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Analysis in $\mathbb{R}^n$: Section 1
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Analysis in $\mathbb{R}^n$ Homework 5

Problem 1 Let X, Y be two sets, $f : X \to Y$ a function, and $A \subset X, B \subset Y$ two sets. Define the *image* of A under f as

$$f(A) := \{y \in Y \;:\; y = f(x) \text{ for some } x \in A\} \subset Y,$$

and the *preimage* of B under f as

$$f^{-1}(B) := \{x \in X \;:\; f(x) \in B\} \subset X.$$

Determine if each of the followings is true for every $A, A' \subset X$ and $B, B' \subset Y$. If so, prove it; otherwise, provide a counterexample.

(a) $f(A \cup A') = f(A) \cup f(A')$,

(b) $f(A \cap A') = f(A) \cap f(A')$,

(c) $f^{-1}(B \cup B') = f^{-1}(B) \cup f^{-1}(B')$,

(d) $f^{-1}(B \cap B') = f^{-1}(B) \cap f^{-1}(B')$,

(e) $f(A) \subset B$ if and only if $A \subset f^{-1}(B)$.

Solution.

(a) We claim this is true. We have

$$\begin{aligned}
& y \in f(A \cup A') \\
\iff & y = f(x) \text{ for some } x \in A \cup A' && \text{(definition)} \\
\iff & y = f(x) \text{ for some } x \in A \text{ or } y = f(x) \text{ for some } x \in A' \\
\iff & y \in \{y' \in Y : f(x) = y' \text{ for some } x \in A\} \text{ or } y \in \{y' \in Y : f(x) = y' \text{ for some } x \in A'\} \\
\iff & y \in \{y' \in Y : f(x) = y' \text{ for some } x \in A\} \cup \{y' \in Y : f(x) = y' \text{ for some } x \in A'\} \\
\iff & y \in f(A) \cup f(A'),
\end{aligned}$$

so $f(A \cup A') = f(A) \cup f(A')$.

(b) We claim this is false. For example, take

$$f = \big\{(a,1), (b,2), (c,2), (d,3)\big\}$$

with $A = \{a, b\}$ and $A' = \{c, d\}$. Then,

$$f(A \cap A') = f(\emptyset) = \emptyset \neq \{2\} = \{1,2\} \cap \{2,3\} = f(A) \cap f(B).$$

(c) We claim this is true. We have

$$\begin{aligned} & x \in f^{-1}(B \cup B') \\ \iff & f(x) \in B \cup B' \\ \iff & f(x) \in B \text{ or } f(x) \in B' \\ \iff & x \in f^{-1}(B) \text{ or } x \in f^{-1}(B') \\ \iff & x \in f^{-1}(B) \cup f^{-1}(B'), \end{aligned}$$

so $f^{-1}(B \cup B') = f^{-1}(B) \cup f^{-1}(B')$.

(d) We claim this is true. We have

$$\begin{aligned} & x \in f^{-1}(B \cap B') \\ \iff & f(x) \in B \cap B' \\ \iff & f(x) \in B \text{ and } f(x) \in B' \\ \iff & x \in f^{-1}(B) \text{ and } x \in f^{-1}(B') \\ \iff & x \in f^{-1}(B) \cap f^{-1}(B'), \end{aligned}$$

so $f^{-1}(B \cap B') = f^{-1}(B) \cap f^{-1}(B')$.

(e) We claim this is true. We will prove the contrapositive for both directions.

($\Rightarrow$) Suppose $a \in A$ but $a \notin f^{-1}(B)$. Then $f(a) \notin B$, so we cannot have $f(A) \subset B$.

($\Leftarrow$) Suppose $y \in f(A)$ but $y \notin B$. So there exists some $a \in A$ such that $y = f(a)$, but because $f(a) \notin B$, we have $a \notin f^{-1}(B)$. So we cannot have $A \subset f^{-1}(B)$.

