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MATH 555: Section H01
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# MATH 555 Homework 7

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**Problem 1** Let $n$ be a positive integer, and compute

$$I_n := \int_{-1}^{1} (1-x^2)^n \, dx.$$

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Define

$$I(m,n) := \int_{-1}^{1} (1-x)^m (1+x)^n \, dx.$$

This is useful as we have

$$I_n = \int_{-1}^{1} (1-x^2)^n \, dx = \int_{-1}^{1} [(1-x)(1+x)]^n \, dx = \int_{-1}^{1} (1-x)^n (1+x)^n \, dx = I(n,n).$$

We will first prove three lemmas.

*Lemma 1:* For all $m, n \geq 0$, we have

$$I(m,n) = \frac{m}{n+1} I(m-1, n+1).$$

Proof: We can use integration by parts with $u = (1-x)^m$, $dv = (1+x)^n \, dx$ to write

$$\begin{aligned}
I(m,n) &= \int_{-1}^{1} (1-x)^m (1+x)^n \, dx \\
&= \left. \frac{(1-x)^m (1+x)^{n+1}}{n+1} \right|_{x=-1}^{1} - \int_{-1}^{1} \frac{(1+x)^{n+1}(-m)(1-x)^{m-1}}{n+1} \, dx && \text{(from described } u, v\text{)} \\
&= \frac{0^m 2^{n+1}}{n+1} - \frac{2^m 0^{n+1}}{n+1} + \int_{-1}^{1} \frac{m(1-x)^{m-1}(1+x)^{n+1}}{n+1} \, dx \\
&= \frac{m}{n+1} \int_{-1}^{1} (1-x)^{m-1} (1+x)^{n+1} \, dx \\
&= \frac{m}{n+1} I(m-1, n+1)
\end{aligned}$$

as desired. □

*Lemma 2:* For all $n \geq 0$, we have

$$I(0,n) = \frac{2^{n+1}}{n+1}.$$

Proof: This is straightfoward with $u$-substitution by choosing $u = 1+x$. We have

$$I(0,n) = \int_{-1}^{1} (1+x)^n \, dx = \int_{0}^{2} u^n \, du = \frac{u^{n+1}}{n+1} = \frac{2^{n+1}}{n+1}.$$

□

*Lemma 3:* For all $1 \le k \le n$, we have

$$I(n,n) = \prod_{i=1}^{k} \left( \frac{n-(i-1)}{n+i} \right) I(n-k, n+k).$$

Proof: We will use induction on $k$. The base case $k = 1$ follows directly from Lemma 1 as

$$I(n,n) = \frac{n}{n+1} I(n-1, n+1) = \frac{n-(1-1)}{n+1} I(n-1, n+1).$$

Now, let $1 < k \le n$ and assume that the claim holds for $k-1$. Then, we can use Lemma 1 to write

$$\begin{aligned}
I(n,n) &= \prod_{i=1}^{k-1} \left( \frac{n-(i-1)}{n+i} \right) I(n-k+1, n+k-1) && \text{(IH)} \\
&= \prod_{i=1}^{k-1} \left( \frac{n-(i-1)}{n+i} \right) \left( \frac{n-k+1}{n+k-1+1} \right) I(n-k+1-1, n+k-1+1) && \text{(Lemma 1)} \\
&= \prod_{i=1}^{k-1} \left( \frac{n-(i-1)}{n+i} \right) \left( \frac{n-(k-1)}{n+k} \right) I(n-k, n+k) && \text{(simplifying)} \\
&= \prod_{i=1}^{k} \left( \frac{n-(i-1)}{n+i} \right) I(n-k, n+k). && \text{(combining product)}
\end{aligned}$$

So by induction, the lemma holds for all $1 \le k \le n$. □

We can now compute $I_n$ by writing

$$\begin{aligned}
I(n,n) &= \prod_{i=1}^{n} \left( \frac{n-(i-1)}{n+i} \right) I(n-n, n+n) && \text{(Lemma 3 with } k = n\text{)} \\
&= \frac{(n)(n+1)\dots(1)}{(n+1)(n+2)\dots(2n)} I(0, 2n) \\
&= \frac{(n!)^2}{(2n)!} I(0, 2n) \\
&= \frac{(n!)^2}{(2n)!} \left( \frac{2^{2n+1}}{2n+1} \right) && \text{(Lemma 2)} \\
&= \frac{2^{2n+1}(n!)^2}{(2n+1)!}.
\end{aligned}$$

