Nathan Bickel
MATH 576: Section H01
Professor: Dr. Linz
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MATH 576 Homework

Problem 1 Analyze the Subtraction-$\{2,5,6\}$ game. What are the $\mathcal{P}$- and $\mathcal{N}$-positions? Justify your answer.

For brevity, in this problem we denote $\underset{\{2,5,6\}}{n}$ by $\underline{n}$. We claim that for all $n \in \mathbb{N}$, we have

$$\underline{n} \in \begin{cases} \mathcal{P} & \text{if } n \mod 11 \in \{0,1,4,8\} \\ \mathcal{N} & \text{otherwise.} \end{cases} \qquad (\star)$$

We proceed by induction on n. For the base case, we check $\underline{0}$ through $\underline{10}$:

- Clearly, we have $\underline{0}, \underline{1} \in \mathcal{P}$ and $\underline{2}, \underline{5}, \underline{6} \in \mathcal{N}$.
- Since $\underline{3-2} = \underline{1} \in \mathcal{P}$, we have $\underline{3} \in \mathcal{N}$.
- The only option in $\underline{4}$ is $\underline{4-2} = \underline{2} \in \mathcal{N}$, so $\underline{4} \in \mathcal{P}$.
- Since $\underline{7-6} = \underline{1} \in \mathcal{P}$, we have $\underline{7} \in \mathcal{N}$.
- The options for $\underline{8}$ are $\underline{8-2} = \underline{6} \in \mathcal{N}$, $\underline{8-5} = \underline{3} \in \mathcal{N}$, and $\underline{8-6} = \underline{2} \in \mathcal{N}$, so $\underline{8} \in \mathcal{P}$.
- Since $\underline{9-5} = \underline{4} \in \mathcal{P}$, we have $\underline{9} \in \mathcal{N}$.
- Since $\underline{10-2} = \underline{8} \in \mathcal{P}$, we have $\underline{10} \in \mathcal{N}$.

For the induction step, let $n \in \mathbb{N}$, $n > 10$, and suppose $(\star)$ holds for all $m < n$. Let $\equiv$ denote equivalence (mod 11). We check that $(\star)$ holds in each case:

- $n \equiv 0$: The options for $\underline{n}$ are $\underline{n-2} \equiv \underline{9} \in \mathcal{N}$, $\underline{n-5} \equiv \underline{6} \in \mathcal{N}$, and $\underline{n-6} \equiv \underline{5} \in \mathcal{N}$ by the IH, so $\underline{n} \in \mathcal{P}$.
- $n \equiv 1$: The options for $\underline{n}$ are $\underline{n-2} \equiv \underline{10} \in \mathcal{N}$, $\underline{n-5} \equiv \underline{7} \in \mathcal{N}$, and $\underline{n-6} \equiv \underline{6} \in \mathcal{N}$ by the IH, so $\underline{n} \in \mathcal{P}$.
- $n \equiv 2$: Since $\underline{n-2} \equiv \underline{0} \in \mathcal{P}$ by Case 0, we have $\underline{n} \in \mathcal{N}$.
- $n \equiv 3$: Since $\underline{n-2} \equiv \underline{1} \in \mathcal{P}$ by Case 1, we have $\underline{n} \in \mathcal{N}$.
- $n \equiv 4$: The options for $\underline{n}$ are $\underline{n-2} \equiv \underline{2} \in \mathcal{N}$, $\underline{n-5} \equiv \underline{10} \in \mathcal{N}$, and $\underline{n-6} \equiv \underline{9} \in \mathcal{N}$, so $\underline{n} \in \mathcal{P}$.
- $n \equiv 5$: Since $\underline{n-5} \equiv \underline{0} \in \mathcal{P}$ by Case 0, we have $\underline{n} \in \mathcal{N}$.
- $n \equiv 6$: Since $\underline{n-6} \equiv \underline{0} \in \mathcal{P}$ by Case 0, we have $\underline{n} \in \mathcal{N}$.
- $n \equiv 7$: Since $\underline{n-6} \equiv \underline{1} \in \mathcal{P}$ by Case 1, we have $\underline{n} \in \mathcal{N}$.
- $n \equiv 8$: The options for $\underline{n}$ are $\underline{n-2} \equiv \underline{6} \in \mathcal{N}$, $\underline{n-5} \equiv \underline{3} \in \mathcal{N}$, and $\underline{n-6} \equiv \underline{2} \in \mathcal{N}$, so $\underline{n} \in \mathcal{P}$.
- $n \equiv 9$: Since $\underline{n-5} \equiv \underline{4} \in \mathcal{P}$ by Case 4, we have $\underline{n} \in \mathcal{N}$.
- $n \equiv 10$: Since $\underline{n-2} \equiv \underline{8} \in \mathcal{P}$ by Case 8, we have $\underline{n} \in \mathcal{N}$.

