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MATH 544: Section H01
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MATH 544 Homework 3

Problem 2.29 Let $\delta > 0$. Assume that $|y - c| < \delta$. Show that

$$|(-5y + 42) - (-5c + 42)| < 5\delta.$$

We can write

$$\begin{aligned} |(-5y + 42) - (-5c + 42)| &= |-5y + 42 + 5c - 42| \\ &= |-5(y - c)| \\ &= 5|y - c| && (|-5| = 5) \\ &< 5\delta. && (\text{because } |y - c| < \delta) \end{aligned}$$

$\square$

Problem 2.30 Show that if $|y - c| < \delta \leq 2$, then

$$|y^4 - c^4| \leq 4(2 + |c|^3)\delta.$$

We first note that we can write

$$\begin{aligned} |y| &= |(y - c) + c| \\ &\leq |y - c| + |c| && (\text{triangle inequality}) \\ &< 2 + |c|. && (|y - c| < \delta \geq 2) \end{aligned}$$

Clearly, we also have $|c| < 2 + |c|$. Consequently, we can write

$$\begin{aligned} |y^4 - c^4| &= |y^3 + y^2c + yc^2 + c^3||y - c| && (\text{factoring}) \\ &< |y^3 + y^2c + yc^2 + c^3|\delta && (|y - c| < \delta) \\ &\leq (|y^3| + |y^2c| + |yc^2| + |c^3|)\delta && (\text{triangle inequality}) \\ &= \left(|y|^3 + |y|^2|c| + |y||c|^2 + |c|^3\right)\delta \\ &< \left((2 + |c|)^3 + (2 + |c|)^2(2 + |c|) + (2 + |c|)(2 + |c|)^2 + (2 + |c|)^3\right)\delta && (|y|, |c| < 2 + |c|) \\ &= 4(2 + |c|)^3\delta. \end{aligned}$$

$\square$

Problem 2.31 Assume $|a - x| < \delta$, $|y - b| < \delta$, and $\delta \leq 1$. Show that

$$|xy - ab| < (1 + |a| + |b|)\delta.$$

We have

$$\begin{aligned}
|xy - ab| &\le \big||xy| - |ab|\big| && \text{(reverse triangle inequality)} \\
&= \big||x||y| - |a||b|\big| \\
&< \big|(|a| + \delta)(|b| + \delta) - |a||b|\big| && \text{(same reasoning as in 2.30)} \\
&= \left||a||b| + |a|\delta + |b|\delta + \delta^2 - |a||b|\right| \\
&= \left||a|\delta + |b|\delta + \delta^2\right| \\
&= \big|(|a| + |b| + \delta)\delta\big| \\
&= (|a| + |b| + \delta)\delta && \text{(all values are non-negative)} \\
&< (|a| + |b| + 1)\delta. && (\delta < 1)
\end{aligned}$$

□

Problem 2.32 Assume $|x - a| < \delta \le |a|/2$. Show

$$\frac{|a|}{2} < |x| < \frac{3|a|}{2}$$

and

$$\left|\frac{1}{x^2} - \frac{1}{a^2}\right| < \frac{10\delta}{|a|^3}.$$

First, we can compute

$$\begin{aligned}
|x| &= |a + x - a| \\
&\le |a| + |x - a| && \text{(triangle inequality)} \\
&< |a| + \frac{|a|}{2} && (|x - a| < \delta \le |a|/2) \\
&= \frac{3|a|}{2},
\end{aligned}$$

and similarly,

$$\begin{aligned}
|a| &= |x + a - x| \\
&\le |x| + |a - x| \\
&< |x| + \frac{|a|}{2} \\
\implies \frac{|a|}{2} &< |x|, && \text{(algebra)}
\end{aligned}$$

