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MATH 575: Section H01
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MATH 575 Homework 9

Problem 1 Recall that Q_d is the d-dimensional hypercube. (The vertices of Q_d are the bit strings of length d, and two vertices are adjacent if and only if their bit strings differ in exactly one bit.) For every $d \geq 1$, give an explicit edge-coloring to prove $\chi'(Q_d) = \Delta(Q_d)$.

Solution.

We claim that Q_d has a proper edge-coloring using d colors for any $d \in \{0, 1, 2 \ldots, d\}$. We will induct on d to show this. First, let $d = 0$. Clearly Q_0 has an edge-coloring using 0 colors because there are no edges. Next, let $d \in \mathbb{N}$, and assume that for all $d' < d$, $Q_{d'}$ has a proper edge-coloring using d' colors. By the way have defined it, Q_d will have two subgraphs G_1' and G_2' that are isomorphic copies of Q_{d-1}. In Q_d, each vertex in G_1' will connect to exactly one vertex in G_2', since we need to extend into another dimension.

Now we can properly edge-color G_1' and G_2' with $d-1$ colors using the proper edge-coloring for Q_{d-1} from the inductive hypothesis. Once we have these edge-colorings, we can color the remaining edges (the ones between G_1' and G_2') with only one color. If it took more, than we would have two edges between G_1' and G_2' incident to one vertex, a contradiction. So we can properly edge-color Q_d with d colors. Each vertex will be neighbors with vertices in all d dimensions, so we have $\Delta(Q_d) = d$. Therefore, we have $\chi'(Q_d) \leq \Delta(Q_d)$, and since $\chi'(G) \geq \Delta(G)$ for any graph G, we have $\chi'(Q_d) = \Delta(Q_d)$. □

Problem 2 Recall that $\alpha'(G)$ denotes the size of a maximum matching in G.

(a) Prove that $\chi'(G) \geq \frac{|E(G)|}{\alpha'(G)}$.

(b) Prove that if G is a k-regular graph with no perfect matching, then $\chi'(G) = \Delta(G) + 1$.

Solution.

(a) Let G be a graph with $\chi'(G) = k$. Then, we can partition the edges into color classes $M_1, M_2, \ldots, M_k$, where each class is a matching in G. Then, we have

$$\begin{aligned}
|E(G)| &= |M_1| + |M_2| + \ldots + |M_k| \\
&\leq \alpha'(G) + \alpha'(G) + \ldots + \alpha'(G) \qquad \text{(each matching has size at most } \alpha'(G) \text{ by definition)} \\
&= k\alpha'(G) \\
&= \chi'(G)\alpha'(G) \\
\implies \chi'(G)\alpha'(G) &\geq |E(G)| \\
\implies \chi'(G) &\geq \frac{|E(G)|}{\alpha'(G)}.
\end{aligned}$$

(b) Let G be a k-regular graph on n vertices. Assume that $\chi'(G) \neq \Delta(G) + 1$. Then,

$$\chi'(G) = \Delta(G) \qquad \text{(Vizing's theorem)}$$

$$
\begin{aligned}
&= k && (G \text{ is } k\text{-regular}) \\
&\geq \frac{|E(G)|}{\alpha'(G)} && (\text{from (a)}) \\
&= \frac{1}{2\alpha'(G)} \sum_{v \in V(G)} d(v) && (\text{Handshaking lemma}) \\
&= \frac{nk}{2\alpha'(G)}. && (G \text{ is } k\text{-regular})
\end{aligned}
$$

So we have

$$k \geq \frac{nk}{2\alpha'(G)} \implies 1 \geq \frac{n}{2\alpha'(G)} \implies \alpha'(G) \geq \frac{n}{2},$$

which is only possible if there is a perfect matching. Therefore, we have from the contrapositive that if G is a k-regular graph with no perfect matching, then $\chi'(G) = \Delta(G) + 1$. □

Problem 3 Chromatic number versus edge-chromatic number

(a) For every $n \geq 4$, construct a graph G on n-vertices with $\chi(G) < \chi'(G)$.

(b) Give a characterization of all connected graphs for which $\chi(G) > \chi'(G)$. That is, prove a statement of the form:

"If G is a connected graph, then $\chi(G) > \chi'(G)$ if and only if ____________."

Solution.

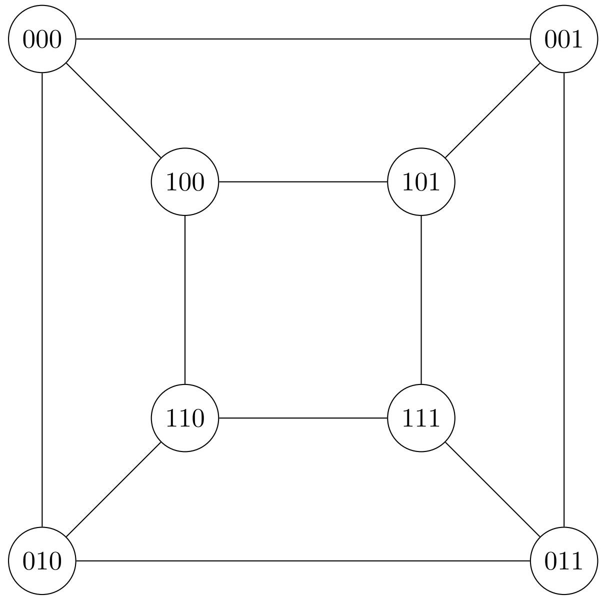
(a) Let $n \in \mathbb{N}$, $n \geq 4$, and consider $G = K_{1,n-1}$. Since G is bipartite, $\chi(G) = 2$, but we will need to color every edge in G a different color since every edge is incident to a single vertex. Thus $\chi(G') = n - 1$, and since $n \geq 4$, we have $2 < n - 1$ and $\chi(G) < \chi'(G)$.

