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MATH 546: Section 001
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MATH 546 Homework 2

In each of the problems below, a set and an operation are given. Using the given set and operation, check whether each of the four group requirements holds or not. Prove your answers.

Problem 1 $G = \{x \in \mathbb{R} \mid x > 1\}$; operation defined by $a \star b = ab - a - b + 2$.

1. Closure holds. Let $a, b \in G$. Then $a > 1$ and $b > 1$, so $ab > 1$ and $-a-b > -2$. Thus, $ab-a-b > -1$, so $a \star b = ab - a - b + 2 > 1$. Therefore, $a \star b \in G$.
2. Identity holds. Consider $e = 2$. Then we have $a \cdot 2 - a - 2 + 2 = 2a - a = a$ and $2 \cdot a - 2 - a + 2 = 2a - a = a$. Since $2 \in G$, 2 is the identity element.
3. Inverses hold. Let $a \in G$, and consider $\frac{a}{a-1}$. Then we have
$$a \cdot \frac{a}{a-1} - a - \frac{a}{a-1} + 2 = \frac{a^2 - a(a-1) - a}{a-1} + 2 = \frac{a^2 - a^2 + a - a}{a-1} + 2 = 0 + 2 = 2 = e$$
and
$$\frac{a}{a-1} \cdot a - \frac{a}{a-1} - a + 2 = \frac{a^2 - a - a(a-1)}{a-1} + 2 = \frac{a^2 - a - a^2 + a}{a-1} + 2 = 0 + 2 = 2 = e,$$
and since $\frac{a}{a-1} > 1$ for all $a > 1$, we have $a^{-1} = \frac{a}{a-1}$ is an inverse in G.
4. Associativity holds. Let $a, b, c \in G$. Then, we have
$$\begin{aligned}(a \star b) \star c &= (ab - a - b + 2) \star c \\ &= (ab - a - b + 2)c - (ab - a - b + 2) - c + 2 \\ &= abc - ac - bc + 2c - ab + a + b - 2 - c + 2 \\ &= abc - ac - bc + c - ab + 2a - a + b - 2 + 2 \\ &= abc - ab - ac + 2a - a - bc + b + c - 2 + 2 \\ &= a(bc - b - c + 2) - a - (bc - b - c + 2) + 2 \\ &= a \star (bc - b - c + 2) \\ &= a \star (b \star c).\end{aligned}$$

So $(G, \star)$ is a group. $\square$

Problem 2 $G = \{x \in \mathbb{Z} \mid x \geq 8\}$; operation defined by $a \star b = \max\{a, b\}$.

1. Closure holds. Let $a, b \in G$. Then $a \geq 8$ and $b \geq 8$, so clearly $\max\{a, b\} \geq 8$. Thus, $a \star b \in G$.
2. Identity holds. Consider $e = 8$, which is in G because $8 \geq 8$. Since all $a \in G$ satisfy $a \geq 8$, we always have $\max\{a, 8\} = \max\{8, a\} = a$. Thus, $e = 8$.

3. Inverses do not hold. Consider any $a \in G$ other than 8, such as $9 \in G$. There is no element b in G that will yield $\max\{9, b\} = 8$, because $9 > 8$.

4. Associativity holds. Let a, b, c. We have

$$\max\Big\{\max\{a,b\}\Big\} = \max\{a,b,c\} = \max\Big\{a, \max\{b,c\}\Big\} :$$

it makes no difference which comparison we make first, because we will always choose the largest of the three numbers.

So $(G, \star)$ is not a group. $\square$

Problem 3 $G = \{x \in \mathbb{R} \mid x \geq 0\}$; operation defined by $a \star b = |a - b|$.

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1. Closure holds. The absolute value of a real number is always a non-negative real number, and since $a - b \in \mathbb{R}$, we have $|a - b| \in G$.

