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Linear Algebra: Section 1
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Linear Algebra Homework 2

Problem 1 For the example from class slightly modified: Let

$$A = \begin{bmatrix} 2 & 3 \\ 4 & 5 \end{bmatrix}$$

Find the echelon form of the matrix and in every step write down the exact row operations performed. Find the elementary matrices that correspond to the row operations and verify they have the same effect. Finally, find the inverse in two ways, first by performing the same row operations on the identity and then by multiplying the elementary matrices used to bring the matrix A in echelon form.

Solution.

1. We compute

$$A = \begin{bmatrix} 2 & 3 \\ 4 & 5 \end{bmatrix} \xrightarrow{A \mapsto \frac{1}{2}\rho_1} \begin{bmatrix} 1 & 3/2 \\ 4 & 5 \end{bmatrix} \xrightarrow{\rho_2 \mapsto \rho_2 - 4\rho_1} \begin{bmatrix} 1 & 3/2 \\ 0 & -1 \end{bmatrix} \xrightarrow{\rho_2 \mapsto -\rho_2} \begin{bmatrix} 1 & 3/2 \\ 0 & 1 \end{bmatrix}$$

$$\begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix} \xleftarrow{\rho_1 \mapsto \rho_1 - \frac{3}{2}\rho_2}$$

The corresponding elementary matrices are

$$E_1 = \begin{bmatrix} 1/2 & 0 \\ 0 & 1 \end{bmatrix}, \quad E_2 = \begin{bmatrix} 1 & 0 \\ -4 & 1 \end{bmatrix}, \quad E_3 = \begin{bmatrix} 1 & 0 \\ 0 & -1 \end{bmatrix}, \quad E_4 = \begin{bmatrix} 1 & -3/2 \\ 0 & 1 \end{bmatrix}.$$

We verify this by observing that

$$\begin{aligned}
E_4 E_3 E_2 E_1 A &= \begin{bmatrix} 1 & -3/2 \\ 0 & 1 \end{bmatrix}\begin{bmatrix} 1 & 0 \\ 0 & -1 \end{bmatrix}\begin{bmatrix} 1 & 0 \\ -4 & 1 \end{bmatrix}\begin{bmatrix} 1/2 & 0 \\ 0 & 1 \end{bmatrix}\begin{bmatrix} 2 & 3 \\ 4 & 5 \end{bmatrix} \\
&= \begin{bmatrix} 1 & 3/2 \\ 0 & -1 \end{bmatrix}\begin{bmatrix} 1 & 0 \\ -4 & 1 \end{bmatrix}\begin{bmatrix} 1/2 & 0 \\ 0 & 1 \end{bmatrix}\begin{bmatrix} 2 & 3 \\ 4 & 5 \end{bmatrix} \\
&= \begin{bmatrix} -5 & 3/2 \\ 4 & -1 \end{bmatrix}\begin{bmatrix} 1/2 & 0 \\ 0 & 1 \end{bmatrix}\begin{bmatrix} 2 & 3 \\ 4 & 5 \end{bmatrix} \\
&= \begin{bmatrix} -5/2 & 3/2 \\ 2 & -1 \end{bmatrix}\begin{bmatrix} 2 & 3 \\ 4 & 5 \end{bmatrix} \qquad (*) \\
&= \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix}.
\end{aligned}$$

Finally, we find the inverse by performing the row operations on I_2:

$$[A \mid I_2] = \left[\begin{array}{cc|cc} 2 & 3 & 1 & 0 \\ 4 & 5 & 0 & 1 \end{array}\right] \xrightarrow{\rho_1 \mapsto \frac{1}{2}\rho_1} \left[\begin{array}{cc|cc} 1 & 3/2 & 1/2 & 0 \\ 4 & 5 & 0 & 1 \end{array}\right] \xrightarrow{\rho_2 \mapsto \rho_2 - 4\rho_1} \left[\begin{array}{cc|cc} 1 & 3/2 & 1/2 & 0 \\ 0 & -1 & -2 & 1 \end{array}\right]$$

