MATH 552: Section 001
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MATH 552 Homework 11*

Problem 8 Rederive the Maclaurin series (4) in Sec. 64 for the function $f(z) = \cos z$ by
(a) using the definition

$$\cos z = \frac{e^{iz} + e^{-iz}}{2}$$

in Sec. 37 and appealing to the Maclaurin series (2) for e^z in Sec. 64;
(b) showing that

$$f^{(2n)}(0) = (-1)^n \text{ and } f^{(2n+1)}(0) = 0 \quad (n = 0, 1, 2, ...).$$

Solution.

(a)

$$\begin{aligned}
\cos z &= \frac{1}{2}(e^{iz} + e^{-iz}) && \text{(definition)}\\
&= \frac{1}{2}\left[\left(1 + \frac{iz}{1!} + \frac{(iz)^2}{2!} + \frac{(iz)^3}{3!} + \frac{(iz)^4}{4!} + ...\right) + \left(1 + \frac{-iz}{1!} + \frac{(-iz)^2}{2!} + \frac{(-iz)^3}{3!} + \frac{(-iz)^4}{4!} + ...\right)\right]\\
& && \text{(using series (2))}\\
&= \frac{1}{2}\left[(1+1) + \left(\frac{iz}{1!} + \frac{-iz}{1!}\right) + \left(\frac{i^2z^2}{2!} + \frac{i^2z^2}{2!}\right) + \left(\frac{i^3z^3}{3!} + \frac{-i^3z^3}{3!}\right) + \left(\frac{i^4z^4}{4!} + \frac{i^4z^4}{4!}\right) + ...\right]\\
& && \text{(rearranging and combining powers)}\\
&= 1 + \frac{i^2z^2}{2!} + \frac{i^4z^4}{4!} + ... && \text{(combining like terms and dividing by 2)}\\
&= 1 - \frac{z^2}{2!} + \frac{z^4}{4!} + ... && \text{(using } i^2 = 1\text{; this is series (4))}
\end{aligned}$$

(b) Using repeated differentiation, we have

$$\frac{d}{dz}[\cos z] = -\sin z \tag{1}$$

$$\frac{d^2}{dz^2}[\cos z] = -\cos z \tag{2}$$

$$\frac{d^3}{dz^3}[\cos z] = \sin z \tag{3}$$

$$\frac{d^4}{dz^4}[\cos z] = \cos z \tag{4}$$

Thus, since the differentiation cycles every 4 times, we have

$$\frac{d^{(n)}}{dz^{(n)}}[\cos z] = \frac{d^{(n \bmod 4)}}{dz^{(n \bmod 4)}}[\cos z].$$

So for $\frac{d^{(n)}}{dz^{(n)}}[\cos z]$, any odd n (which $2n+1$ will be for any $n \in \mathbb{N}$) will result in either $-\sin z$ or $\sin z$. Since $\pm \sin 0 = 0$, $f^{(2n+1)}(0) = 0$.

Additionally, for $\frac{d^{(n)}}{dz^{(n)}}[\cos z]$, any even n (which $2n$ will be for any $n \in \mathbb{N}$) will result in $\cos z$ if it is divisible by 4 or $-\cos z$ if it is not. So starting from $f^{(0)}$, the even powers will start at $\cos z$ and then switch back and forth from negative to positive. So $f^{(2n)}(0)$ will start at 1 and then switch back and forth from positive to negative. Since $(-1)^n$ has the same behavior, $f^{(2n)}(0) = (-1)^n$.

Using the coefficients, it follows that

$$\cos z = 1 - \frac{z^2}{2!} + \frac{z^4}{4!} + ...$$

Problem 11 Show that when $0 < |z| < 4$,

$$\frac{1}{4z - z^2} = \frac{1}{4z} + \sum_{n=0}^{\infty} \frac{z^n}{4^{n+2}}.$$

Solution.

