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MATH 544: Section H01
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MATH 544 Homework 7

Problem 1 Let $A = \begin{pmatrix} 2 & 1 & 3 \\ 4 & 3 & 5 \\ 3 & 3 & 3 \end{pmatrix}$. Compute bases for Row$(A)$, Col$(A)$, and Null$(A)$, and find nullity$(A^{\mathrm{T}})$.

Solution.

We observe that

$$\begin{aligned} A &= \begin{pmatrix} 2 & 1 & 3 \\ 4 & 3 & 5 \\ 3 & 3 & 3 \end{pmatrix} \\ &\sim \begin{pmatrix} 2 & 1 & 3 \\ 0 & 1 & -1 \\ 3 & 3 & 3 \end{pmatrix} && (\rho_2 \mapsto \rho_2 - 2\rho_1) \\ &\sim \begin{pmatrix} 2 & 1 & 3 \\ 0 & 1 & -1 \\ 0 & \frac{3}{2} & -\frac{3}{2} \end{pmatrix} && (\rho_3 \mapsto \rho_3 - \tfrac{3}{2}\rho_1) \\ &\sim \begin{pmatrix} 2 & 1 & 3 \\ 0 & 1 & -1 \\ 0 & 0 & 0 \end{pmatrix} && (\rho_3 \mapsto \rho_3 - \tfrac{3}{2}\rho_2) \\ &\sim \begin{pmatrix} 2 & 0 & 4 \\ 0 & 1 & -1 \\ 0 & 0 & 0 \end{pmatrix} && (\rho_1 \mapsto \rho_1 - \rho_2) \\ &\sim \begin{pmatrix} 1 & 0 & 2 \\ 0 & 1 & -1 \\ 0 & 0 & 0 \end{pmatrix}, && (\rho_1 \mapsto \tfrac{1}{2}\rho_1) \end{aligned}$$

which is in reduced row-echelon form. Let

$$X = \left\{ \begin{pmatrix} 1 \\ 0 \\ 2 \end{pmatrix}, \begin{pmatrix} 0 \\ 1 \\ -1 \end{pmatrix} \right\}, Y = \left\{ \begin{pmatrix} 2 \\ 4 \\ 3 \end{pmatrix}, \begin{pmatrix} 1 \\ 3 \\ 3 \end{pmatrix} \right\}, Z = \left\{ \begin{pmatrix} -2 \\ 1 \\ 1 \end{pmatrix} \right\}.$$

From the methods we have shown in class, we have shown that X, Y, Z are linearly independent and $\text{Row}(A) = \text{Span}(X), \text{Col}(A) = \text{Span}(Y), \text{Null}(A) = \text{Span}(Z)$. So X is a basis for Row(A), Y is a basis for Col(A), and Z is a basis for Null(A). By the theorem we have shown in class, we have

$$3 = \text{rank}(A^T) + \text{nullity}(A^T) = \text{rank}(A) + \text{nullity}(A^T) = 2 + \text{nullity}(A^T),$$

so nullity$(A^T) = 1$. $\square$

Problem 2 Determine whether the set B is a basis for the vector space V. If B fails to be a basis, briefly explain why it fails.

(a) $B = \{x^2 - 5x + 3,\, 3x^2 - 7x + 5,\, x^2 - x + 1\}$, $V = \mathbb{R}_2[x]$.

(b) $B = \left\{ \begin{pmatrix} 1 & 2 \\ 2 & 1 \end{pmatrix}, \begin{pmatrix} 3 & 1 \\ 0 & 3 \end{pmatrix}, \begin{pmatrix} 2 & 2 \\ 1 & 1 \end{pmatrix}, \begin{pmatrix} 3 & 3 \\ 3 & 4 \end{pmatrix} \right\}$, $V = \mathrm{Mat}_{2\times 2}(\mathbb{R})$.

