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MATH 574: Section H01
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October 23, 2022

MATH 574 Homework 9

Collaboration: I discussed some of the problems with Jackson Ginn, Sam Maloney, Jack Hyatt, Chance Storey, Emma Devine, Siri Avula, and Miriam Rozin.

Problem 1 Find a closed form for the generating function for the following sequences. (By a **closed form** we mean an algebraic expression not involving a summation over a range of values or the use of ellipses. For instance, the closed form of the generating function $\sum_{k=0}^{n} x^k$ is $\frac{1}{1-x}$.)
(a) $1, 2, 4, 8, 16, 32, \ldots$
(b) $\binom{7}{0}, 2^1\binom{7}{1}, 2^2\binom{7}{2}, 2^3\binom{7}{3}, 2^4\binom{7}{4}, 2^5\binom{7}{5}, \ldots$
(c) $1, -1, 1, -1, 1, -1, \ldots$
(d) $1, 0, 1, 0, 1, 0, 1, 0, \ldots$

Solution.

(a) The generating function is $g(x) = 1 + 2x + 4x^2 + 8x^3 + \ldots$. The coefficients are powers of 2, so we can write

$$g(x) = \sum_{k=0}^{\infty} (2x)^k.$$

This power series is equal to $g(x) = \frac{1}{1-2x}$.

(b) The generating function is $g(x) = \binom{7}{0} + 2x\binom{7}{1} + 2^2x^2\binom{7}{2} + 2^3x^3\binom{7}{3} + \ldots$, so we can write

$$g(x) = \sum_{k=0}^{\infty} \binom{7}{k}(2x)^k.$$

By the extended binomial theorem, this series is equal to $g(x) = (1+2x)^7$.

(c) The generating function is $g(x) = 1 - x + x^2 - x^3 + \ldots$, so we can write

$$g(x) = \sum_{k=0}^{\infty} (-x)^k.$$

This power series is equal to $g(x) = \frac{1}{1-(-x)} = \frac{1}{x+1}$.

(d) The generating function is $g(x) = 1 + x^2 + x^4 + x^6 + \ldots$, so we can write

$$g(x) = \sum_{k=0}^{\infty} (x^2)^k.$$

This power series is equal to $g(x) = \frac{1}{1-x^2}$.

Problem 2 If $g(x)$ is the generating function for the sequence $\{a_k\}$, what is the generating function for:
(a) $2a_0, 2a_1, 2a_2, 2a_3, \ldots$
(b) $a_5, a_6, a_7, a_8, a_9, \ldots$
(c) $a_1, 2a_2, 3a_3, 4a_4, \ldots$
(d) $a_0 + a_1, a_1 + a_2, a_2 + a_3, a_3 + a_4, \ldots$

Solution.

By definition, $g(x) = a_0 + a_1x + a_2x^2 + a_3x^3 + \ldots$, so we can manipulate this to obtain generating functions for related sequences.

(a) Multiplying, we can write $2g(x) = 2a_0 + 2a_1x + 2a_2x^2 + 2a_3x^3 + \ldots$, so the generating function is $2g(x)$.

(b) Since $g(x)$ is an infinite sum, we can subtract off the terms we aren't interested in and divide appropriately. So the generating function for this sequence is

$$\frac{g(x) - a_4x^4 - a_3x^3 - a_2x^2 - a_1x - a_0}{x^5}.$$

(c) Differentiating term by term, we observe that $g'(x) = a_1 + 2a_2x + 3a_3x^2 + \ldots$. This is the function for the sequence we are interested in, so the generating function is $\frac{d}{dx}\left[g(x)\right]$.

