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MATH 544 Homework 2

Let $\mathbb{F}$ be a field. Then an ordering of $\mathbb{F}$ is a subset $\mathbb{F}_+$, called the set of positive elements, such that the following hold.

1. (Closure) If $a, b \in \mathbb{F}_+$, then $a + b \in \mathbb{F}_+$ and $ab \in \mathbb{F}_+$.
2. (Trichotomy) For any $a \in \mathbb{F}$ exactly one of the following holds:

$$a \in \mathbb{F}_+$$
$$a = 0$$
$$-a \in \mathbb{F}_+.$$

For all problems that follow, we will assume all elements are in an ordered field $\mathbb{F}$ with positive elements $\mathbb{F}_+$.

Problem 2.10 Prove that if $a, b \in \mathbb{F}$, then exactly one of the following holds:

$$a < b, \quad a = b, \quad a > b.$$

Consider $b - a$, which is in $\mathbb{F}$ by closure. By trichotomy, there are three options:

Case 1: $b - a \in \mathbb{F}_+$. Then $a < b$ by definition.

Case 2: $b - a = 0$. Then $a = b$.

Case 3: $-(b - a) \in \mathbb{F}_+$. Then $a - b \in \mathbb{F}_+$, so $b < a$. Thus $a > b$.

So each possibility is covered exactly once. □

Problem 2.11 Prove that if $a < b$ and $b < c$ then $a < c$.

By definition, $(b - a), (c - b) \in \mathbb{F}_+$. By closure, $(c - b) + (b - a) \in \mathbb{F}_+$, so $c - a \in \mathbb{F}_+$. Thus, $a < c$.

We also note that if $a < b$ and $b \leq c$, then we still have $a < c$. If $b < c$ this follows from above, and if $b = c$, then $a < c$ because $a < b$. □

Problem 2.12 Prove that if $a < b$ and $c < d$ then $a + c < b + d$.

By definition, $(b - a), (d - c) \in \mathbb{F}_+$. By closure, $(b - a) + (d - c) \in \mathbb{F}_+$, so by associativity and commutativity $(b + d) - (a + c) \in \mathbb{F}+$. Thus, $a + c < b + d$. □

Problem 2.13 Prove that if $a < b$ and $c > 0$, then $ac < bc$.

By definition, $b - a, c \in \mathbb{F}_+$. By closure, $c \cdot (b - a) \in \mathbb{F}_+$, so by distributivity $bc - ac \in \mathbb{F}_+$. Thus $ac < bc$. □

Problem 2.14 Prove that if $a < b$ and $c < 0$, then $ac > bc$.

By definition, $b - a, -c \in \mathbb{F}_+$. By closure, $-c \cdot (b - a) \in \mathbb{F}_+$, so by distributivity $ac - bc \in \mathbb{F}_+$. Thus $bc < ac$ and so $ac > bc$. □

Problem 2.15 Prove that if $a < b$ and $c \leq d$ then $a + c < b + d$.

We will first prove a lemma: if $e \leq f$, then $e + g \leq f + g$. If $e = f$, then $e + g = f + g$, so $e + g \leq f + g$. If $e < f$, then $f - e \in \mathbb{F}_+$, and rewriting this $(f + g) - (e + g) \in \mathbb{F}_+$. So $e + g < f + g$ and thus $e + g \leq f + g$.

We will now prove the claim. First, since $b - a \in \mathbb{F}_+$, we can write $(b+c) - (a+c) \in \mathbb{F}_+$ to see that $a+c < b+c$. Next, since $c \leq d$, from the lemma $b + c \leq b + d$. By transitivity, we conclude from $a + c < b + c \leq b + d$ that $a + c < b + d$. □

Problem 2.16 Prove that if $0 < a < b$ and $0 < c \leq d$, then $ac < bd$.

We will first prove a lemma: if $e \leq f$ and $g > 0$, then $eg \leq fg$. If $e = f$, then $eg = fg$, so $eg \leq fg$. If $e < f$, then $eg < fg$ by problem 2.13 and thus $eg \leq fg$.

