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MATH 300: Section H01
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MATH 300 Homework 11

Problem 1

(a) $a_1 = 2, a_2 = 6, a_3 = 10, a_4 = 14, a_5 = 18$

(b) The sequence can be defined recursively as follows: $a_1 = 2$ and for $n \in \mathbb{N}$, $n \geq 1$, $a_{n+1} = a_n + 4$.

Problem 2

(a) The sequence defined by $a_n = 2n + 1$ for all $n \in \mathbb{N}$, $n \geq 1$: 3, 5, 7, 9, 11...

(b) The sequence defined by $b_n = 5 + 2(n-2)^3$ for all $n \in \mathbb{N}$, $n \geq 1$: 3, 5, 7, 21, 59...

(c) The sequence defined by $c_n = 2\sin\left(\frac{\pi(x-2)}{2}\right) + 5$ for all $n \in \mathbb{N}$, $n \geq 1$: 3, 5, 7, 5, 3...

Problem 3 For all $n \in \mathbb{N}$, $n \geq 1$:

(a) $a_n = 2 + n^2$

(b) $b_n = 3 + 4n$

(c) $c_n = 3^{n-1} - 1$

(d) $d_n = \dfrac{(2n-1)!}{2^{n-1}(n-1)!}$

(e) $e_n = \left\lceil \dfrac{\sqrt{8n+1} - 1}{2} \right\rceil \bmod 2$

(f) $f_n = 2^{2^{n-1}}$

Problem 4

(a) We claim that for the sequence, we have $(\forall n \in \mathbb{N})\left[a_n = 5^n + 2(-2)^n + 3(-1)^n\right]$.

First, we check the values for $n \in \{0, 1, 2\}$.

$n = 0$: $5^0 + 2(-2)^0 + 3(-1)^0 = 1 + 2 + 3 = 6 = a_0$.

$n = 1$: $5^1 + 2(-2)^1 + 3(-1)^1 = 5 - 4 - 3 = -2 = a_1$.

$n = 2$: $5^2 + 2(-2)^2 + 3(-1)^2 = 25 + 8 + 3 = 36 = a_2$.

So the claim holds for the base cases of the recursive formula.

Next, let $n \geq 2$. Assume $(\forall k \in \mathbb{N})\left[k \leq n \Rightarrow a_k = 5^k + 2(-2)^k + 3(-1)^k\right]$.

By the recursive definition, $a_{n+1} = 2a_n + 13a_{n-1} + 10a_{n-2}$.

Since $n \geq 2$, $0 \leq n-2 < n-1 < n \leq n$, so the induction hypothesis holds for $n-2$, $n-1$, and n.

Using the recursive definition and substituting using the induction hypothesis,

$$a_{n+1} = 2(5^n + 2(-2)^n + 3(-1)^n) + 13(5^{n-1} + 2(-2)^{n-1} + 3(-1)^{n-1}) + 10(5^{n-2} + 2(-2)^{n-2} + 3(-1)^{n-2})$$

$$
\begin{aligned}
&= 2(5)^n + 4(-2)^n + 6(-1)^n + \frac{13}{5}(5)^n + \frac{13\cdot 2}{-2}(-2)^n + \frac{13\cdot 3^n}{-1} + \frac{10}{25}(5)^n + \frac{10\cdot 2}{4}(-2)^n + \frac{10\cdot 3}{1}(-1)^n \\
&= (5)^n\left(2 + \frac{13}{5} + \frac{2}{5}\right) + (-2)^n(4 - 13 + 5) + (-1)^n(6 - 39 + 30) \\
&= (5)(5)^n + 2(-2)(-2)^n + 3(-1)(-1)^n.
\end{aligned}
$$

Therefore $a_{n+1} = 5^{n+1} + 2(-2)^{n+1} + 3(-1)^{n+1}$. So if the claim holds for n, it also holds for $n+1$, and the claim is true by the PSMI. $\square$

(b) We claim that that if $n \in \mathbb{Z}$, $n \geq 8$, then any postage of n¢ can be made using only 3¢ and 5¢ stamps.

First, we check the values for $n \in \{8, 9, 10\}$.

An 8¢ stamp can be made using 1 3¢ stamp and 1 5¢ stamp.

A 9¢ stamp can be made using 3 3¢ stamps.

A 10¢ stamp can be made using 2 5¢ stamps.

So the claim holds for integers between 8 and 10 inclusive.

Then, let $n \geq 10$. Assume that for every k¢, if $8 \leq k \leq n$, then a postage of k stamps can be made using only 3¢ and 5¢ stamps.

We observe that $n+1 = (n-2)+3$. Since $n \geq 10$, $8 \leq n-2 \leq n$. So by the induction hypothesis, a postage of $(n-2)$¢ can be made using only 3¢ and 5¢ stamps. Thus, a postage of $(n+1)$¢ can be made with the 3¢ and 5¢ stamps that make up $n-2$¢ and then adding 1 3¢ stamp.