$$\xrightarrow{\rho_2 \mapsto -\rho_2} \left[\begin{array}{cc|cc} 1 & 3/2 & 1/2 & 0 \\ 0 & 1 & 2 & -1 \end{array}\right] \xrightarrow{\rho_1 \mapsto \rho_1 - \frac{3}{2}\rho_2} \left[\begin{array}{cc|cc} 1 & 0 & -5/2 & 3/2 \\ 0 & 1 & 2 & -1 \end{array}\right],$$

so $A^{-1} = \begin{bmatrix} -5/2 & 3/2 \\ 2 & -1 \end{bmatrix}$.

We note this is is equivalent to $E_4 E_3 E_2 E_1$ as computed in $(*)$

Problem 2 Find the echelon form of the following matrices, furthermore for the square matrices state if they have an inverse and if they do find that inverse.

$$A_1 = \begin{bmatrix} 1 & 2 & 1 & 1 \\ 3 & 0 & 0 & 4 \\ 1 & -4 & -2 & 2 \end{bmatrix}, A_2 = \begin{bmatrix} 1 & 2 & 3 \\ 2 & -3 & 2 \\ 3 & 1 & 1 \end{bmatrix}, A_3 = \begin{bmatrix} 1 & 1 & 2 \\ 0 & 2 & 3 \\ 1 & 3 & 5 \end{bmatrix}$$

Solution.

2. We compute

$$A_1 = \begin{bmatrix} 1 & 2 & 1 & 1 \\ 3 & 0 & 0 & 4 \\ 1 & -4 & -2 & 2 \end{bmatrix} \xrightarrow{\rho_2 \mapsto \rho_2 - 3\rho_1} \begin{bmatrix} 1 & 2 & 1 & 1 \\ 0 & -6 & -3 & 1 \\ 1 & -4 & -2 & 2 \end{bmatrix} \xrightarrow{\rho_2 \mapsto -\frac{1}{6}\rho_2} \begin{bmatrix} 1 & 2 & 1 & 1 \\ 0 & 1 & 1/2 & -1/6 \\ 1 & -4 & -2 & 2 \end{bmatrix}$$

$$\xrightarrow{\rho_3 \mapsto \rho_3 - \rho_1} \begin{bmatrix} 1 & 2 & 1 & 1 \\ 0 & 1 & 1/2 & -1/6 \\ 0 & -6 & -3 & 1 \end{bmatrix} \xrightarrow{\rho_3 \mapsto \rho_3 + 6\rho_2} \begin{bmatrix} 1 & 2 & 1 & 1 \\ 0 & 1 & 1/2 & -1/6 \\ 0 & 0 & 0 & 0 \end{bmatrix} \xrightarrow{\rho_1 \mapsto \rho_1 - 2\rho_2} \begin{bmatrix} 1 & 0 & 0 & 4/3 \\ 0 & 1 & 1/2 & -1/6 \\ 0 & 0 & 0 & 0 \end{bmatrix},$$

$$[A_2 | I_3] = \left[\begin{array}{ccc|ccc} 1 & 2 & 3 & 1 & 0 & 0 \\ 2 & -3 & 2 & 0 & 1 & 0 \\ 3 & 1 & 1 & 0 & 0 & 1 \end{array}\right] \xrightarrow{\rho_2 \mapsto \rho_2 - 2\rho_1} \left[\begin{array}{ccc|ccc} 1 & 2 & 3 & 1 & 0 & 0 \\ 0 & -7 & -4 & -2 & 1 & 0 \\ 3 & 1 & 1 & 0 & 0 & 1 \end{array}\right]$$

