Nathan Bickel
MATH 544: Section H01
Professor: Dr. Boylan
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MATH 544 Homework 9

Problem 1

(a) Find A, $B \in \mathrm{Mat}_{2\times 2}(\mathbb{R})$ such that $\det(A+B) \neq \det(A) + \det(B)$.

(b) Find C, $D \in \mathrm{Mat}_{2\times 2}(\mathbb{R})$ such that $\det(C+D) = \det(C) + \det(D)$.

Note. This problem illustrates how det is not *additive* in general. In contrast, it is *multiplicative*: for all A, $B \in \mathrm{Mat}_{n\times n}(\mathbb{R})$, we have $\det(AB) = \det(A)\det(B)$.

Solution.

(a) Let $A = \begin{pmatrix} 1 & 0 \\ 1 & 0 \end{pmatrix}$ and $B = \begin{pmatrix} 1 & 1 \\ 0 & 0 \end{pmatrix}$. Then, we have

$$\det(A+B) = \begin{vmatrix} 2 & 1 \\ 1 & 0 \end{vmatrix} = -1 \neq 0 = 0 + 0 = \begin{vmatrix} 1 & 0 \\ 1 & 0 \end{vmatrix} + \begin{vmatrix} 1 & 1 \\ 0 & 0 \end{vmatrix} = \det(A) + \det(B).$$

(b) Let $C = D = \begin{pmatrix} 0 & 0 \\ 0 & 0 \end{pmatrix}$. Then, we have

$$\det(C+D) = \begin{vmatrix} 0 & 0 \\ 0 & 0 \end{vmatrix} = 0 = 0 + 0 = \begin{vmatrix} 0 & 0 \\ 0 & 0 \end{vmatrix} + \begin{vmatrix} 0 & 0 \\ 0 & 0 \end{vmatrix} = \det(C) + \det(D).$$

Problem 2 Compute the following determinants, as instructed.

(a) Use a Laplace expansion (of your choice) to compute $\begin{vmatrix} 2 & 1 & -1 & 2 \\ 3 & 0 & 0 & 1 \\ 2 & 1 & 2 & 0 \\ 3 & 1 & 1 & 2 \end{vmatrix}$.

(b) Let $A = \begin{pmatrix} 1 & 3 & 1 & 2 \\ 1 & 3 & 1 & 3 \\ -1 & 2 & 1 & 2 \\ 2 & 4 & -2 & -2 \end{pmatrix}$

i. Compute $|A|$ by using elementary *row* operations to convert A to an *upper triangular* matrix (a matrix with all zeros *below* the main diagonal).

ii. Compute $|A|$ by using elementary *column* operations to convert A to a *lower triangular* matrix (a matrix with all zeros *above* the main diagonal).

Solution.

(a) We have

$$\begin{aligned}
\begin{vmatrix} 2 & 1 & -1 & 2 \\ 3 & 0 & 0 & 1 \\ 2 & 1 & 2 & 0 \\ 3 & 1 & 1 & 2 \end{vmatrix} &= 3\begin{vmatrix} 1 & -1 & 2 \\ 1 & 2 & 0 \\ 1 & 1 & 2 \end{vmatrix} - \begin{vmatrix} 2 & 1 & -1 \\ 2 & 1 & 2 \\ 3 & 1 & 1 \end{vmatrix} \\
&= -3\left(2\begin{vmatrix} 1 & 2 \\ 1 & 1 \end{vmatrix} + 2\begin{vmatrix} 1 & -1 \\ 1 & 2 \end{vmatrix}\right) + \left(2\begin{vmatrix} 1 & 2 \\ 1 & 1 \end{vmatrix} - \begin{vmatrix} 2 & 2 \\ 3 & 1 \end{vmatrix} - \begin{vmatrix} 2 & 1 \\ 3 & 1 \end{vmatrix}\right) \\
&= -3\big(2(1-2) + 2(2+1)\big) + \big(2(1-2) - (2-6) - (2-3)\big) \\
&= -3(-2+6) + (-2+4+1) \\
&= -3(4) + 3 = -9.
\end{aligned}$$