Therefore,

$$I_n := \int_{-1}^{1} (1-x^2)^n \, dx = \frac{2^{2n+1}(n!)^2}{(2n+1)!}.$$

□

**Problem 2** Let $f : [0,1] \to \mathbb{R}$ be a continuous function with

$$f(x) + f(1-x) = 1.$$

Compute

$$\int_0^1 f(x)x(1-x) \, dx.$$

We can use $u$-substitution with $u = 1 - x$ to write

$$\int_0^1 f(1-x)x(1-x) = \int_1^0 -f(u)(1-u)(1-(1-u))\,du = \int_0^1 f(u)u(1-u)\,du,$$

so with a change of variables we have

$$\int_0^1 f(x)x(1-x)dx = \int_0^1 f(1-x)x(1-x). \tag{1}$$

Then, we can compute

$$\begin{aligned}
\int_0^1 f(x)x(1-x)\,dx &= \frac{1}{2}\left[\int_0^1 f(x)x(1-x)\,dx + \int_0^1 f(x)x(1-x)\,dx\right] && \text{(real number property)} \\
&= \frac{1}{2}\left[\int_0^1 f(x)x(1-x)\,dx + \int_0^1 f(1-x)x(1-x)\,dx\right] && \text{(from (1))} \\
&= \frac{1}{2}\int_0^1 \big(f(x)+f(1-x)\big)\,x(1-x)\,dx && \text{(combining integrals)} \\
&= \frac{1}{2}\int_0^1 x(1-x)\,dx && (f(x)+f(1-x)=1 \text{ by assumption}) \\
&= \frac{1}{2}\int_0^1 (x-x^2)\,dx \\
&= \frac{1}{2}\left[\frac{x^2}{2} - \frac{x^3}{3}\right]_{x=0}^1 \\
&= \frac{1}{2}\left(\frac{1}{2} - \frac{1}{3}\right) = \frac{1}{12}.
\end{aligned}$$

$\square$

**Problem 6.1** Prove that the series $\sum_{k=1}^{\infty} a_k$ converges if and only if for all $\varepsilon > 0$ there is an $N$ such that

$$N \le m < n \implies |a_{m+1} + a_{m+2} + \cdots + a_n| < \varepsilon.$$

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Let $\langle A_n \rangle_{n=1}^{\infty}$ be defined by $A_n = \sum_{k=1}^{n} a_k$. Then for all $m < n$, we can write

$$|A_n - A_m| = |a_{m+1} + a_{m+2} + \cdots + a_n|\,. \tag{1}$$

By definition, $\sum_{k=1}^{\infty} a_k$ converges if and only if $\langle A_n \rangle$ converges. Since $\mathbb{R}$ is complete, this convergence happens if and only if $\langle A_n \rangle$ is Cauchy. By definition, this is true if and only for all $\varepsilon > 0$ there is an $N$ such that $N \le m < n \implies |A_n - A_m| < \varepsilon$. From equation 1, this holds if and only if

$$N \le m < n \implies |a_{m+1} + a_{m+2} + \cdots + a_n| < \varepsilon.$$

Therefore, the statement holds. $\square$

**Problem 6.2** Let $f : [1, \infty) \to [0, \infty)$ be a monotone decreasing non-negative function. Let $a_k = f(k)$ and let

$$A_n = \sum_{k=1}^{n} a_k$$

be the $n$-th partial sum of the series $\sum\limits_{k=1}^{\infty} a_k$. Prove that

$$\int_1^n f(x)\,dx \le A_n \le f(1) + \int_1^n f(x)\,dx.$$

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Let $1 \le k < n$ and let $x \in [k, k+1]$. Since $f$ is monotone decreasing, we have

$$f(k) \ge f(x) \ge f(k+1).$$

We will show the inequalities separately. First, we can write

$$\begin{aligned}
\int_k^{k+1} f(x)\,dx &\le \int_k^{k+1} f(k)\,dx && \text{(since } f(x) \le f(k) \text{ on the interval)} \\
&= f(k)(k+1-k) && (f(k) \text{ is constant}) \\
&= a_k && (f(k) = a_k \text{ by definition}) \\
\implies \sum_{k=1}^{n-1} \int_k^{k+1} f(x)\,dx &\le \sum_{k=1}^{n-1} a_k \\
\implies \int_1^n f(x)\,dx &\le A_{n-1} \\
&\le A_n. && (\langle a_k \rangle \text{ is non-negative so } \langle A_n \rangle \text{ is monotone increasing})
\end{aligned}$$

