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MATH 576: Section H01
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MATH 576 Homework 7

Problem 1 Let m be a positive integer. Let $G = \{0 \mid *m\}$. Prove that $G \gtrless *m$, but $G > *k$ whenever k is a nonnegative integer and $k \neq m$.

We first show that $G - *m = \{0 \mid *m\} - *m = \{0 \mid *m\} + *m \in \mathcal{N}$. If L goes first, they can move to $\{0 \mid *m\}$. Then, R must move to $*m$, and then L can win by moving to $*0 = 0$. On the other hand, if R goes first, they can win immediately by moving to $*m + *m = *0 = 0$. So $G + *m \in \mathcal{N}$ and thus $G \gtrless *m$.

Now, let k be a non-negative integer with $m \neq k$. We show that $G - *k = \{0 \mid *m\} - *k = \{0 \mid *m\} + *k \in \mathcal{L}$. If L goes first, they can move $\{0 \mid *m\}$ as before. Then, R must move to $*m$, and then L can win by moving to $*0$. However, consider if R goes first. If they move to $\{0 \mid *m\} + *(k')$ for some $0 < k' < k$, L can move to $\{0 \mid *m\}$ and win as before. If R moves to $\{0 \mid *m\}$, L can move to 0 and win. Finally, if R moves to $*m + *k$, L can move to $*m + *m = 0$ if $m \leq k$ and $*k + *k = 0$ if $m > k$ and win in either case. So $G - *k \in \mathcal{L}$ and thus $G > *k$. □

Problem 2 Let x_1 and x_2 be numbers with $x_1 > x_2 > 0$. Determine the Left and Right stops and the confusion interval of the game

$$\pm x_1 \pm x_2.$$

From Theorem 14.1.1, we have $\pm x_1 \pm x_2 = \{\{x_1 + x_2 \mid x_1 - x_2\} \mid \{-x_1 + x_2 \mid -x_1 - x_2\}\}$. So we have

$$\begin{aligned} L(\pm x_1 \pm x_2) &= L(\{\{x_1 + x_2 \mid x_1 - x_2\} \mid \{-x_1 + x_2 \mid -x_1 - x_2\}\}) \\ &= R(\{x_1 + x_2 \mid x_1 - x_2\}) \\ &= L(x_1 - x_2) \\ &= x_1 - x_2 \end{aligned}$$

and

$$\begin{aligned} R(\pm x_1 \pm x_2) &= R(\{\{x_1 + x_2 \mid x_1 - x_2\} \mid \{-x_1 + x_2 \mid -x_1 - x_2\}\}) \\ &= L(\{-x_1 + x_2 \mid -x_1 - x_2\}) \\ &= R(-x_1 + x_2) \\ &= -x_1 + x_2. \end{aligned}$$

We now check if $\pm x_1 \pm x_2 - (x_1 - x_2) \in \mathcal{N}$. We note that if L goes first, they will move to $\{x_1 + x_2 \mid x_1 - x_2\} - x_1 + x_2$, and R will respond by moving to $x_1 - x_2 - x_1 + x_2 = 0$ and winning. So the position is in $\mathcal{P}^R$, and thus it is not true that $\pm x_1 \pm x_2 \gtrless x_1 - x_2$. Similar reasoning shows that $\pm x_1 \pm x_2 - (-x_1 + x_2) \notin \mathcal{N}$: if R goes first, they will move to $\{-x_1 + x_2 \mid -x_1 - x_2\} + x_1 - x_2$, and L will respond by moving to $-x_1 + x_2 + x_1 - x_2 = 0$ and winning. So it is not true that $\pm x_1 \pm x_2 \gtrless -x_1 + x_2$.

Therefore, the confusion interval of $\pm x_1 \pm x_2$ is $(-x_1 + x_2, x_1 - x_2)$. □

Problem 3 Let $x > 0$ be a number and let m be a positive integer. Prove that $+_x \ngtrless *m$. On the other hand, if $m \geq 2$, give an example of a game $G > 0$ such that $+_G > *m$.

We first show that $+_x - *m = +_x + *m = \{0 \mid \{0 \mid -x\}\} + *m \in \mathcal{N}$. If L moves first, they can win by moving to $+_x \in \mathcal{L}$. If R moves first, they can move to $\{0 \mid -x\} + *m$. Then, if L moves to $0 + *m = *m$, R can win by moving to 0, and if L moves to $\{0 \mid -x\} + *k$ for some $k < m$, R can win by moving to $-x + *k$, which we have showed in a previous homework is in R. So $+_x \ngtrless *m$.

We claim that for all $m \geq 2$, we have $+_\uparrow > *m$. We show that

$$+_\uparrow - *m = +_\uparrow + *m = \{0 \mid \{0 \mid -\uparrow\}\} + *m = \{0 \mid \{0 \mid \downarrow\}\} + *m = \{0 \mid \{0 \mid \{* \mid 0\}\}\} + *m \in \mathcal{L}.$$

If L moves first, they can win by taking all m tokens and moving to $+_\uparrow \in \mathcal{L}$. So $+_\uparrow - *m \in \mathcal{N}^L$.