(b)

i. We have

$$\begin{aligned}
\begin{vmatrix} 1 & 3 & 1 & 2 \\ 1 & 3 & 1 & 3 \\ -1 & 2 & 1 & 2 \\ 2 & 4 & -2 & -2 \end{vmatrix} &= \begin{vmatrix} 1 & 3 & 1 & 2 \\ 0 & 0 & 0 & 1 \\ -1 & 2 & 1 & 2 \\ 2 & 4 & -2 & -2 \end{vmatrix} && (\rho_2 \mapsto \rho_2 - \rho_1) \\
&= \begin{vmatrix} 1 & 3 & 1 & 2 \\ 0 & 0 & 0 & 1 \\ 0 & 5 & 2 & 4 \\ 2 & 4 & -2 & -2 \end{vmatrix} && (\rho_3 \mapsto \rho_3 + \rho_1) \\
&= \begin{vmatrix} 1 & 3 & 1 & 2 \\ 0 & 0 & 0 & 1 \\ 0 & 5 & 2 & 4 \\ 0 & -2 & -4 & -6 \end{vmatrix} && (\rho_4 \mapsto \rho_4 - 2\rho_1) \\
&= -\begin{vmatrix} 1 & 3 & 1 & 2 \\ 0 & -2 & -4 & -6 \\ 0 & 5 & 2 & 4 \\ 0 & 0 & 0 & 1 \end{vmatrix} && (\rho_2 \leftrightarrow \rho_4) \\
&= -\begin{vmatrix} 1 & 3 & 1 & 2 \\ 0 & -2 & -4 & -6 \\ 0 & 0 & -8 & -11 \\ 0 & 0 & 0 & 1 \end{vmatrix} && (\rho_3 \mapsto \rho_3 + \tfrac{5}{2}\rho_2) \\
&= -(-2)\begin{vmatrix} 1 & 3 & 1 & 2 \\ 0 & 1 & 2 & 3 \\ 0 & 0 & -8 & -11 \\ 0 & 0 & 0 & 1 \end{vmatrix} && (\rho_2 \mapsto -\tfrac{1}{2}\rho_2)
\end{aligned}$$

$$= -(-2)(-8)\begin{vmatrix} 1 & 3 & 1 & 2 \\ 0 & 1 & 2 & 3 \\ 0 & 0 & 1 & \frac{11}{8} \\ 0 & 0 & 0 & 1 \end{vmatrix} \qquad (\rho_3 \mapsto -\tfrac{1}{8}\rho_3)$$

$$= -16,$$

since the determinant of the remaining row-equivalent matrix is 1 (as it can be row-reduced to the identity by a series of linear combinations of rows).

ii. We have

$$\begin{vmatrix} 1 & 3 & 1 & 2 \\ 1 & 3 & 1 & 3 \\ -1 & 2 & 1 & 2 \\ 2 & 4 & -2 & -2 \end{vmatrix} = \begin{vmatrix} 1 & 0 & 1 & 2 \\ 1 & 0 & 1 & 3 \\ -1 & 5 & 1 & 2 \\ 2 & -2 & -2 & -2 \end{vmatrix} \qquad (c_2 \mapsto c_2 - 3c_1)$$

$$= \begin{vmatrix} 1 & 0 & 0 & 2 \\ 1 & 0 & 0 & 3 \\ -1 & 5 & 2 & 2 \\ 2 & -2 & -4 & -2 \end{vmatrix} \qquad (c_3 \mapsto c_3 - c_1)$$

$$= \begin{vmatrix} 1 & 0 & 0 & 0 \\ 1 & 0 & 0 & 1 \\ -1 & 5 & 2 & 4 \\ 2 & -2 & -4 & -6 \end{vmatrix} \qquad (c_4 \mapsto c_4 - 2c_1)$$