(d) We observe that:

$$\begin{aligned}
g(x) &= a_0 + a_1x + a_2x^2 + a_3x^3 + \ldots && \text{(definition)}\\
\frac{g(x) - a_0}{x} &= a_1 + a_2x + a_3x^2 + a_4x^3 + \ldots && \text{(manipulating } g(x)\text{)}\\
g(x) + \frac{g(x) - a_0}{x} &= (a_0 + a_1) + (a_1 + a_2)x + (a_2 + a_3)x^2 + (a_3 + a_4)x^3 + \ldots && \text{(adding 2 series)}
\end{aligned}$$

Since the coefficients yield the sequence we are interested in, our generating function is

$$g(x) + \frac{g(x) - a_0}{x}.$$

Problem 3 Let $\{a_k\}$ be the sequence with $a_k = (k+1)(k+2)$ for all $n \geq 0$. Find a closed form for the generating function $f(x) = \sum_{k=0}^{n} a_kx^k$.

Solution.

We can write the generating function as:

$$\begin{aligned}
g(x) &= \sum_{k=0}^{\infty} a_kx^k && \text{(definition)}\\
&= \sum_{k=0}^{\infty} (k+1)(k+2)x^k && \text{(using } a_k = (k+1)(k+2)\text{)}\\
&= \sum_{k=0}^{\infty} \frac{d^2}{dx^2}\left[x^{k+2}\right] && \\
&= \frac{d^2}{dx^2}\left[\sum_{k=0}^{\infty} x^{k+2}\right] && \text{(sum rule of derivatives)}\\
&= \frac{d^2}{dx^2}\left[\sum_{k=2}^{\infty} x^{k}\right] && \text{(adjusting bounds)}\\
&= \frac{d^2}{dx^2}\left[\sum_{k=0}^{\infty} \left(x^{k}\right) - x - 1\right] && \text{(readjusting bounds)}\\
&= \frac{d^2}{dx^2}\left[\sum_{k=0}^{\infty} \left(x^{k}\right)\right] - \frac{d^2}{dx^2}[x] - \frac{d^2}{dx^2}[1] && \text{(sum rule of derivatives)}
\end{aligned}$$

$$
\begin{aligned}
&= \frac{d^2}{dx^2}\left[\frac{1}{1-x}\right] - \frac{d^2}{dx^2}[x] - \frac{d^2}{dx^2}[1] && \text{(evaluating power series)}\\
&= \frac{d}{dx}\left[(1-x)^{-2}\right] - \frac{d}{dx}[1] - \frac{d}{dx}[0]\\
&= 2(1-x)^{-3} - 0 - 0\\
&= \frac{2}{(1-x)^3}.
\end{aligned}
$$

Problem 4 For each of these generating functions, provide a closed formula for the sequence it determines. I.e., give a closed form for the coefficient of x^k for each k.
(a) $\frac{3x^2}{1+9x}$
(b) $(1+x^2)^4$
(c) $e^{4x} + e^{-4x}$ *Hint: you may use that the power series for e^x is $e^x = \sum_{k=0}^{\infty} \frac{x^k}{k!}$.*

Solution.

(a) We can write

$$
\begin{aligned}
\frac{3x^2}{1+9x} &= 3x^2\left(\frac{1}{1-(-9x)}\right)\\
&= 3x^2 \sum_{k=0}^{\infty} (-9x)^k\\
&= \sum_{k=0}^{\infty} 3(-9)^k x^{k+2}\\
&= \sum_{k=2}^{\infty} 3(-9)^{k-2} x^k. && \text{(adjusting bounds)}
\end{aligned}
$$

So the k^{th} term of the sequence the generating function determines is $3(-9)^{k-2}$ for $k \geq 2$ and 0 otherwise.

(b) We can use the extended binomial theorem to write

$$(1+x^2)^4 = \sum_{k=0}^{\infty} \binom{4}{k} x^{2k}.$$

Since we only need to sum up to 4, we can rewrite this as

$$\sum_{k \in \{0,2,4,6,8\}} \binom{4}{k/2} x^k.$$

So the k^{th} term of the sequence the generating function determines is $\binom{4}{k/2}$ for $k \in \{0, 2, 4, 6, 8\}$ and 0 otherwise.

