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MATH 575: Section H01
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MATH 575 Homework 5

Collaboration:

Problem 1 A *permutation matrix* is a square matrix with entries in $\{0, 1\}$ such that each row and each column have exactly one 1. Prove that a square matrix of nonnegative integers can be expressed as a sum of k permutation matrices if and only if all rows and column sums equal k. (Hint: induct on k and use Hall's Theorem.)

Solution.

We will induct on k. First, let $k = 1$. If a square matrix of nonnegative integers can be expressed as a sum of 1 permutation matrix, then it is itself a permutation matrix and all rows and columns must sum to 1 by definition. Conversely, if a square matrix of nonnegative integers has all rows and column sums equal to 1, then each row and column must contain exactly one 1 and the rest 0s, which is the definition of a permutation matrix. So the claim holds for $k = 1$.

Next, let $k \in \mathbb{N}$, $k > 1$, and assume that for all $k' < k$, a square matrix of nonnegative integers can be expressed a sum of k' permutation matrices if and only if all row and column sums equal k'.

($\Rightarrow$) Let $M \in \text{Mat}_{n\times n}(\mathbb{Z}_{\geq 0})$, and suppose M can be expressed as a sum of k permutation matrices $M_1, M_2, \ldots, M_k$. Consider M before we have added M_k. By the induction hypothesis, every row and column has sum $k-1$. Since every row and column sum in M_k is 1, adding M_k back will result in every row and column sum being k.

($\Leftarrow$) Let $M \in \text{Mat}_{n\times n}(\mathbb{Z}_{\geq 0})$ such that all the rows and columns sum to k. We construct a bipartite graph $G = R \cup C$, where the vertices in R are the rows in M, the vertices in C are the columns in M, and there is an edge between $r \in R$ and $c \in C$ if the r, c-entry in M is positive.

If we can find a perfect matching, we can construct a permutation matrix M' where ${M'}_{rc}$ is 1 if rc is in the matching and 0 otherwise. Then, $M - M'$ has all row and column sums equal to $k-1$, and by the induction hypothesis it can be expressed as the sum of $k-1$ permutation matrices. So M can be expressed as the sum of k permutation matrices.

Suppose we do not have a perfect matching. Then, Hall's condition is violated, so we have some $S \subseteq R$ such that $|S| > |N(S)|$. Since a row r has an edge with column c if $M_{rc} \neq 0$, this means that there are only $|N(S)| < |S|$ columns that are non-zero within the rows of S. Since every row adds up to k, these non-zero columns have a sum $\geq |S|k$ (there could be numbers in other rows of these columns). But this is a contradiction, because there are $|N(S)| < |S|$, and thus at least one of the columns has sum greater than k. Therefore, we have a perfect matching, and M can be expressed as the sum of k permutation matrices.

Therefore, a square matrix of nonnegative integers can be expressed as a sum of k permutation matrices if and only if all rows and column sums equal k. □

Problem 2 Recall that $\alpha'(G)$ denotes the maximum size of a matching in G and $\beta(G)$ denotes the minimum size of a vertex cover of G.

(a) Prove that for every graph G, $\beta(G) \leq 2\alpha'(G)$.
(b) For every $k \in \mathbb{N}$, construct a graph G with $\alpha'(G) = k$ and $\beta(G) = 2k$.

Solution.

(a) Let M be a maximum matching in G, and let S be the set of all vertices saturated by M. Then, S must be a vertex cover. If it is not, then there is an edge xy not covered, and by our definition of S neither x nor y is saturated by M. But then M is not maximum because we can add xy. Since we have two vertices in S for every edge in M, we have $|S| \leq 2|M|$. We could have a smaller vertex cover but not a larger matching since M is maximum, so we have $\beta(G) \leq |S| \leq 2|M| = 2\alpha'(G)$.

(b) One such graph is a K_{2k+1}. Since every vertex is connected, we can algorithmically create a matching by adding pairs of vertices not yet in the matching one at a time until we have k pairs and one vertex (which, although it is compatible with every other vertex in the graph, is doomed to be forever alone). So $\alpha'(K_{2k+1}) = k$.

We can obtain a vertex cover of K_{2k+1} by removing one vertex from the set of vertices. Then, all of its neighbors will be in the vertex cover, so all the edges will be covered. However, we cannot remove more than one vertex, because the edge between them will not be covered. So $\beta(K_{2k+1}) = 2k$.