$$= \begin{vmatrix} 1 & 0 & 0 & 0 \\ 1 & 0 & 0 & 1 \\ -1 & 0 & 2 & 4 \\ 2 & 8 & -4 & -6 \end{vmatrix} \qquad (c_2 \mapsto c_2 - \tfrac{5}{2}c_3)$$

$$= -\begin{vmatrix} 1 & 0 & 0 & 0 \\ 1 & 1 & 0 & 0 \\ -1 & 4 & 2 & 0 \\ 2 & -6 & -4 & 8 \end{vmatrix} \qquad (c_2 \leftrightarrow c_4)$$

$$= -2(8)\begin{vmatrix} 1 & 0 & 0 & 0 \\ 1 & 1 & 0 & 0 \\ -1 & 4 & 1 & 0 \\ 2 & -6 & -2 & 1 \end{vmatrix} \qquad (c_4 \mapsto \tfrac{1}{8}c_4)$$

$$= -16,$$

since the determinant of the remaining row-equivalent matrix is 1 (as it can be row-reduced to the identity by a series of linear combinations of rows).

Problem 3

(a) Let A, $B \in \mathrm{Mat}_{n\times n}(\mathbb{R})$, and suppose that B is invertible. Show that

$$\det(BAB^{-1}) = \det(A).$$

Note. It turns out that $\mathrm{tr}(BAB^{-1}) = \mathrm{tr}(A)$ as well, where tr is the trace (I am not asking you to prove this). The expression BAB^{-1} is *conjugation* of A by B. The statement of the problem and its companion statement for the trace say that determinant and trace are *conjugation invariant* functions $\text{Mat}_{n\times n}(\mathbb{R}) \longrightarrow \mathbb{R}$.

(b) A matrix $A \in \text{Mat}_{n\times n}(\mathbb{R})$ is *orthogonal* if and only if $A^{\text{T}}A = I_n$. Let A be an orthogonal matrix. Show that $\det(A) \in \{\pm 1\}$.

Note. We denote the set of $n \times n$ orthogonal matrices by $O_n(\mathbb{R}) = \{A \in \text{Mat}_{n\times n}(\mathbb{R}) : A^{\text{T}}A = I_n\}$. We denote the *special orthogonal* matrices by $SO_n(\mathbb{R}) = \{A \in O_n : \det(A) = 1\} \subseteq O_n$. It turns out that

i. $SO_2(\mathbb{R}) = \left\{ \begin{pmatrix} \cos\theta & -\sin\theta \\ \sin\theta & \cos\theta \end{pmatrix} : \theta \in [0, 2\pi) \right\}$ and

ii. $A \in SO_n(\mathbb{R})$ for $n \in \{2, 3\}$ if and only if the linear transformation $T_A : \mathbb{R}^n \longrightarrow \mathbb{R}^n$ defined for all $\vec{v} \in \mathbb{R}^n$ by $T_A(\vec{v}) = A\vec{v}$ is a *rotation*. A map $T : \mathbb{R}^n \longrightarrow \mathbb{R}^n$ is a rotation if and only if $T(\vec{0}) = \vec{0}$, $|\vec{v} - \vec{w}| = |T(\vec{v}) - T(\vec{w})|$ for all $\vec{v}, \vec{w} \in \mathbb{R}^n$ (T preserves distance), and T is orientation-preserving (for example, it rotates $\mathbb{R}^2$ counter-clockwise).

(c) A matrix $A \in \text{Mat}_{n\times n}(\mathbb{R})$ is *idempotent* if and only if $A^2 = A$. What possible values can $\det(A)$ have?

Solution.

