Nathan Bickel
MATH 701: Section 001
Professor: Dr. Tsai
October 22, 2024

MATH 701 Homework 6

Problem 3.3.4 Let C be a normal subgroup of the group A and let D be a normal subgroup of the group B. Prove that $(C \times D) \trianglelefteq (A \times B)$ and $(A \times B)/(C \times D) \cong (A/C) \times (B/D)$.

Let $(a, b) \in A \times B$ and $(c, d) \in C \times D$. Then we have $(a, b)(c, d)(a, b)^{-1} = (aca^{-1}, dbd^{-1})$. Since $C \trianglelefteq A$ and $D \trianglelefteq B$, we have $aca^{-1} \in C$ and $dbd^{-1} \in D$, so $(a, b)(c, d)(a, b)^{-1} \in (C \times D)$. Thus, $(C \times D) \trianglelefteq (A \times B)$.

Let $\varphi : (A/C) \times (B/D) \to (A \times B)/(C \times D)$ by $\varphi\left((aC, bD)\right) = (a, b)(C \times D)$.

To see that φ is well-defined, let $a, a' \in A$ and $b, b' \in B$ such that $aC = a'C$ and $bD = b'D$. Let $(x, y) \in (a, b)(C \times D)$. Then $x = ac$ and $y = bd$ for some $c \in C$ and $d \in D$. Since $aC = a'C$, we have $a = a'c'$ for some $c' \in C$, so we have $x = a'(c'c)$ and using the same reasoning for b we have $y = b'(d'd)$ for some $d' \in D$. So $(x, y) \in (a', b')(C \times D)$. Using similar reasoning backward we can conclude that $(a, b)(C \times D) = (a', b')(C \times D)$.

To see that φ is injective, let $aC, a'C \in (A/C)$ and $bD, b'D \in (B/D)$ such that $\varphi\left((aC, bD)\right) = \varphi\left((a'C, b'D)\right)$. So $(a, b)(C \times D) = (a', b')(C \times D)$. So $(a, b) = (a', b')(c', d')(c, d)$ for some $(c', d') \in C \times D$, so $a = a'c'c$ and $b = b'd'd$, so $a \in a'C$ and $b \in b'D$. The backwards reasoning also holds, so we have $aC = a'C$ and $bD = b'D$, so $(aC, bD) = (a'C, b'D)$. So φ is injective.

To see that φ is surjective, let $(a, b)(C \times D) \in (A \times B)/(C \times D)$. Then $\varphi(aC, bD) = (a, b)(C \times D)$.

To see that φ is a homomorphism, let $(aC, bD), (a'C, b'D) \in (A/C) \times (B/D)$. Then we have

$$\begin{aligned}
\varphi\left((aC, bD)(a'C, b'D)\right) &= \varphi\left((aa'C), (bb'D)\right) \\
&= (aa', bb')(C \times D) \\
&= (a, b)(a', b')(C \times D) \\
&= (a, b)(C \times D)(a', b')(C \times D) \\
&= \varphi\left((aC, bD)\right)\varphi\left((a'C, b'D)\right),
\end{aligned}$$

so φ is a homomorphism.

So φ is an isomorphism, and therefore $(A/C) \times (B/D) \cong (A \times B)/(C \times D)$. $\square$

(Note that we could have also used the first isomorphism theorem, which would have been much easier...)

Problem 3.3.7 Let M and N be normal subgroups of G such that $G = MN$. Prove that

$$G/(M \cap N) \cong (G/M) \times (G/N).$$

[Draw the lattice.]

Define $\varphi : G \to (G/M) \times (G/N)$ by $\varphi(g) = (gM, gN)$. Then we have $\ker \varphi := \{g : \varphi(g) = (M, N)\}$. We have that $gM = M$ iff $g \in M$ and $gN = N$ iff $g \in N$, so $\varphi(g) = (gM, gN) = (M, N)$ iff $g \in M \cap N$. So $\ker \varphi = M \cap N$. Since we clearly have $\varphi(G) = (G/M) \times (G/N)$, from the first isomorphism theorem we have

$$G/(M \cap N) = G/\ker(\varphi) \cong \varphi(G) = (G/M) \times (G/N).$$

□

Problem 3.4.1 Prove that if G is an abelian simple group then $G \cong Z_p$ for some prime p. [Do not assume G is a finite group.]

