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SCHC 501: Section H01
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September 1, 2024

SCHC 501 Homework 3

Problem 1

(a) Determine the properties of the following relations on the set of all people. In each case, make the strongest possible statement, e.g. call a relation irreflexive whenever possible rather than non-reflexive.

- (i) is a child of
- (ii) is a brother of
- (iii) is a descendant of
- (iv) is an uncle of (assuming that one may marry one's aunt or uncle)

(b) Which of your answers would be changed if these relations were defined in the set of all male human beings?

(a)

- (i) The relation is irreflexive, since no one is their own child. It is asymmetric, since no one is a child of their parent. It is non-symmetric, since a grandparent of a child is not a parent of the child (except in morally bad cases).
- (ii) The relation is irreflexive, since no one is a brother of oneself. It is non-symmetric, since there are cases where Bob is a brother of Alice but Alice is not a brother of Bob, but there are also cases where Alex and Bob are brothers of each other. It is transitive, since if Alex is a brother of Bob and Bob is a brother of Charlie, then Alex and Charlie have the same parents and thus Alex is a brother of Charlie (excluding cases with half-siblings).
- (iii) The relation is irreflexive, since no one is a descendent of themselves. It is asymmetric, since no one can be a descendant of their descendants. It is transitive, since if Alice is a descendant of Bob and Bob is a descendant of Charlie, Alice is a descendant of Charlie (one can follow the family tree "up through" Bob to Charlie from Alice).
- (iv) The relation is non-reflexive, non-symmetric, and non-transitive: none of the usual properties hold, but morally bad cases exist to prevent irreflexivity, asymmetry, and intransitivity.

(b) Relation (i) would be intransitive and relation (ii) would be symmetric.

Problem 2 Investigate the properties of each of the following relations. If any one is an equivalence relation, indicate the partition it induces on the appropriate set. (If you do not know the concepts, try to find some reasonable assumptions, state them explicitly, and do the exercise based on those).

(a) $M = \{\langle x, y\rangle \mid x \text{ and } y \text{ are a minimal pair of utterances in English}\}$

(b) $C = \{\langle x, y\rangle \mid x \text{ and } y \text{ are phones of English in complementary distribution}\}$

(c) $F = \{\langle x, y\rangle \mid x$ and y are phones of English in free variation$\}$

(d) $A = \{\langle x, y\rangle \mid x$ and y are allophones of the same English phoneme$\}$

(e) Q is the relation defined by 'X is a set having the same number of members as Y' in some appropriate collection of sets.

(a) We will assume a minimal pair of utterances in English is two utterances that differ by a single sound. We have that M is irreflexive since a word does not differ from itself. It is symmetric since the order isn't relevant. It is non-transitive since ⟨"cat", "cut"⟩ and ⟨"cat", "cut"⟩ are in M (and there are some words that are in M).

(b) We will assume two phones are in complementary distribution if one would never appear in the same phonetic context as the other. We have that C is irreflexive since a word is used in the same context as itself, and it is symmetric since the order isn't relevant. It is non-transitive for the same reason that inequality is non-transitive.

(c) We will assume two phones are in free distribution if they correspond to two pronunciations of the same word. We have that F is an equivalence relation since the pronunciations of words can be partitioned into equivalence classes corresponding to words. Thus, F is reflexive, symmetric, and transitive.

(d) We will assume two phonemes are allophones of the same English phoneme if the allophones are two pronunciations of the same phoneme. Thus, A is an equivalence relations since the allophones can be partitioned into equivalence classes corresponding to phonemes. Thus, A is reflexive, symmetric, and transitive.

(e) This is clearly an equivalence relation, so Q is reflexive, symmetric, and transitive.

Problem 3 Let $A = \{1, 2, 3, 4\}$.

(a) Determine the properties of each of the following relations, its inverse, and its complement. If any of the relations happens to be an equivalence relation, show the partition that is induced on A.

$$R_1 = \{\langle 1,1\rangle, \langle 2,1\rangle, \langle 3,4\rangle, \langle 2,2\rangle, \langle 3,3\rangle, \langle 4,4\rangle, \langle 4,1\rangle\}$$
$$R_2 = \{\langle 3,4\rangle, \langle 1,2\rangle, \langle 1,4\rangle, \langle 2,3\rangle, \langle 2,4\rangle, \langle 1,3\rangle\}$$
$$R_3 = \{\langle 2,4\rangle, \langle 3,1\rangle, \langle 3,4\rangle, \langle 2,2\rangle, \langle 1,3\rangle, \langle 4,3\rangle, \langle 4,2\rangle\}$$
$$R_4 = \{\langle 1,1\rangle, \langle 2,4\rangle, \langle 1,3\rangle, \langle 2,2\rangle, \langle 3,1\rangle, \langle 4,4\rangle, \langle 3,3\rangle, \langle 4,2\rangle\}$$

(b) Give the equivalence relation that induces the following partition on A: $P = \{\{1\}, \{2,3\}, \{4\}\}$.