$$\xrightarrow{\rho_3 \mapsto \rho_3 - 3\rho_1} \left[\begin{array}{ccc|ccc} 1 & 2 & 3 & 1 & 0 & 0 \\ 0 & -7 & -4 & -2 & 1 & 0 \\ 0 & -5 & -8 & -3 & 0 & 1 \end{array}\right] \xrightarrow{\rho_2 \mapsto -\frac{1}{7}\rho_2} \left[\begin{array}{ccc|ccc} 1 & 2 & 3 & 1 & 0 & 0 \\ 0 & 1 & 4/7 & 2/7 & -1/7 & 0 \\ 0 & -5 & -8 & -3 & 0 & 1 \end{array}\right]$$

$$\xrightarrow{\rho_3 \mapsto \rho_3 + 5\rho_2} \left[\begin{array}{ccc|ccc} 1 & 2 & 3 & 1 & 0 & 0 \\ 0 & 1 & 4/7 & 2/7 & -1/7 & 0 \\ 0 & 0 & -36/7 & -11/7 & -5/7 & 1 \end{array}\right] \xrightarrow{\rho_3 \mapsto -\frac{7}{36}\rho_3} \left[\begin{array}{ccc|ccc} 1 & 2 & 3 & 1 & 0 & 0 \\ 0 & 1 & 4/7 & 2/7 & -1/7 & 0 \\ 0 & 0 & 1 & 11/36 & 5/36 & -7/36 \end{array}\right]$$

$$\xrightarrow{\rho_1 \mapsto \rho_1 - 2\rho_2} \left[\begin{array}{ccc|ccc} 1 & 0 & 13/7 & 3/7 & 2/7 & 0 \\ 0 & 1 & 4/7 & 2/7 & -1/7 & 0 \\ 0 & 0 & 1 & 11/36 & 5/36 & -7/36 \end{array}\right] \xrightarrow{\rho_1 \mapsto \rho_1 - \frac{13}{7}\rho_3} \left[\begin{array}{ccc|ccc} 1 & 0 & 0 & -5/36 & 1/36 & 13/36 \\ 0 & 1 & 4/7 & 2/7 & -1/7 & 0 \\ 0 & 0 & 1 & 11/36 & 5/36 & -7/36 \end{array}\right]$$

$$\xrightarrow{\rho_2 \mapsto \rho_2 - \frac{4}{7}\rho_3} \left[\begin{array}{ccc|ccc} 1 & 0 & 0 & -5/36 & 1/36 & 13/36 \\ 0 & 1 & 0 & 1/9 & -2/9 & 1/9 \\ 0 & 0 & 1 & 11/36 & 5/36 & -7/36 \end{array}\right],$$

which shows $A_2^{-1} = \begin{bmatrix} -5/36 & 1/36 & 13/36 \\ 1/9 & -2/9 & 1/9 \\ 11/36 & 5/36 & -7/36 \end{bmatrix}$ and

$$A_3 = \begin{bmatrix} 1 & 1 & 2 \\ 0 & 2 & 3 \\ 1 & 3 & 5 \end{bmatrix} \xrightarrow{\rho_3 \mapsto \rho_3 - \rho_1} \begin{bmatrix} 1 & 1 & 2 \\ 0 & 2 & 3 \\ 0 & 2 & 3 \end{bmatrix} \xrightarrow{\rho_3 \mapsto \rho_3 - \rho_2} \begin{bmatrix} 1 & 1 & 2 \\ 0 & 2 & 3 \\ 0 & 0 & 0 \end{bmatrix}$$

$$\xrightarrow{\rho_2 \mapsto \frac{1}{2}\rho_2} \begin{bmatrix} 1 & 1 & 2 \\ 0 & 1 & 3/2 \\ 0 & 0 & 0 \end{bmatrix} \xrightarrow{\rho_1 \mapsto \rho_1 - \rho_2} \begin{bmatrix} 1 & 0 & 1/2 \\ 0 & 1 & 3/2 \\ 0 & 0 & 0 \end{bmatrix}$$

Since A_3 has an echelon form not equal to I_3, it does not have an inverse.

Problem 3 Find all the solutions to systems

$$A_1x = B, A_2x = B, A_3x = B$$

where A_1, A_2, A_3 are given in the previous exercise and $B = \begin{bmatrix} 0 \\ 0 \\ 0 \end{bmatrix}$. Same for $B = \begin{bmatrix} 1 \\ 1 \\ 0 \end{bmatrix}$.