Problem 3 Matchings and vertex covers
(a) Use the König–Egerváry Theorem to prove every bipartite graph G has a matching of size at least $\frac{|E(G)|}{\Delta(G)}$.
(b) Use (a) to conclude that every subgraph of $K_{n,n}$ with more than $(k-1)n$ edges has a matching of size at least k.

Solution.

(a) We note this is not true for an entirely disconnected graph, since it has 0 edges. So we will only consider graphs with $\Delta(G) \geq 1$. We will induct on n, the number of vertices in G. First, let $n = 2$. The only possible graph with edges is a K_2, which has a matching of size $1 = \frac{1}{1} = \frac{|E(G)|}{\Delta(G)}$. So the claim holds for $n = 2$.

Next, let $n \in \mathbb{N}$, $n > 2$, and suppose that for all $n' < n$, a graph G' on n' vertices has a matching of size at least $\frac{|E(G')|}{\Delta(G')}$. Let $G = X \cup Y$ be a bipartite graph on n vertices, and let $v \in V(G)$ with $d(v) = \Delta(G)$. We now consider $G' = G - v$, and observe the following:

1. $|E(G')| = |E(G)| - \Delta(G)$, since $d(v) = \Delta(G)$ and thus we are removing $\Delta(G)$ edges from the edge set.

2. $1 \leq \Delta(G') \leq \Delta(G)$. Since v had largest degree, the maximum degree will certainly not increase, and it could be the case that no other vertex had as large of a degree, in which case the maximum degree will decrease. (It could be the case that v was incident to every edge in the graph, in which case $\Delta(G') = 0$. But then we would have had $\frac{|E(G)|}{\Delta(G)} = 1$ and a matching of size 1, and we would be done. So we will assume $\Delta(G') > 0$).

3. We have a matching M' in G' such that $|M'| \geq \frac{|E(G')|}{\Delta(G')}$, by the induction hypothesis. Then, by the König–Egerváry Theorem, we have a vertex cover S' such that $|S'| \leq \frac{|E(G')|}{\Delta(G')}$.

Now, we add back v, and choose M to be a maximum matching of G. Suppose (toward contradiction) that

$$\begin{aligned} |M| &< \frac{|E(G)|}{\Delta(G)} && \text{(assumption)} \\ &= \frac{|E(G')| + \Delta(G)}{\Delta(G)} && \text{(from 2)} \end{aligned}$$

$$
\begin{aligned}
&= \frac{|E(G')|}{|\Delta(G)|} + 1 \\
&\leq \frac{|E(G')|}{\Delta(G')} + 1. && \text{(from 2)}
\end{aligned}
$$

Then, by the König–Egerváry Theorem, a minimum vertex cover S must satisfy

$$
\begin{aligned}
|S| &> \frac{|E(G')|}{\Delta(G')} + 1 \\
\implies |S| - 1 &\geq \frac{|E(G')|}{\Delta(G')} \\
&\geq |S'| && \text{(from 3)} \\
\implies |S| &> |S'| + 1.
\end{aligned}
$$

But $S' \cup \{v\}$ is a vertex cover, a contradiction because $|S| > |S' \cup \{v\}| = |S'| + 1$ and we chose S to be minimal.

(b) Let G be a subgraph of $K_{n,n}$ with $|E(G)| > (k-1)n$. Since G is bipartite with and each bipartition has size n, we have that $\Delta(G) < n$. Let M be a maximum matching of G. Then, we have

$$
\begin{aligned}
|M| &\geq \frac{|E(G)|}{\Delta(G)} && \text{(from (a))} \\
&> \frac{(k-1)n}{\Delta(G)} && \text{(from assumption)} \\
&> \frac{(k-1)n}{n} && \text{(since } \Delta(G) < n\text{)} \\
&= k - 1.
\end{aligned}
$$

So $|M| > k - 1$, and thus we must have a matching with size at least k.

Problem 4 Recall that $\alpha(G)$ is the size of a largest independent set in G.
(a) Prove that if G is an n-vertex bipartite graph, then $\alpha(G) \geq n/2$.
(b) Prove that for any n-vertex graph G, $\alpha(G) \geq \frac{n}{\Delta(G)+1}$.
(c) Show that (b) is best possible by constructing for every pair of non-negative integers r, s a graph G with $\alpha(G) = r$, $\Delta(G) = s$, and $\alpha(G) = \frac{n}{\Delta(G)+1}$.

Solution.

