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MATH 546: Section
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October 17, 2023

MATH 546 Homework 6

Problem 1 List all possible orders of elements of the group $G = S_8$. For each possible order, give an example of an element that has that order. Explain why no other orders are possible.

Solution.

For any $\sigma \in S_8$, we can find the disjoint cycle decomposition $\sigma = \tau_1\tau_2 \dots \tau_k$ such that $1 \le k \le 8$, where τ_i has length ℓ_i, $1 \le \ell_i \le 8$ (we are including cycles of length one). Then, we have $o(\sigma) = \text{lcm}(s_1, s_2, \dots, s_k)$. Since every element must be represented, we have $s_1 + s_2 + \cdots + s_k = 8$.

We can find the partitions of 8 using combinatorics, find the set of unique elements used in the partition, and then take the lcm of these elements. For each partition, we will give an example element.

Partition	**Set**	lcm	**Example Element with Order** lcm
$1+1+1+1+1+1+1+1$	$\{1\}$	1	(1)(2)(3)(4)(5)(6)(7)(8)
$2+1+1+1+1+1+1$	$\{1,2\}$	2	(1 2)(3)(4)(5)(6)(7)(8)
$2+2+1+1+1+1$	$\{1,2\}$	2	(1 2)(3 4)(5)(6)(7)(8)
$2+2+2+1+1$	$\{1,2\}$	2	(1 2)(3 4)(5 6)(7)(8)
$2+2+2+2$	$\{2\}$	2	(1 2)(3 4)(5 6)(7 8)
$3+1+1+1+1+1$	$\{1,3\}$	3	(1 2 3)(4)(5)(6)(7)(8)
$3+3+1+1$	$\{1,3\}$	3	(1 2 3)(4 5 6)(7)(8)
$4+1+1+1+1$	$\{1,4\}$	4	(1 2 3 4)(5)(6)(7)(8)
$4+2+1+1$	$\{1,2,4\}$	4	(1 2 3 4)(5 6)(7)(8)
$4+2+2$	$\{2,4\}$	4	(1 2 3 4)(5 6)(7 8)
$4+4$	$\{4\}$	4	(1 2 3 4)(5 6 7 8)
$5+1+1+1$	$\{1,5\}$	5	(1 2 3 4 5)(6)(7)(8)
$3+2+1+1+1$	$\{1,2,3\}$	6	(1 2 3)(4 5)(6)(7)(8)
$3+2+2+1$	$\{1,2,3\}$	6	(1 2 3)(4 5)(6 7)(8)
$3+3+2$	$\{2,3\}$	6	(1 2 3)(4 5 6)(7 8)
$6+1+1$	$\{1,6\}$	6	(1 2 3 4 5 6)(7)(8)
$6+2$	$\{2,6\}$	6	(1 2 3 4 5 6)(7 8)
$7+1$	$\{1,7\}$	7	(1 2 3 4 5 6 7)(8)
8	$\{8\}$	8	(1 2 3 4 5 6 7 8)
$5+2+1$	$\{1,2,5\}$	10	(1 2 3 4 5)(6 7)(8)
$4+3+1$	$\{3,4\}$	12	(1 2 3 4)(5 6 7)(8)
$5+3$	$\{3,5\}$	15	(1 2 3 4 5)(6 7 8)

So the possible orders are $1, 2, 3, 4, 5, 6, 7, 8, 10, 12, 15$. There cannot be any other cycles because we have listed all lengths of cycles that add up to 8 and calculated their orders using the lcm of the partition. $\square$

Problem 2 Let $\tau \in S_n$ be the cycle $(1\ 2\ \dots\ k)$. Prove that for all $\sigma \in S_n$,

$$\sigma\tau\sigma^{-1} = (\sigma(1)\ \sigma(2)\ \dots\ \sigma(k)).$$

Solution.

We observe from the defintion of τ that for all $1 \le j \le k$,

$$\tau(j) = (j+1) \mod k,$$

and for all $k < j \le n$,

$$\tau(j) = j.$$

Let $j \in \{1, 2, \dots, n\}$. If $1 \le j \le k$, we can use associativity and inverses to write

$$\sigma\tau\sigma^{-1}(\sigma(j)) = \sigma\tau(j) = \sigma((j+1) \mod k).$$

Similarly, if $k < j \le n$, we can write

$$\sigma\tau\sigma^{-1}(\sigma(j)) = \sigma\tau(j) = \sigma(j).$$

So $\sigma(j)$ is a cycle of length k for $j \le k$ under this permutation, and maps to itself for $j > k$. Thus, we have

$$\sigma\tau\sigma^{-1} = (\sigma(1)\ \sigma(2)\ \dots\ \sigma(k)).$$

$\square$

Problem 3 Let $G = S_4$, and let

$$H = \{\sigma \in S_4 \mid \sigma^2 = e\}$$

where e denotes the identity element of S_4.

(a) List all the elements of H.