Solution.

3. We first consider $B = \begin{bmatrix} 0 \\ 0 \\ 0 \end{bmatrix}$. Since no row operations will affect B, the row-echelon form matrices will form equivalent systems.

For $A_1x = B$, consider $$[A_1|B] = \left[\begin{array}{cccc|c} 1 & 2 & 1 & 1 & 0 \\ 3 & 0 & 0 & 4 & 0 \\ 1 & -4 & -2 & 2 & 0 \end{array}\right] \xrightarrow{\text{work for 2}} \left[\begin{array}{cccc|c} 1 & 0 & 0 & 4/3 & 0 \\ 0 & 1 & 1/2 & -1/6 & 0 \\ 0 & 0 & 0 & 0 & 0 \end{array}\right].$$

So the solution set is $$x = \left\{ \begin{bmatrix} 0 \\ -1/2 \\ 1 \\ 0 \end{bmatrix} x_3 + \begin{bmatrix} -4/3 \\ 1/6 \\ 0 \\ 1 \end{bmatrix} x_4 : x_3, x_4 \in \mathbb{R} \right\}.$$

For $A_2x = B$, consider $$[A_2|B] = \left[\begin{array}{ccc|c} 1 & 2 & 3 & 0 \\ 2 & -3 & 2 & 0 \\ 3 & 1 & 1 & 0 \end{array}\right] \xrightarrow{\text{work for 2}} \left[\begin{array}{ccc|c} 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & 0 \end{array}\right].$$

So the solution set is $$x = \left\{ \begin{bmatrix} 0 \\ 0 \\ 0 \end{bmatrix} \right\}.$$

For $A_3x = B$, consider $$[A_3|B] = \left[\begin{array}{ccc|c} 1 & 1 & 2 & 0 \\ 0 & 2 & 3 & 0 \\ 1 & 3 & 5 & 0 \end{array}\right] \xrightarrow{\text{work for 3}} \left[\begin{array}{ccc|c} 1 & 0 & 1/2 & 0 \\ 0 & 1 & 3/2 & 0 \\ 0 & 0 & 0 & 0 \end{array}\right].$$

So the solution set is $$x = \left\{ \begin{bmatrix} -1/2 \\ -3/2 \\ 1 \end{bmatrix} x_3 : x_3 \in \mathbb{R} \right\}.$$

Now we consider $B = \begin{bmatrix} 1 \\ 1 \\ 0 \end{bmatrix}$. We will need to recompute the row operations.

For $A_1x = B$, consider $$[A_1|B] = \left[\begin{array}{cccc|c} 1 & 2 & 1 & 1 & 1 \\ 3 & 0 & 0 & 4 & 1 \\ 1 & -4 & -2 & 2 & 0 \end{array}\right] \xrightarrow{\rho_2 \mapsto \rho_2 - 3\rho_1} \left[\begin{array}{cccc|c} 1 & 2 & 1 & 1 & 1 \\ 0 & -6 & -3 & 1 & -2 \\ 1 & -4 & -2 & 2 & 0 \end{array}\right]$$

$$\xrightarrow{\rho_2 \mapsto -\frac{1}{6}\rho_2} \left[\begin{array}{cccc|c} 1 & 2 & 1 & 1 & 1 \\ 0 & 1 & 1/2 & -1/6 & 1/3 \\ 1 & -4 & -2 & 2 & 0 \end{array}\right] \xrightarrow{\rho_3 \mapsto \rho_3 - \rho_1} \left[\begin{array}{cccc|c} 1 & 2 & 1 & 1 & 1 \\ 0 & 1 & 1/2 & -1/6 & 1/3 \\ 0 & -6 & -3 & 1 & -1 \end{array}\right]$$

