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MATH 575: Section H01
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MATH 575 Homework 4

Collaboration:

Problem 1 Let M be a maximum matching of a graph G, and let M' be a *maximal* matching of G. Prove that $|M'| \geq \frac{|M|}{2}$.

Solution.

We will induct on $n = |V(G)|$. First, let $n = 1$. Then, there is no matching since there are no edges, so clearly the claim holds.

Next, let $n \in \mathbb{N}$, G be a graph on n vertices, M be a maximum matching on G, and M' be a maximal matching on G. Assume that for all graphs G' on $n' < n$ vertices, any maximum matching N on G' and maximal matching N' on G' will satisfy $|N'| \geq \frac{|N|}{2}$.

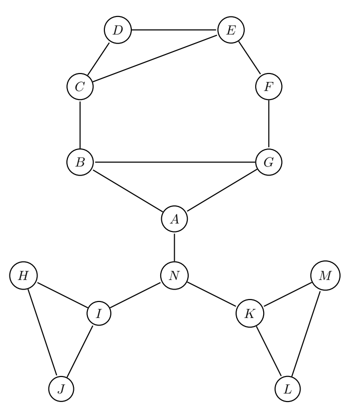
Choose $x, y \in V(G)$ such that $xy \in E(M')$ (if this isn't possible, then we have an empty graph and the property holds), and construct an induced subgraph G' on $V(G) - \{x, y\}$. Let N' be the matching on G' obtained from removing xy from M', and let N be a maximum matching on G'. Since we have only removed xy in both G' and N', N' will be maximal. Then by the induction hypothesis, since $n - 2 < n$, we have $|N'| \geq \frac{|N|}{2}$.

We defined N' to have one fewer edge than M', so we have $|N'| = |M'| - 1$. If x and y are incident to separate edges in M, we could have $|N| = |M| - 2$, but since we have only removed two vertices, the maximum matching in G' will not decrease by more than two edges from that in G. So we have $|N| \geq |M| - 2$. Therefore, by our induction hypothesis we have

$$|M'| - 1 = |N'| \geq \frac{|N|}{2} \geq \frac{|M| - 2}{2} = \frac{|M|}{2} - 1,$$

which implies that $|M'| \geq \frac{|M|}{2}$. So by strong induction, this property holds for all graphs. $\square$

Problem 2 Consider the following graph and the matching given by the edges $M = \{CE, BG, HI, KM\}$.



(a) Starting from the matching M, use a series of augmenting paths to find a matching in the graph of size 6. Write down the vertices for each augmenting path you use.
(b) Is the matching you found in part (a) maximum? If so, explain why. If not, find a larger matching.

Solution.

(a) First, since A and F are not saturated by the matching, $\{A, B, G, F\}$ is an M-augmenting path. So let $M = \{CE, AB, GF, HI, KM\}$. Next, since J and N are not saturated by the matching, $\{J, H, I, N\}$ is an M-augmenting path. So let $M = \{CE, AB, GF, JH, IN, KM\}$.

(b) This matching is maximum. The only two vertices not saturated by the matching are D and L, and no path between them is M-augmenting. This is because AN and NK are both cut-edges so we must pass through them. Since they are next to each other and neither is in the matching, no path between D and L can be M-alternating.

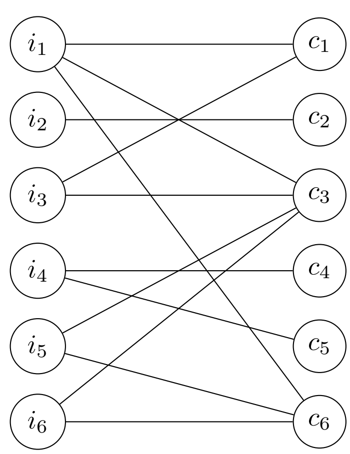
Problem 3 6 instructors $i_1, i_2, \ldots, i_6$ must be assigned to teach 6 classes $c_1, c_2, \ldots, c_6$, 1 class per instructor. The table below shows an "x" if a teacher has taught a class in a previous semester. Suppose each teacher would prefer to teach a class they have taught in the past.

	c_1	c_2	c_3	c_4	c_5	c_6
i_1	x		x			x
i_2		x				
i_3	x		x			
i_4				x	x	
i_5			x			x
i_6			x			x

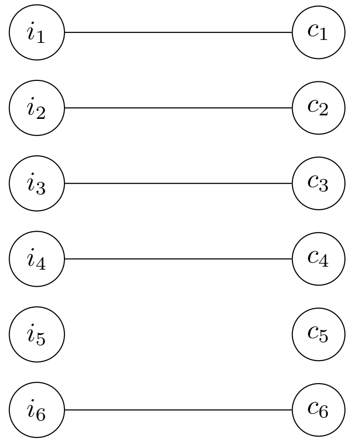
(a) Construct a bipartite graph G such that a matching in G corresponds to a (partial) assignment of instructors to classes.
(b) Find a maximum matching of G from part(a). Can every instructor be assigned a class of their preference? If not, find a subset of the instructors that violate Hall's condition.

Solution.

(a)



(b) This is a maximum matching of G:



Unfortunately, not every instructor can be assigned a class of their preference, because

$$|S| = |\{i_1, i_3, i_5, i_6\}| > |\{c_1, c_3, c_6\}| = |N(S)|$$

is a violation of Hall's condition.