2. Identity holds. Consider $e = 0$, which is in G because it is non-negative, and let $a \in G$. Then, we have $a \star e = |a - 0| = |a| = a$ since $a > 0$. Similarly, we have $e \star a = |0 - a| = |-a| = |a| = a$.

3. Inverses hold. Let $a \in G$. Then, $a^{-1} = a$, because $|a - a| = 0 = e$.

4. Associativity does not hold. For example, consider $a = 3$, $b = 2$, and $c = 1$. Then, we have

$$(3 \star 2) \star 1 = \Big||3 - 2| - 1\Big| = |1 - 1| = 0 \neq 2 = |3 - 1| = \Big|3 - |2 - 1|\Big| = 3 \star (2 \star 1).$$

So $(G, \star)$ is not a group. $\square$

Problem 4 $G = \{a + b\sqrt{2} \mid a, b \in \mathbb{Q}, a \neq 0 \text{ or } b \neq 0\}$; operation defined by usual multiplication.

Solution.

1. Closure holds. Let $x, y \in G$ with $x = a + b\sqrt{2}$ and $y = c + d\sqrt{2}$. Then,

$$xy = \left(a + b\sqrt{2}\right)\left(c + d\sqrt{2}\right) = ac + ad\sqrt{2} + bc\sqrt{2} + bd\left(\sqrt{2}\right)^2 = (ac + 2bd) + (ad + bc)\sqrt{2}.$$

Since $a, b, c, d \in \mathbb{Q}$ and we do not have $a = b = 0$ or $c = d = 0$, we have $(ac + 2bd) \in \mathbb{Q}$ and $(ad + bc) \in \mathbb{Q}$ without $ac + 2bd = ad + bc = 0$. Thus, $xy \in G$.

2. Identity holds. Since $e = 1 + 0\sqrt{2} \in G$, and 1 is the identity element for multiplication, 1 works in G.

3. Inverses hold. Let $x = a + b\sqrt{2} \in G$, and consider

$$y = \frac{a}{a^2 - 2b^2} - \left(\frac{b}{a^2 - 2b^2}\right)\sqrt{2}.$$

Then, we have

$$\begin{aligned}
yx &= xy && \text{(commutativity of multiplication)} \\
&= \left(a + b\sqrt{2}\right)\left(\frac{a}{a^2 - 2b^2} - \frac{b\sqrt{2}}{a^2 - 2b^2}\right) \\
&= \frac{a^2}{a^2 - 2b^2} - \frac{ab\sqrt{2}}{a^2 - 2b^2} + \frac{ab\sqrt{2}}{a^2 - 2b^2} - \frac{2b^2}{a^2 - 2b^2} && \text{(distributing)}
\end{aligned}$$

$$= \frac{a^2 - 2b^2}{a^2 - 2b^2}$$
$$= 1 = e.$$

4. Associativity holds. This is known because the operation is standard multiplication.

So $(G, \star)$ is a group. □

Problem 5 G is the set of all the *affine functions* $f_{m,b} : \mathbb{R} \to \mathbb{R}$, $f_{m,b}(x) = mx + b$, where $m, b \in \mathbb{R}$ and $m \neq 0$; operation defined by composition of functions.

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1. Closure holds. Let $f(x) = mx + b, g(x) = m'x + b' \in G$. Then
$$f \circ g = m(m'x + b') + b = (m'm)x + (b' + b) \in G.$$
2. Identity holds. Consider $e = x = 1x + 0, f = mx + b \in G$. Then
$$f \circ e = m(x) + b = f = mx + b = e \circ f.$$
3. Inverses hold. Let $f(x) = mx + b$, and consider $g(x) = \frac{1}{m}x + \frac{-b}{m} \in G$. Then
$$f \circ g = m\left(\frac{x - b}{m}\right) + b = x - b + b = x = e = x = \frac{mx}{m} = \frac{(mx + b) - b}{m} = g \circ f.$$
4. Associativity holds. This is known because the operation is function composition.

So $(G, \star)$ is a group. □