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MATH 574: Section H01
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MATH 574 Homework 2

Collaboration: I discussed some of the problems with Jackson Ginn.

Problem 1 A club has 20 members.
(a) In how many ways can we choose a president and a vice president of the club?
(b) In how many ways can we choose 4 people to serve as an executive committee of the club?
(c) Suppose the club has 5 freshmen, 5 sophomores, 5 juniors, and 5 seniors. In how many ways can we choose an executive committee of 4 people provided that at least one freshman must be on the committee?

Solution.

(a) Assuming the president and vice-president must be different people, there are 20 options for the president and then 19 options for the vice-president. By the product rule, then, there are $20 \times 19 = 380$ ways we can choose this.

(b) This can be represented by $\binom{20}{4} = 4,485$.

(c) There are $\binom{15}{4}$ ways to choose a committee with no freshmen (since there are 15 upperclassmen). Thus, the number of ways to choose a committee with 4 people with at least one freshman is $\binom{20}{4} - \binom{15}{4} = 3,480$.

Problem 2 The English language contains 21 consonants and 5 vowels. How many strings of 5 lowercase letters of the English alphabet contain
(a) no vowels?
(b) exactly 1 vowel?
(c) exactly 2 vowels?
(d) at least 3 vowels?

Solution.

(a) Since there are 21 consonants, and 5 spots, by the product rule there are $21^5 = 4,084,101$ strings with no vowels.

(b) We need one spot with a vowel and 4 spots with consonants. There are $\binom{5}{1}$ ways to choose where the vowel goes, 5 ways to choose which vowel is used, and 21 ways to choose what the consonant will be for the other 4 spots. Thus, there are $\binom{5}{1}(5)(21)^4 = 4,862,025$ strings with exactly 1 vowel.

(c) Using similar reasoning as (b), there are $\binom{5}{2}(5)^2(21)^3 = 2,315,250$ strings with exactly two vowels.

(d) Using similar reasoning as (b) and (c), there are

$$\sum_{n=3}^{5} \binom{5}{n} (5)^n (21)^5 - n = 620,000$$

strings with at least 3 vowels.

Problem 3 How many permutations of the alphabet `ABCDEFGHIJKLMNOPQRSTUVWXYZ`
(a) have all the vowels in the beginning?

(b) contain the string $LMNOP$?
(c) contain the strings ABC and DEF?
(d) do not end with the string XYZ?

Solution.

(a) There are 5! ways to arrange the vowels at the beginning, and 21! ways to arrange the consonants after. So there are 5!21! permutations of the alphabet with all the vowels in the beginning.

(b) We can treat $LMNOP$ and the other 21 letters as 22 blocks to be moved around. Thus, there are 22! strings that contain the string $LMNOP$.

(c) Similar to (b), we can treat ABC, DEF, and the other 20 letters as 22 blocks to be moved around. Thus, there are 22! strings that contain ABC and DEF.

(d) We first find the number of strings that do end with XYZ. There are no other restrictions for the other 23 letters, so there are 23! permutations that end in XYZ. Since there are 26! permutations of the alphabet, there are $26! - 23!$ permutations of the alphabet not ending in XYZ.

Problem 4 Two strings are called anagrams if one string can be obtained from the other by rearranging its letters.
(a) How many anagrams does the string `ABCDEFG` have?
(b) How many anagrams does the string `PUPPIES` have?
(c) How many anagrams does the string `MISSISSIPPI` have?

Solution.

(a) Since there are no duplicate letters, there are simply $7! = 5040$ anagrams.

(b) There are 7! permutations of `PUPPIES`, but since the 3 `P`'s are indistinguishable, there will be $3! = 6$ duplicates of each anagram since there are because the `P`'s can be switched around without changing anything. Thus, by the division rule, there are $7!/3! = 840$ anagrams.

(c) Using similar reasoning to (b), since there are 4 `I`'s, 4 `S`'s, and 2 `P`'s, there are $11!/(4!4!2!) = 34,650$ anagrams.

Problem 5 How many different solutions (x, y, z, w) of the equation $x + y + z + w = 25$ are there such that
(a) x, y, z, and w are all non-negative integers?
(b) x, y, z, and w are all positive integers?
(c) x, y, z, and w are all positive integers greater than 1?

Solution.

We can treat each variable as a distinguishable box, and each "1" as an indistinguishable object. For example, if we have $(x, y, z, w) = (5, 5, 5, 10)$, then there are 5 objects in x, y, and z and 10 objects in w. By Theorem 2 from the textbook, there are $\binom{n+r-1}{n-1}$ ways to place r indistinguishable objects in n distinguishable boxes.

(a) Since there no restrictions on how many "objects" can go in each variable, there are simply $\binom{4+25-1}{4-1} = 3276$ solutions.

