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MATH 576: Section H01
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MATH 576 Homework 8

Problem 1 Show that for a positive integer n, the Toppling Dominoes position

$$B\ \underbrace{R \ldots R}_{n+2 \text{ times}}\ B = +_n.$$

Let $R^k := \underbrace{R \ldots R}_{k \text{ times}}$. We note that we have proved in class that $R^k = -k$.

We first prove a lemma: for all $0 \le r < s$, we claim we have $\{0 \mid -r\} > s$ and $\{0 \mid -r\} > \{0 \mid -s\}$. To see $\{0 \mid -r\} > s$, note that $\{0 \mid -r\} + s \in \mathcal{L}$: if it is L's turn they can move to $s > 0$, and if it is R's turn their only hope is to move is to $-r + s > 0$. To see $\{0 \mid -r\} > \{0 \mid -s\}$, consider $\{0 \mid -r\} + \{s \mid 0\}$. If it L's turn they can move to $\{0 \mid -r\} + s$ and then R's only hope is to move to $-r + s > 0$. If it is R's turn and they move to $-r + \{s \mid 0\}$, L can respond by moving to $-r + s > 0$, and if R instead moves to $\{0 \mid -r\}$, L wins by moving to 0. So $\{0 \mid -r\} + \{s \mid 0\} \in \mathcal{L}$ and thus $\{0 \mid -r\} > \{0 \mid -s\}$.

We now prove a second lemma: we claim that for all $k \in \mathbb{N}$, we have

$$B\,R^k = \begin{cases} 1 & \text{if } k = 0, \\ * & \text{if } k = 1, \\ \{0 \mid -(k-1)\} & \text{if } k \ge 2. \end{cases} \tag{$\star$}$$

We have $B\,R^0 = B = 1$ by definition, and we have

$$B\,R^1 = B\,R = \{0, R \mid 0, B\} = \{0, -1 \mid 0, 1\} = \{0 \mid 0\} = *$$

by the dominated removal theorem. We prove the $k \ge 2$ piece by strong induction on k. We have

$$B\,R^2 = B\,R\,R = \{0, R\,R \mid B, B\,R, 0, R\} = \{0, -2 \mid 1, *, 0, -1\} = \{0 \mid -1\} = \{0 \mid -(2-1)\}$$

by the dominated removal theorem (as $* > -1$). Now, let $k > 2$, and suppose that for all $k' < k$, we have $B\,R^{k'} = \{0 \mid -(k'-1)\}$. Now, we write

$$\begin{aligned} B\,R^k &= \left\{0, R^k \mid B, B\,R, B\,R^2, \ldots, B\,R^{k-1}, 0, R, R^2, \ldots, R^{k-1}\right\} \\ &= \left\{0, k \mid 1, *, \{0 \mid -1\}, \ldots, \{0 \mid -(k-2)\}, 0, -1, -2, \ldots, -(k-1)\right\} && \text{(induction hypothesis)} \\ &= \left\{0, k \mid 1, 0, -1, -2, \ldots, -(k-1)\right\} && (\{0 \mid -(i-1)\} > -i \text{ for all } 1 \le i \le k-2 \text{ by first lemma}) \\ &= \left\{0 \mid -(k-1)\right\}, && \text{(applying dominated removal again)} \end{aligned}$$

so $(\star)$ holds.

We now prove that $B\,R^{n+2}\,B = \{0 \mid \{0 \mid -n\}\} = +_n$ for all $n \ge 1$. Since the position is symmetric, we will only write the options with a blue domino on the left (each symmetric option with the blue domino on the right is isomorphic and thus equal, so it is dominated and can be removed by the dominated removal theorem). We have

$$\begin{aligned}
B\,R^{n+2}\,B &= \left\{0, B\,R^{n+2} \mid B, B\,R, B\,R^2, \dots, B\,R^{n+1}\right\} \\
&= \{0, \{0 \mid -(n+1)\} \mid 1, *, \{0 \mid -1\}, \dots, \{0 \mid -n\}\} && \text{(by second lemma)} \\
&= \{0, \{0 \mid -(n+1)\} \mid \{0 \mid -n\}\} && (\{0 \mid -i\} > \{0 \mid -n\} \text{ for all } i < n \text{ by first lemma}) \\
&= \{0 \mid \{0 \mid -n\}\} && (\{0 \mid -(n+1)\} \text{ reversible through } -(n+1) \text{ as } -(n+1) < 0 < B\,R^{n+2}\,B) \\
&= +_n. && \text{(by definition)}
\end{aligned}$$

□

Problem 2 Show that $\uparrow + \uparrow + * = \{0 \mid \uparrow\}$ in canonical form.