(a) From the note in problem 1, we have that for all $A, B \in \text{Mat}_{n\times n}(\mathbb{R})$, we have $\det(AB) = \det(A)\det(B)$. Using this,

$$\begin{aligned}
\det(BAB^{-1}) &= \det(BA)\det(B^{-1}) \\
&= (\det(B)\det(A))\det(B^{-1}) \\
&= \det(B)\det(B^{-1})\det(A) && \text{(associativity/commutativity)} \\
&= \det(BB^{-1})\det(A) \\
&= \det(I_n)\det(A) \\
&= \det(A). && (\det(I_n) = 1 \text{ by definition})
\end{aligned}$$

□

(b) Let A be an orthogonal matrix. Then, we have

$$\begin{aligned}
A^TA &= I_n && \text{(definition)} \\
\implies \det(A^TA) &= \det(I_n) \\
\implies 1 &= \det(A^TA) && (\det(I_n) = 1) \\
&= \det(A^T)\det(A) \\
&= \det(A)\det(A) && (\det(A) = \det(A^T)) \\
\implies \det(A)^2 &= 1 \\
\implies \det(A) &\in \{\pm 1\}.
\end{aligned}$$

□

(c) Let A be an idempotent matrix. Then, we have

$$\begin{aligned} A^2 &= A && \text{(definition)} \\ \implies \det(A^2) &= \det(A) \\ \implies \det(A)\det(A) &= \det(A) \\ \implies \det(A) &= 1 \text{ or } \det(A) = 0. \end{aligned}$$

Therefore, $\det(A) \in \{0, 1\}$. □

Problem 4 Use the determinant to find all values $\lambda \in \mathbb{R}$ such that $A_\lambda = \begin{pmatrix} \lambda & 1 & 1 \\ 1 & \lambda & 1 \\ 1 & 1 & \lambda \end{pmatrix}$ is **not invertible**.

Solution.

We have

$$\begin{aligned} \begin{vmatrix} \lambda & 1 & 1 \\ 1 & \lambda & 1 \\ 1 & 1 & \lambda \end{vmatrix} &= \lambda \begin{vmatrix} \lambda & 1 \\ 1 & \lambda \end{vmatrix} - \begin{vmatrix} 1 & 1 \\ 1 & \lambda \end{vmatrix} + \begin{vmatrix} 1 & \lambda \\ 1 & 1 \end{vmatrix} \\ &= \lambda(\lambda^2 - 1) - (\lambda - 1) + (1 - \lambda) \\ &= \lambda^3 - 3\lambda + 2. \end{aligned}$$

Since A_λ is not invertible precisely when $\det(A) = 0$, we can find the roots of $\lambda^3 - 3\lambda + 2$. One can use the cubic equation, the rational root test, or other nonsense to conclude that the roots are 1 and -2, so A_{-2} and A_1 are the only non-invertible matrices.

Problem 5 For each matrix A, compute its characteristic polynomial, $p_A(t)$, and its eigenvalues.

(a) $A = \begin{pmatrix} 13 & -16 \\ 9 & -11 \end{pmatrix}$.

(b) $A = \begin{pmatrix} 3 & -1 & -1 \\ -12 & 0 & 5 \\ 4 & -2 & -1 \end{pmatrix}$.

(c) $A = \begin{pmatrix} 2 & 4 & 4 \\ 0 & 1 & -1 \\ 0 & 1 & 3 \end{pmatrix}$.

Solution.

(a) We have

$$\begin{aligned} p_A(t) &= \begin{vmatrix} 13 - t & -16 \\ 9 & -11 - t \end{vmatrix} \\ &= (13 - t)(-11 - t) - (-16)(9) \\ &= -143 - 13t + 11t + t^2 + 144 \end{aligned}$$

$$\begin{aligned}
&= t^2 - 2t + 1 \\
&= (t-1)^2,
\end{aligned}$$

which has root 1 with multiplicity 2. So 1 is an eigenvalue of A.