(c) How many distinct partitions of A are possible?

(a) We determine the properties of each:

- R_1 is reflexive, anti-symmetric, and non-transitive, so R_1 is not an equivalence relation. We have

$$(R_1)^{-1} = \{\langle 1,1\rangle, \langle 1,2\rangle, \langle 4,3\rangle, \langle 2,2\rangle, \langle 3,3\rangle, \langle 4,4\rangle, \langle 1,4\rangle\}$$

and

$$(R_1)' = \{\langle 1,2\rangle, \langle 1,3\rangle, \langle 1,4\rangle, \langle 2,3\rangle, \langle 2,4\rangle, \langle 3,1\rangle, \langle 3,2\rangle, \langle 4,2\rangle, \langle 4,3\rangle\}.$$

- R_2 is irreflexive, asymmetric, and transitive, so R_2 is not an equivalence relation (although it is a total order since it is connected.). We have

$$(R_2)^{-1} = \{\langle 4,3\rangle, \langle 2,1\rangle, \langle 4,1\rangle, \langle 3,2\rangle, \langle 4,2\rangle, \langle 3,1\rangle\}$$

and

$$(R_2)' = \{\langle 1,1\rangle, \langle 2,1\rangle, \langle 2,2\rangle, \langle 3,1\rangle, \langle 3,2\rangle, \langle 3,3\rangle, \langle 4,4\rangle\}.$$

- R_3 is non-reflexive, symmetric, and non-transitive, so R_3 is not an equivalence relation. We have

$$(R_3)^{-1} = \{\langle 4,2\rangle, \langle 1,3\rangle, \langle 4,3\rangle, \langle 2,2\rangle, \langle 3,1\rangle, \langle 3,4\rangle, \langle 2,4\rangle\}$$

(so $R_3 = (R_3)^{-1}$) and

$$(R_3)' = \{\langle 1,1\rangle, \langle 1,2\rangle, \langle 1,4\rangle, \langle 2,1\rangle, \langle 2,3\rangle, \langle 3,2\rangle, \langle 3,3\rangle, \langle 4,1\rangle, \langle 4,4\rangle\}.$$

- R_4 is reflexive, symmetric, and transitive, so it is an equivalence relation. We have

$$(R_4)^{-1} = \{\langle 1,1\rangle, \langle 4,2\rangle, \langle 3,1\rangle, \langle 2,2\rangle, \langle 1,3\rangle, \langle 4,4\rangle, \langle 3,3\rangle, \langle 2,4\rangle\}$$

(so $R_4 = (R_4)^{-1}$) and

$$(R_4)' = \{\langle 1,2\rangle, \langle 1,4\rangle, \langle 2,1\rangle, \langle 2,3\rangle, \langle 3,2\rangle, \langle 3,4\rangle, \langle 4,1\rangle, \langle 4,3\rangle\}.$$

(b) The equivalence relation that induces P on A is

$$\{\langle 1,1\rangle, \langle 2,2\rangle, \langle 2,3\rangle, \langle 3,2\rangle, \langle 3,3\rangle, \langle 4,4\rangle\}.$$

(c) There are 15 distinct partitions of A: we list them below.

1. $\big\{\{1,2,3,4\}\big\}$
2. $\big\{\{1\},\{2,3,4\}\big\}$
3. $\big\{\{2\},\{1,3,4\}\big\}$
4. $\big\{\{3\},\{1,2,4\}\big\}$
5. $\big\{\{4\},\{1,2,3\}\big\}$
6. $\big\{\{1,2\},\{3,4\}\big\}$
7. $\big\{\{1,3\},\{2,4\}\big\}$
8. $\big\{\{1,4\},\{2,3\}\big\}$
9. $\big\{\{1\},\{2\},\{3,4\}\big\}$
10. $\big\{\{1\},\{3\},\{2,4\}\big\}$
11. $\big\{\{1\},\{4\},\{2,3\}\big\}$
12. $\big\{\{2\},\{3\},\{1,4\}\big\}$
13. $\big\{\{2\},\{4\},\{1,3\}\big\}$
14. $\big\{\{3\},\{4\},\{1,2\}\big\}$
15. $\big\{\{1\},\{2\},\{3\},\{4\}\big\}$

Problem 4 What is wrong with the following reasoning that reflexivity is a consequence of symmetry and transitivity? "If $\langle x, y\rangle \in R$, then $\langle y, x\rangle \in R$, since we assume R is symmetric. If both $\langle x, y\rangle$ and $\langle y, x\rangle$ are in R, then $\langle x, x\rangle$ must be in R by transitivity.

The problem is there may some x not in the domain of R but is in the set R is on. For example, for $A = \{1, 2, 3\}$, the relation $R = \{\langle 1, 1\rangle, \langle 1, 2\rangle, \langle 2, 1\rangle, \langle 2, 2\rangle\}$ is symmetric and transitive but not reflexive because $\langle 3, 3\rangle \notin R$: in this case, 3 is in A but not the domain of R.