We first show that $\uparrow + \uparrow + * = \{0 \mid \uparrow\}$. Consider $G := \{0 \mid \uparrow\} - \uparrow - \uparrow - * = \{0 \mid \uparrow\} + \downarrow + \downarrow + *$. It suffices to show $G \in \mathcal{P}$.

If L moves first:

- L should not move in G to $0 + \downarrow + \downarrow + *$, as we have shown in class that $2 \cdot \downarrow < *$ (and more generally that $k \cdot \downarrow < *$ for all $k \geq 2$).
- L should not move in G to $\{0 \mid \uparrow\} + * + \downarrow + * = \{0 \mid \uparrow\} + \downarrow$, as R can win by moving to $\uparrow + \downarrow = 0$.
- L should not move in G to $\{0 \mid \uparrow\} + \downarrow + \downarrow$, as R can win by moving to $\uparrow + \downarrow + \downarrow = \downarrow$.

So L has no winning moves as $\mathcal{N}$ext player.

If R moves first:

- R should not move in G to $\uparrow + \downarrow + \downarrow + * = \downarrow + *$, as we have shown in class that $\downarrow + * \in \mathcal{N}$.
- R should not move in G to $\{0 \mid \uparrow\} + \downarrow + *$, as L can move to $\{0 \mid \uparrow\} + * + * = \{0 \mid \uparrow\}$ and then R must move to $\uparrow > 0$.
- R should not move in G to $\{0 \mid \uparrow\} + \downarrow + \downarrow$, as L can move to $\{0 \mid \uparrow\} + \downarrow + *$. Then, if R moves to $\uparrow + \downarrow + * = *$, L can win by moving to 0. If R instead moves to $\{0 \mid \uparrow\} + *$, L can win by moving to $\{0 \mid \uparrow\}$ and forcing R to move to $\uparrow > 0$. Finally, if R moves to $\{0 \mid \uparrow\} + \downarrow$, L can move to $\{0 \mid \uparrow\} + *$, and then R's only options are $\uparrow + * \in \mathcal{N}$ and $\{0 \mid \uparrow\}$, the second of which L can respond to by moving to 0 and winning.

So R has no winning moves as $\mathcal{N}$ext player. Thus, $G \in \mathcal{P}$.

Clearly, $\{0 \mid \uparrow\}$ has no dominated options, as each player has only one option. So it suffices to check that 0 is not reversible through $\uparrow$ and $\uparrow$ is not reversible through 0. Clearly, 0 is not reversible through $\uparrow$, as 0 has no right options at all. Also, $\uparrow$ is not reversible through 0: while 0 is indeed a left option of $\uparrow$, it is not true that $0 \geq \uparrow + \uparrow + *$, as we have proved in class that $2 \cdot \uparrow > *$. Therefore, we have that $G = \{0 \mid \uparrow\}$ in canonical form. □

Problem 3 List 5 distinct games which have birthday 2.

We have that 0 has birthday 0, and thus $\{0 \mid\} = 1$ and $\{\mid 0\} = -1$ both have birthday 1 by definition. So $2 := \{1 \mid\}$, $-2 := \{\mid -1\}$, $\frac{1}{2} := \{0 \mid 1\}$, $-\frac{1}{2} := \{-1 \mid 0\}$, and $\pm 1 := \{1 \mid -1\}$ all have birthday 2, and clearly all are distinct.

Problem 4 Show that the following Toads-and-Frogs position has value $\{\{2 \mid 1\} \mid 0\}$:

T	T	T	T		F

We will write the above position as $T\,T\,T\,T\,O\,F$ and other positions using the same convention. We will compute positions that will be relevant:

1. We have $O\,T\,T\,T\,T\,F = \{\,|\,\} = 0$.

2. Using #1, we have $T\,O\,T\,T\,T\,F = \{O\,T\,T\,T\,T\,F \mid \} = \{0 \mid \} = 1$.

3. Using #2, we have $T\,T\,O\,T\,T\,F = \{T\,O\,T\,T\,T\,F \mid \} = \{1 \mid \} = 2$.

4. We have $T\,T\,T\,F\,O\,T = 0$ by the Death Leap Principle.

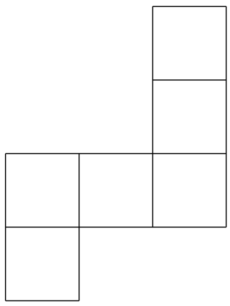
5. Using #4, we have $T\,T\,T\,F\,T\,O = \{T\,T\,T\,F\,O\,T \mid \} = \{0 \mid \} = 1$.