To show the other equality, we can write

$$\begin{aligned}
f(k+1) &= \int_k^{k+1} f(k+1)\,dx && (f(k+1) \text{ is constant}) \\
&\le \int_k^{k+1} f(x)\,dx && \text{(since } f(k+1) \le f(x) \text{ on the interval)} \\
\implies \sum_{k=1}^{n-1} f(k+1) &\le \sum_{k=1}^{n-1} \int_k^{k+1} f(x)\,dx \\
\implies \sum_{k=2}^{n} f(k) &\le \int_1^n f(x)\,dx && \text{(adjusting/combining bounds)} \\
\implies f(1) + \sum_{k=2}^{n} f(k) &\le f(1) + \int_1^n f(x)\,dx \\
\implies A_n &\le f(1) + \int_1^n f(x)\,dx. && \left(A_n = \sum_{k=1}^{n} a_k = f(1) + \sum_{k=2}^{n} f(k)\right)
\end{aligned}$$

Therefore,

$$\int_1^n f(x)\,dx \le A_n \le f(1) + \int_1^n f(x)\,dx$$

holds. □

**Problem 6.3** (Integral Test) Let $f : [1, \infty) \to [0, \infty)$ be a monotone decreasing non-negative function. Let $a_k = f(k)$ and let

$$A_n = \sum_{k=1}^{n} a_k$$

be the $n$-th partial sum of the series $\sum_{k=1}^{\infty} a_k$. Prove that

$$\sum_{k=1}^{\infty} a_k < \infty \iff \lim_{n\to\infty} \int_1^n f(x)\,dx \text{ exists and is finite.}$$

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($\Rightarrow$) Let $A := \sum_{k=1}^{\infty} a_k$, and $n \in \mathbb{N}$. From Problem 6.2 we have

$$\int_1^n f(x)\,dx \le A_n.$$

Further, $A_n \le A$ since $\langle a_k \rangle$ is non-negative, so

$$\int_1^n f(x)\,dx \le A$$

for all $n$. Then the sequence of integrals bounded above by $A$, and since it is monotone increasing since $f(x) \ge 0$, by a theorem we proved last semester it converges to a finite limit. Thus,

$$\lim_{n\to\infty} \int_1^n f(x)\,dx \text{ exists and is finite.}$$

($\Leftarrow$) Suppose

$$L := \lim_{n\to\infty} \int_1^n f(x)\,dx$$

exists and is finite. By again applying Problem 6.2 and using similar reasoning to the forward direction, we can bound $\langle A_n \rangle_{n=1}^{\infty}$ above with $f(1) + L$. So $\langle A_n \rangle$ is a monotone increasing sequence that is bounded above, and therefore it converges to a finite limit. Thus,

$$\sum_{k=1}^{\infty} a_k < \infty.$$

$\square$

**Problem 6.4** Use the Integral Test to prove that for any real number $p > 0$, the series

$$\sum_{k=1}^{\infty} \frac{1}{k^p}$$

converges if and only if $p > 1$.

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Let $f_p(x) := \frac{1}{x^p} = x^{-p}$.

($\Rightarrow$) We will prove the contrapositive. Suppose $p \le 1$.

Case 1: $p = 1$. Then

$$\lim_{n\to\infty} \int_1^n f_1(x)\,dx = \lim_{n\to\infty} \int_1^n \frac{dx}{x} = \lim_{n\to\infty} \big[\ln(x)\big]_{x=1}^{n} = \infty,$$

since we showed in class that $\ln(x)$ is surjective and increasing. So by the Integral Test, since the integral diverges, $\sum_{k=1}^{\infty} \frac{1}{k^p}$ diverges as well.