We will now prove the claim. By 2.13, we have $ac < bc$ since $c > 0$. By transitivity, $b > 0$ from $0 < a < b$, so by the lemma $bc \leq bd$. By transitivity, we conclude from $ac < bc \leq bd$ that $ac < bd$. □

Problem 2.17 Prove that if $a_1, a_2, \ldots, a_n > 0$ then $a_1 a_2 \ldots a_n > 0$ and $a_1 + a_2 + \cdots + a_n > 0$. Thus if $a > 0$ then $a^n, na > 0$.

We will induct on n.

Base Case: $n = 1$ is clear.

Induction Step: Let $n \in \mathbb{N}$, $n > 1$, and assume that $a_1 a_2 \ldots a_{n-1} > 0$ and $a_1 + a + 2 + \cdots + a_{n-1} > 0$. So we have $a_1 a_2 \ldots a_{n-1}, a_n \in \mathbb{F}_+$, and by closure of multiplication, we have $(a_1 a_2 \ldots a_{n-1}) a_n \in \mathbb{F}_+$, so $a_1 a_2 \ldots a_n > 0$. Similarly, we have $a_1 + a_2 + \cdots + a_{n-1}, a_n \in \mathbb{F}_+$, and by closure of addition, we have $(a_1 + a_2 + \cdots + a_{n-1}) + a_n \in \mathbb{F}_+$, so $a_1 + a_2 + \cdots + a_n > 0$.

So the claim holds for all n. It follows from repeated multiplication and addition that $a^n, na > 0$. □

Problem 2.18 If $a \neq 0$, then $a^2 > 0$. In particular $1 = 1^2$ is positive.

Case 1: $a > 0$. Then $a^2 > 0$ follows from problem 2.17.

Case 2: $a < 0$. Then $-a \in \mathbb{F}_+$. By closure, $(-a)(-a) \in \mathbb{F}_+$, so $a^2 \in \mathbb{F}_+$ and thus $a^2 > 0$.

Problem 2.19 Let $a_1, a_2, \ldots, a_n \in \mathbb{F}$. Then

$$a_1{}^2 + a_2{}^2 + \cdots + a_n{}^2 \geq 0$$

with equality if and only if $a_1 = a_2 = \cdots = a_n = 0$.

Case 1: $a_1 = a_2 = \cdots = a_n = 0$. Since $0^2 = 0$, clearly $a_1{}^2 + a_2{}^2 + \cdots + a_n{}^2 = 0$.

Case 2: For some $1 \leq k \leq n$, there are indices i_j such that $a_{i_1}, a_{i_2}, \ldots, a_{i_k}$ are non-zero for all j. Since all other values are $0^2 = 0$, we have

$$a_1{}^2 + a_2{}^2 + \cdots + a_n{}^2 = a_{i_1}{}^2 + a_{i_2}{}^2 + \cdots + a_{i_k}{}^2.$$

From problem 2.18, we have $a_{i_1}{}^2, a_{i_2}{}^2, \ldots, a_{i_k}{}^2 > 0$, and then by problem 2.17, $a_{i_1}{}^2 + a_{i_2}{}^2 + \cdots + a_{i_k}{}^2 > 0$. So we have $a_1{}^2 + a_2{}^2 + \cdots + a_n{}^2 > 0$, and thus equality never holds in this case. □

Problem 2.20 Prove that:

(a) If $a > 0$, then $1/a > 0$.

(b) If $a < 0$, then $1/a < 0$.

(a) Suppose (toward contradiction) that $a > 0$ and $\frac{1}{a} \leq 0$.

Case 1: $\frac{1}{a} = 0$. But then $a \cdot \frac{1}{a} = a \cdot 0 = 0$, a contradiction since $a \cdot \frac{1}{a} = 1$ by definition.

Case 2: $\frac{1}{a} < 0$. Then by definition we have $a, -\frac{1}{a} \in \mathbb{F}_+$, and so by closure $a \cdot -\frac{1}{a} \in \mathbb{F}_+$. Thus, $-1 \in \mathbb{F}_+$, a contradiction.

(b) We will use a similar approach. Suppose that $a < 0$ and $\frac{1}{a} \geq 0$.