(b) Since there must be an object in each variable (since each must be at least 1), we set 4 objects aside and calculate $\binom{4+21-1}{4-1} = 2024$ solutions.

(c) Since there must be two objects in each variable (since each must be at least 2), we set 8 objects aside and calculate $\binom{4+17-1}{4-1} = 1140$ solutions.

Problem 6 Prove the Binomial Theorem using induction on n. You are allowed to use Pascal's identity.
The Binomial Theorem. Let x and y be variables, and let n be a nonnegative integer. Then

$$(x+y)^n = \sum_{i=0}^{n} \binom{n}{i} x^{n-i} y^i.$$

Solution.

First, let $n = 0$. Then,

$$(x+y)^0 = 0 = \binom{0}{0} x^{0-0} y^0 = \sum_{i=0}^{0} \binom{0}{i} x^{0-i} y^i.$$

So the claim holds for $n = 0$. Next, let $n \in \mathbb{N} \cup \{0\}$. Assume that

$$(x+y)^n = \sum_{i=0}^{n} \binom{n}{i} x^{n-i} y^i.$$

We claim that

$$(x+y)^{n+1} = \sum_{i=0}^{n+1} \binom{n+1}{i} x^{n+1-i} y^i.$$

Using our induction hypothesis, this can be shown as follows:

$$\begin{aligned}
\sum_{i=0}^{n+1} \binom{n+1}{i} x^{n+1-i} y^i &= \binom{n+1}{0} x^{n+1-0} y^0 + \sum_{i=1}^{n} \left[\binom{n+1}{i} x^{n+1-i} y^i \right] + \binom{n+1}{n+1} x^{n+1-(n+1)} y^{n+1} \\
&= x^{n+1} + \sum_{i=1}^{n} \left[\binom{n+1}{i} x^{n+1-i} y^i \right] + y^{n+1} && \text{(splitting sum and simplifying)} \\
&= x^{n+1} + \sum_{i=1}^{n} \left(\left[\binom{n}{i} + \binom{n}{i-1} \right] x^{n+1-i} y^i \right) + y^{n+1} && \text{(Pascal's identity)} \\
&= x^{n+1} + \sum_{i=1}^{n} \left[\binom{n}{i} x^{n+1-i} y^i \right] + \sum_{i=1}^{n} \left[\binom{n}{i-1} x^{n+1-i} y^i \right] + y^{n+1} && \text{(distributing)} \\
&= x^{n+1} + \sum_{i=1}^{n} \left[\binom{n}{i} x^{n+1-i} y^i \right] + \sum_{i=0}^{n-1} \left[\binom{n}{i} x^{n-i} y^{i+1} \right] + y^{n+1} && \text{(adjusting bounds)} \\
&= x^{n+1} + x \sum_{i=1}^{n} \left[\binom{n}{i} x^{n-i} y^i \right] + y \sum_{i=0}^{n-1} \left[\binom{n}{i} x^{n-i} y^i \right] + y^{n+1} && \text{(redistributing)} \\
&= x \left(\binom{n}{0} x^{n-0} y^0 + \sum_{i=1}^{n} \left[\binom{n}{i} x^{n-i} y^i \right] \right) + y \left(\sum_{i=0}^{n-1} \left[\binom{n}{i} x^{n-i} y^i \right] + \binom{n}{n} x^{n-n} y^n \right) \\
&= x \left(\sum_{i=0}^{n} \left[\binom{n}{i} x^{n-i} y^i \right] \right) + y \left(\sum_{i=0}^{n} \left[\binom{n}{i} x^{n-i} y^i \right] \right) && \text{(combining sum)} \\
&= (x+y) \sum_{i=0}^{n} \left[\binom{n}{i} x^{n-i} y^i \right] && \text{(regrouping)} \\
&= (x+y)(x+y)^n && \text{(induction hypothesis)} \\
&= (x+y)^{n+1}.
\end{aligned}$$

So if the claim holds for n, it also holds for $n+1$. Therefore, we have that for all non-negative integers n,

$$(x+y)^n = \sum_{i=0}^{n} \binom{n}{i} x^{n-i} y^i.$$

Problem 7 Use the binomial theorem to find the coefficient of $x^a y^b$ in the expansion of $(2x^3 - 4y^2)^7$ where
(a) $a = 9$, $b = 8$.
(b) $a = 18$, $b = 2$.
(c) $a = 0$, $b = 14$.

Solution.

We have from the binomial theorem that

$$\Big((2x^3) + (-4y^2)\Big)^7 = \sum_{i=0}^{7} \binom{7}{i} (2x^3)^{7-i} (-4y^2)^i = \sum_{i=0}^{7} \binom{7}{i} 2^{7-i} (-4)^i x^{21-3i} y^{2i}.$$

Thus, for any a or b, we can solve $21 - 3i = a$ or $2i = b$ for i, and the coefficient will be $\binom{7}{i} 2^{7-i} (-4)^i$.

