MATH 552: Section 001
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January 18, 2022

MATH 552 Homework 1ˆ

Problem 1.5.6 Using the fact that $|z_1 - z_2|$ is the distance between two points z_1 and z_2, give a geometric argument that $|z-1| = |z+i|$ represents the line through the origin whose slope is -1.

Solution.

Using this definition, z can be any point equidistant from $(1, 0)$ and $(0, -1)$ in the complex plane. This implies that all possible z will lie on a straight line, and this line will be the perpendicular bisector of the line connecting the two points. The line connecting the two points has midpoint $\left(\frac{1+0}{2}, \frac{0-1}{2}\right) = \left(\frac{1}{2}, -\frac{1}{2}\right)$ and slope $\frac{-1-0}{0-1} = 1$. The perpendicular bisector will pass through the midpoint with a perpendicular slope.

$$\begin{aligned}
m &= -\frac{1}{1} = -1 && \text{(perpendicular slope is the opposite reciprocal)}\\
y - \left(-\frac{1}{2}\right) &= -1\left(x - \frac{1}{2}\right) && \text{(using point-slope form)}\\
y &= -x && \text{(simplifying)}\\
y(0) &= 0 && \text{(passes through the origin)}
\end{aligned}$$

Thus, the line passes through the origin with slope -1.

Problem 1.6.7 Show that $|\mathrm{Re}(2 + \overline{z} + z^3)| \leq 4$ when $|z| \leq 1$.

Solution.

$$\begin{aligned}
|\mathrm{Re}(2+\overline{z}+z^3)| = |\mathrm{Re}\,2 + \mathrm{Re}\,\overline{z} + \mathrm{Re}\,z^3| \leq |\mathrm{Re}\,2| + |\mathrm{Re}\,\overline{z}| + |\mathrm{Re}\,z^3| && \text{(triangle inequality)}\\
|\mathrm{Re}\,2| = 2 && \text{(2 is purely real)}\\
|\mathrm{Re}\,z| = |\mathrm{Re}\,\overline{z}| \leq |z| = |\overline{z}| && \text{(by definition of conjugate)}\\
|z| \leq 1 && \text{(given condition)}\\
\implies |\mathrm{Re}\,z| \leq |z| \leq 1 && \text{(real part cannot exceed modulus)}\\
\implies |\mathrm{Re}\,z^3| \leq |z^3| \leq |z| \leq 1 && \text{(cubed fraction becomes smaller)}\\
\implies |\mathrm{Re}\,z| + |\mathrm{Re}\,z^3| \leq 2 && \text{(since both parts are less than 1)}\\
\implies 2 + |\mathrm{Re}\,\overline{z}| + |\mathrm{Re}\,z^3| \leq 4 && \text{(adding 2 to both sides)}\\
\implies |\mathrm{Re}(2+\overline{z}+z^3)| \leq |\mathrm{Re}\,2| + |\mathrm{Re}\,\overline{z}| + |\mathrm{Re}\,z^3| \leq 4 && \text{(reapplying triangle inequality)}
\end{aligned}$$

Thus, $|z| \leq 1 \Rightarrow |\mathrm{Re}(2 + \overline{z} + z^3)| \leq 4$.

Problem 1.6.15 Follow the steps below to give an algebraic derivation of the triangle inequality

$$|z_1 + z_2| \leq |z_1| + |z_2|.$$

(a) Show that

$$|z_1 + z_2|^2 = (z_1 + z_2)(\overline{z_1} + \overline{z_2}) = z_1\overline{z_1} + (z_1\overline{z_2} + \overline{z_1\overline{z_2}}) + z_2\overline{z_2}).$$

(b) Point out why

$$z_1\overline{z_2} + \overline{z_1\overline{z_2}} = 2\,\mathrm{Re}(z_1\overline{z_2}) \leq 2|z_1||z_2|.$$

(c) Use the results in parts (a) and (b) to obtain the inequality

$$|z_1 + z_2|^2 \leq (|z_1| + |z_2|)^2,$$

and note how the triangle inequality follows.

Solution.