$\rho_3 \mapsto \rho_3$

$$+(?)\rho_2 \quad \left[\begin{array}{cccc|c} 1 & 2 & 1 & 1 & 1 \\ 0 & 1 & 1/2 & -1/6 & -1/3 \\ 0 & 0 & 0 & 0 & 1 \end{array}\right] \xrightarrow{\rho_1 \mapsto \rho_1 - 2\rho_2} \left[\begin{array}{cccc|c} 1 & 0 & 0 & 4/3 & 1/3 \\ 0 & 1 & 1/2 & -1/6 & 1/3 \\ 0 & 0 & 0 & 0 & 1 \end{array}\right] \xrightarrow{\rho_1 \mapsto \rho_1 - \frac{1}{3}\rho_3} \left[\begin{array}{cccc|c} 1 & 0 & 0 & 4/3 & 0 \\ 0 & 1 & 1/2 & -1/6 & 1/3 \\ 0 & 0 & 0 & 0 & 1 \end{array}\right]$$

$$\xrightarrow{\rho_2 \mapsto \rho_2 - \frac{1}{3}\rho_2} \left[\begin{array}{cccc|c} 1 & 0 & 0 & 4/3 & 0 \\ 0 & 1 & 1/2 & -1/6 & 0 \\ 0 & 0 & 0 & 0 & 1 \end{array}\right]$$

Since there is a pivot in the last column, the system has no solutions.

For $A_2 x = B$, we computed A_2^{-1}, so $x = A_2^{-1} B$

$$= \begin{bmatrix} -5/36 & 1/36 & 13/36 \\ 1/9 & -2/9 & 1/9 \\ 11/36 & 5/36 & -7/36 \end{bmatrix} \begin{bmatrix} 1 \\ 1 \\ 0 \end{bmatrix} = \begin{bmatrix} -1/9 \\ -1/9 \\ 4/9 \end{bmatrix}.$$

For $A_3 x = B$, consider $[A_3|B] = \left[\begin{array}{ccc|c} 1 & 1 & 2 & 1 \\ 0 & 2 & 3 & 1 \\ 1 & 3 & 5 & 0 \end{array}\right] \xrightarrow{\rho_3 \mapsto \rho_3 - \rho_1} \left[\begin{array}{ccc|c} 1 & 1 & 2 & 1 \\ 0 & 2 & 3 & 1 \\ 0 & 2 & 3 & -1 \end{array}\right]$

$$\xrightarrow{\rho_3 \mapsto \rho_3 - \rho_2} \left[\begin{array}{ccc|c} 1 & 1 & 2 & 1 \\ 0 & 2 & 3 & 1 \\ 0 & 0 & 0 & -2 \end{array}\right] \xrightarrow{\rho_2 \mapsto \frac{1}{2}\rho_2} \left[\begin{array}{ccc|c} 1 & 1 & 2 & 1 \\ 0 & 1 & 3/2 & 1/2 \\ 0 & 0 & 0 & -2 \end{array}\right] \xrightarrow{\rho_1 \mapsto \rho_1 - \rho_2} \left[\begin{array}{ccc|c} 1 & 0 & 1/2 & 1/2 \\ 0 & 1 & 3/2 & 1/2 \\ 0 & 0 & 0 & -2 \end{array}\right]$$

$$\xrightarrow{\rho_3 \mapsto -\frac{1}{2}\rho_3} \left[\begin{array}{ccc|c} 1 & 0 & 1/2 & 1/2 \\ 0 & 1 & 3/2 & 1/2 \\ 0 & 0 & 0 & 1 \end{array}\right] \xrightarrow{\rho_1 \mapsto \rho_1 - \frac{1}{2}\rho_3} \left[\begin{array}{ccc|c} 1 & 0 & 1/2 & 0 \\ 0 & 1 & 3/2 & 1/2 \\ 0 & 0 & 0 & 1 \end{array}\right] \xrightarrow{\rho_2 \mapsto \rho_2 - \frac{1}{2}\rho_3} \left[\begin{array}{ccc|c} 1 & 0 & 1/2 & 0 \\ 0 & 1 & 3/2 & 0 \\ 0 & 0 & 0 & 1 \end{array}\right].$$

Since there is a pivot in the last column the system has no solutions.