(b) We have

$$\begin{aligned}
p_A(t) &= \begin{vmatrix} 3-t & -1 & -1 \\ -12 & -t & 5 \\ 4 & -2 & -1-t \end{vmatrix} \\
&= (3-t)\begin{vmatrix} -t & 5 \\ -2 & -1-t \end{vmatrix} + \begin{vmatrix} -12 & 5 \\ 4 & -1-t \end{vmatrix} - \begin{vmatrix} -12 & -t \\ 4 & -2 \end{vmatrix} \\
&= (3-t)(t^2+t+10) + (12t+12-20) - (4t+24) \\
&= -t^3 + 2t^2 - 7t + 30 + 12t - 8 - 4t - 24 \\
&= -t^3 + 2t^2 + t - 2 \\
&= -t^2(t-2) + 1(t-2) \\
&= -(t^2-1)(t-2) \\
&= -(t+1)(t-1)(t-2),
\end{aligned}$$

which has roots $-1, 1, 2$, each with multiplicity 1. So $-1, 1, 2$ are the eigenvalues of A.

(c) We have

$$\begin{aligned}
p_A(t) &= \begin{vmatrix} 2-t & 4 & 4 \\ 0 & 1-t & -1 \\ 0 & 1 & 3-t \end{vmatrix} \\
&= (2-t)\begin{vmatrix} 1-t & -1 \\ 1 & 3-t \end{vmatrix} && \text{(Laplace expansion down column 1)} \\
&= (2-t)(t^2-4t+3+1) \\
&= -(t-2)(t^2-4t+4) \\
&= -(t-2)^3,
\end{aligned}$$

which has root 2 with multiplicity 3. So 2 is the eigenvalue of A.

Problem 6 Suppose that $A \in \text{Mat}_{n\times n}(\mathbb{R})$ is invertible and that A and A^{-1} both have *integer* entries. Show that $\det(A) \in \{\pm 1\}$.

(It may be helpful to recall that $\det(A^{-1}) = \frac{1}{\det(A)}$.)

Solution.

Since A and A^{-1} has integer entries, it follows that $\det(A)$ and $\det(A^{-1})$ are integers since the set of integers is closed under addition, subtraction, and multiplication. Since $\det(A^{-1}) = \frac{1}{\det(A)}$, we must have that $\frac{1}{\det(A)}$ is an integer. This is possible if and only if the denominator is 1 or -1, so $\det(A) \in \{\pm 1\}$. $\square$

Problem 7

(a) Let a, b, $c \in \mathbb{R}$. Show that $\begin{vmatrix} 1 & a & a^2 \\ 1 & b & b^2 \\ 1 & c & c^2 \end{vmatrix} = (c-a)(c-b)(b-a)$.

(b) **Challenging.** Prove by induction that

$$\begin{vmatrix} 1 & a_1 & \cdots & a_1^{n-1} \\ 1 & a_2 & \cdots & a_2^{n-1} \\ \vdots & \vdots & \cdots & \vdots \\ 1 & a_n & \cdots & a_n^{n-1} \end{vmatrix} = \prod_{1 \le r < s \le n} (a_s - a_r).$$

Note. Part (a) is the 3×3 *Vandermonde* determinant; part (b) is the $n \times n$ version.

(a) We have

$$\begin{aligned}
\begin{vmatrix} 1 & a & a^2 \\ 1 & b & b^2 \\ 1 & c & c^2 \end{vmatrix} &= \begin{vmatrix} 1 & a & a^2 \\ 1 & b & b^2 \\ 0 & c-a & c^2-a^2 \end{vmatrix} && (\rho_3 \mapsto \rho_3 - \rho_1) \\
&= \begin{vmatrix} 1 & a & a^2 \\ 0 & b-a & b^2-a^2 \\ 0 & c-a & c^2-a^2 \end{vmatrix} && (\rho_2 \mapsto \rho_2 - \rho_1) \\
&= \begin{vmatrix} b-a & b^2-a^2 \\ c-a & c^2-a^2 \end{vmatrix} \\
&= (b-a)(c^2-a^2) - (c-a)(b^2-a^2) \\
&= (b-a)(c+a)(c-a) - (c-a)(b-a)(b+a) \\
&= (c-a)(b-a)(c+a-b-a) && \text{(factoring)} \\
&= (c-a)(b-a)(c-b).
\end{aligned}$$