Let G be an abelian simple group. Since every subgroup of an abelian group is normal, and G is simple, there are no subgroups other than 1 and G. So for every $a \in G$ with $a \neq 1$, we have $\langle a \rangle = G$, so we have that G is cyclic.

It is known that every cyclic group is isomorphic to either $\mathbb{Z}$ or Z_n for some n. We know that $\mathbb{Z}$ is not simple ($2\mathbb{Z}$ is a normal subgroup, for instance), so $G \cong Z_p$ for some $p \in \mathbb{N}$. If p is not prime, then there would be an element $a \in Z_n \setminus \{\overline{0}\}$ with $\langle a \rangle \neq Z_p$, so p must be prime. □

Problem 3.4.5 Prove that subgroups and quotient groups of a solvable group are solvable.

Let G be a solvable group with

$$1 = G_0 \trianglelefteq G_1 \trianglelefteq G_2 \trianglelefteq \ldots \trianglelefteq G_s = G$$

such that G_{i+1}/G_i is abelian for all $i \in \{0, 1, \ldots, s-1\}$.

Let $H \leq G$. Then we claim that we have the chain

$$1 = H_0 \trianglelefteq H_1 \trianglelefteq H_2 \trianglelefteq \ldots \trianglelefteq H_s = H,$$

where $H_i = G_i \cap H$. We first show that $H_i \trianglelefteq H_{i+1}$ for all i. Let $x \in H_{i+1}$ and $h \in H_i$. Since $x \in H_{i+1} = G_{i+1} \cap H$ we have $x, h \in H$, so we have $xhx^{-1} \in H$. Since $G_i \trianglelefteq G_{i+1}$ and $x \in G_{i+1}$ we have $xhx^{-1} \in G_i$. So $xhx^{-1} \in H_i$ and thus $G_i \trianglelefteq G_{i+1}$. We next show that H_{i+1}/H_i is abelian for all i. Let $x, y \in H_{i+1}$. Then we have

$$\begin{aligned}
(xy)H_i &= (xy)(G_i \cap H) \\
&= (xy)G_i \cap (xy)H \\
&= (xG_i)(yG_i) \cap H && (x, y \in H) \\
&= (yG_i)(xG_i) \cap (yx)H && (G_{i+1}/G \text{ is abelian}) \\
&= (yx)G_i \cap (yx)H \\
&= (yx)(G_i \cap H) \\
&= (yx)H_i,
\end{aligned}$$

so H_{i+1}/H_i is abelian. Thus, H is solvable.

Now, let $N \trianglelefteq G$ and G/N. Then we claim we have the chain

$$1 = G_0N/N \trianglelefteq G_1N/N \trianglelefteq G_2N/N \trianglelefteq \cdots \trianglelefteq G_sN/N = G/N.$$

We first show that $G_iN/N \trianglelefteq G_{i+1}N/N$ for all i. Since $G_i \trianglelefteq G_{i+1}$, we must have $G_iN \trianglelefteq G_{i+1}N$ and $N \leq G_iN$, $N \trianglelefteq G_{i+1}N$. So by the third isomorphism theorem, $G_iN/N \trianglelefteq G_{i+1}N/N$. Also by the third isomorphism theorem we have $(G_{i+1}N/N)/(G_iN/N) \cong G_{i+1}N/G_iN$, which is abelian since G_{i+1}/G_i is abelian. Thus, G/N is solvable. □

Problem 3.4.11 Prove that if H is a nontrivial normal subgroup of the solvable group G then there is a nontrival subgroup A of H with $A \trianglelefteq G$ and A abelian.

Let G be a solvable group with

$$1 = G_0 \trianglelefteq G_1 \trianglelefteq G_2 \trianglelefteq \cdots \trianglelefteq G_s = G$$

such that G_{i+1}/G_i is abelian for all $i \in \{0, 1, \ldots, s-1\}$, $G_i \trianglelefteq G$ for all i (which is possible by another exercise) and $G_1 \neq G_0$ (which is possible because G has the nontrivial subgroup H). Let $H \trianglelefteq G$ be nontrivial, and consider $A := G_1 \cap H$, which is a subgroup of H.

