Nathan Bickel
MATH 575: Section H01
Professor: Dr. Luo
February 28, 2023

MATH 575 Homework 6

Collaboration: I discussed some of the problems with Jack, Chance, and Sam.

Problem 1 Let $n \geq 3$, and let G be an n-vertex graph. Prove that if $\kappa(G) = k$, then there exists $v \in V(G)$ such that $\kappa(G - v) = k - 1$. (We proved already (Midterm 1 Practice Problems) that $\kappa(G - v) \geq \kappa(G) - 1$ for all $v \in V(G)$. You may use this fact without repeating the proof.)

Solution.

Let G be a graph on n vertices with $\kappa(G) = k$. Then, we have a separating set S of size k. Now, consider $G - v$ for some $v \in S$. Since S was a separating set, $S - \{v\}$ must be a separating set of $G - v$, and thus $\kappa(G - v) \leq k - 1$. Since we have proved that $\kappa(G - v) \geq k - 1$, we have $\kappa(G - v) = k - 1$. $\square$

Problem 2 Let G be a graph on $n \geq 3$ vertices. Prove that G is 2-connected if and only if for every three distinct vertices $x, y_1, y_2 \in V(G)$, there exists a y_1, y_2-path that passes through x.

Solution.

($\Leftarrow$) Let G be a graph that is not 2-connected. Then, G is either 0- or 1-connected. If it is 0-connected, then there are at least two components, so there exist a pair of vertices with no path between them at all. If it 1-connected, then we have a vertex v such that $G - v$ has two components G_1 and G_2. Let $y_1, y_2 \in V(G_1)$ and $x \in V(G_2)$. Then, any path in G between x and y_1 or y_2 has to pass through v. But then no path from y_1 to y_2 can contain x, since we must pass through v to get to x and again to get back to y_2, and paths must contain no repeated vertices.

($\Rightarrow$) Let G be a 2-connected graph, and let x, y_1, and y_2 be three distinct vertices in G. We have shown in class that for every pair of vertices in G, there exist two internally disjoint paths. Thus, we have an y_1, x-path P_1 and two internally disjoint x, y_2-paths P_2 and ${P_2}'$.

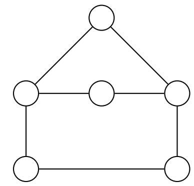
Case 1: P_1 is internally disjoint with either P_2 or ${P_2}'$. Then we can simply travel from y_1 to x along P_1 and then from x to y_2 along the internally disjoint path to obtain a y_1, y_2-path that passes through x.

Case 2: P_1 intersects with both P_2 and ${P_2}'$. Without loss of generality, assume that P_1 intersects with P_2 before it intersects with P_2'. Then, we can travel along P_1 until we first intersect with P_2, and then travel (backward) along P_2 until we get to x. Now, we can travel from x to y_2 along ${P_2}'$ to obtain a y_1, y_2-path that passes through x: since P_1 intersects with P_2 before ${P_2}'$, ${P_2}'$ will be disjoint from the part of P_1 we traversed. $\square$

Problem 3 Let G be an n-vertex graph. A *Hamiltonian cycle* in G is a cycle of length n, i.e., a cycle that covers all vertices of G. We say G is *Hamiltonian* if it contains a Hamiltonian cycle.
(a) Prove or disprove: if G is 2-connected, then G is Hamiltonian.
(b) Prove or disprove: if G is Hamiltonian, then G is 2-connected.

Solution.

(a) This is false. For example, the graph below is 2-connected but has no Hamiltonian cycle:



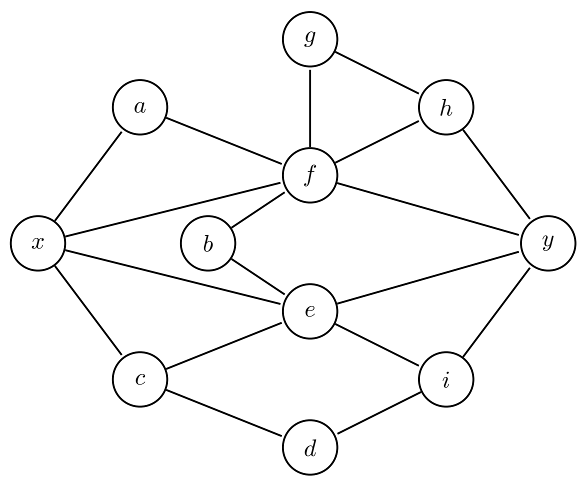
(b) This is true. Suppose that G is Hamiltonian. Then, we can partition G into a cycle C, and edges $e_1, e_2, \ldots, e_k$ incident on vertices in C. Since these edges are all ears of C, we have that $C \cup e_1 \cup e_2 \cup \ldots \cup e_k$ is an ear decomposition of G, and so by Whitney G is 2-connected.

Problem 4 Let G be a k-connected graph and suppose A and B are disjoint subsets of $V(G)$ with $|A|, |B| \geq k$. Prove there exists k pairwise-disjoint A, B-paths. (An A, B-path is a path with one endpoint in A and one endpoint in B.)

Solution.

Construct a graph G' by adding a vertex u with $N(u) = A$ and by adding a vertex v with $N(v) = B$. Since $|A|, |B| \geq k$, we have from the expansion lemma that G' is k-connected. By Menger's theorem, then, there exist k internally disjoint u, v-paths. Since every u, v-path must pass through A and B, these k paths are pairwise-disjoint A, B-paths. □

Problem 5 Let G be the graph below.



(a) Determine $\kappa(x, y)$ and give an example of an x, y-cut of size $\kappa(x, y)$.
(b) Determine $\kappa'(x, y)$ and give an example of an x, y-disconnecting set of size $\kappa'(x, y)$
Hint: use the dual problems to give a short proof of optimality.

Solution.

(a) We observe that $\{d, e, f\}$ is an x, y-cut of size 3, and that we have the internally disjoint x, y-paths:

1. x, a, f, g, h, y
2. x, e, y

3. x, c, d, i, y

So by Menger's theorem, we have $3 \leq \kappa(x, y) = \lambda(x, y) \geq 3 \implies \kappa(x, y) = 3$.

(b) We observe that $\{xa, xf, xe, xc\}$ is an x, y-disconnecting set of size 4, and that we have the pairwise edge-disjoint x, y-paths:

1. xa, af, fg, gh, hy
2. xf, fy
3. xe, ey
4. xc, cd, di, iy

So by "Menger II", we have $4 \leq \kappa'(x, y) = \lambda'(x, y) \geq 4 \implies \kappa'(x, y) = 4$.