(a) Since $21 - 3i = 9$ and $2i = 8$ are both satisfied by $i = 4$, the coefficient is $\binom{7}{4} 2^{7-4} (-4)^4 = 71680$.

(b) Since $21 - 3i = 18$ and $2i = 2$ are both satisfied by $i = 1$, the coefficient is $\binom{7}{1} 2^{7-1} (-4)^1 = -1792$.

(c) Since $21 - 3i = 0$ and $2i = 14$ are both satisfied by $i = 7$, the coefficient is $\binom{7}{7} 2^{7-7} (-4)^7 = -16384$.

Problem 8 Use the Binomial Theorem to prove the following: if n and k are integers such that $1 \le k \le n$, then

$$\sum_{k=0}^{n} \binom{n}{k} (-5)^k = \begin{cases} 4^n & \text{if } n \text{ is even,} \\ -4^n & \text{if } n \text{ is odd.} \end{cases}$$

Solution.

We first rewrite the sum as

$$\sum_{k=0}^{n} \binom{n}{k} (-1)^k 5^k.$$

Because adding two numbers with the same parity yields an even number, subtracting a number from an even number will yield a number with the same parity. Thus, if n is even, we can replace $(-1)^k$ by $(-1)^{n-k}$ since $(-1)^k$ is equal for any k with the same parity. So we can write

$$\sum_{k=0}^{n} \binom{n}{k} (-1)^{n-k} 5^k,$$

which by the binomial theorem is equal to $(-1+5)^n = 4^n$. Since subtracting a number from an odd number will yield a number with opposite parity, if n is odd we can replace $(-1)^k$ with $-(-1)^{n-k}$. So we can write

$$-\sum_{k=0}^{n} \binom{n}{k} (-1)^{n-k} 5^k,$$

which by the binomial theorem is equal to $-(-1+5)^n = -4^n$. Therefore, if n and k are integers such that $1 \le k \le n$, then

$$\sum_{k=0}^{n} \binom{n}{k} (-5)^k = \begin{cases} 4^n & \text{if } n \text{ is even,} \\ -4^n & \text{if } n \text{ is odd.} \end{cases}$$

Problem 9 Give a double counting proof of the following: if n and k are integers with $1 \leq k \leq n$, then $k\binom{n}{k} = n\binom{n-1}{k-1}$.

We observe that algebraically,

$$k\binom{n}{k} = \frac{kn!}{k!(n-k)!} = \frac{n!}{(k-1)!(n-k)!} = \frac{n(n-1)!}{(k-1)!(n-1-(k-1))!} = n\binom{n-1}{k-1}.$$

Combinatorially, we can represent $n\binom{n-1}{k-1}$ as the following: Choose a number a from $\{1, 2, 3, ...n\}$. Then, for each of these numbers, choose $k-1$ numbers from $\{1, 2, 3, .., n\} - \{a\}$. Since there are n choices for a, and for each a there are $\binom{n-1}{k-1}$ choices, by the product rule there are $n\binom{n-1}{k-1}$ ways to do this.

We can represent $k\binom{n}{k}$ in the same way. Since $\binom{n}{k}$ represents the number of ways to choose groups of k from $\{1, 2, 3, ...n\}$, and there are k ways to choose which number will be a from each group, by the product rule there are $k\binom{n}{k}$ ways to do this.

So both count the same situation, and thus they are equal.

Problem 10 Prove the following by induction on k: if n and k are integers with $1 \leq k \leq n$, then $\sum_{i=0}^{k}\binom{n+i}{i} = \binom{n+k+1}{k}$.

Solution.

First, let $k = 1$. Then,

$$\sum_{i=0}^{1}\binom{n+i}{i} = \binom{n+0}{0} + \binom{n+1}{1} = 1 + n + 1 = \binom{n+1+1}{1}.$$

So the claim holds for $k = 1$. Next, let $k \in \mathbb{N}$. Assume we have

$$\sum_{i=0}^{k}\binom{n+i}{i} = \binom{n+k+1}{k}.$$

Then,

$$\begin{aligned}
\sum_{i=0}^{k+1}\binom{n+i}{i} &= \sum_{i=0}^{k}\binom{n+i}{i} + \binom{n+k+1}{k+1} && \text{(splitting sum)}\\
&= \binom{n+k+1}{k} + \binom{n+k+1}{k+1} && \text{(induction hypothesis)}\\
&= \binom{n+k+1+1}{k+1}. && \text{(Pascal's identity)}
\end{aligned}$$

So if the claim holds for k, it also holds for $k+1$. Therefore, for $1 \leq k \leq n$,

$$\sum_{i=0}^{k}\binom{n+i}{i} = \binom{n+k+1}{k}.$$