(c) $B = \left\{ \begin{pmatrix} 6 & 0 \\ 0 & 1 \\ 3 & 3 \end{pmatrix}, \begin{pmatrix} 0 & 1 \\ 5 & 5 \\ 4 & 1 \end{pmatrix}, \begin{pmatrix} 1 & 1 \\ 1 & 1 \\ 1 & 1 \end{pmatrix}, \begin{pmatrix} 2 & 2 \\ 1 & 7 \\ 0 & 3 \end{pmatrix} \right\}$, $V = \mathrm{Mat}_{3\times 2}(\mathbb{R})$.

(d) $B = \left\{ \begin{pmatrix} 2 \\ 1 \\ 3 \end{pmatrix}, \begin{pmatrix} 1 \\ 2 \\ 1 \end{pmatrix}, \begin{pmatrix} 1 \\ 1 \\ 4 \end{pmatrix}, \begin{pmatrix} 1 \\ 5 \\ 1 \end{pmatrix} \right\}$, $V = \mathbb{R}^3$.

(e) $B = \left\{ \begin{pmatrix} 1 \\ 1 \\ 2 \end{pmatrix}, \begin{pmatrix} 2 \\ 2 \\ 0 \end{pmatrix}, \begin{pmatrix} 3 \\ 4 \\ -1 \end{pmatrix} \right\}$, $V = \mathbb{R}^3$.

Solution.

(a) This is not a basis, because B is not linearly independent. In particular, we have

$$(3x^2 - 7x + 5) - 2(x^2 - x + 1) = 3x^2 - 2x^2 - 7x + 2x + 5 - 3 = x^2 - 5x + 3 \in B.$$

(b) This is not a basis, because B is not linearly independent. In particular, we have

$$2\begin{pmatrix} 1 & 2 \\ 2 & 1 \end{pmatrix} + \begin{pmatrix} 3 & 1 \\ 0 & 3 \end{pmatrix} - \begin{pmatrix} 2 & 2 \\ 1 & 1 \end{pmatrix} = \begin{pmatrix} 3 & 3 \\ 3 & 4 \end{pmatrix} \in B.$$

(c) This is not a basis, because $\dim(\mathrm{Mat}_{3\times 2}(\mathbb{R})) = 3 \times 2 = 6$ but this set has 4 vectors.

(d) This is not a basis, because $\dim(\mathbb{R}^3) = 3$ but this set has 4 vectors.

(e) Since we have $|B| = 3 = \dim(R^3)$, it suffices to show that B is linearly independent. We create a matrix A where the columns of A are vectors in B, and find its reduced row-echelon form:

$$\begin{aligned}
A &= \begin{pmatrix} 1 & 2 & 3 \\ 1 & 2 & 4 \\ 2 & 0 & 1 \end{pmatrix} \\
&\sim \begin{pmatrix} 1 & 2 & 3 \\ 0 & 0 & 1 \\ 2 & 0 & 1 \end{pmatrix} && (\rho_2 \mapsto \rho_2 - \rho_1) \\
&\sim \begin{pmatrix} 1 & 2 & 3 \\ 0 & 0 & 1 \\ 0 & -4 & -5 \end{pmatrix} && (\rho_3 \mapsto \rho_3 - \rho_1)
\end{aligned}$$

$$\sim \begin{pmatrix} 1 & 2 & 3 \\ 0 & 0 & 1 \\ 0 & -4 & -5 \end{pmatrix} \qquad (\rho_3 \mapsto \rho_3 - \rho_1)$$

$$\sim \begin{pmatrix} 1 & 2 & 3 \\ 0 & 0 & 1 \\ 0 & 1 & \frac{5}{4} \end{pmatrix} \qquad (\rho_3 \mapsto -\tfrac{1}{4}\rho_3)$$

$$\sim \begin{pmatrix} 1 & 2 & 3 \\ 0 & 0 & 1 \\ 0 & 1 & 0 \end{pmatrix} \qquad (\rho_3 \mapsto \rho_3 - \tfrac{5}{4}\rho_2)$$

$$\sim \begin{pmatrix} 1 & 0 & 3 \\ 0 & 0 & 1 \\ 0 & 1 & 0 \end{pmatrix} \qquad (\rho_1 \mapsto \rho_1 - 2\rho_3)$$

$$\sim \begin{pmatrix} 1 & 0 & 0 \\ 0 & 0 & 1 \\ 0 & 1 & 0 \end{pmatrix} \qquad (\rho_1 \mapsto \rho_1 - 3\rho_2)$$

$$\sim \begin{pmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{pmatrix}. \qquad (\rho_2 \leftrightarrow \rho_3)$$

Since we have row-reduced A to the identity, B is linearly independent and thus B is a basis.

