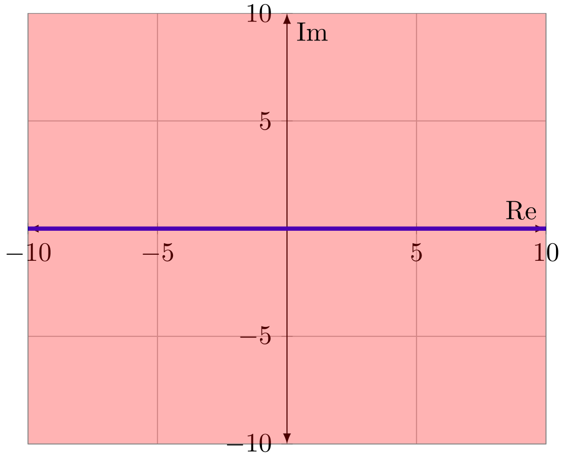
MATH 552: Section 001
Professor: Dr. Miller
January 28, 2022

MATH 552 Homework Set 3*

Problem 1.12.4bd In each case, sketch the closure of the set:
(b) $|\operatorname{Re} z| < |z|$
(d) $\operatorname{Re}(z^2) > 0$

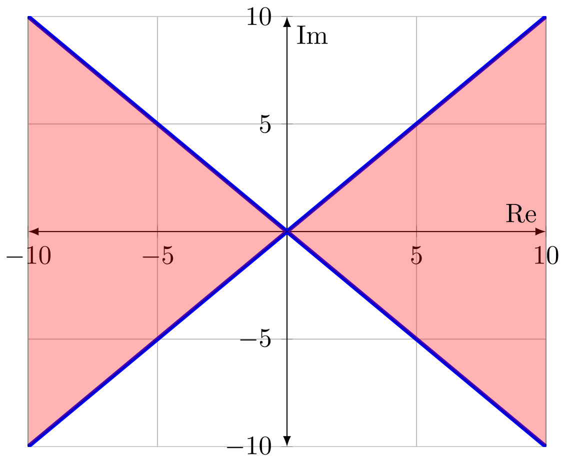
Solution.

(b) $|\operatorname{Re} z| < |z|$ everywhere except when z is a real number:



The closure is shown with the interior in red and the boundary in blue.

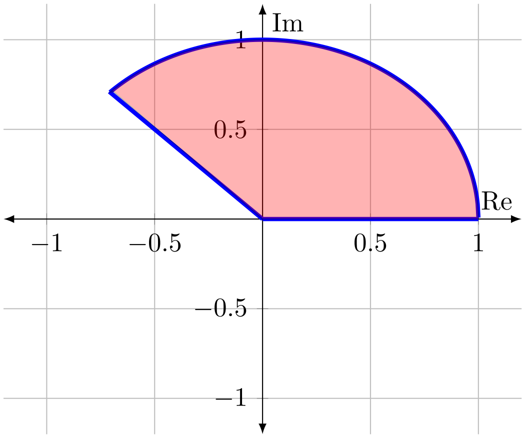
(d) Let $z = x + yi$. Then, $z^2 = x^2 - y^2 + 2xyi$, and $\operatorname{Re}(z^2) = x^2 - y^2$. $\operatorname{Re}(z^2) > 0$, so $y^2 < x^2$:



The closure is shown with the interior in red and the boundary in blue.

Problem 2.14.8b Sketch the region onto which the sector $r \leq 1, 0 \leq \theta \leq \pi/4$ is mapped by the transformation $w = z^3$.

Solution. The magnitude of any point in this interval will be cubed, so any r in $[0,1]$ will stay in $[0,1]$. The angle will be multiplied by 3, so $[0,\pi/4]$ is transformed to $[0,3\pi/4]$.



The sector is mapped to the region shown above.

Problem 2.18.11b Show that when

$$T(z) = \frac{az+b}{cz+d} \qquad (ad-bc \neq 0),$$

$$\lim_{z\to\infty} T(z) = \frac{a}{c} \text{ and } \lim_{z\to -d/c} T(z) = \infty \text{ if } c \neq 0.$$

Solution.

By Theorem 2.17.2,

$$\lim_{z\to 0} T\left(\frac{1}{z}\right) = \frac{a}{c} \implies \lim_{z\to\infty} T(z) = \frac{a}{c}.$$

$$\begin{aligned}
T\left(\frac{1}{z}\right) &= \frac{\frac{a}{z}+b}{\frac{c}{z}+d} && \text{(substituting } z = \tfrac{1}{z}\text{)}\\
T\left(\frac{1}{z}\right) &= \frac{\frac{a+bz}{z}}{\frac{c+dz}{z}} && \text{(rearranging)}\\
T\left(\frac{1}{z}\right) &= \frac{a+bz}{c+dz} && \text{(using } \tfrac{z}{z} = 1 \text{ (not interested in } z = 0\text{))}\\
\lim_{z\to 0} \frac{a+bz}{c+dz} &= \frac{a}{c} && \text{(direct substitution)}\\
\lim_{z\to\infty} T(z) &= \frac{a}{c} && \text{(using the theorem)}
\end{aligned}$$

By Theorem 2.17.1,

$$\lim_{z\to -d/c} \frac{1}{T(z)} = 0 \implies \lim_{z\to -d/c} T(z) = \infty.$$

$$\frac{1}{T(z)} = \frac{1}{\frac{az+b}{cz+d}}$$

$$\frac{1}{T(z)} = \frac{cz+d}{az+b}$$

$$\lim_{z \to -d/c} \frac{cz+d}{az+b} = \frac{c\left(-\frac{d}{c}\right)+d}{a\left(-\frac{d}{c}\right)+b} \qquad \text{(direct substitution)}$$

$$\lim_{z \to -d/c} \frac{cz+d}{az+b} = \frac{-d+d}{-\frac{ad}{c}+b}$$

$$\lim_{z \to -d/c} \frac{cz+d}{az+b} = -\frac{0}{\frac{ad-bc}{c}}$$

$$\lim_{z \to -d/c} \frac{cz+d}{az+b} = \frac{0}{ad-bc}$$

$$\lim_{z \to -d/c} \frac{cz+d}{az+b} = 0 \qquad \text{(when } ad-bc \neq 0\text{, which is specified in the domain)}$$

$$\lim_{z \to -d/c} T(z) = \infty \qquad \text{(by the theorem)}$$

Problem Supplemental A Show how the mapping $w = e^z$ transforms the box $-1 \leq x \leq 1, -\ln 2 \leq y \leq \ln 3$. Here ln denotes the real natural logarithm.

Solution. Let $z = x + iy$. Then, $w = e^x e^{iy}$, and the rectangle will be mapped to a polar rectangle. As shown below, r will be restricted to $[e^{-1}, e]$ and θ will be restricted to $[-\ln 2, \ln 3]$. This is because the e^x factor affects the magnitude of a complex number in polar form, and the e^{iy} factor affects the angle, with the angle being equal to y.