so $\frac{|a|}{2} < |x| < \frac{3|a|}{2}$. From this, we have

$$\begin{aligned}
\left|\frac{1}{x^2} - \frac{1}{a^2}\right| &= \left|\frac{a^2 - x^2}{x^2a^2}\right| \\
&= \left|\frac{(a + x)(a - x)}{x^2a^2}\right| \\
&= \frac{1}{|x|^2|a|^2}|a + x||a - x|
\end{aligned}$$

$$
\begin{aligned}
&< \frac{1}{|x|^2|a|^2}|a+x|\delta && (|a-x|<\delta)\\
&\le \frac{\delta}{|x|^2|a|^2}(|a|+|x|) && \text{(triangle inequality)}\\
&\le \frac{\delta}{|x|^2|a|^2}\left(|a|+\frac{3|a|}{2}\right) && (|x|<\tfrac{3|a|}{2})\\
&< \frac{\delta}{|x|^2|a|^2}\left(\frac{5|a|}{2}\right)\\
&= \frac{5\delta}{2|x|^2|a|}\\
&< \frac{5\delta}{2|a|\left(\frac{|a|}{2}\right)^2} && (\tfrac{|a|}{2}<|x|)\\
&= \frac{5\delta}{\frac{2|a|^3}{4}}\\
&= \frac{10\delta}{|a|^3}.
\end{aligned}
$$

Problem 2.33 Let $a, b \in \mathbb{F}$. Then define

$$\max(a,b) = \begin{cases} a, & \text{if } a \ge b; \\ b, & \text{if } b > a. \end{cases}$$

$$\min(a,b) = \begin{cases} a, & \text{if } a \le b; \\ b, & \text{if } b < a. \end{cases}$$

Prove that

$$\max(a,b) = \frac{a+b+|a-b|}{2}$$
$$\min(a,b) = \frac{a+b-|a-b|}{2}.$$

Without loss of generality, assume $a \ge b$, which implies that $\max\{a,b\} = a$, $\min\{a,b\} = b$, and $|a-b| = a-b$. Then, we have

$$\frac{a+b+|a-b|}{2} = \frac{a+b+a-b}{2} = \frac{2a}{2} = a = \max\{a,b\}$$

and

$$\frac{a+b-|a-b|}{2} = \frac{a+b-a+b}{2} = \frac{2b}{2} = b = \min\{a,b\}.$$

□

Problem 2.34 Solve the following inequalities:

(a) $5x - 9 < 7x + 21$.

(b) $x^2 - 10x + 9 < 16$.

(c) $\frac{x+2}{x-2} \le 5$.

We can use algebra as a result of the properties of inequalities we proved to find:

(a)

$$\begin{aligned}
&5x - 9 < 7x + 21 \\
\iff &-9 < 2x + 21 \\
\iff &-30 < 2x \\
\iff &x > -15.
\end{aligned}$$

(b)

$$\begin{aligned}
&x^2 - 10x + 9 < 16 \\
\iff &x^2 - 10 + 25 < 32 \\
\iff &(x-5)^2 < 32 \\
\iff &|x-5|^2 < 32 \\
\iff &|x-5| < 4\sqrt{2} \\
\iff &5 - 4\sqrt{2} < x < 5 + 4\sqrt{2}.
\end{aligned}$$

(c) Case 1: $x - 2 > 0$. Then

$$\begin{aligned}
&\frac{x+2}{x-2} \leq 5 \\
\iff &x + 2 \leq 5(x-2) \\
\iff &x + 2 \leq 5x - 10 \\
\iff &2 \leq 4x - 10 \\
\iff &12 \leq 4x \\
\iff &x \geq 3.
\end{aligned}$$

Case 2: $x - 2 = 0$. Then $\frac{x+2}{x-2}$ is undefined, so clearly $x = 2$ does not satisfy the inequality.

Case 3: $x - 2 < 0$. Then $x < 2$.

So $x \in (-\infty, 2) \cup [3, \infty) \iff \frac{x+2}{x-5} \leq 5$.