Case 1: $\frac{1}{a} = 0$. Then $a \cdot \frac{1}{a} = a \cdot 0 = 0$, a contradiction.

Case 2: $\frac{1}{a} > 0$. Then we have $-a, \frac{1}{a} \in \mathbb{F}_+$, and so by closure $-a \cdot \frac{1}{a} = -1 \in \mathbb{F}_+$, a contradiction. □

Problem 2.21 If $0 < a < b$, then $1/b < 1/a$.

By definition, we have $a, b - a \in \mathbb{F}_+$, and by transitivity we have $b \in \mathbb{F}_+$. By closure, $ab \in \mathbb{F}_+$, and so by problem 2.20 we have $\frac{1}{ab} \in \mathbb{F}_+$. By closure, then, we have $(b - a)\left(\frac{1}{ab}\right) \in \mathbb{F}_+$, which we can rewrite as $\frac{1}{a} - \frac{1}{b} \in \mathbb{F}_+$. Thus, $\frac{1}{b} < \frac{1}{a}$. □

Problem 2.22 For $a \in \mathbb{F}$,

$$|a| \geq 0$$

with equality if and only if $a = 0$.

By trichotomy, there are three cases:

Case 1: $a > 0$. Then $|a| = a$, so $|a| > 0$.

Case 2: $a = 0$. Then $|a| = 0$ by definition.

Case 3: $a < 0$. Then $-a \in \mathbb{F}_+$, and since $|a| = -a$, we have $|a| \in \mathbb{F}_+$ and thus $|a| > 0$.

So $|a| \geq 0$ for all $a \in \mathbb{F}$ (because in all cases we have $|a| > 0$ or $|a| = 0$), with equality only where $a = 0$. □

Problem 2.23 Prove that for $a \in \mathbb{F}$ we have $a \leq |a|$.

Case 1: $a > 0$. Then $|a| = a$.

Case 2: $a = 0$. Then $|a| = 0$, so $|a| = a$.

Case 3: $a < 0$. Then $-a \in \mathbb{F}_+$ by definition, and since $|a| = -a$, we have $|a| \in \mathbb{F}_+$. By closure, $|a| + (-a) \in \mathbb{F}_+$, so $|a| - a \in \mathbb{F}_+$. Thus, $a < |a|$.

So $a \leq |a|$ in every case. □

Problem 2.24 Prove that for $a \in \mathbb{F}$ we have $a^2 = |a|^2$.

Case 1: $a > 0$. Then $|a| = a$, so clearly $a^2 = |a|^2$.

Case 2: $a = 0$. Then $|a| = 0$, so clearly $a^2 = 0^2 = |a|^2$.

Case 3: $a < 0$. Then $|a| = -a$, and we have $a^2 = (-a) \cdot (-a) = |a| \cdot |a| = |a|^2$. $\square$

Problem 2.25 If $a, b \in \mathbb{F}$, then the following are equivalent:

(a) $|a| = |b|$,

(b) $a = \pm b$,

(c) $a^2 = b^2$.

$(a) \implies (b)$: Assume $|a| = |b|$.

Case 1: $a \geq 0$ and $b \geq 0$. Then $a = |a| = |b| = b$, so $a = b$.

Case 2: $a < 0$ and $b < 0$. Then $-a = |a| = |b| = -b$, so taking the additive inverse of each side yields $a = b$.

Case 3: $a \geq 0$ and $b < 0$. Then $a = |a| = |b| = -b$, so $a = -b$.

Case 4: $a < 0$ and $b \geq 0$. Then $-a = |a| = |b| = b$, which shows $a = -b$.

$(b) \implies (c)$: Assume $a = \pm b$.

Case 1: $a = b$. Then $a^2 = b^2$.

Case 2: $a = -b$. Then $a^2 = a \cdot a = (-b) \cdot (-b) = b^2$, so $a^2 = b^2$.

$(c) \implies (a)$: Assume $a^2 = b^2$. From problem 2.24, we can then write $|a|^2 = a^2 = b^2 = |b|^2$. Since, from problem 2.22, we have $|a| \geq 0$ and $|b| \geq 0$, this implies that $|a| = |b|$. $\square$