Problem 3 Consider the subset of $\mathbb{R}^4$ given by $X = \left\{ \vec{v}_1 = \begin{pmatrix} 1 \\ 1 \\ 1 \\ 1 \end{pmatrix}, \vec{v}_2 = \begin{pmatrix} 0 \\ 1 \\ 3 \\ -1 \end{pmatrix} \right\}$.

Find vectors $\vec{v}_3$ and $\vec{v}_4$ such that $\{\vec{v}_1, \vec{v}_2, \vec{v}_3, \vec{v}_4\}$ is a basis of $\mathbb{R}^4$.

Possible strategy: Consider the set $S = \{\vec{v}_1, \vec{v}_2, \vec{e}_1, \vec{e}_2, \vec{e}_3, \vec{e}_4\}$, where the $\vec{e}$'s are the natural basis vectors for $\mathbb{R}^4$. Note that we must have $\mathrm{Span}(S) = \mathbb{R}^4$. To conclude, compute a basis for $\mathrm{Span}(S)$ in the usual way.

Note: This is a general strategy for how to **extend** a set of linearly independent vectors to a basis.

Solution.

Let $X' = \{\vec{v_1}, \vec{v_2}, \vec{e_1}, \vec{e_2}, \vec{e_3}, \vec{e_4}\}$, where $e_i \in \mathbb{R}^4$ is defined by a 1 in position i and zeroes elsewhere. Since $\{\vec{e_1}, \vec{e_2}, \vec{e_3}, \vec{e_4}\}$ is a basis for $\mathbb{R}^4$ and thus spans it, X' also spans $\mathbb{R}^4$. So it suffices to find a subset of X' that is linearly independent. Let $A = (\vec{v_1} \mid \vec{v_2} \mid \vec{e_1} \mid \vec{e_2} \mid \vec{e_3} \mid \vec{e_4})$. We can now use the method from class to find a linearly independent set that spans $\mathrm{Col}(A)$. We observe that

$$A = \begin{pmatrix} 1 & 0 & 1 & 0 & 0 & 0 \\ 1 & 1 & 0 & 1 & 0 & 0 \\ 1 & 3 & 0 & 0 & 1 & 0 \\ 1 & -1 & 0 & 0 & 0 & 1 \end{pmatrix}$$

$$\sim \begin{pmatrix} 1 & 0 & 1 & 0 & 0 & 0 \\ 0 & 1 & -1 & 1 & 0 & 0 \\ 1 & 3 & 0 & 0 & 1 & 0 \\ 1 & -1 & 0 & 0 & 0 & 1 \end{pmatrix} \qquad (\rho_2 \mapsto \rho_2 - \rho_1)$$

$$\sim \begin{pmatrix} 1 & 0 & 1 & 0 & 0 & 0 \\ 0 & 1 & -1 & 1 & 0 & 0 \\ 0 & 3 & -1 & 0 & 1 & 0 \\ 1 & -1 & 0 & 0 & 0 & 1 \end{pmatrix} \qquad (\rho_3 \mapsto \rho_3 - \rho_1)$$

$$\sim \begin{pmatrix} 1 & 0 & 1 & 0 & 0 & 0 \\ 0 & 1 & -1 & 1 & 0 & 0 \\ 0 & 3 & -1 & 0 & 1 & 0 \\ 0 & -1 & -1 & 0 & 0 & 1 \end{pmatrix} \qquad (\rho_4 \mapsto \rho_4 - \rho_1)$$

