infty} \frac{(n!)^2 2^{2n+1}\sqrt{2n}}{(2n+1)!K} && \text{(value of } I_n \text{ as calculated)} \\
&= \lim_{n\to\infty} \frac{(n!)^2 2^{2n+1}\sqrt{2n}\left(n^{n+\frac{1}{2}}\right)}{(2n+1)!(n!)e^n} && \text{(using second asymptotic formula)} \\
&= \lim_{n\to\infty} \frac{(n!)2^{2n+\frac{3}{2}} n^{n+1}}{(2n+1)!e^n} && \text{(rearranging)} \\
&= \lim_{n\to\infty} \frac{K n^{n+\frac{1}{2}} e^{-n} 2^{2n+\frac{3}{2}} n^{n+1}}{K(2n+1)^{2n+\frac{3}{2}} e^{-2n-1} e^n} && \text{(using second formula again)} \\
&= \lim_{n\to\infty} \frac{K n^{2n+\frac{3}{2}} 2^{2n+\frac{3}{2}} e^{-n}}{K(2n+1)^{2n+\frac{3}{2}} e^{-n-1}} && \text{(rearranging)} \\
&= \lim_{n\to\infty} \frac{(2n)^{2n+\frac{3}{2}} e}{(2n+1)^{2n+\frac{3}{2}}} && \text{(cancelling/grouping)}
\end{aligned}$$

$$
\begin{aligned}
&= e \lim_{n\to\infty} \left(\frac{2n}{2n+1}\right)^{2n+\frac{3}{2}} \\
&= e \lim_{n\to\infty} \left(\frac{2n+1-1}{2n+1}\right)^{2n+1} \lim_{n\to\infty} \sqrt{\frac{2n}{2n+1}} && \text{(splitting limit)} \\
&= e \lim_{n\to\infty} \left(1 - \frac{1}{2n+1}\right)^{2n+1} && \text{(second limit is 1 by L'Hôpital)} \\
&= e(e^{-1}) && \text{(limit is well-known)} \\
&= 1.
\end{aligned}
$$

Therefore,

$$I_n \sim \frac{K}{\sqrt{2n}}.$$

Since $\sim$ is an equivalence relation, this combined with the first asymptotic equation yields

$$\frac{K}{\sqrt{2n}} \sim I_n \sim \frac{\sqrt{\pi}}{\sqrt{n}} = \frac{\sqrt{2\pi}}{\sqrt{2n}} \implies \frac{K}{\sqrt{2n}} \sim \frac{\sqrt{2\pi}}{\sqrt{2n}}.$$

So if there is any justice, $K = \sqrt{2\pi}$. ☐

Problem 1.4 If $\lim\limits_{n\to\infty} a_n = \infty$, show $\lim\limits_{n\to\infty} \frac{1}{a_n} = 0$.

Since $\lim\limits_{n\to\infty} a_n = \infty$, for all real M there exists $N \geq 0$ such that

$$n \geq N \implies a_n \geq M.$$

Let $\varepsilon > 0$, and choose N such that $n \geq N \implies a_n > \frac{1}{\varepsilon}$. Then a_n is positive since $\frac{1}{\varepsilon} > 0$, so

$$a_n > \frac{1}{\varepsilon} \implies \frac{1}{a_n} < \varepsilon \implies \left|\frac{1}{a_n} - 0\right| < \varepsilon.$$

Therefore, by definition $\left\langle \frac{1}{a_n} \right\rangle$ converges to 0.

Problem 1.5 Let $\langle a_n \rangle_{n=1}^{\infty}$ be a monotone sequence. Prove that $\lim\limits_{n\to\infty} a_n$ exists, but might have the value ∞ or $-\infty$.

Without loss of generality, suppose $\langle a_n \rangle$ is monotone increasing (if not, replace $\langle a_n \rangle$ by $\langle -a_n \rangle_{n=1}^{\infty}$).

Case 1: $\langle a_n \rangle$ is bounded above. Then we have shown before that $\langle a_n \rangle$ converges.

Case 2: $\langle a_n \rangle$ is not bounded above. Then for all $M \in \mathbb{R}$, there exists some $N \in \mathbb{N}$ such that $a_N \geq M$. But then since $\langle a_n \rangle$ is monotone increasing, we have $a_n \geq a_N \geq M$ for all $n \geq N$, so

$$n \geq N \implies a_N \geq M.$$

But then by definition,

$$\lim_{n\to\infty} a_n = \infty.$$

So in either case, the limit exists. ☐

Problem 1.6 Prove that if $a_n > 0$ and $\lim\limits_{n\to\infty} a_n = 0$, then $\lim\limits_{n\to\infty} \frac{1}{a_n} = \infty$.

