MATH 552: Section 001
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MATH 552 Homework 2*

Problem 1.9.6 Show that if $\operatorname{Re} z_1 > 0$ and $\operatorname{Re} z_2 > 0$, then

$$\operatorname{Arg} z_1 z_2 = \operatorname{Arg} z_1 + \operatorname{Arg} z_2,$$

where principal arguments are used.

Solution. Without loss of generality for $|z_1| \neq |z_2| \neq 1$, let $z_1 = e^{i\theta_1}$ and $z_2 = e^{i\theta_2}$.

$$\begin{aligned}
z_1 z_2 &= e^{i\theta_1} e^{i\theta_2} = e^{i(\theta_1+\theta_2)} && \text{(using exponent rules)} \\
\arg z_1 z_2 &= \arg z_1 + \arg z_2 && \text{(using } \arg(e^{i\theta}) = \theta \text{ and substituting from above line)} \\
\operatorname{Re} z_1 > 0 \Rightarrow -\frac{\pi}{2} &< \theta_1 < \frac{\pi}{2} && \text{(must be to the right of y-axis)} \\
\operatorname{Re} z_2 > 0 \Rightarrow -\frac{\pi}{2} &< \theta_2 < \frac{\pi}{2} \\
\implies -\pi &< \theta_1 + \theta_2 < \pi \\
\operatorname{Arg} z_1 z_2 &= \operatorname{Arg} z_1 + \operatorname{Arg} z_2 && \text{(since all angles must be in } (-\pi, \pi]\text{)}
\end{aligned}$$

The equality is shown.

Problem 1.11.6 Find the four zeros of the polynomial $z^4 + 4$, one of them being

$$z_0 = \sqrt{2} e^{i\pi/4} = 1 + i.$$

Then use those zeros to factor $z^2 + 4$ into quadratic factors with real coefficients.

Solution.

$$\begin{aligned}
z^4 + 4 &= 0 \\
z^4 &= -4 \\
z^4 &= 4e^{i\pi} && \text{(converting to polar form)} \\
z &= (4e^{i\pi})^{\frac{1}{4}} \\
z &= 4^{\frac{1}{4}} e^{i(\frac{\pi}{4} + \frac{2\pi k}{4})}, k \in \{-2, -1, 0, 1\} && \text{(values chosen to keep } -\pi < \theta \leq \pi\text{)} \\
z &= \{\sqrt{2} e^{-i\frac{3\pi}{4}}, \sqrt{2} e^{-i\frac{\pi}{4}}, \sqrt{2} e^{i\frac{\pi}{4}}, \sqrt{2} e^{i\frac{3\pi}{4}}\} \\
z &= \{-1-i, 1-i, 1+i, -1+i\} && \text{(converting back to rectangular)} \\
z^4 + 4 &= (z - (-1-i))(z - (-1+i))(z - (1-i))(z - (1+i)) && \text{(using solutions to form linear factors)}
\end{aligned}$$

We can take pairs of the factors of $z^4 + 4 = 0$ and multiply them together to get real factors.

$$\begin{aligned}
(z^2 - z(-1+i) - z(-1-i) + (-1-i)(-1+i)) &\qquad \text{(first pair chosen because constants are conjugates)} \\
(z^2 + z - zi + z + zi + (-1)^2 - i^2) & \\
(z^2 + 2z + 2) & \\
(z^2 - z(1+i) - z(1-i) + (1-i)(1+i)) &\qquad \text{(choosing other pair)} \\
(z^2 - z - zi - z + zi + 1^2 - i^2) & \\
(z^2 - 2z + 2) & \\
z^4 + 4 = (z^2 + 2z + 2)(z^2 - 2z + 2) &\qquad \text{(combining the real factors)}
\end{aligned}$$