(b) Let G be a graph such that $\chi(G) > \chi'(G)$. By the greedy algorithm and Vizing's theorems, we have $\chi(G) \leq \Delta(G) + 1$ and $\chi'(G) \in \{\Delta(G), \Delta(G) + 1\}$ respectively. Because of these three restrictions, we can conclude that we have $\chi(G) = \Delta(G) + 1$ and $\chi'(G) = \Delta(G)$. Since $\chi(G) = \Delta(G) + 1$, we have from Brooks that G is either an odd cycle or a complete graph. However, we have shown in class that both odd cycles and complete graphs on an odd number of vertices are Class 2 graphs, contradicting $\chi'(G) = \Delta(G)$. Complete graphs on an even number of vertices are $\Delta(G)$-edge-colorable, so this is the only possible option. Therefore, if G is a connected graph, then $\chi(G) > \chi'(G)$ if and only if G is a complete graph on an even number of vertices. □

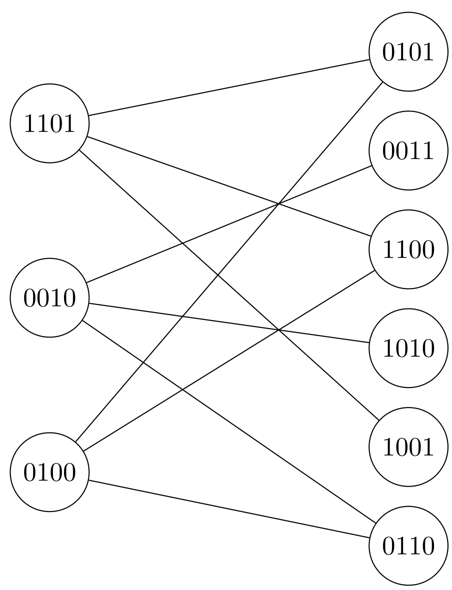
Problem 4 Determine for which values d the hypercube Q_d is planar and for which values it is non-planar.

Solution.

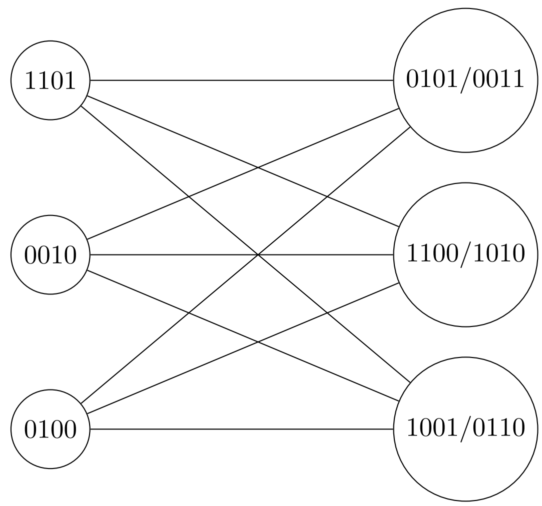
We observe that this is a planar embedding of Q_3:



So Q_3 is planar, and since Q_3 contains Q_0, Q_1, and Q_2 as subgraphs, these are also planar. Next, we observe that the following is a subgraph of Q_4:

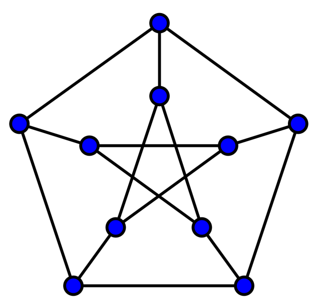


We can then perform three edge contractions and obtain a minor of Q_4:



Notably, this minor of Q_4 is a $K_{3,3}$, so by Wagner's Theorem, Q_4 is not planar. Since Q_d for $d \geq 4$ contains Q_4 as a subgraph, Q_d is not planar for $d \geq 4$. Therefore, Q_d is planar if and only if $d \in \{0, 1, 2, 3\}$. $\square$

Problem 5 Use Euler's Formula to prove that the Petersen graph (drawn below) is not planar. *Do not use Wagner's Theorem or Kuratowski's Theorem.*



Solution.

Let G be the Petersen graph, and assume to the contrary that G is planar. Then, Euler's formula holds, and since we have 10 vertices and 15 edges, we must have 7 faces to satisfy $n - e + f = 2$. Also, we observe that a shortest cycle in G has length 5, so there are no closed walks of length 3 or length 4. Since each edge is counted twice and we must use at least 5 edges for each face, we have $5f \leq 2e$. Substituting from Euler's formula, we have $35 = 5(7) = 5f \leq 2e = 2(15) = 30$, a contradiction. $\square$