6. Using #3 and #5, we have $T\,T\,T\,O\,T\,F = \{T\,T\,O\,T\,T\,F \mid T\,T\,T\,F\,T\,O\} = \{2 \mid 1\}$.

7. We have $T\,T\,T\,T\,F\,O = 0$ by the Death Leap Principle.

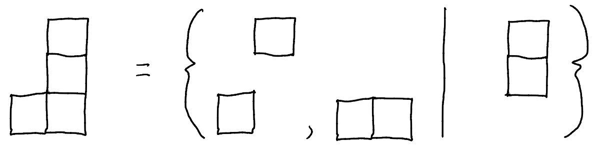
Therefore, using #6 and #7, we have $T\,T\,T\,T\,O\,F = \{T\,T\,T\,O\,T\,F \mid T\,T\,T\,T\,F\,O\} = \{\{2 \mid 1\} \mid 0\}$. $\square$

Problem 5 Determine the value of the following Domineering position:

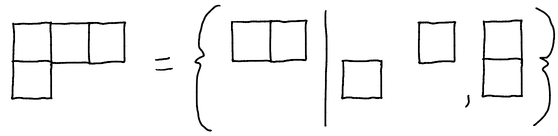


We ask the reader as they turn the page to forgive our laziness in not wanting to typeset Domineering positions. In our defense, doing so seemed unbearably tedious.

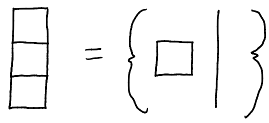
We first compute some positions that will be relevant. We have:

- 

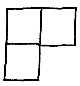
  $$= \{0, -1 \mid 1\} = \{0 \mid 1\} = \tfrac{1}{2}.$$

- 

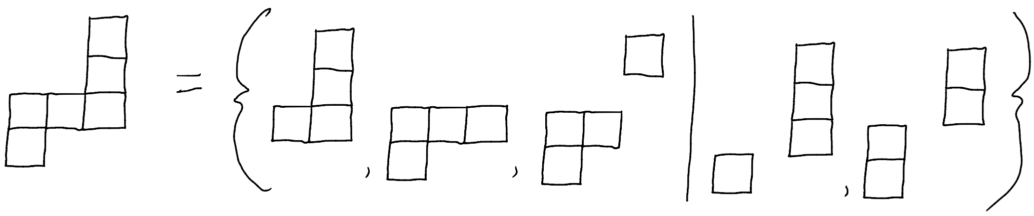
  $$= \{-1 \mid 0, 1\} = \{-1 \mid 0\} = -\tfrac{1}{2}.$$

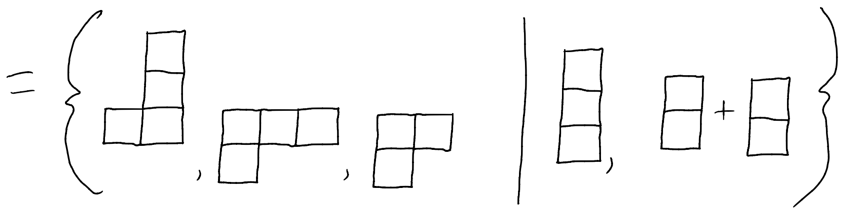
- 

  $$= \{0 \mid \} = 1.$$

-  $= \star$ as we proved in class.

Then, we can write:





$$= \left\{\tfrac{1}{2}, -\tfrac{1}{2}, \star \,\middle|\, 1, 1+1\right\}$$

$$= \left\{\tfrac{1}{2}, \star, -\tfrac{1}{2} \,\middle|\, 1, 2\right\}$$

$$= \left\{\tfrac{1}{2} \,\middle|\, 1\right\}$$

$$= \tfrac{3}{4}.$$