$$\sim \begin{pmatrix} 1 & 0 & 1 & 0 & 0 & 0 \\ 0 & 1 & -1 & 1 & 0 & 0 \\ 0 & 0 & 2 & -3 & 1 & 0 \\ 0 & -1 & -1 & 0 & 0 & 1 \end{pmatrix} \qquad (\rho_3 \mapsto \rho_3 - 3\rho_2)$$

$$\sim \begin{pmatrix} 1 & 0 & 1 & 0 & 0 & 0 \\ 0 & 1 & -1 & 1 & 0 & 0 \\ 0 & 0 & 2 & -3 & 1 & 0 \\ 0 & 0 & -2 & 1 & 0 & 1 \end{pmatrix} \qquad (\rho_4 \mapsto \rho_4 + \rho_2)$$

$$\sim \begin{pmatrix} 1 & 0 & 1 & 0 & 0 & 0 \\ 0 & 1 & -1 & 1 & 0 & 0 \\ 0 & 0 & 2 & -3 & 1 & 0 \\ 0 & 0 & 0 & -2 & 1 & 1 \end{pmatrix} \qquad (\rho_4 \mapsto \rho_4 + \rho_3)$$

$$\sim \begin{pmatrix} 1 & 0 & 1 & 0 & 0 & 0 \\ 0 & 1 & -1 & 1 & 0 & 0 \\ 0 & 0 & 1 & -\frac{3}{2} & \frac{1}{2} & 0 \\ 0 & 0 & 0 & -2 & 1 & 1 \end{pmatrix} \qquad (\rho_3 \mapsto \tfrac{1}{2}\rho_3)$$

$$\sim \begin{pmatrix} 1 & 0 & 1 & 0 & 0 & 0 \\ 0 & 1 & -1 & 1 & 0 & 0 \\ 0 & 0 & 1 & -\frac{3}{2} & \frac{1}{2} & 0 \\ 0 & 0 & 0 & 1 & -\frac{1}{2} & -\frac{1}{2} \end{pmatrix}, \qquad (\rho_4 \mapsto -\tfrac{1}{2}\rho_4)$$

which is in row-echelon form. So the leading ones are in the first, second, third, and fourth column, and thus $\text{Col}(A) = \text{Span}(B)$ for $B = \{\vec{v_1}, \vec{v_2}, \vec{e_1}, \vec{e_2}\}$, and B is linearly independent. Since $\mathbb{R}^4 \subseteq \text{Col}(A)$, we have that

$$B = \left\{ \begin{pmatrix} 1 \\ 1 \\ 1 \\ 1 \end{pmatrix}, \begin{pmatrix} 0 \\ 1 \\ 3 \\ -1 \end{pmatrix}, \begin{pmatrix} 1 \\ 0 \\ 0 \\ 0 \end{pmatrix}, \begin{pmatrix} 0 \\ 1 \\ 0 \\ 0 \end{pmatrix} \right\}$$

is a basis for $\mathbb{R}^4$.

Problem 4 Let $A \in \text{Mat}_{n\times n}(\mathbb{R})$. Show that there exists $c_0, c_1, \ldots, c_{n^2} \in \mathbb{R}$, **not all zero**, such that $c_0 I_n + c_1 A + c_2 A^2 + \cdots + c_{n^2} A^{n^2} = O_{n\times n}$. (What is the dimension of $\text{Mat}_{n\times n}(\mathbb{R})$?)

This problem shows that there exists a polynomial $p(x) \in \mathbb{R}[x]$ with degree at most n^2 such that $p(A) = O_{n\times n}$.

Solution.