Problem 4 Show that if the product of two $n \times n$ matrices $A \cdot B$ is invertible, then so are the factors A and B.

Solution.

We have shown in class that AB is invertible if and only if $(AB)\vec{x} = \vec{0}$ has only the trivial solution $\vec{x} = \vec{0}$. By associativity, $A(B\vec{x}) = \vec{0}$ has only the trivial solution. If we let $\vec{y} := B\vec{x}$, then $A\vec{y} = \vec{0}$ has only the trivial solution $\vec{y} = \vec{0}$, so A is invertible. Now, since $AB\vec{x} = \vec{0}$ has only the trivial solution, so does $A^{-1}AB\vec{x} = A^{-1}\vec{0}$, and thus so does $B\vec{x} = \vec{0}$. So B is also invertible. □

Problem 5 Let A be an $m \times n$ matrix and B an $m \times k$ matrix. Show that for an arbitrary system of linear equations $A \cdot X = B$ where X is an $n \times k$ matrix

1. If the system of equations $A \cdot X = B$ has more than one solution then it has infinitely many. Hint: Start with $B = 0$ and assume you have two solutions, see if you can generate more. Then try to connect this idea in the case you have at least two solutions for a general B.

2. For the case where A is a square matrix, show that if $A \cdot X = B$ has a unique solution for some particular column vector B, then it has a unique solution for all B.

Solution.

1. Suppose there exist distinct X_1, X_2 such that $AX_1 = B$ and $AX_2 = B$. Then, we observe that since we have $AX_1 + AX_2 = 2B$, $\frac{X_1+X_2}{2}$ is also a distinct solution to $AX = B$. Using this, we can define an infinite sequence $X_n = \frac{X_1+X_{n-1}}{2}$ for $n \geq 3$ where $AX_n = B$ for all $n \in \mathbb{N}$. Also, since the entries of X_n are getting closer and closer to the entries of X_1, no matrix is repeated, so this sequence generates infinite solutions to $AX = B$.

2. Suppose $AX = B$ has a unique solution for some particular column vector B. Then, the system $AX = B$ is equivalent to the system with the equations $X_1 = B_1, X_2 = B_2, \ldots, X_n = B_n$, which is the system $I_n X = B$. But then A must be row-equivalent to the identity, which we have shown is equivalent to A being invertible and $AX = B$ having a unique solution for all B.

Problem 6 In this exercise we will examine a more constructive connection between solvability of $A \cdot x = b$ for an $n \times n$ matrix A and a column vector b and its inverse. Assume that

$$A \cdot x = b$$

is uniquely solvable for all b. Ignore for this exercise the fact that we showed this implies that A has an inverse. Find specific choices of $b_1, \ldots, b_n$ such if $x_1, \ldots, x_n$ are the corresponding solutions, then the matrix with columns x_i

$$\begin{bmatrix} x_1 & \ldots & x_n \end{bmatrix}$$

is the right inverse of A (which we showed also yields the left inverse as well). Hint:

- We are looking for a matrix B such that $A \cdot B = I_n$. Think of

  $$\begin{bmatrix} b_1 & \ldots & b_n \end{bmatrix}$$

  where each b_i is its column.

- What do we want $A \cdot b_i$ to equal?

Solution.

Let $\vec{e_i}$ be a vector in $\mathbb{R}^n$ with all components 0 except component i. Then, $B = [\vec{x_1} \mid \vec{x_2} \mid \cdots \mid \vec{x_n}]$ where $A\vec{x_i} = \vec{e_i}$ for all $i \in \{1, 2, \ldots, n\}$ is a right inverse of A. This is because $I_n = [\vec{e_1} \mid \vec{e_2} \mid \cdots \mid \vec{e_n}]$, so $AB = I_n$ by the definition of matrix multiplication and thus B is a right inverse of A.