(a) Let $z_1 = a + bi$ and $z_2 = c + di$. Then, $\overline{z_1} = a - bi$ and $z_2 = c - di$.

$$\begin{aligned}
(z_1 + z_2)(\overline{z_1} + \overline{z_2}) &= ((a + c) + (b + d)i)((a + c) - (b + d)i) && \text{(distributing)} \\
(z_1 + z_2)(\overline{z_1} + \overline{z_2}) &= (a + c)^2 - (b + d)^2 i^2 = (a + c)^2 + (b + d)^2 && \text{(using difference of perfect squares)} \\
\mathrm{Re}(z_1 + z_2) &= a + c \\
\mathrm{Im}(z_1 + z_2) &= b + d \\
(a + c)^2 + (b + d)^2 &= |z_1 + z_2|^2 && \text{(Pythagorean theorem)} \\
\overline{z_1}z_2 &= (a - bi)(c + di) = (ac + bd) + i(ad - bc) \\
z_1\overline{z_2} &= (a + bi)(c - di) = (ac + bd) + i(bc - ad) \\
\overline{z_1\overline{z_2}} &= (ac + bd) - i(bc - ad) && \text{(negating imaginary part)} \\
\overline{z_1}z_2 &= \overline{z_1\overline{z_2}} \\
(z_1 + z_2)(\overline{z_1} + \overline{z_2}) &= z_1\overline{z_1} + (z_1\overline{z_2} + \overline{z_1}z_2) + z_2\overline{z_2} && \text{(distributing)} \\
|z_1 + z_2|^2 = (z_1 + z_2)(\overline{z_1} + \overline{z_2}) &= z_1\overline{z_1} + (z_1\overline{z_2} + \overline{z_1\overline{z_2}}) + z_2\overline{z_2} && \text{(using } \overline{z_1}z_2 = \overline{z_1\overline{z_2}}\text{)}
\end{aligned}$$

(b)

$$\begin{aligned}
z_1\overline{z_2} + \overline{z_1}z_2 &= (a + bi)(c - di) + (a - bi)(c + di) && \text{(using } \overline{z_1}z_2 = \overline{z_1\overline{z_2}} \text{ from (a))} \\
z_1\overline{z_2} + \overline{z_1}z_2 &= (ac + bd) + (bc - ad)i + (ac + bd) - (bc - ad)i && \text{(distributing)} \\
z_1\overline{z_2} + \overline{z_1}z_2 &= 2(ac + bd) && \text{(cancelling like terms)} \\
\mathrm{Re}(z_1\overline{z_2}) &= ac + bd && \text{(using same distribution as above)} \\
z_1\overline{z_2} + \overline{z_1\overline{z_2}} &= 2\,\mathrm{Re}(z_1\overline{z_2}) && \text{(doubling previous line)}
\end{aligned}$$

The equality is thus true. The inequality is true geometrically, because complex multiplication results in the magnitudes being multiplied. The two sides are equal if both z_1 and $\overline{z_2}$ are strictly real, and the left side will be less if either one has an imaginary component.

(c)

$$
\begin{aligned}
|z_1 + z_2|^2 &\leq |z_1|^2 + |z_2|^2 + 2|z_1||z_2| && \text{(distributing)} \\
\text{Assume } |z_1 + z_2|^2 &\leq |z_1|^2 + |z_2|^2 + (z_1\overline{z_2} + \overline{z_1\overline{z_2}}) && \text{(using result from (b))} \\
\implies z_1\overline{z_1} + z_2\overline{z_2} + (z_1\overline{z_2} + \overline{z_1\overline{z_2}}) &\leq |z_1|^2 + |z_2|^2 + (z_1\overline{z_2} + \overline{z_1\overline{z_2}}) && \text{(using result from(a))} \\
\implies z_1\overline{z_1} + z_2\overline{z_2} &\leq |z_1|^2 + |z_2|^2 && \text{(subtracting from both sides)}
\end{aligned}
$$

The last line is equal, because $z\overline{z} = |z|^2$ by the definition of a conjugate. Thus, since the inequality is satisfied, the proposition is true. The triangle inequality follows from taking the square root of both sides.