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MATH 576: Section H01
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MATH 576 Homework 6

Problem 1 Show that the Toppling Dominoes position

$$\underbrace{B\,R}_{m \text{ times}} = *m.$$

Let $m \in \mathbb{N}$. Let $T_m := \underbrace{B\,R}_{m \text{ times}}$. It suffices to show that $T_m + *m \in \mathcal{P}$.

First, we claim that for any $k > 0$, we have $T_m\,B + *k \in \mathcal{L}$. To see this, we note that since R cannot topple the whole row of dominoes given any number of turns, L can win by taking all the tokens in k and leaving R with no moves, or by toppling the remaining dominoes once all the tokens in $*k$ are gone, depending on R's move. Symmetric reasoning shows that $R\,T_m + *k \in \mathcal{R}$ for all $k > 0$.

We observe that L can only move in T_m to $T_{m'}\,R$ or to $T_{m'}$ for some $m' < m$, and R can only move in T_m to $B\,T_{m'}$ or to $T_{m'}$. Thus, the previous paragraph shows that if the $\mathcal{N}$ext player has a winning strategy in $T_m + *m$, it is by moving to $T_{m'} + *m$ or $T_m + *(m')$ for some $m' < m$.

However, the $\mathcal{P}$revious player can then adopt a version of the Tweedledum-Tweedledee strategy: when the $\mathcal{N}$ext player moves to $T_{m'} + *m$ or to $T_m + (m')$, the $\mathcal{P}$revious player can move to $T_{m'} + *(m')$. That $T_m + *m \in \mathcal{P}$ then follows easily by strong induction, and therefore $T_m = *m$. □

Problem 2 Determine the value of the game $\{0, -10, \frac{3}{2}, -\frac{7}{8} \mid 2, 100, 3, 5, \frac{25}{4}\}$.

We proved in class that any partisan game is equivalent to the game where its dominated options are removed, so this game is equivalent to $\{\frac{3}{2} \mid 2\}$ since $\frac{3}{2} \geq 0 \geq \frac{7}{8} \geq -10$ and $2 \leq 3 \leq 5 \leq \frac{25}{4} \leq 100$. Therefore, by the Simplest Number Theorem, this game is equivalent to $\frac{7}{4}$. □

Problem 3 Let a be a number and let m be a nonnegative integer. Use the number avoidance theorem to determine the outcome class of the game $a + *m$ for any choice of a and m.

We claim we have

$$a + *m \in \begin{cases} \mathcal{L} & \text{if } a > 0, \\ \mathcal{R} & \text{if } a < 0, \\ \mathcal{P} & \text{if } a = m = 0, \\ \mathcal{N} & \text{otherwise.} \end{cases}$$

Case 1: $a > 0$. Then if L goes first, they can win by moving to 0 in $*m$ and leaving $a + 0 = a \in \mathcal{L}$. If R goes first, any winning move for them is in $*m$ by the number avoidance theorem. However, L can still win by taking all the tokens R did not take in $*m$ and leaving $a \in \mathcal{L}$, or by moving to some non-negative option in a if R took all the tokens. So we have $a + *m \in \mathcal{N}^L \cap \mathcal{P}^L = \mathcal{L}$.

Case 2: $a < 0$. We have $a + *m = a - *m = -(-a + *m) \in \mathcal{R}$ since $-a + *m \in \mathcal{L}$ by Case 1.

Case 3: $a = m = 0$. Then $a + *m = 0 + 0 = 0$, which we have proved in class is true only if $a + *m \in \mathcal{P}$.

Case 4: $a = 0$ and $m > 0$. Then $a + *m = 0 + *m = *m$, which we have proved in class is in $\mathcal{N}$. $\square$

Problem 4 Let n, x and y be numbers with $x > y$. Prove that

$$n \pm x \pm y = \{\{n + x + y \mid n + x - y\} \mid \{n - x + y \mid n - x - y\}\}.$$

It suffices to show that

$$\{\{n + x + y \mid n + x - y\} \mid \{n - x + y \mid n - x - y\}\} - (n \pm x \pm y) \in \mathcal{P}. \qquad (\star)$$

Case 1: $x > y$. Then by Theorem 14.1.1, we have $-(\pm x \pm y) = \pm x \pm y = \{\{x+y \mid x-y\} \mid \{-x+y \mid -x-y\}\}$, so we will show

$$\{\{n + x + y \mid n + x - y\} \mid \{n - x + y \mid n - x - y\}\} + \{\{x + y \mid x - y\} \mid \{-x + y \mid -x - y\}\} - n \in \mathcal{P}.$$

First, suppose L is the $\mathcal{N}$ext player. By the number avoidance theorem, any winning strategy L has is on the non-numbers, so we consider L's two options.

