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MATH 300: Section H01
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March 19, 2022

MATH 300 Homework Extra Credit Problems

Problem 1

(a) We claim this is true. Assume f and $f \circ g$ are both injective. By definition of $f \circ g$ being injective, for all c_1 and c_2, $f(g(c_1)) = f(g(c_2)) \Leftrightarrow c_1 = c_2$. By definition of f being injective, $f(g(c_1)) = f(g(c_2)) \Leftrightarrow g(c_1) = g(c_2)$. So $g(c_1) = g(c_2) \Rightarrow f(g(c_1)) = f(g(c_2)) \Rightarrow c_1 = c_2$, and $g(c_1) = g(c_2) \Rightarrow c_1 = c_2$. By definition, then, g is injective.

(b) We claim this is false. For example, take $f : \mathbb{R} \to \mathbb{R}_{\geq 0}$ where $f(x) = x^2$ and $g : \mathbb{R}_{\geq 0} \to \mathbb{R}$ where $g(x) = \sqrt{x}$. Then, $f \circ g : \mathbb{R}_{\geq 0} \to \mathbb{R}_{\geq 0}$ where $f \circ g = f(g(x)) = (\sqrt{x})^2 = x$. Since $f \circ g$ is the identity function from $\mathbb{R}_{\geq 0}$ to $\mathbb{R}_{\geq 0}$, it is injective and surjective. This is because no single output is assigned to two different inputs (since the output equals the input, it can clearly only be assigned to one input), so it is injective, and for any $c \in \mathbb{R}_{\geq 0}$, $(f \circ g)(c) = c$, so it is surjective. Since the square root is uniquely defined for every real positive number, g is injective. However, f is not injective, because $f(-1) = f(1) = 1$ but $-1 \neq 1$. Since $f \circ g$ and g are both injective and f is not, the property does not hold.

(c) We claim this is false using the same counterexample as (b). In this example, $f \circ g$ is injective but it is not true that f and g are both injective (because f is not), so the property does not hold.

(d) We claim this is false using the same counterexample as (b). In this example, f is surjective because for any $c \in \mathbb{R}_{\geq 0}$, $f(\sqrt{c}) = c$. However, g is not surjective because there is no $a \in \mathbb{R}_{\geq 0}$ such that $g(a) = -1$ (or any other negative number), even though $-1 \in \mathbb{R}$, because a square root yields only non-negative numbers. So f and $f \circ g$ are both surjective, but g is not surjective, so the property does not hold.

(e) We claim this is true. Assume g and $f \circ g$ are both surjective. Then, assume $c \in C$. Since $f \circ g$ is a surjective function and is defined from A to C, $(\exists a \in A)[(f \circ g)(a) = c]$. So $f(g(a)) = c$, and since g is a function from A to B, $g(a) \in B$. Thus, there exists a $b \in B$ such that $f(b) = c$ by choosing $b = g(a)$, so by definition f is surjective.

(f) We claim this is false using the same counterexample as (b). In this example, $f \circ g$ is surjective but is not true that f and g are both surjective (because g is not), so the property does not hold.

Problem 2

The example in problem 1, part (b) has this property. As shown, $f \circ g$ is both injective and surjective and thus is a bijection, but f is not a bijection because it is not injective and g is not a bijection because it is not surjective.

I have neither given nor received unauthorized help on these problems.