(a) Let $G = X \cup Y$, and WLOG assume $|X| \geq |Y|$. Then, $|X|$ has at least $n/2$ vertices, and by definition it is an independent set. So $\alpha(G) \geq |X| \geq n/2$.

(b) Let $k < n$, and assume G is k-regular. If it is not, then after adding edges to make it k-regular, any independent set we find will still be independent after removing the edges we added. Let S be a maximum independent set in G. Then,

1. $G = S \cup N(S)$. If it is not, then there is a vertex v in neither S nor $N(S)$. But then, we can add v to S, because there is no edge between S and v, so S wasn't maximum.

2. From 1, $|S| + |N(S)| = n$, since S and $N(S)$ are a partition of G.

3. We can bound $N(S)$ with $|N(S)| \leq k|S|$, since equality holds when every vertex in S has k neighbors and none of these neighbors are common.

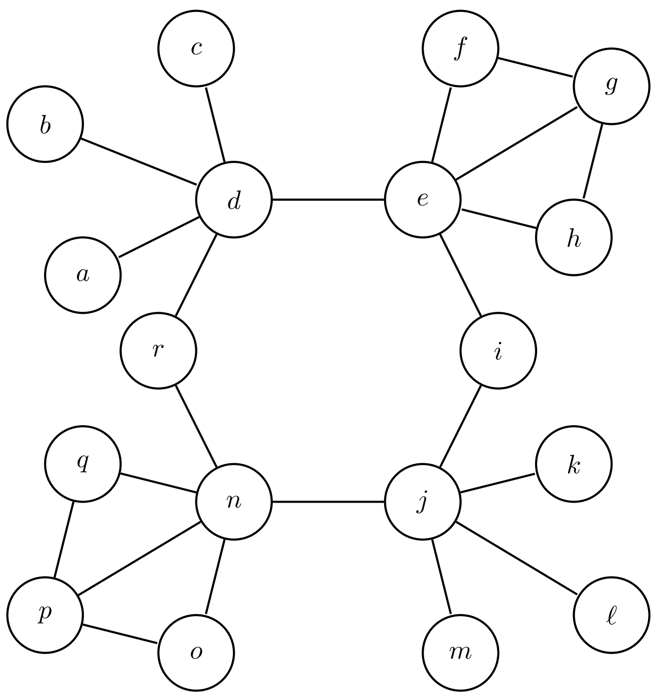
So we have

$$
\begin{aligned}
|N(S)| &\leq k|S| && \text{(from 3)} \\
\implies |S| + |N(S)| &\leq |S| + k|S| \\
\implies n &\leq |S| + k|S| && \text{(from 2)} \\
\implies n &\leq (k+1)|S| \\
\implies |S| &\geq \frac{n}{k+1}.
\end{aligned}
$$

Since $k = \Delta(G)$ and S is an independent set, we have $\alpha(G) = \frac{n}{\Delta(G)+1}$.

(c) Let $r, s \in \mathbb{Z}_{\geq 0}$ and let G be r disconnected copies of K_{s+1}. Then, $\Delta(G) = s + 1 - 1 = s$. We can choose a vertex from each copy, and this will be an independent set because none of the copies are connected. However, suppose we have an independent set of $r + 1$ or greater. Then by PHP, two of the vertices will be in the same component, and since the component is complete there will be an edge between the two vertices, a contradiction. So $\alpha(G) = r$.

Problem 5 Consider the graph G below.



(a) Find a matching of size 6 in G.
(b) Prove that your matching in part (a) is maximum using the Berge–Tutte Formula.
(c) Prove that your matching in part (a) is maximum using the König–Egerváry Theorem.

Solution.

(a) We have $M = \{ad, ei, fg, jk, nr, op\}$ is a matching of size 6.

(b) Let $S = \{d, j\}$. We observe that $o(G - S) = 8$, with the odd components having vertices:

1. $\{a\}$

2. $\{b\}$
3. $\{c\}$
4. $\{e, f, g, h, i\}$
5. $\{k\}$
6. $\{\ell\}$
7. $\{m\}$
8. $\{m, o, p, q, r\}$

Then, the deficiency of set S is $8 - 2 = 6$. Since M leaves only the 6 vertices $\{b, c, h, \ell, m, q\}$ unsaturated, by the Berge–Tutte Formula, M is maximum.

(c) Let $S = \{d, e, g, j, n, p\}$. Then, S is a vertex cover, so we have found a vertex cover equal with size equal to that of our matching, and therefore by the König–Egerváry Theorem, M is maximum.