(b) Decide whether H is a subgroup of S_4 or not. Justify your answer.

Solution.

(a) Since $e^2 = e$, $e \in H$. This is the only element with order 1, so all other elements will have order 2. This happens when the lcm of the lengths of all cycles is 2, so any element of S (other than e) will either have one cycle of length 2 and two cycles of length 1, or two cycles of length 2 (this is the only way to add numbers to get to 4 with lcm 2).

We will list the elements (we note that $(a\ b) = (b\ a)$ since this means a and b map to each other):

- $(1)(2)(3)(4)$
- $(1\ 2)(3)(4)$
- $(1\ 3)(2)(4)$
- $(1\ 4)(2)(3)$
- $(2\ 3)(1)(4)$

- (2 4)(1)(3)
- (3 4)(1)(2)
- (1 2)(3 4)
- (1 3)(2 4)
- (1 4)(2 3).

(b) This is not a subgroup. In particular, closure fails. Consider

$$\sigma_1 = (1\ 2),\ \sigma_2 = (2\ 3),$$

which are both in H as shown in (a). Then we have

$$\begin{aligned}\sigma_1\sigma_2(1) &= \sigma_1(1) = 2,\\ \sigma_1\sigma_2(2) &= \sigma_1(3) = 3,\\ \sigma_1\sigma_2(3) &= \sigma_1(2) = 1,\\ \sigma_1\sigma_2(4) &= \sigma_1(4) = 4.\end{aligned}$$

So we can write

$$\sigma_1\sigma_2 = (1\ 2\ 3)(4).$$

But $\operatorname{lcm}(3,1) = 3$, so the order of $\sigma_1\sigma_2 = 3$ and thus $(\sigma_1\sigma_2)^2 \neq e$. Therefore, $\sigma_1\sigma_2 \notin H$, so H is not a subgroup. □

Problem 4 Recall that for a group G, $Z(G)$ means the set

$$\{x \in G \mid ax = xa\ \forall a \in G\}.$$

(a) Find $Z(G)$ for $G = S_3$. Prove your answer.

(b) If $G = S_4$ and $x = (1\ 2\ 3\ 4)$, prove that $x \notin Z(G)$.

Solution.

(a) We have

$$S_3 = \{(1), (1\ 2), (2\ 3), (1\ 3), (1\ 2\ 3), (1\ 3\ 2)\}.$$

We can check each element:

- Since we have proven S_n is a group and shown (1) is the identity, it must commute with everything. So this is in $Z(G)$.
- Let $\sigma_1 = (1\ 2), \sigma_2 = (2\ 3)$. Then $\sigma_1\sigma_2(1) = 2$ but $\sigma_2\sigma_1(1) = 3$, so (1 2) and (2 3) do not commute. Thus neither commute with everything, so they are not in $Z(G)$.
- Let $\sigma_1 = (1\ 3)$, $\sigma_2 = (1\ 2\ 3)$. Then $\sigma_1\sigma_2(1) = 2$ but $\sigma_2\sigma_1(1) = 1$, so (1 3) and (1 2 3) do not commute. Thus neither are in $Z(G)$.
- Let $\sigma_1 = (1\ 3)$, $\sigma_2 = (1\ 3\ 2)$. Then $\sigma_1\sigma_2(1) = 1$ but $\sigma_2\sigma_1(1) = 2$, so (1 3 2) does not commute with everything. Thus it is not in $Z(G)$.

We have now checked every element. Therefore, $Z(G) = \{(1)\}$.

(b) We have $x = (1\ 2\ 3\ 4)$. Consider $\sigma = (1\ 2) \in S_4$. Then $x\sigma(1) = 3$ but $\sigma x(1) = 1$. So x does not commute with everything in S_4, and therefore $x \notin Z(G)$.

Problem 5 Let $G = S_4$. Assume that H is a subgroup of S_4 such that every cycle of length 2 in S_4 belongs to H. Prove that H must be equal to the entire S_4.

Solution.

We will first prove a lemma: any permutation in S_n can be written as the product of cycles of length 2 (which we will call a transposition) in S_n. Since we have shown that every permutation is the product of disjoint cycles, it suffices to show that any cycle can be expressed as a product of transpositions. The identity (1) can be expressed as (1 2)(2 1), and for any other cycle, we can write

$$(a_1\ a_2\ \ldots\ a_k) = (a_1\ a_2)(a_2\ a_3)\ldots(a_{k-1}\ a_k).$$

Thus, any permutation can be written as the product of transpositions.

We now prove $H = S_4$. Since H is a subgroup of S_4, $H \subseteq S_4$ is obvious. To show $S_4 \subseteq H$, let $\sigma \in S_4$. By the lemma, σ is the product of transpositions in S_4. Since H contains all transpositions in S_4 by definition, we have from closure that $\sigma \in H$. So $S_4 \subseteq H$, and therefore $H = S_4$. $\square$