Suppose to the contrary that the only solution to $c_0 I_n + c_1 A + c_2 A^2 + \ldots + c_{n^2} A^{n^2}$ is

$$c_0 = c_1 = c_2 = \cdots = c_{n^2} = 0.$$

Then, the dependency equation for the set $I = \{I_n, A, A^2, \ldots, A^{n^2}\}$ has only the trivial solution, and by definition I is a linearly independent subset of $\mathrm{Mat}_{n\times n}(\mathbb{R})$. Since $|I| = n^2 + 1$, we have shown in class that we must have $\dim(\mathrm{Mat}_{n\times n}(\mathbb{R})) \geq n^2 + 1$, a contradiction because we have shown $\dim(\mathrm{Mat}_{n\times n}(\mathbb{R})) = n^2$. $\square$

Problem 5 Let $V = \mathrm{Span}\left\{ \vec{v}_1 = \begin{pmatrix} 1 \\ 0 \\ 1 \\ 1 \end{pmatrix}, \vec{v}_2 = \begin{pmatrix} 2 \\ 1 \\ 1 \\ 2 \end{pmatrix} \right\}$, and let $W = \mathrm{Span}\left\{ \vec{w}_1 = \begin{pmatrix} 0 \\ 1 \\ 1 \\ 0 \end{pmatrix}, \vec{w}_2 = \begin{pmatrix} 2 \\ 0 \\ 1 \\ 2 \end{pmatrix} \right\}$. This problem will lead you through a computation of bases and dimensions for the subspaces $V \cap W$ and $V + W$.

Note: The vectors in V and W are linearly independent, so we have $\dim(V) = \dim(W) = 2$. The dimensions that you obtain for $V \cap W$ and $V + W$ should satisfy

$$4 = 2 + 2 = \dim(V) + \dim(W) = \dim(V + W) + \dim(V \cap W).$$

(a) Find $\dim(V + W)$ and a basis for $V + W$.

Note: To find a basis for $V + W$, it suffices to find a basis for $\mathrm{Span}\{\vec{v}_1, \vec{v}_2, \vec{w}_1, \vec{w}_2\}$. The general fact that we are using here (which you do not have to prove) is that $\mathrm{Span}\{\vec{a}_1, \ldots, \vec{a}_n\} + \mathrm{Span}\{\vec{b}_1, \ldots, \vec{b}_m\} = \mathrm{Span}\{\vec{a}_1, \ldots, \vec{a}_n, \vec{b}_1, \ldots, \vec{b}_m\}$.

(b) Find a homogeneous system $B\vec{x} = \vec{0}$ such that $\mathrm{Null}(B) = W$.

Note: As an example to show how to proceed, we find a homogeneous system $A\vec{x} = \vec{0}$ such that $\mathrm{Null}(A) = V$ as follows. We observe that $\vec{u} = \begin{pmatrix} a \\ b \\ c \\ d \end{pmatrix} \in V$ if and only if the system $(\vec{v}_1 | \vec{v}_2)\vec{x} = \vec{u}$ is consistent. One can row-reduce the augmented matrix for the system to obtain

$$\left(\begin{array}{cc|c} 1 & 2 & a \\ 0 & 1 & b \\ 1 & 1 & c \\ 1 & 2 & d \end{array}\right) \longrightarrow \left(\begin{array}{cc|c} 1 & 2 & a \\ 0 & 1 & b \\ 0 & 0 & c - a + b \\ 0 & 0 & d - a \end{array}\right).$$

Therefore, the system is consistent if and only if $-a + b + c = 0$ and $-a + d = 0$. These equations imply that

$$V = \left\{ \begin{pmatrix} a \\ b \\ c \\ d \end{pmatrix} : -a + b + c = 0 \text{ and } -a + d = 0 \right\} = \mathrm{Null}\begin{pmatrix} -1 & 1 & 1 & 0 \\ -1 & 0 & 0 & 1 \end{pmatrix}.$$

(c) Use the results from part (a) (the matrix A that I computed and the matrix B that you computed) to find $\dim(V \cap W)$ and a basis for $V \cap W$.

Note: Part (b) shows that $\vec{u} \in V \cap W$ if and only if $\vec{u} \in \text{Null}(A) \cap \text{Null}(B)$. Therefore, we have $\vec{u} \in V \cap W$ if and only if it satisfies both $A\vec{x} = \vec{0}$ and $B\vec{x} = \vec{0}$. To conclude, it suffices to compute a basis for Null $\left(\frac{A}{B}\right)$, where $\left(\frac{A}{B}\right)$ is the matrix obtained by "placing A on top of B".

Solution.