Let $M \in \mathbb{R}$. We want to find an $N \in \mathbb{N}$ such that

$$n \geq N \implies \frac{1}{a_n} \geq M.$$

Since $\langle a_n \rangle$ converges to 0, there exists an N such that

$$n \geq N \implies a_n - 0 < \frac{1}{M}$$

(we can drop the absolute value signs since $a_n > 0$). Since

$$a_n < \frac{1}{M} \implies \frac{1}{a_n} > M,$$

this choice of N works. □

Problem 1.7 Let $\langle a_n \rangle_{n=1}^{\infty}$ be a sequence in $\mathbb{R}$. For each n, set

$$A_n = \sup\{a_n, a_{n+1}, a_{n+2}, \ldots\}.$$

(a) Show that the sequence $\langle A_n \rangle_{n=1}^{\infty}$ is monotone decreasing.

(b) Show that $\lim\limits_{n\to\infty} A_n$ exists (but might be either ∞ or $-\infty$.)

(a) Let $n \in \mathbb{N}$. Then we have

$$A_n = \sup\{a_n, a_{n+1}, a_{n+2}, \ldots\}, \quad A_{n+1} = \sup\{a_{n+1}, a_{n+2}, \ldots\}.$$

Suppose toward contradiction that $A_{n+1} > A_n$. Then there must be some element $x \in \{a_{n+1}, a_{n+2}, \ldots\}$ with $A_n < x < A_{n+1}$ (otherwise, A_n would be the supremum of $\{a_{n+1}, a_{n+2}, \ldots\}$). We have that $x = a_i$ for some $i \geq n+1$. Since $i \geq n+1 > n$, we also have $x \in \{a_n, a_{n+1}, a_{n+2}, \ldots\}$. But since we choose x so that $A_n < x$, A_n cannot be an upper-bound of $\{a_n, a_{n+1}, a_{n+2}\}$, a contradiction. □

(b) It is possible that $A_n = \infty$ for all $n \in \mathbb{N}$ (this is the case for $\langle n \rangle_{n=1}^{\infty}$, for example). Then clearly,

$$\lim_{n\to\infty} A_n = \infty.$$

Otherwise, once $A_k < \infty$ for some k, the subsequence $\langle A_n \rangle_{n=k}^{\infty}$ is a sequence of all real numbers since $\langle A_n \rangle_{n=1}^{\infty}$ is monotone decreasing from (a). Since $\langle A_n \rangle_{n=k}^{\infty}$ is monotone, its limit exists from Problem 1.5. Therefore, $\lim\limits_{n\to\infty} A_n$ exists in both cases.

Problem 1.8 Find the lim sup and lim inf for the following sequences $\langle a_n \rangle_{n=1}^{\infty}$.

(a) $a_n = (-1)^n$.

(b) $a_n = \sin(n)$. You can assume that

$$\sin[\mathbb{N}] := \{\sin(n) : n \in \mathbb{N}\}$$

is dense in $[-1, 1]$: between any $-1 \leq \alpha < \beta \leq 1$ there is an n (in fact infinitely many) with $\alpha < \sin(n) < \beta$.

For both parts, define $\langle S_n \rangle_{n=1}^{\infty}$ and $\langle I_n \rangle_{n=1}^{\infty}$ by

$$S_n := \sup\{a_n, a_{n+1}, a_{n+2}, \ldots\}, \quad I_n := \inf\{a_n, a_{n+1}, a_{n+2}, \ldots\}.$$

(a) Clearly $-1 \leq a_n \leq 1$ for all $n \in \mathbb{N}$.

We first show that $\limsup a_n = 1$. Let $n \in \mathbb{N}$. Since $a_k \leq 1$ for all $k \geq n$ (since it holds for all k), we have $S_n \leq 1$. Also, since $\max\{a_n, a_{n+1}\} = 1$, we have $S_n \geq 1$. Thus, $S_n = 1$ for all n. So

$$\limsup a_n = \lim_{n\to\infty} S_n = 1.$$

We similarly show that $\liminf a_n = -1$. Let $n \in \mathbb{N}$. Since $a_k \geq -1$ for all $k \geq n$, we have $S_n \geq -1$. Also, since $\min\{a_n, a_{n+1}\} = -1$, we have $I_n \leq 1$. Thus, $I_n = -1$ for all n. So

$$\liminf a_n = \lim_{n\to\infty} I_n = -1.$$

(b) Since $-1 \leq \sin(x) \leq 1$ for all $x \in \mathbb{R}$, we have $-1 \leq a_n \leq 1$ for all $n \in \mathbb{N}$.

We will show that $\limsup a_n = 1$. Let $n \in \mathbb{N}$. Since $a_k \leq 1$ for all $k \geq n$, we have $S_n \leq 1$. Let $\varepsilon > 0$. Since there are infinitely many $k \in \mathbb{N}$ with $1 - \varepsilon < \sin(k) \leq 1$, there is an $m \geq n$ with $a_m > 1 - \varepsilon$. Since this holds for every $\varepsilon > 0$, $S_n \geq 1$. Thus, $S_n = 1$ for all n. So

$$\limsup a_n = \lim_{n\to\infty} S_n = 1.$$

Nearly identical reasoning can be used to show that $I_n = -1$ for all n, and thus

$$\liminf a_n = \lim_{n\to\infty} I_n = -1.$$