Let $a \in A$ and $g \in G$. We have $a \in H$, so since $H \trianglelefteq G$ we have $gag^{-1} \in H$. We also have $a \in G_1$, so since $G_1 \trianglelefteq G$ we have $gag^{-1} \in G_1$. So $gag^{-1} \in G_1 \cap H = A$. Thus, $A \trianglelefteq G$. We have that $G_1/G_0 = G_1/1$ is abelian, so A is abelian. □

Problem 3.4.12 Prove (without using the Feit-Thompson Theorem) that the following are equivalent:

(i) Every group of odd order is solvable.

(ii) The only simple groups of odd order are those of prime order.

(i) $\Rightarrow$ (ii): Let G be a simple group of odd order. By (i), G is solvable. The only possible chain is $1 = G_0 \trianglelefteq G_1 = G$, and since G_1/G_0 must be abelian we have that $G \cong G/1 = G_1/G_0$ is abelian. From Problem 3.4.1, then, $G \cong Z_p$ for some prime p, so G has prime order.

(ii) $\Rightarrow$ (i): Let G be a group of odd order. We proceed by induction on $|G|$. For the base case, if $|G| = 1$, then G is trivially solvable. Now, let $|G| > 1$ be odd, and suppose that for all groups of odd order G' with $|G'| < |G|$, G' is solvable.

Case 1: G is simple. Then from (ii), G has prime order, so G is cyclic and thus G is abelian. So G is solvable.

Case 2: G is not simple. Then G has a nontrivial normal subgroup N. By Lagrange's theorem, $|N| \mid |G|$, so $|N|$ is odd, and since N is non-trivial, we have $|N| < |G|$. So N is solvable by the induction hypothesis. This also holds for G/N, so G/N is solvable. So there exist chains

$$1 = N_0 \trianglelefteq N_1 \trianglelefteq N_2 \trianglelefteq \cdots \trianglelefteq N_s = N$$

and

$$1 = G_0N/N \trianglelefteq G_1N/N \trianglelefteq G_2N/N \trianglelefteq \cdots \trianglelefteq G_tN/N = G/N,$$

and by the third isomorphism theorem $N = G_0 \trianglelefteq G_1 \trianglelefteq G_2 \trianglelefteq \cdots \trianglelefteq G_t = G$ up to isomorphism. So we have

$$1 = N_0 \trianglelefteq N_1 \trianglelefteq N_2 \trianglelefteq \cdots \trianglelefteq N_s = N = G_0 \trianglelefteq G_1 \trianglelefteq G_2 \trianglelefteq \cdots \trianglelefteq G_t = G.$$

So G is solvable. □

Problem 3.5.2 Prove that σ^2 is an even permutation for every permutation σ.

Let σ be a permutation. Then σ^2 can be written as the composition of twice the number of transpositions that compose σ, which is an even number. So σ^2 is even.

Problem 3.5.3 Prove that S_n is generated by $\{(i\, i+1) \mid 1 \leq i \leq n-1\}$.

Let $\tau = (a\, b) \in S_n$ be a transposition. Suppose $a < b$ (if $b < a$, write $(b\, a)$, and if $a = b$, $(a\, b) = e$, which is in any generated set). We prove by induction on $b - a$ that $(a\, b)$ can be written as a product of transpositions in the given set. The base case, $b - a = 1$, is trivial. Now suppose $b - a > 1$ and that $(a\ (b-1))$ can be generated by the set. Then, it is easy to see that

$$((b-1)\ b)(a\ (b-1))((b-1)\ b) = (a\,b),$$

so the claim holds.

Thus, any transposition in S_n can be generated by the set. Since every permutation can be written as the product of transpositions, the set generated S_n. $\square$

Problem 3.5.4 Show that $S_n = \langle (1\,2), (1\,2\,3 \cdots n) \rangle$ for all $n \geq 2$.

We can generate $(i\ (i+1))$ for any $i \in \{1, 2, \ldots, n-1\}$ by

$$(1\,2\,3 \cdots n)^{i-1}(1\,2)(1\,2\,3 \cdots n)^{-(i-1)},$$

so by Problem 3.5.3 this set generates S_n. $\square$

Problem 3.5.5 Show that if p is prime, $S_p = \langle \sigma, \tau \rangle$ where σ is any transposition and τ is any p-cycle.