(a) To find a basis for $V + W$, it suffices to find a basis for $\text{Span}\{\vec{v_1}, \vec{v_2}, \vec{w_1}, \vec{w_2}\}$. Let $A = (\vec{v_1} \mid \vec{v_2} \mid \vec{w_1} \mid \vec{w_2})$. We can now use the method from class to find a linearly independent set that spans $\text{Col}(A)$. We observe that

$$A = \begin{pmatrix} 1 & 2 & 0 & 2 \\ 0 & 1 & 1 & 0 \\ 1 & 1 & 1 & 1 \\ 1 & 2 & 0 & 2 \end{pmatrix}$$

$$\sim \begin{pmatrix} 1 & 2 & 0 & 2 \\ 0 & 1 & 1 & 0 \\ 0 & -1 & 1 & -1 \\ 1 & 2 & 0 & 2 \end{pmatrix} \qquad (\rho_3 \mapsto \rho_3 - \rho_1)$$

$$\sim \begin{pmatrix} 1 & 2 & 0 & 2 \\ 0 & 1 & 1 & 0 \\ 0 & -1 & 1 & -1 \\ 0 & 0 & 0 & 0 \end{pmatrix} \qquad (\rho_4 \mapsto \rho_4 - \rho_1)$$

$$\sim \begin{pmatrix} 1 & 2 & 0 & 2 \\ 0 & 1 & 1 & 0 \\ 0 & 0 & 2 & -1 \\ 0 & 0 & 0 & 0 \end{pmatrix} \qquad (\rho_3 \mapsto \rho_3 + \rho_2)$$

$$\sim \begin{pmatrix} 1 & 2 & 0 & 2 \\ 0 & 1 & 1 & 0 \\ 0 & 0 & 1 & -\frac{1}{2} \\ 0 & 0 & 0 & 0 \end{pmatrix}, \qquad (\rho_3 \mapsto \tfrac{1}{2}\rho_3)$$

which is in row-echelon form. So the leading ones are in the first, second, and third column, and thus $\text{Col}(A) = \text{Span}(B)$ for $B = \{\vec{v_1}, \vec{v_2}, \vec{w_1}\}$, and B is linearly independent. Since $V + W \subseteq \text{Col}(A)$, we have that

$$B = \left\{ \begin{pmatrix} 1 \\ 0 \\ 1 \\ 1 \end{pmatrix}, \begin{pmatrix} 2 \\ 1 \\ 1 \\ 2 \end{pmatrix}, \begin{pmatrix} 0 \\ 1 \\ 1 \\ 0 \end{pmatrix} \right\}$$

is a basis for $V + W$. Since $|B| = 3$, $\dim(V + W) = 3$.

(b) We observe that

$$\begin{pmatrix} a \\ b \\ c \\ d \end{pmatrix} \in W \iff \exists x_1, x_2 \in \mathbb{R} : \begin{pmatrix} a \\ b \\ c \\ d \end{pmatrix} = x_1 \begin{pmatrix} 0 \\ 1 \\ 1 \\ 0 \end{pmatrix} + x_2 \begin{pmatrix} 2 \\ 0 \\ 1 \\ 2 \end{pmatrix} \iff \left\{ \begin{array}{ccccc} & & 2x_2 & = & a \\ x_1 & & & = & b \\ x_1 & + & x_2 & = & c \\ & & 2x_2 & = & d \end{array} \right. \text{ is consistent.}$$

Let $B = (\vec{w_1} \mid \vec{w_2})$. Then, we can represent the above situation by augmenting the solution to B and observing that