Problem 4 In a bipartite graph $G = X \cup Y$, the *deficiency* of a set $S \subseteq X$ is

$$\operatorname{def}(S) = \max\{0, |S| - |N(S)|\}.$$

Prove that a maximum matching in G has size $|X| - \max_{S \subseteq X} \operatorname{def}(S)$. (Hint: Form a bipartite graph G' that has a matching that saturates X if and only if G has a matching of the desired size, and prove that G' satisfies Hall's Condition.)

Solution.

Let $S \subseteq X$ be a set of largest deficiency in G, and M be a maximum matching in G. Assume that $|M| \neq |X| - \operatorname{def}(S)$.

- Case 1: $\operatorname{def}(S) = 0$. Then, no subset of X is larger than its neighborhood, so Hall's condition is satisfied. Then, $|M| = |X|$, contradicting our assumption.
- Case 2: $\operatorname{def}(S) > 0$. Then, $|M| \neq |X| - |S| + |N(S)|$ by definition of deficiency (and our assumption).
  - Case 2.1: $|M| > |X| - |S| + |N(S)|$. Then, even if M saturates every vertex in $X - S$, the matching will have to include more edges between S and $N(S)$ than there are vertices in $N(S)$, a contradiction.
  - Case 2.2: $|M| < |X| - |S| + |N(S)|$.
    * Case 2.2.1: M saturates $N(S)$. Let $M' \subseteq M$ be a maximum matching on $(X - S) \cup (Y - N(S))$. Since $|M| = |N(S)|$, we have $|M'| < |X| - |S|$, so M' does not cover $X - S$. Thus, Hall's condition is violated, and there must exist $S' \subseteq X - S$ such that $|S'| > |N(S')|$. We observe that

$$\begin{aligned} |S \cup S'| - |N(S \cup S')| &= |S| + |S'| - |N(S) \cup N(S')| \\ &= |S| + |S'| - |N(S)| - |N(S')| + |N(S \cap S')| \quad \text{(inclusion-exclusion)} \\ &\geq |S| - |N(S)| + |S'| - |N(S')| \end{aligned}$$

$$> |S| - |N(S)|. \qquad (|S'| - |N(S')| \text{ is positive})$$

But then $S \cup S'$ has larger deficiency than S, a contradiction because we chose S to have largest deficiency.

* Case 2.2.2: M does not saturate $N(S)$. Since M is maximum, this means there is no matching covering $N(S)$, and thus Hall's condition must not hold. So there exists some $T \subseteq N(S)$ such that $|T| > |N(T)|$. We note that we have $N(T) \subseteq X$, and thus

$$|T| > |N(T)| \geq |N(T) \cap S| \implies |T| - |N(T) \cap S| > 0.$$

Consequently, we observe that

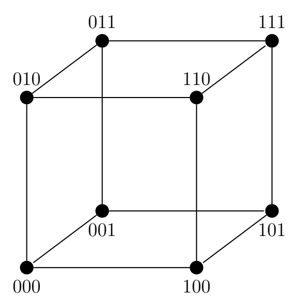
$$\begin{aligned} |S - N(T)| - |N(S - N(T))| &= |S| - |N(T) \cap S| - (|N(S)| - |T|) \\ &= |S| - |N(S)| + |T| - |N(T) \cap S| && \text{(rearranging)} \\ &> |S| - |N(S)|. && (\text{since } |T| - |N(T) \cap S| > 0) \end{aligned}$$

So by removing $N(T)$ from S and updating $N(S)$ accordingly, we have found a set $(S - N(T))$ with larger deficiency than S, a contradiction because we chose S to have largest deficiency.

Since each case is a contradiction, therefore, we must have $|M| = |X| - \text{def}(S)$. □

Problem 5 For $d \in \mathbb{N}$, the d-dimensional hypercube Q_d is the 2^d-vertex graph in which every vertex is a binary string of length d, and two vertices are adjacent if their corresponding strings differ in exactly one coordinate.

Prove that for $d \geq 2$, Q_d has at least $2^{2^{d-2}}$ perfect matchings.



Solution.

We will induct on d. First, let $d = 2$, and we have $V(Q_d) = \{00, 01, 10, 11\}$. Then, we can match 00 to 01 and 10 to 11, or 00 to 10 or 01 to 11. So we have $2^{2^{d-2}} = 2^{2^{2-2}} = 2^{2^0} = 2^1 = 2$ perfect matchings, and the claim holds for $d = 2$.

Next, let $d \in \mathbb{N}$, $d \geq 2$, and assume that we have k perfect matchings where $k = 2^{2^{d-2}}$. Now, consider Q_{d+1}. We can think of this as two copies ${Q_d}'$ and ${Q_d}''$ of Q_d connected by edges in the $(d+1)^{\text{th}}$ dimension. So ${Q_d}'$ and ${Q_d}''$ have k matchings, and we can choose any of the k matchings for both of them. By the product rule of combinatorics, then, there are k^2 ways to choose these k matchings in ${Q_d}'$ and ${Q_d}''$.

Thus, there are at least

$$k^2 = \left(2^{2^{d-2}}\right)^2 = 2^{(2)\left(2^{d-2}\right)} = 2^{2^{d+1-2}}$$

choices of matchings, and since these saturate all vertices, they are perfect matchings. So if the claim holds for d, it holds for $d + 1$, and therefore it is true for all n. □