$$\begin{aligned}
\frac{1}{4z - z^2} &= \frac{1}{z(4-z)} \\
&= \frac{1}{4z} + \frac{1}{4(4-z)} && \text{(using partial fractions)} \\
&= \frac{1}{4z} + \frac{1}{4}\left(\frac{1/4}{1 - z/4}\right) && \text{(multiplying fraction by } \tfrac{1/4}{1/4}\text{)} \\
&= \frac{1}{4z} + \frac{1}{16}\left(\frac{1}{1 - \frac{z}{4}}\right) && \text{(writing in form of power series)} \\
&= \frac{1}{4z} + \frac{1}{16}\sum_{n=0}^{\infty}\left(\frac{z}{4}\right)^n && \text{(replacing with power series)} \\
&= \frac{1}{4z} + \sum_{n=0}^{\infty}\frac{z^n}{4^{n+2}} && \text{(bringing constant inside sum)}
\end{aligned}$$

Problem 3 Find the Laurent series that represents the function $f(z)$ in Example 1, Sec. 68, when $1 < |z| < \infty$.

Solution.

$$\begin{aligned}
f(z) &= \frac{1}{z(1+z^2)} \\
&= \frac{1}{z}\left(\frac{1}{1+z^2}\right)\left(\frac{1/z^2}{1/z^2}\right) \\
&= \frac{1}{z^3}\left(\frac{1}{\frac{1}{z^2} + 1}\right) && \text{(multiplying fractions)} \\
&= \frac{1}{z^3}\left(\frac{1}{1 - (-\frac{1}{z^2})}\right) && \text{(writing in power series form: } |z| > 1 \Rightarrow \left|\tfrac{1}{z^2}\right| < 1\text{)} \\
&= \frac{1}{z^3}\left(1 + \left(-\frac{1}{z^2}\right) + \left(-\frac{1}{z^2}\right)^2 + \left(-\frac{1}{z^2}\right)^3 + ...\right) && \text{(power series expansion)} \\
&= \frac{1}{z^3}\left(1 - \frac{1}{z^2} + \frac{1}{z^4} - \frac{1}{z^6} + ...\right) \\
&= \frac{1}{z^3} - \frac{1}{z^5} + \frac{1}{z^7} - \frac{1}{z^9} + ...
\end{aligned}$$

Problem 4 Give two Laurent series expansions in powers of z for the function

$$f(z) = \frac{1}{z^2(1-z)},$$

and specify the regions in which those expansions are valid.

The function has singularities at $z = 0$ and $z = 1$. So $f(z)$ has a Laurent series that is valid in the region $0 < |z| < 1$ and another series valid in the region $1 < |z| < \infty$.

For $0 < |z| < 1$:

$$\begin{aligned}
f(z) &= \frac{1}{z^2(1-z)} \\
&= \frac{1}{z^2}\left(\frac{1}{1-z}\right) && \text{(writing in form of power series)} \\
&= \frac{1}{z^2}(1 + z + z^2 + z^3 + z^4 + ...) && \text{(power series expansion: } |z| < 1\text{)} \\
&= \frac{1}{z^2} + \frac{1}{z} + 1 + z + z^2 + ...
\end{aligned}$$

For $1 < |z| < \infty$:

$$\begin{aligned}
f(z) &= \frac{1}{z^2(1-z)} \\
&= \frac{1}{z^2}\left(\frac{1}{1-z}\right)\left(\frac{1/z}{1/z}\right) \\
&= \frac{1}{z^3}\left(\frac{1}{\frac{1}{z} - 1}\right) \\
&= -\frac{1}{z^3}\left(\frac{1}{1 - \frac{1}{z}}\right) && \text{(writing in form of power series: } |z| > 1 \Rightarrow \left|\tfrac{1}{z}\right| < 1\text{)} \\
&= -\frac{1}{z^3}\left(1 + \left(-\frac{1}{z}\right) + \left(-\frac{1}{z}\right)^2 + \left(-\frac{1}{z}\right)^3 + \left(-\frac{1}{z}\right)^4 + ...\right) && \text{(power series expansion)} \\
&= -\frac{1}{z^3}\left(1 - \frac{1}{z} + \frac{1}{z^2} - \frac{1}{z^3} + \frac{1}{z^4} + ...\right) \\
&= -\frac{1}{z^3} + \frac{1}{z^4} - \frac{1}{z^5} + \frac{1}{z^6} - \frac{1}{z^7} + ...
\end{aligned}$$