Let $a_0, a_1, \ldots, a_{p-1} \in \{0, 1, \ldots, p-1\}$ be distinct such that $\sigma = (a_0\ a_i)$ for some $i \in \{1, 2, \ldots, p-1\}$ and $\tau = (a_0\ a_1\ a_2\ \cdots\ a_{p-1})$. For all $k \in \mathbb{Z}$, let $\overline{k}$ denote $k \mod p$. Let $j \in \mathbb{Z}$ such that $\overline{ij} = 1$, which is guaranteed to exist since p is prime. Then we have

$$\tau^i \sigma \tau^{-i} = (a_i\ a_{\overline{2i}}), \tau^{2i}\sigma\tau^{-2i} = (a_{\overline{2i}}, a_{\overline{3i}}), \ldots, \tau^{ij}\sigma\tau^{-ij} = (a_{\overline{i(j-1)}}\ a_{\overline{ij}}) = (a_{\overline{i(j-1)}}\ a_1).$$

So all of these transpositions are in $\langle \sigma, \tau \rangle$. We can then write

$$(a_0\ a_1) = (a_{\overline{i(j-1)}}\ a_1)(a_{\overline{i(j-2)}}\ a_{\overline{i(j-1)}}) \cdots (a_{\overline{2i}}, a_{\overline{3i}})(a_i\ a_{\overline{2i}})(a_0\ a_i),$$

so $(a_0\ a_1) \in \langle \sigma, \tau \rangle$. We can use the same reasoning as in Problem 3.5.4 to write that

$$\langle (a_0\ a_1), (a_0\ a_1\ a_2\ \cdots\ a_{p-1}) \rangle = S_p.$$

$\square$

Problem 3.5.9 Prove that the (unique) subgroup of order 4 in A_4 is normal and is isomorphic to V_4.

The unique subgroup of order 4 in A_4 is $H := \{(1), (1\,2)(3\,4), (1\,3)(2\,4), (1\,4)(2\,3)\}$. This is (1) together with all possible products of disjoint transpositions (up to reordering) in A_4: all other elements are 3-cycles. Let $\sigma \in A_4$ and $\tau = (a\ b)(c\ d)$ in this subgroup. Then $\sigma(a\ b)(c\ d)\sigma^{-1} = (\sigma(a)\ \sigma(b))(\sigma(c)\ \sigma(d))$. If $\tau = (1)$, then $\sigma\tau\sigma^{-1} = (1)$, and otherwise a, b, c, d are pairwise distinct, which means $\sigma(a), \sigma(b), \sigma(c), \sigma(d)$ are pairwise distinct. So $\sigma\tau\sigma^{-1} \in H$ since it is the product of disjoint cycles, and thus H is normal in A_4.

We note that A_4 is not cyclic since all elements except the identity have order 2. The only other group of order 4 up to isomorphism is V_4, so $A_4 \cong V_4$. $\square$

Problem 3.5.12 Prove that A_n contains a subgroup isomorphic to S_{n-2} for each $n \geq 3$.

Let $\varphi : S_{n-2} \to A_n$ by, for all $\sigma \in S_{n-2}$,

$$\varphi(\sigma) := \begin{cases} \sigma & \sigma \text{ is even} \\ \sigma(n-1\ n) & \sigma \text{ is odd} \end{cases}.$$

Then, we have that $H := \varphi(S_{n-2}) \leq A_n$. Then $\varphi : S_{n-2} \to H$ is surjective by definition. To see that it is injective, let $\sigma, \tau \in S_{n-2}$ such that $\varphi(\sigma) = \varphi(\tau)$. if $\varphi(\sigma) = \sigma(n-1\ n)$ and $\varphi(\tau) = \tau(n-1\ n)$. Then $\sigma = \tau$. Otherwise, $\varphi(\sigma) = \sigma$ and $\varphi(\tau) = \tau$. Then $\sigma = \tau$. So φ is injective.

It is easy to show that φ is a homomorphism: since $(n-1\ n)$ will be disjoint from any $\sigma, \tau \in S_{n-2}$, so $(n-1\ n)$ will commute with σ and τ, which is enough to show the homomorphism property by cases. So φ is an isomorphism, and thus $S_{n-2} \cong H$. $\square$