If R moves first to $\{0 \mid \{* \mid 0\}\} + *m$, L can move to $\{0 \mid \{* \mid 0\}\} + *$. Then, if R moves to $\{0 \mid \{* \mid 0\}\}$, L wins by moving to 0, and if R moves to $\{* \mid 0\} + *$, L wins by moving to $* + * = 0$. If R instead moves first to $+_\uparrow + *k$ for some $k < m$, L is winning if $k = 0$ and L can move to $+_\uparrow \in \mathcal{L}$ if $k > 0$. So $+_\uparrow - *m \in \mathcal{P}^L$.

Thus, $+_\uparrow - *m \in \mathcal{N}^L \cap \mathcal{P}^L = \mathcal{L}$, and so $+_\uparrow > *m$. □

Problem 4 Let n be a positive integer and let x and y be numbers with $x > y > 0$. Prove that

$$n \cdot +_x < +_y.$$

We first prove a lemma: we claim for all $n \geq 0$, we have $n \cdot +_x + \{y \mid 0\} \in \mathcal{N}^R$. We prove this by induction on n. The base case follows from $\{y \mid 0\} \in \mathcal{N}^R$ (R wins by moving to 0). Let $n > 0$, and suppose

$$(n-1) \cdot +_x + \{y \mid 0\} \in \mathcal{N}^R.$$

Suppose R moves first in $n \cdot +_x + \{y \mid 0\}$. R can move to $(n-1) \cdot +_x + \{0 \mid -x\} + \{y \mid 0\}$. Then, L's only hope is to move to $(n-1) \cdot +_x + \{y \mid 0\}$, for if they do not, R can move to $-x$ and gain an insurmountable advantage (as $-x$ is less than the sum of y and any finite number of infinitesimals). But $(n-1) \cdot +_x + \{y \mid 0\} \in \mathcal{N}^R$ by the induction hypothesis, so R is winning and thus $n \cdot +_x + \{y \mid 0\} \in \mathcal{N}^R$.

We now show that $n \cdot +_x + -_y = n \cdot +_x - +_y \in \mathcal{R}$. We proceed by induction on n. The base case follows from $-_y \in \mathcal{R}$. Let $n > 0$, and suppose we have $(n-1) \cdot +_x + -_y \in \mathcal{R}$. We have

$$n \cdot +_x + -_y = n \cdot \{0 \mid \{0 \mid -x\}\} + \{\{y \mid 0\} \mid 0\}.$$

Suppose L moves first. They should not move to $n \cdot +_x + \{y \mid 0\}$ by the lemma, but all their other options are to $0 + (n-1) \cdot \{0 \mid \{0 \mid -x\}\} + \{\{y \mid 0\} \mid 0\} = (n-1) \cdot +_x + -_y$, which is in $\mathcal{R}$ by the induction hypothesis. So $n \cdot +_x + -_y \in \mathcal{P}^R$.

Suppose R moves first. They can move to $\{0 \mid -x\} + (n-1) \cdot +_x + -_y$. Then, L's only hope is to move to $(n-1) \cdot +_x + -_y$ for the same reason as in the lemma: if they do not, R can move to $-x$ in $\{0 \mid -x\}$ and gain an insurmountable advantage. But $(n-1) \cdot +_x + -_y \in \mathcal{R}$ by the induction hypothesis, so L is losing regardless. So $n \cdot +_x + -_y \in \mathcal{N}^R$.

Therefore, $n \cdot +_x + -_y \in \mathcal{P}^R \cap \mathcal{N}^R = \mathcal{R}$, and we have $n \cdot +_x < +_y$. □

Problem 5 Let $G = \{-_1 \mid \{1 \mid 0\}, 1\}$. Explain why the argument below showing that $L(G) = 0$ and $R(G) = 1$ is wrong and give an argument correctly determining what $L(G)$ and $R(G)$ are.

Proof. We have $L(G) = R(-_1) = 0$, since $-_1$ is an infinitesimal. Similarly, $R(G) = \min\{L(\{1 \mid 0\}), L(1)\}$. But $L(1) = 1$ and $L(\{1 \mid 0\}) = R(1) = 1$, so

$$R(G) = \min\{1, 1\} = 1.$$

□

The definition of $L(G)$ and $R(G)$ state that $L(G) = R(G) = G$ if G is a number and have an alternate definition otherwise. The argument assumes the alternate definition holds, which is not necessarily true if G is a number.

In fact, G is a number: in particular, $G \in \mathcal{P}$, so $G = 0$. To see this, note that if L moves first, they must move to $-_1 \in \mathcal{R}$. Also, if R moves first, they must move to $1 \in \mathcal{L}$ or to $\{1 \mid 0\}$, and if they move to $\{1 \mid 0\}$, L can win by moving to $1 \in \mathcal{L}$. So $G \in \mathcal{P}^R \cap \mathcal{P}^L = \mathcal{P}$, so we have $G = 0$ and therefore $L(G) = R(G) = 0$.□