(c) We can write

$$
\begin{aligned}
e^{4x} + e^{-4x} &= \sum_{k=0}^{\infty}\left[\frac{(4x)^k}{k!}\right] + \sum_{k=0}^{\infty}\left[\frac{(-4x)^k}{k!}\right] && \text{(given)}\\
&= \sum_{k=0}^{\infty} \frac{4^k x^k + (-4)^k x^k}{k!} && \text{(combining sum)}\\
&= \sum_{k=0}^{\infty} \frac{4^k + (-4)^k}{k!} x^k. && \text{(factoring)}
\end{aligned}
$$

So the k^{th} term of the sequence the generating function determines is $\frac{4^k+(-4)^k}{k!}$.

Problem 5 Find the coefficient of x^{12} in the power series of each of the following functions.
(a) $x/(1+3x)$
(b) $1/(1-2x)^8$

Solution.

(a) We can write

$$\begin{aligned}\frac{x}{1+3x} &= x\left(\frac{1}{1-(-3x)}\right)\\ &= x\sum_{k=0}^{\infty}(-3x)^k\\ &= \sum_{k=0}^{\infty}(-3)^k x^{k+1}\\ &= \sum_{k=1}^{\infty}(-3)^{k-1}x^k. && \text{(readjusting bounds)}\end{aligned}$$

So the coefficient of x^{12} comes from the $k=12$ term in the series, and thus the coefficient is $(-3)^{12-1} = -177147$.

(b) We can write

$$\begin{aligned}\frac{1}{(1-2x)^8} &= (1+(-2x))^{-8}\\ &= \sum_{k=0}^{\infty}\binom{-8}{k}(-2x)^k && \text{(extended binomial theorem)}\\ &= \sum_{k=0}^{\infty}\binom{-8}{k}(-2)^k x^k.\end{aligned}$$

So the coefficient of x^{12} comes from the $k=12$ term in the series, and thus the coefficient is $\binom{-8}{12}(-2)^{12} = 50388 \times 4096$.

Problem 6 Prove using generating functions that the number of ways to distribute n cookies among k children such that each child receives at least 2 cookies is $\binom{n-k-1}{k-1}$.

Solution.

Let $\{a_m\}$ be a sequence where a_m represents the number of valid ways to give one child m cookies. For any given child, there are 0 valid ways to give them 0 cookies, 0 valid ways to give them 1 cookie, and 1 valid way to give them m cookies for $m \in \mathbb{N}$, $m \geq 2$. So $\{a_m\} = 0, 0, 1, 1, 1 \ldots$, and the resulting generating function for the number of ways to give cookies to the child is

$$\begin{aligned}a(x) &= x^2 + x^3 + x^4 + x^5 \ldots\\ &= \sum_{m=2}^{\infty} x^m\\ &= \sum_{m=0}^{\infty}(x^m) - x - 1 && \text{(adjusting bounds)}\end{aligned}$$

$$
\begin{aligned}
&= \frac{1}{1-x} - x - 1 \\
&= \frac{1 - x(1-x) - 1(1-x)}{1-x} \\
&= \frac{x^2}{1-x}.
\end{aligned}
$$

Let $\{b_m\}$ be a sequence where b_m represents the number of valid ways to distribute m cookies to k children. Since there are k children, we need the product of the generating functions for k children, which is

$$
b(x) = \left(\frac{x^2}{1-x}\right)^k = x^{2k}(1-x)^{-k}.
$$

This is the generating function for $\{b_m\}$, and we are interested in b_n. We can write

$$
\begin{aligned}
b(x) &= x^{2k}(1+(-x))^{-k} \\
&= x^{2k} \sum_{m=0}^{\infty} \binom{-k}{m} (-x)^m \\
&= \sum_{m=0}^{\infty} \binom{-k}{m} (-1)^m x^{2k+m}.
\end{aligned}
$$