Problem Rudin 1 Suppose f is a real function defined on $\mathbb{R}^1$ which satisfies

$$\lim_{h \to 0} [f(x+h) - f(x-h)] = 0$$

for every $x \in \mathbb{R}^1$. Does this imply that f is continuous?

Solution.

This is false. For example, consider the piecewise function

$$f(x) = \begin{cases} 1 & \text{if } x = 0 \\ x & \text{otherwise} \end{cases}.$$

Then, $\lim_{x \to 0} f(x) = 0 \neq 1 = f(0)$, so f is not continuous. However, since we have

$$\lim_{h \to 0} f(x+h) = 0 \text{ and } \lim_{h \to 0} f(x-h) = 0,$$

it is true that

$$\lim_{h \to 0} [f(x+h) - f(x-h)] = 0.$$

□

Problem Rudin 2 If f is a continuous mapping of a metric space X into a metric space Y, prove that

$$f(\overline{E}) \subset \overline{f(E)}$$

for every set $E \subset X$. ($\overline{E}$ denotes the closure of E.) Show, by an example, that $f(\overline{E})$ can be a proper subset of $\overline{f(E)}$.

Solution.

Let $y \in f(\overline{E})$. Then, there exists some $x \in \overline{E}$ such that $y = f(x)$. We have shown that this implies there exists some sequence $\{x_n\} \subset E$ that converges to x. Since we have that f is continuous at x, we also have that $f\left(\{x_n\}\right)$ converges to $f(x) = y$ from a result in class. Thus, since $f\left(\{x_n\}\right)$ is a sequence in $f(E)$ that converges to y, y is a limit point of $f(E)$ and $y \in \overline{f(E)}$. Therefore, $f(\overline{E}) \subset \overline{f(E)}$.

Additionally, this inclusion can be strict. For example, consider the function $f(x) = e^x$. Then, since $\mathbb{R}$ is closed, we have $f(\overline{\mathbb{R}}) = f(\mathbb{R}) = (0, \infty)$, but $\overline{f(\mathbb{R})} = [0, 1)$. □

Problem Rudin 3 Let f be a continuous real function on a metric space X. Let $Z(f)$ (the *zero set* of f) be the set of all $p \in X$ at which $f(p) = 0$. Prove that $Z(f)$ is closed.

Solution.

Case 1: The codomain of f does not contain 0. Then $Z(p) = \emptyset$, so it is closed because it is finite.

Case 2: The codomain of f contains 0. We have shown in class that f is continuous if and only if $f^{-1}(F)$ is closed for every closed set F in the codomain. Since $\{0\}$ is a closed set in the codomain of f, and $f^{-1}(\{0\}) = Z(p)$, we must have that $Z(p)$ is closed. □

Problem Rudin 4 Let f and g be continuous mappings of a metric space X into a metric space Y, and let E be a dense subset of X (E is *dense* in X if $\overline{E} = X$). Prove that $f(E)$ is dense in $f(X)$. If $g(p) = f(p)$ for all $p \in E$, prove that $g(p) = f(p)$ for all $p \in X$. (In other words, a continuous mapping is determined by its values on a dense subset of its domain.)

Solution.

We will show a double-inclusion to prove $\overline{f(E)} = f(X)$.

($\subset$) By $\overline{f(E)}$ we mean the closure in $f(X)$, so this is given.

($\supset$) From problem 2, we have $f(X) = f(\overline{E}) \subset \overline{f(E)}$.

Next, let $p \in X$, and assume that $f(e) = g(e)$ for all $e \in E$. Since $X = \overline{E}$, there exists some $\{p_n\} \in E$ such that $\{p_n\}$ converges to p. We have shown in class that since f and g are continuous, $f(\{p_n\})$ converges to $f(p)$ and $g(\{p_n\})$ converges to $g(p)$. Since $f(p_n) = g(p_n)$ for all $n \in \mathbb{N}$ (as $p_n \in E$ for all $n \in \mathbb{N}$), $f(\{p_n\})$ and $g(\{p_n\})$ must converge to the same point. Thus, $f(p) = g(p)$. □