$$\begin{aligned}
&\left(\begin{array}{cc|c} 0 & 2 & a \\ 1 & 0 & b \\ 1 & 1 & c \\ 0 & 2 & d \end{array}\right) \\
\sim &\left(\begin{array}{cc|c} 1 & 1 & c \\ 1 & 0 & b \\ 0 & 2 & a \\ 0 & 2 & d \end{array}\right) && (\rho_1 \leftrightarrow \rho_3) \\
\sim &\left(\begin{array}{cc|c} 1 & 0 & b \\ 1 & 1 & c \\ 0 & 2 & a \\ 0 & 2 & d \end{array}\right) && (\rho_1 \leftrightarrow \rho_2) \\
\sim &\left(\begin{array}{cc|c} 1 & 0 & b \\ 0 & 1 & c-b \\ 0 & 2 & a \\ 0 & 2 & d \end{array}\right) && (\rho_2 \mapsto \rho_2 - \rho_1) \\
\sim &\left(\begin{array}{cc|c} 1 & 0 & b \\ 0 & 1 & c-b \\ 0 & 2 & a \\ 0 & 0 & d-a \end{array}\right) && (\rho_4 \mapsto \rho_4 - \rho_3) \\
\sim &\left(\begin{array}{cc|c} 1 & 0 & b \\ 0 & 1 & c-b \\ 0 & 0 & a-2c+2b \\ 0 & 0 & d-a \end{array}\right), && (\rho_3 \mapsto \rho_3 - 2\rho_2)
\end{aligned}$$

which is in reduced row-echelon form. So the system is consistent if and only if $a-2c+2b = 0$ and $d-a = 0$, which we can represent with a homogenous system

$$\begin{pmatrix} 1 & 0 & -2 & 2 \\ -1 & 0 & 0 & 1 \end{pmatrix} \begin{pmatrix} a \\ b \\ c \\ d \end{pmatrix} = \begin{pmatrix} 0 \\ 0 \end{pmatrix}.$$

So we have $\text{Null}(B) = W$, where $B = \begin{pmatrix} 1 & 0 & -2 & 2 \\ -1 & 0 & 0 & 1 \end{pmatrix}$.

(c) It suffices to compute a basis for

$$\text{Null}\begin{pmatrix} \frac{A}{B} \end{pmatrix} = \text{Null}\begin{pmatrix} -1 & 1 & 1 & 0 \\ -1 & 0 & 0 & 1 \\ 1 & 0 & -2 & 2 \\ -1 & 0 & 0 & 1 \end{pmatrix}.$$

We observe that

$$\begin{aligned}
\left(\frac{A}{B} \mid \vec{0}\right) &= \left(\begin{array}{cccc|c} -1 & 1 & 1 & 0 & 0 \\ -1 & 0 & 0 & 1 & 0 \\ 1 & 0 & -2 & 2 & 0 \\ -1 & 0 & 0 & 1 & 0 \end{array}\right) \\
&\sim \left(\begin{array}{cccc|c} 1 & -1 & -1 & 0 & 0 \\ -1 & 0 & 0 & 1 & 0 \\ 1 & 0 & -2 & 2 & 0 \\ -1 & 0 & 0 & 1 & 0 \end{array}\right) && (\rho_1 \mapsto -\rho_1) \\
&\sim \left(\begin{array}{cccc|c} 1 & -1 & -1 & 0 & 0 \\ 0 & -1 & -1 & 1 & 0 \\ 1 & 0 & -2 & 2 & 0 \\ -1 & 0 & 0 & 1 & 0 \end{array}\right) && (\rho_2 \mapsto \rho_2 + \rho_1) \\
&\sim \left(\begin{array}{cccc|c} 1 & -1 & -1 & 0 & 0 \\ 0 & -1 & -1 & 1 & 0 \\ 0 & 0 & -2 & 3 & 0 \\ -1 & 0 & 0 & 1 & 0 \end{array}\right) && (\rho_3 \mapsto \rho_3 + \rho_4) \\
&\sim \left(\begin{array}{cccc|c} 1 & -1 & -1 & 0 & 0 \\ 0 & -1 & -1 & 1 & 0 \\ 0 & 0 & -2 & 3 & 0 \\ 0 & -1 & -1 & 1 & 0 \end{array}\right) && (\rho_4 \mapsto \rho_4 + \rho_1) \\
&\sim \left(\begin{array}{cccc|c} 1 & -1 & -1 & 0 & 0 \\ 0 & -1 & -1 & 1 & 0 \\ 0 & 0 & -2 & 3 & 0 \\ 0 & 0 & 0 & 0 & 0 \end{array}\right) && (\rho_4 \mapsto \rho_4 - \rho_2) \\
&\sim \left(\begin{array}{cccc|c} 1 & -1 & -1 & 0 & 0 \\ 0 & 1 & 1 & -1 & 0 \\ 0 & 0 & -2 & 3 & 0 \\ 0 & 0 & 0 & 0 & 0 \end{array}\right) && (\rho_2 \mapsto -\rho_2) \\
&\sim \left(\begin{array}{cccc|c} 1 & -1 & -1 & 0 & 0 \\ 0 & 1 & 1 & -1 & 0 \\ 0 & 0 & 1 & -\frac{3}{2} & 0 \\ 0 & 0 & 0 & 0 & 0 \end{array}\right) && (\rho_3 \mapsto -\rho_3) \\
&\sim \left(\begin{array}{cccc|c} 1 & 0 & 0 & -1 & 0 \\ 0 & 1 & 1 & -1 & 0 \\ 0 & 0 & 1 & -\frac{3}{2} & 0 \\ 0 & 0 & 0 & 0 & 0 \end{array}\right) && (\rho_1 \mapsto \rho_1 + \rho_2)
\end{aligned}$$

