MATH 552: Section 001
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MATH 552 Homework 1*

Problem 1.2.2 Show that
(a) Re(iz) = -Im z
(b) Im$(iz) = \operatorname{Re} z$

Solution. Let $z = a + bi$.

$$\begin{aligned} iz &= ai + bi^2 \\ iz &= -b + ai && \text{(Using } i^2 = -1\text{)} \end{aligned}$$

(a)

$$\begin{aligned} \operatorname{Re}(iz) &= -b \\ \operatorname{Im} z &= b \\ -b &= -b && \text{(Negating } \operatorname{Im} z\text{)} \end{aligned}$$

Thus, $\operatorname{Re}(iz) = -\operatorname{Im} z$.

(b)

$$\begin{aligned} \operatorname{Im}(iz) &= a \\ \operatorname{Re} z &= a \\ a &= a \end{aligned}$$

Thus, $\operatorname{Im}(iz) = \operatorname{Re} z$.

Problem 1.2.11 Solve the equation $z^2 + z + 1 = 0$ for $z = (x, y)$ by writing

$$(x, y)(x, y) + (x, y) + (1, 0) = (0, 0)$$

and then solving a pair of simultaneous equations in x and y.
Suggestion: Use the fact that no real number x satisfies the given equation to show that $y \neq 0$.

Solution. The discriminant $1^2 - 4(1)(1) = -3$ is less than 0, so the equation's 2 solutions must have a nonzero complex component. The solutions can be found in 2 ways:

(1) System of equations:

$$
\begin{aligned}
(x+iy)^2 + (x+iy) + (1+0i) &= (0+0i) \\
x^2 + 2xyi + i^2y^2 + x + iy + 1 + 0i &= 0 + 0i && \text{(distributing)} \\
(x^2 - y^2 + x + 1) + i(2xy + y) &= 0 + i(0) && \text{(combining like terms)} \\
&\begin{cases} x^2 - y^2 + x + 1 = 0 \\ i(2xy + y) = i(0) \end{cases} && \text{(creating a system of equations)} \\
i(2xy + y) &= i(0) && \text{(considering the second equation)} \\
2x + 1 &= 0 && \text{(dividing both sides by } iy\text{)} \\
x &= -\frac{1}{2} \\
\left(-\frac{1}{2}\right)^2 - y^2 - \frac{1}{2} + 1 &= 0 && \text{(substituting } x = -\tfrac{1}{2} \text{ into the first equation)} \\
y^2 &= \frac{3}{4} && \text{(rearranging)} \\
y &= \pm\frac{\sqrt{3}}{2}
\end{aligned}
$$

Thus, the solutions are $z = (-\frac{1}{2}, \frac{\sqrt{3}}{2})$ and $(-\frac{1}{2}, -\frac{\sqrt{3}}{2})$.

(2) Quadratic equation:

$$
\begin{aligned}
z &= \frac{-b \pm \sqrt{b^2 - 4ac}}{2a} \\
a, b, c &= 1 && \text{(assigned from coefficients from equation)} \\
z &= \frac{-1 \pm \sqrt{1^2 - 4(1)(1)}}{2(1)} && \text{(substituting values)} \\
z &= -\frac{1}{2} \pm i\frac{\sqrt{3}}{2} && \text{(simplifying)}
\end{aligned}
$$

Thus, the solutions are $z = (-\frac{1}{2}, \frac{\sqrt{3}}{2})$ and $(-\frac{1}{2}, -\frac{\sqrt{3}}{2})$, agreeing with the result from (1).

Problem 1.5.6 Verify that $\sqrt{2}|z| \geq |\operatorname{Re} z| + |\operatorname{Im} z|$.

Solution. Define $L(\theta) = |\operatorname{Re} z| + |\operatorname{Im} z|$, where z has a fixed modulus $|z|$ and variable argument $\theta, 0 \leq \theta \geq \frac{\pi}{2}$ in the Argand plane.

The first derivative test can be used to find where $L(\theta)$ takes its maximum. Since θ is restricted to the first quadrant, the absolute value signs can be disregarded.

$$\begin{aligned}
L(\theta) &= |z|\cos\theta + |z|\sin\theta && \text{(writing } L(\theta) \text{ in terms of } \theta) \\
\frac{dL}{d\theta} &= |z|\cos\theta - |z|\sin\theta \\
0 = \frac{dL}{d\theta} &= |z|\cos\theta - |z|\sin\theta && \text{(solutions will be critical points)} \\
|z|\sin\theta &= |z|\cos\theta \\
\sin\theta &= \cos\theta \\
\theta &= \frac{\pi}{4}
\end{aligned}$$

Since $\frac{dL}{d\theta}$ is positive on $[0, \frac{\pi}{4})$ and negative on $(\frac{\pi}{4}, \frac{\pi}{2}]$, $\theta = \frac{\pi}{4}$ is a maximum on $[0, \frac{\pi}{2}]$.

Then, choose a z with $\operatorname{Arg}(z) = \frac{\pi}{4}$. In this case $|\operatorname{Re} z| = |\operatorname{Im} z|$. Let $s = |\operatorname{Re} z|$.

$$\begin{aligned}
|\operatorname{Re} z| + |\operatorname{Im} z| &= 2s && \text{(as both equal s)} \\
\sqrt{2}|z| &= \sqrt{2}\sqrt{s^2 + s^2} && \text{(by Pythagorean theorem)} \\
\sqrt{2}|z| &= 2s && \text{(simplifying)}
\end{aligned}$$

Thus, when $\operatorname{Arg}(z) = \frac{\pi}{4}$, $\sqrt{2}|z| = |\operatorname{Re} z| + |\operatorname{Im} z|$. Since, in the first quadrant, $\theta = \frac{\pi}{4}$ is the maximum value taken by $|\operatorname{Re} z| + |\operatorname{Im} z|$, $\sqrt{2}|z| \geq |\operatorname{Re} z| + |\operatorname{Im} z|$ in the first quadrant. Similar reasoning can be used in the other three quadrants by symmetry, so the identity is always true.