Case 2: $p < 1$. Then

$$\lim_{n\to\infty} \int_1^n f_p(x)\,dx = \lim_{n\to\infty} \int_1^n x^{-p}\,dx = \lim_{n\to\infty} \left[\frac{x^{1-p}}{1-p}\right]_{x=1}^{n} = \frac{1}{1-p} \lim_{n\to\infty} \left[e^{(1-p)\ln(x)}\right]_{x=1}^{n}. \tag{1}$$

We showed in class that $e^u$ is surjective and increasing if $u$ is positive. Since $\ln(x)$ is positive for $x \geq 1$ and $1-p$ is positive for $p < 1$, $(1-p)\ln(x)$ is positive in this case. Thus,

$$\lim_{n\to\infty} e^{(1-p)\ln(n)} = \infty,$$

so the integral diverges. So by the Integral Test, $\sum\limits_{k=1}^{\infty} \frac{1}{k^p}$ diverges as well.

So for all $p \leq 1$, the sum does not converge.

($\Leftarrow$) Suppose $p > 1$. Then we can again use equation 1 to evaluate the integral. However, we now have $1-p < 0$, so $e^{(1-p)\ln(x)}$ is now decreasing. Since it is bounded below by 0,

$$\lim_{n\to\infty} e^{(1-p)\ln(n)}$$

exists and is finite. Therefore, the interval converges, so by the Integral Test, $\sum\limits_{k=1}^{\infty} \frac{1}{k^p}$ also converges. $\square$

**Problem 6.5** Use the Integral Test to show

$$\sum_{k=2}^{\infty} \frac{1}{n(\ln(n))^p}$$

converges if and only if $p > 1$.

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Let

$$f_p(x) := \frac{1}{x(\ln(x))^p} = \frac{(\ln(x))^{-p}}{x}.$$

Then, we can use $u$-substitution with $u = \ln(x)$ to write

$$\int_2^n f_p(x)\,dx = \int_2^n \frac{(\ln(x))^{-p}}{x}\,dx = \int_{\ln(2)}^{\ln(n)} u^{-p}\,du.$$

Since

$$\lim_{n\to\infty} n = \lim_{n\to\infty} \ln(n) = \infty,$$

we have that

$$\lim_{n\to\infty} \int_{\ln(2)}^{\ln(n)} u^{-p}\,du \text{ converges if and only if } \lim_{n\to\infty} \int_1^n u^{-p} \text{ converges,}$$

and we proved in Problem 6.4 that the second integral converges if and only if $p > 1$.

**Problem 6.6** Let $\sum\limits_{k=1}^{\infty} a_k$ and $\sum\limits_{k=1}^{\infty} b_k$ be two series of non-negative terms. Assume there is a constant $C > 0$ such that

$$a_k \leq C b_k$$

for all $k$. Prove that

(a) If $\sum\limits_{k=1}^{\infty} b_k$ converges, so does $\sum\limits_{k=1}^{\infty} a_k$.

(b) If $\sum\limits_{k=1}^{\infty} a_k$ diverges, so does $\sum\limits_{k=1}^{\infty} b_k$.

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**(a)** Suppose $\sum_{k=1}^{\infty} b_k$ converges. Let $\varepsilon > 0$. From Problem 6.1 (and since both sequences have non-negative terms), there exists $N$ such that for all $n > m \geq N$,

$$b_{m+1} + b_{m+2} + \cdots + b_n < \frac{\varepsilon}{C}. \tag{1}$$

Let $n > m \geq N$. Then

$$\begin{aligned} a_{m+1} + a_{m+2} + \cdots + a_n &\leq Cb_{m+1} + Cb_{m+2} + \cdots + Cb_n \\ &< C\left(\frac{\varepsilon}{C}\right) && \text{(equation 1)} \\ &= \varepsilon. \end{aligned}$$

From Problem 6.1, this implies that $\sum_{k=1}^{\infty} a_k$ converges.

**(b)** This follows immediately from **(a)**, as it is the contrapositive. □

**Problem 6.7** Let $\sum_{k=1}^{\infty} a_k$ and $\sum_{k=1}^{\infty} b_k$ be two series of positive terms. Assume that

$$L := \lim_{k\to\infty} \frac{a_k}{b_k}$$

exists. Prove that

(a) $\sum_{k=1}^{\infty} b_k < \infty$ implies $\sum_{k=1}^{\infty} a_k < \infty$

(b) If $L \neq 0$ and $\sum_{k=1}^{\infty} a_k < \infty$, then $\sum_{k=1}^{\infty} b_k < \infty$.[1]

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Let $\langle q_k \rangle_{k=1}^{\infty}$ be defined for all $k$ by

$$q_k = \frac{a_k}{b_k}.$$

Since $\langle q_k \rangle$ converges to $L$, it is bounded, so there exists $C$ such that $q_k \leq C$ for all $k$.