Therefore, $(\star)$ holds in all cases. $\square$

Problem 2 Consider a Kayles game with position $\underline{1}\,\underline{2}\ \underline{4}\,\underline{5}\ \ \underline{8}\,\underline{9}\,\underline{10}\,\underline{11}\,\underline{12}$. Is this game a $\mathcal{P}$-position or a $\mathcal{N}$-position? Justify your answer.

Let G be the given Kayles position. Consider the three Kayles games G_1, G_2, G_3 where G_1 and G_2 are Kayles games with 2 pins and G_3 is a Kayles game with 5 pins. There is a natural correspondence between G and $G' := (G_1 + G_2) + G_3$, and it is clear that $G' \in \mathcal{N}$ if and only if $G \in \mathcal{N}$.

We proved in class that for any impartial game H, we have $H+H \in \mathcal{P}$. Since $G_1 = G_2$, we have $G_1+G_2 \in \mathcal{P}$. Since G_3 is a Kayles game with a positive number of pins, we have $G_3 \in \mathcal{N}$. Therefore, by Problem 4, we have $G' \in \mathcal{N}$, so the given game is a $\mathcal{N}$-position. $\square$

Problem 3 Find and prove the outcome class for the game $\underset{\{1,3,5\}}{8} + \underset{\{2,5,6\}}{8}$.

Clearly, we have $\underset{\{1,3,5\}}{0} \in \mathcal{P}$ and $\underset{\{1,3,5\}}{1}, \underset{\{1,3,5\}}{3}, \underset{\{1,3,5\}}{5} \in \mathcal{N}$. Then, we can compute that $\underset{\{1,3,5\}}{2}, \underset{\{1,3,5\}}{4}$, and $\underset{\{1,3,5\}}{6}$ each have only options in $\mathcal{N}$, so each of these in $\mathcal{P}$. Then, $\underset{\{1,3,5\}}{7} \in \mathcal{N}$ because removing 1 token (or indeed 3 or 5 tokens) yields an option in $\mathcal{P}$. This shows that $\underset{\{1,3,5\}}{8} \in \mathcal{P}$ since it only has options in $\mathcal{N}$. We proved in Problem 1 that we also have $\underset{\{2,5,6\}}{8} \in \mathcal{P}$. Therefore, from a result in class we have $\underset{\{1,3,5\}}{8} + \underset{\{2,5,6\}}{8} \in \mathcal{P}$. $\square$

Problem 4 Prove that if the game G is in $\mathcal{P}$ and the game H is in $\mathcal{N}$, then the game $G + H$ is in $\mathcal{N}$.

Since H is an $\mathcal{N}$-position, there is an option H' from H in $\mathcal{P}$. From a result in class, we have $G + H' \in \mathcal{P}$ since $G \in \mathcal{P}$ and $H' \in \mathcal{P}$. Therefore, since $G + H$ has $G + H'$ as an option in $\mathcal{P}$, we have $G + H \in \mathcal{N}$. $\square$

Problem 5 Two players play the game $G + H$, where G is the game $\underset{\{1,2,3\}}{400}$ and H is the game of Kayles with 1001 pins in a row. Which player has a winning strategy? Justify your answer.

We proved in class that $\underset{\{1,2,3\}}{n}$ is in $\mathcal{P}$ if $n \equiv 0 \pmod 4$, so G is in $\mathcal{P}$ since $400 \equiv 0 \pmod 4$. We also proved in class that the game of Kayles is in $\mathcal{N}$ for any positive number of pins, so H is in $\mathcal{N}$. Therefore, by Problem 4, $G + H$ is in $\mathcal{N}$. $\square$