MATH 552: Section 001
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March 11, 2022

MATH 552 Homework 8*

Problem 42.4 According to definition (2), Sec. 42, of definite integrals of complex-valued functions of a real variable,

$$\int_0^\pi e^{(1+i)x}\ dx = \int_0^\pi e^x \cos x\ dx + i\int_0^\pi e^x \sin x\ dx.$$

Evaluate the two integrals on the right here by evaluating the single integral on the left and then using the real and imaginary parts of the value found.

Solution.

$$\begin{aligned}
\int_0^\pi e^{(1+i)x}\ dx &= \frac{1}{1+i}e^{(1+i)x}\bigg|_0^\pi \\
&= \frac{1}{1+i}e^{(1+i)x} - \frac{1}{1+i} \\
&= \frac{e^\pi e^{i\pi}}{1+i} - \frac{1}{1+i} && \text{(restating exponent)} \\
&= \frac{-e^\pi}{1+i} - \frac{1}{1+i} && \text{(using } e^{i\pi} = -1\text{)} \\
&= \frac{-e^\pi - 1}{1+i}\left(\frac{1-i}{1-i}\right) && \text{(multiplying by conjugate)} \\
&= \frac{-e^\pi + ie^\pi - 1 + i}{2} \\
&= -\frac{1}{2}(e^\pi + 1) + i\frac{1}{2}(e^\pi + 1)
\end{aligned}$$

Equation the real and imaginary parts, then,

$$\int_0^\pi e^x \cos x\ dx = -\frac{1}{2}(e^\pi + 1)$$

and

$$i\int_0^\pi e^x \sin x\ dx = \frac{i}{2}(e^\pi + 1).$$

Problem 43.5 Suppose that a function $f(z)$ is analytic at a point $z_0 = z(t_0)$ lying on a smooth arc $z = z(t)$ $(a \le t \le b)$. Show that if $w(t) = f[z(t)]$, then

$$w'(t) = f'[z(t)]z'(t)$$

when $t = t_0$.

Solution.

We write $f[z(t)] = u(x, y) + iv(x, y)$ and $z(t) = x(t) + iy(t)$, so that

$$w(t) = u[x(t), y(t)] + iv[x(t), y(t)].$$

We then apply the chain rule in calculus for functions of two real variables to write

$$w'(t) = (u_x(x,y)x'(t) + u_y(x,y)y'(t)) + i(v_x(x,y)x'(t) + v_y(x,y)y'(t)),$$

and differentiate $z(t)$ as

$$z'(t) = x'(t) + iy'(t).$$

Rearranging,

$$w'(t) = x'(t)[u_x(x,y) + iv_x(x,y)] + y'(t)[u_y(x,y) + iv_y(x,y)].$$

The Cauchy-Riemann equations tell us that

$$f'[z(t)] = u_x(x,y) + iv_x(x,y) = -i[u_y(x,y) + iv_y(x,y)],$$

so

$$u_x(x,y) + iv_x(x,y) = f'[z(t)] \text{ and } u_y(x,y) + iv_y(x,y) = if'[z(t)].$$

Using these identities to make substitutions,

$$w'(t) = x'(t)f'[z(t)] + iy'(t)f'[z(t)] = f'[z(t)][x'(t) + iy'(t)].$$

From $z'(t) = x'(t) + iy'(t)$, then,

$$w'(t) = f'[z(t)]z'(t).$$

Problem 46.10 With the aid of the result in Exercise 2, Sec. 42, evaluate the integral

$$\int_C z^m \overline{z}^n \, dz,$$

where m and n are integers and C is the unit circle $|z| = 1$, taken counterclockwise.

Solution.

Let $z = e^{i\theta}$, which will constrain z to a magnitude of 1. Then, since $\frac{dz}{d\theta} = ie^{i\theta}$,

$$\int_C z^m \overline{z}^n dz = \int_0^{2\pi} (e^{i\theta})^m (e^{-i\theta})^n ie^{i\theta} d\theta = \int_0^{2\pi} ie^{i\theta(m-n+1)} d\theta.$$

First, assume $m - n + 1 = 0$:

$$\int_0^{2\pi} ie^{i\theta(m-n+1)} d\theta = \int_0^{2\pi} i d\theta = i\theta|_0^{2\pi} = 2\pi i.$$

Then, assume $m - n + 1 \neq 0$ (and consequentially, we can multiply by $\frac{m-n+1}{m-n+1}$):

$$\begin{aligned}
\int_0^{2\pi} ie^{i\theta(m-n+1)} d\theta &= \frac{1}{m-n+1} \int_0^{2\pi} i(m-n+1)e^{i\theta(m-n+1)} d\theta && \left(\text{multiplying by } \tfrac{m-n+1}{m-n+1}\right) \\
&= \frac{1}{m-n+1} \left[e^{i\theta(m-n+1)} \right]_0^{2\pi} \\
&= \frac{1}{m-n+1} \left[e^{i[2\pi(m-n+1)]} - e^{i(0)} \right] \\
&= \frac{1}{m-n+1} (1-1) = 0 && \left(\text{using } (\forall k \in \mathbb{Z})(e^{2\pi k i} = 1)\right)
\end{aligned}$$

So the integral evaluates to 2π when $n = m + 1$ and evaluates to 0 otherwise.