Since we are interested in the coefficient of b_n, we want $2k + m = n$ which is satisfied by $m = n - 2k$. So the coefficient is

$$
\begin{aligned}
\binom{-k}{n-2k}(-1)^{n-2k} &= (-1)^{n-2k} \frac{(-k)(-k-1)(-k-2)\dots(-k-(n-2k-1))}{(n-2k)!} \\
&= (-1)^{n-2k} \frac{(k-n+1)(k-n+2)\dots(-k-1)(-k)}{(n-2k)!} && \text{(commutative)} \\
&= (-1)^{n-2k} \frac{(-1)^{n-2k}(n-k-1)(n-k-2)\dots(k+1)(k)}{(n-2k)!} \\
&= \frac{(n-k-1)(n-k-2)\dots(k+1)(k)}{(n-2k)!} && \text{(negatives cancel)} \\
&= \frac{(n-k-1)!}{(k-1)!(n-2k)!} \\
&= \frac{(n-k-1)!}{(k-1)!((n-k-1)-(k-1))!} && (n-2k = n-k-1-k+1) \\
&= \binom{n-k-1}{k-1}.
\end{aligned}
$$

Therefore, there are $\binom{n-k-1}{k-1}$ ways to distribute n cookies to k children if every child needs at least 2 cookies. □

Problem 7 Use generating functions to solve the recurrence relation $a_k = 2a_{k-1} - 7$ with $a_0 = 1$.

Solution.

Let $g(x)$ be the generating function for $\{a_k\}$. We observe the following:

$$
\begin{aligned}
g(x) &= \sum_{k=0}^{\infty} a_k x^k && \text{(by definition)} \\
\implies xg(x) &= \sum_{k=0}^{\infty} a_k x^{k+1}
\end{aligned}
$$

$$
\begin{aligned}
&= \sum_{k=1}^{\infty} a_{k-1}x^k \\
\implies 2xg(x) - g(x) &= \sum_{k=1}^{\infty} 2a_{k-1}x^k - \sum_{k=0}^{\infty} a_k x^k \\
&= \sum_{k=1}^{\infty} \left[(2a_{k-1} - a_k)\, x^k \right] - a_0 x^0 && \text{(combining sums)} \\
\implies 2xg(x) - g(x) + 1 &= \sum_{k=1}^{\infty} (2a_{k-1} - a_k)\, x^k && (a_0 = 1) \\
&= \sum_{k=1}^{\infty} (7)x^k && (2a_{k-1} - a_k = 7) \\
&= \sum_{k=0}^{\infty} \left(7x^k\right) - 7 && \text{(adjusting bounds)} \\
&= \frac{7}{1-x} - 7 \\
\implies (2x-1)g(x) &= \frac{7}{1-x} - 8 \\
&= \frac{8x-1}{1-x} \\
\implies g(x) &= \frac{8x-1}{(1-x)(2x-1)} \\
&= \frac{7}{1-x} + \frac{6}{2x-1} && \text{(using partial fractions)} \\
&= 7\left(\frac{1}{1-x}\right) - 6\left(\frac{1}{1-2x}\right) \\
&= \sum_{k=0}^{\infty} 7x^k - \sum_{k=0}^{\infty} 6(2x)^k \\
&= \sum_{k=0}^{\infty} 7x^k - 6(2)^k x^k && \text{(combining sums)} \\
&= \sum_{k=0}^{\infty} \left(7 - 6(2)^k\right) x^k.
\end{aligned}
$$

Thus, the coefficient of the x^k term is $7 - 6(2)^k$. Since this is a generating function, this means that $a_k = 7 - 6(2)^k$.