L could move to $\{n + x + y \mid n + x - y\} + \{\{x + y \mid x - y\} \mid \{-x + y \mid -x - y\}\} - n$. Then, R can move to $\{n+x+y \mid n+x-y\} + \{-x+y \mid -x-y\} - n$. If L then moves to $n+x+y+\{-x+y \mid -x-y\} - n$, R can move to $n + x + y - x - y - n = 0$ and win. But if L instead moves to $\{n + x + y \mid n + x - y\} - x + y - n$, R can move to $n + x - y - x + y - n = 0$ and also win. So this move is losing for L.

Thus, L's only hope is to move to $\{\{n+x+y \mid n+x-y\} \mid \{n-x+y \mid n-x-y\}\} + \{x+y \mid x-y\} - n$. Then, R can move to $\{n-x+y \mid n-x-y\} + \{x+y \mid x-y\} - n$. If L then moves to $n-x+y+\{x+y \mid x-y\} - n$, R can move to $n-x+y+x-y-n = 0$ and win. But if L instead moves to $\{n-x+y \mid n-x-y\}+x+y-n$, R can move to $n - x - y + x + y - n = 0$ and also win. So this move is also losing for L.

Thus, we have $\{\{n + x + y \mid n + x - y\} \mid \{n - x + y \mid n - x - y\}\} - (n \pm x \pm y) \in \mathcal{P}^L$.

Next, suppose R is the $\mathcal{N}$ext player. By the number avoidance theorem, any winning strategy R has is on the non-numbers, so we consider R's two options.

R could move to $\{n - x + y \mid n - x - y\} + \{\{x + y \mid x - y\} \mid \{-x + y \mid -x - y\}\} - n$. Then, L can move to $\{n - x + y \mid n - x - y\} + \{x + y \mid x - y\} - n$. If R then moves to $n - x - y + \{x + y \mid x - y\} - n$, L can move to $n - x - y + x + y - n = 0$ and win. But if R instead moves to $\{n - x + y \mid n - x - y\} + x - y - n$, L can move to $n - x + y + x - y - n = 0$ and also win. So this move is losing for R.

Thus, R's only hope is to move to $\{\{n+x+y \mid n+x-y\} \mid \{n-x+y \mid n-x-y\}\}+\{-x+y \mid -x-y\}-n$. Then, L can move to $\{n+x+y \mid n+x-y\}+\{-x+y \mid -x-y\}-n$. If R then moves to $n+x-y+\{-x+y \mid -x-y\}-n$, L can move to $n+x-y-x+y-n = 0$ and win. But if R instead moves to $\{n+x+y \mid n+x-y\}-x-y-n$, L can move to $n + x + y - x - y - n = 0$ and also win. So this move is also losing for R.

Thus, we have $\{\{n + x + y \mid n + x - y\} \mid \{n - x + y \mid n - x - y\}\} - (n \pm x \pm y) \in \mathcal{P}^R$ and so $(\star)$ holds.

Case 2: $x = y$. Then $n \pm x \pm y = n \pm x \pm x = n \pm x - \pm x = n + 0 = n$ by Tweedledum-Tweedledee, and since $x + y = x + x = 2x$ and $x - y = x - x = 0$, it suffices to show $\{\{n + 2x \mid n\} \mid \{n \mid n - 2x\}\} - n \in \mathcal{P}$.

If L goes first, their only possible win is to move to $\{n + 2x \mid n\} - n$, but then R can move to $n - n = 0$ and win. If R goes first, their only possible win is to move to $\{n \mid n - 2x\} - n$, but then L can move to $n - n = 0$ and win. So this is a $\mathcal{P}$-position and thus $(\star)$ holds.

Therefore, since $(\star)$ holds in both cases, we have

$$n \pm x \pm y = \{\{n + x + y \mid n + x - y\} \mid \{n - x + y \mid n - x - y\}\}.$$

$\square$

Problem 5 Determine the outcome class of the game $\{7 \mid 5\} + \{-7 \mid -11\} + \{8 \mid 2\}$.

We proved in class that we can write

$$\{7 \mid 5\} + \{-7 \mid -11\} + \{8 \mid 2\} = 6 \pm 1 - 9 \pm 2 + 5 \pm 3 = \pm 3 \pm 2 \pm 1 + 2.$$

If L goes first, they can play on ± 3, taking the position to $3 \pm 2 \pm 1 + 2 = \pm 2 \pm 1 + 5$. R's best move is then to play on ± 2, taking the position to $-2 \pm 1 + 5 = \pm 1 + 3$. L can then move to $1 + 3 = 4 \in \mathcal{L}$, so the game is in $\mathcal{N}^L$. On the other hand, if R goes first, they can play on ± 3, taking the position to $-3 \pm 2 \pm 1 + 2 = \pm 2 \pm 1 - 1$. L's best move is then to play on ± 2, taking the position to $2 \pm 1 - 1 = \pm 1 + 1$. R can then move to $-1 + 1 = 0$ and win, so the game is in $\mathcal{N}^R$.

Therefore, the game is an $\mathcal{N}$-position. $\square$