Therefore, a postage $(n+1)$¢ can be made using only 3¢ and 5¢ stamps. So if the claim holds for n, it also holds for $n+1$, and the claim is true by the PSMI. $\square$

(c) We claim that we have $(\forall n \in \mathbb{Z}^+)\left[\sum_{i=0}^{n}(2i+1)^2 = \frac{(n+1)(2n+1)(2n+3)}{3}\right]$.

First, let $n = 1$. The claim holds for $n = 1$ because

$$\sum_{i=0}^{1}(2i+1)^2 = (2(0)+1)^2 + (2(1)+1)^2 = 1^2 + 3^2 = 10 = (2)(5) = \frac{(2)(3)(5)}{3} = \frac{(1+1)(2(1)+1)(2(1)+3)}{3}.$$

Next, let $n \in \mathbb{Z}^+$. Assume $\sum_{i=0}^{n}(2i+1)^2 = \frac{(n+1)(2n+1)(2n+3)}{3}$. Then,

$$
\begin{aligned}
\sum_{i=0}^{n}(2i+1)^2 &= \frac{(n+1)(2n+1)(2n+3)}{3} \\
\sum_{i=0}^{n}(2i+1)^2 + (2(n+1)+1)^2 &= \frac{(n+1)(2n+1)(2n+3)}{3} + (2(n+1)+1)^2 \\
\sum_{i=0}^{n+1}(2i+1)^2 &= \frac{(n+1)(2n+1)(2n+3) + 3(2n+3)^2}{3} \\
&= \frac{(2n+3)[(n+1)(2n+1) + 3(2n+3)]}{3} \\
&= \frac{(2n+3)[2n^2 + 9n + 10]}{3} \\
&= \frac{(2n+3)(2n^2 + 9n + 10)}{3}
\end{aligned}
$$

$$= \frac{(2n+3)(2n+5)(n+2)}{3}.$$

Therefore, $\sum_{i=0}^{n+1}(2i+1)^2 = \frac{((n+1)+1)(2(n+1)+1)(2(n+1)+3)}{3}$. So if the claim holds for n, it also holds for $n+1$, and the claim is true by the PMI. $\square$

(d) We claim that for the sequence, we have $(\forall n \in \mathbb{N})\left[b_n = (-1)^n\right]$.

First, we check the values for $n \in \{0,1\}$.

$n = 0 : (-1)^0 = 1 = b_0$.

$n = 1 : (-1)^1 = -1 = b_1$.

So the claim holds for the base cases of the recursive formula.

Next, let $n \geq 1$. Assume $(\forall k \in \mathbb{N})\left[k \leq n \Rightarrow b_k = (-1)^k\right]$.

By the recursive definition, $b_{n+1} = b_n + 2b_{n-1}$.

Since $n \geq 1$, $0 \leq n-1 < n \leq n$, so the induction hypothesis holds for $n-1$ and n.

Using the recursive definition and substituting using the induction hypothesis,

$$\begin{aligned}
b_{n+1} &= b_n + 2b_{n-1} \\
&= (-1)^n + 2(-1)^{n-1} \\
&= (-1)^n + \frac{2(-1)^n}{-1} \\
&= (-1)^n(1-2) \\
&= (-1)^n(-1)
\end{aligned}$$

Therefore, $b_{n+1} = (-1)^{n+1}$. So if the claim holds for n, it also holds for $n+1$, and the claim is true by the PSMI. $\square$

(e) We claim that we have $(\forall n \in \mathbb{N})\left[28|(3^{3n} - (-1)^n)\right]$.

First, let $n = 0$. Since $3^{3n} - (-1)^0 = 1 - 1 = 0$ and $28(0) = 0$, $28|(3^{3(0)} - (-1)^0)$ and the claim holds for $n = 0$.

Then, let $n \in \mathbb{N}$. Assume $28|(3^{3n} - (-1)^n)$. Then $(\exists k)\left[28k = 3^{3n} - (-1)^n\right]$.

We observe that

$$\begin{aligned}
3^{3n+3} - (-1)^{n+1} &= 3^3(3)^{3n} + (-1)^n && \text{(using } (-1)^{n+1} = -(-1)^n\text{)} \\
&= 3^3\left(28k + (-1)^n\right) - (-1)^n && \text{(using induction hypothesis that } 3^{3n} = 28k + (-1)^n\text{)} \\
&= 3^3(28k) + (-1)^n(3^3+1) && \text{(distributing/combining like terms)} \\
&= 28(27k) + 28(-1)^n \\
&= 28(27k - (-1)^{n+1}).
\end{aligned}$$

Since $27k - (-1)^{n+1}$ is the sum and product of integers, it is an integer. Therefore, since $3^{3(n+1)} - (-1)^{n+1}$ can be written as an integer times 28, $28|(3^{3(n+1)} - (-1)^{n+1}$. So if the claim holds for n, it also holds for $n+1$, and the claim is true by the PMI. $\square$