Problem 8 Let a_n denote the sum of the first n squares, i.e., $a_n = 0^2 + 1^2 + 2^2 + \ldots + n^2$.
(a) Give a recurrence relation for $\{a_n\}$.
(b) Use part (a) to show that the generating function for $\{a_n\}$ is

$$g(x) = (x^2 + x)/(1 - x)^4.$$

Hint: you may use that $\sum_{n=1}^{\infty} n^2 x^{n-1} = \frac{1+x}{(1-x)^3}$ *(see homework 5, problem 6(a), you do not need to reprove this).*
(c) Use part (b) to find an explicit formula for the sum $1^2 + 2^2 + \ldots + n^2$.

Solution.

(a) Since a_n is the sum of the first n squares, it is also the sum of the first $n-1$ squares and n^2. Thus, $a_n = a_{n-1} + n^2$. Since the sum of no squares is 0, we define $a_0 = 0$.

(b) We observe the following:

$$
\begin{aligned}
g(x) &= \sum_{k=0}^{\infty} a_k x^k \\
\implies xg(x) &= \sum_{k=0}^{\infty} a_k x^{k+1} \\
&= \sum_{k=1}^{\infty} a_{k-1} x^k \\
\implies g(x) - xg(x) &= \sum_{k=0}^{\infty} a_k x^k - \sum_{k=1}^{\infty} a_{k-1} x^k \\
&= \sum_{k=1}^{\infty} \left(a_k x^k - a_{k-1} x^k\right) + a_0 \\
&= \sum_{k=1}^{\infty} \left[\left(a_k - a_{k-1}\right) x^k\right] + 0 \\
&= \sum_{k=1}^{\infty} (k^2) x^k && (a_k - a_{k-1} = k^2) \\
&= x \sum_{k=1}^{\infty} k^2 x^{k-1} \\
&= x\left(\frac{1+x}{(1-x)^3}\right) && \text{(given)} \\
\implies (1-x)g(x) &= \frac{x+x^2}{(1-x)} \\
\implies g(x) &= \frac{x^2+x}{(1-x)^4}. \quad \square
\end{aligned}
$$

(c) We find the power series of $g(x)$:

$$
\begin{aligned}
g(x) &= \frac{x^2+x}{(1-x)^4} \\
&= (x^2+x)(1+(-x))^{-4} \\
&= (x^2+x)\sum_{k=0}^{\infty} \binom{-4}{k} (-x)^k && \text{(extended binomial theorem)} \\
&= x^2 \sum_{k=0}^{\infty} \binom{-4}{k} (-1)^k x^k + x \sum_{k=0}^{\infty} \binom{-4}{k} (-1)^k x^k && \text{(distributing)} \\
&= \sum_{k=0}^{\infty} \binom{-4}{k} (-1)^k x^{k+2} + x \sum_{k=0}^{\infty} \binom{-4}{k} (-1)^k x^{k+1} \\
&= \sum_{k=2}^{\infty} \binom{-4}{k-2} (-1)^{k-2} x^k + \sum_{k=1}^{\infty} \binom{-4}{k-1} (-1)^{k-1} x^k \\
&= x + \sum_{k=2}^{\infty} \left[\binom{-4}{k-2} (-1)^{k-2} x^k + \binom{-4}{k-1} (-1)^{k-1} x^k\right] \\
&= x + \sum_{k=2}^{\infty} \left[\binom{-4}{k-2} (-1)^{k-2} + \binom{-4}{k-1} (-1)^{k-1}\right] x^k.
\end{aligned}
$$

Since $g(x)$ is the generating function, we have $a_1 = 1$ and for $n \geq 2$,

$$a_n = \binom{-4}{n-2}(-1)^{n-2} + \binom{-4}{n-1}(-1)^{n-1}.$$

Using the result from (6) for negative binomials, we can write this more simply as

$$a_k = \binom{n+1}{n-2} + \binom{n+2}{n-1}.$$

With some simplification using the definition of binomial coefficients, we obtain the fairly well known

$$\sum_{k=1}^{n} k^2 = \frac{n(n+1)(2n+1)}{6}.$$