$$\sim \left(\begin{array}{cccc|c} 1 & 0 & 0 & -1 & 0 \\ 0 & 1 & 0 & \frac{1}{2} & 0 \\ 0 & 0 & 1 & -\frac{3}{2} & 0 \\ 0 & 0 & 0 & 0 & 0 \end{array}\right), \qquad (\rho_2 \mapsto \rho_2 - \rho_3)$$

which is in reduced row-echelon form. So we have

$$\text{Null}\begin{pmatrix} A \\ B \end{pmatrix} = \text{Span}\left\{\begin{pmatrix} 1 \\ -\frac{1}{2} \\ \frac{3}{2} \\ 0 \end{pmatrix}\right\}$$

is a basis for $V \cap W$, and thus $\dim(V \cap W) = 1$. □

Problem 7 Let $n \geq 1$ and let $B_n = \{f(x) \in \mathbb{R}_n[x] : f'(x) = f'(-x)\}$, where the prime notation denotes the derivative. It is a fact that B_n is a subspace of $\mathbb{R}_n[x]$. **You do not have to prove this.**

(a) (**Warm-up**): Compute $\dim(B_1)$ and $\dim(B_2)$, and find bases for B_1 and B_2.

(b) Compute $\dim(B_n)$, and describe a basis for B_n.

Solution.

(b) Let

$$f(x) = c_n x^n + c_{n-1}x^{n-1} + \cdots + c_1 x + c_0 = \sum_{k=0}^{n} c_k x^k.$$

Then, differentiating term by term, we have

$$f'(x) = \sum_{k=0}^{n} kc_k x^{k-1}, \quad f'(-x) = \sum_{k=0}^{n} kc_k(-x)^{k-1}.$$

Since $f'(x) = f'(-x)$, we have $kc_k x^{k-1} = kc_k(-x)^{k-1}$ for all $k \in \{0, 1, \ldots, n\}$ and all $x \in \mathbb{R}$. When $2 \mid (k-1)$, this is true for all c_k, but when $2 \mid k$, we must have $c_k = 0$ unless $k = 0$. So we can write

$$f(x) = c_0 + c_1 x + c_3 x^3 + c_5 x^5 + \cdots + g(x),$$

where $g(x) = \begin{cases} c_n x^n & \text{if } n \text{ is odd} \\ c_{n-1}x^{n-1} & \text{if } n \text{ is even} \end{cases}$. So a basis for B_n is

$$S = \{1, x, x^3, x^5, \ldots, g(x)\},$$

and we have

$$|S| = \dim(B_n) = \left\lceil \frac{n}{2} \right\rceil + 1.$$

(a) From (b), we have $\{1, x\}$ is a basis for both B_1 and B_2, and so $\dim(B_1) = \dim(B_2) = 2$.