(a) Suppose $\sum_{k=1}^{\infty} b_k$ converges. For all $k$ we have

$$\frac{a_k}{b_k} = q_k \leq C \implies a_k \leq Cb_k,$$

so from Problem 6.6 $\sum_{k=1}^{\infty} a_k$ converges.

(b) Suppose $L \neq 0$ and $\sum_{k=1}^{\infty} a_k$ converges. Since $L, C > 0$, we have

$$\lim_{k\to\infty} \frac{b_k}{a_k} = \frac{1}{L}, \quad \frac{b_k}{a_k} \leq \frac{1}{C} \text{ for all } k.$$

Then, from **(a)**, we have that $\sum_{k=1}^{\infty} b_k$ converges. □

[1] I'm assuming this is what the problem was meant to be: what is written in the notes is simply the contrapositive of (a) and Corollary 6.18 only follows from this version.

**Problem 4** For what values of $p$ do the following series converge?

$$\textbf{(a)} \sum_{k=2}^{\infty} \frac{1}{k(\ln(k))^p} \quad \textbf{(b)} \sum_{k=2}^{\infty} \frac{1}{k\ln(k)(\ln(\ln(k)))^p}$$

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**(a)** From Problem 6.5, this converges for all $p \in (1, \infty)$.

**(b)** Let

$$f_p(x) := \frac{1}{x\ln(x)(\ln(x))^p} = \frac{(\ln(x))^{-p}}{x\ln(x)}.$$

Then, we can use $u$-substitution with $u = \ln(\ln(x))$ to write

$$\int_2^n f_p(x)\,dx = \int_2^n \frac{(\ln(x))^{-p}}{x\ln(x)}\,dx = \int_{\ln(\ln(2))}^{\ln(\ln(n))} u^{-p}\,du.$$

Since

$$\lim_{n\to\infty} n = \lim_{n\to\infty} \ln(\ln(n)) = \infty,$$

we have that

$$\lim_{n\to\infty} \int_{\ln(\ln(2))}^{\ln(\ln(n))} u^{-p}\,du \text{ converges if and only if } \lim_{n\to\infty} \int_1^n u^{-p} \text{ converges,}$$

and we proved in Problem 6.4 that the second integral converges if and only if $p \in (1, \infty)$. □

**Problem 5** Show that

$$\sum_{k=1}^{\infty} \left(\frac{1}{k} - \frac{1}{k+x}\right)$$

converges for all real numbers $x \notin \{-1, -2, -3, \ldots\}$.

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Let $x \notin \{-1, -2, -3, \ldots\}$. Then, we have

$$\sum_{k=1}^{\infty} \left(\frac{1}{k} - \frac{1}{k+x}\right) = \sum_{k=1}^{\infty} \frac{x}{k(k+x)}.$$

Case 1: $x = 0$. Then the series is $0 + 0 + 0 + \ldots$, so clearly it converges to 0.

Case 2: $x \neq 0$. We note from Problem 6.4 that

$$\sum_{k=1}^{\infty} \frac{1}{k^2}$$

converges. With this in mind, we can write

$$\begin{aligned}
\lim_{k\to\infty} \frac{\frac{x}{k(k+x)}}{\frac{1}{k^2}} &= \lim_{k\to\infty} \frac{k^2 x}{k(k+x)} \\
&= \lim_{k\to\infty} \frac{kx}{k+x} \\
&= \lim_{k\to\infty} \frac{x}{1} && \text{(L'Hôpital's rule)} \\
&= x \neq 0.
\end{aligned}$$

We can then conclude from Problem 6.7 that since

$$\sum_{k=1}^{\infty} \frac{1}{k^2} < \infty \text{ and } \lim_{k\to\infty} \frac{\frac{x}{k(k+x)}}{\frac{1}{k^2}} \text{ exists and is non-zero,}$$

we also have

$$\sum_{k=1}^{\infty} \frac{x}{k(k+x)} < \infty.$$

Therefore,

$$\sum_{k=1}^{\infty} \left(\frac{1}{k} - \frac{1}{k+x}\right)$$

converges for all real numbers $x \notin \{-1, -2, -3, \ldots\}$. $\square$