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MATH 544: Section H01
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MATH 544 Homework 8

Problem 1 Explain why each of the functions $T : V \to W$ is not a linear transformation. **Be brief.**

(a) $T : \mathbb{R}^2 \to \mathbb{R}^2$ defined for all $\begin{pmatrix} a \\ b \end{pmatrix} \in \mathbb{R}^2$ by $T\begin{pmatrix} a \\ b \end{pmatrix} = \begin{pmatrix} e^{x_1} \\ x_2 \end{pmatrix}$.

(b) $T : \mathrm{Mat}_{2\times 2}(\mathbb{R}) \to \mathbb{R}$ defined for all $M = \begin{pmatrix} a & b \\ c & d \end{pmatrix} \in \mathrm{Mat}_{2\times 2}(\mathbb{R})$ by $T(M) = (a-d)(b-c)$.

(c) $T : \mathbb{R}_2[x] \to \mathrm{Mat}_{2\times 2}(\mathbb{R})$ defined for all $p(x) = a_2x^2 + a_1x + a_0 \in \mathbb{R}_2[x]$ by $T(p(x)) = \begin{pmatrix} a_0+1 & a_1+1 \\ a_2+1 & 0 \end{pmatrix}$.

Solution.

(a) We do not have $\vec{0_V} \in \ker(T)$ because

$$T\begin{pmatrix} 0 \\ 0 \end{pmatrix} = \begin{pmatrix} 1 \\ 0 \end{pmatrix} \neq \begin{pmatrix} 0 \\ 0 \end{pmatrix}.$$

(b) This is not linear. In particular,

$$T\left(\begin{pmatrix} 0 & 1 \\ 0 & 1 \end{pmatrix} + \begin{pmatrix} 1 & 0 \\ 1 & 0 \end{pmatrix}\right) = T\begin{pmatrix} 1 & 1 \\ 1 & 1 \end{pmatrix} = 0 \neq -2 = -1 - 1 = T\begin{pmatrix} 0 & 1 \\ 0 & 1 \end{pmatrix} + T\begin{pmatrix} 1 & 0 \\ 1 & 0 \end{pmatrix}.$$

(c) We do not have $\vec{0_V} \in \ker(T)$ because

$$T(0) = T(0x^2 + 0x + 0) = \begin{pmatrix} 0+1 & 0+1 \\ 0+1 & 0 \end{pmatrix} = \begin{pmatrix} 1 & 1 \\ 1 & 0 \end{pmatrix} \neq \begin{pmatrix} 0 & 0 \\ 0 & 0 \end{pmatrix}.$$

Problem 2 Define, for all $\vec{x} \in \mathbb{R}^3$, the function $T\begin{pmatrix} x \\ y \\ z \end{pmatrix} = \begin{pmatrix} x+2y-z \\ y+z \\ x+y-2z \end{pmatrix}$.

(a) Show that T is a linear transformation. (There are at least two ways to do this.)

(b) Compute $\mathrm{rank}(T)$, and find a basis for $\mathrm{Im}(T)$, the image of T.

(c) Compute $\mathrm{nullity}(T)$, and find a basis for $\ker(T) = \{\vec{v} \in \mathbb{R}^3 \mid T(\vec{v}) = \vec{0}\}$.

Solution.

(a) Let
$$A = \begin{pmatrix} 1 & 2 & -1 \\ 0 & 1 & 1 \\ 1 & 1 & -2 \end{pmatrix}.$$
Then, we have $T = T_A$, where $T_A : \mathbb{R}^3 \to \mathbb{R}^3$ and for all $\vec{x} \in \mathbb{R}^3$, $T(\vec{x}) = A\vec{x}$. Since we have shown T_A is a linear transformation for all matrices A, T is a linear transformation.

(b) Let $\vec{e_1}, \vec{e_2}, \vec{e_3} \in \mathbb{R}^3$ such that the i^{th} component $\vec{e_i}$ is 1 and all other components are zero. Then, $\{\vec{e_1}, \vec{e_2}, \vec{e_3}\}$ is a basis for $\mathbb{R}^3$, and so we have
$$\operatorname{Im}(T) = \operatorname{Span}\{T(\vec{e_1}), T(\vec{e_2}), T(\vec{e_3})\} = \operatorname{Span}\left\{ \begin{pmatrix} 1 \\ 0 \\ 1 \end{pmatrix}, \begin{pmatrix} 2 \\ 1 \\ 1 \end{pmatrix}, \begin{pmatrix} -1 \\ 1 \\ -2 \end{pmatrix} \right\}.$$
So we have found a spanning set of $\operatorname{Im}(T)$, and now we can remove vectors until it is linearly independent. We construct a matrix A where the columns are the vectors in the spanning set, and perform Gauss-Jordan Elimination:
$$A = \begin{pmatrix} 1 & 2 & -1 \\ 0 & 1 & 1 \\ 1 & 1 & -2 \end{pmatrix} \sim \begin{pmatrix} 1 & 0 & -3 \\ 0 & 1 & 1 \\ 0 & 0 & 0 \end{pmatrix}.$$
Since the leading ones are in columns 1 and 2 in the reduced row-echelon form, we have that
$$B = \left\{ \begin{pmatrix} 1 \\ 0 \\ 1 \end{pmatrix}, \begin{pmatrix} 2 \\ 1 \\ 1 \end{pmatrix} \right\}$$
is a basis for $\operatorname{Im}(T)$. Therefore, $\operatorname{rank}(T) = \dim(\operatorname{Im}(T)) = |B| = 2$.

(c) We want to find all vectors $\vec{x} \in \mathbb{R}^3$ such that $T(\vec{x}) = \vec{0}$. From (a), this is equivalent to the solution set of $A\vec{x} = \vec{0}$, so we would like to find $\operatorname{Null}(A)$. From $\operatorname{rref}(A)$ as we found in (b), we conclude that
$$\ker(A) = \operatorname{Null}(A) = \operatorname{Span}\left\{ \begin{pmatrix} 3 \\ -1 \\ 1 \end{pmatrix} \right\}$$
and thus $\beta = \left\{ \begin{pmatrix} 3 \\ -1 \\ 1 \end{pmatrix} \right\}$ is a basis for $\ker(A)$. Therefore, $\operatorname{nullity}(T) = \dim(\ker(T)) = |\beta| = 1$.

Problem 3

(a) Let m, $n \geq 1$, and suppose that $T : \mathbb{R}^n \to \mathbb{R}^m$ is a linear transformation. Let $\{\vec{e}_1, \ldots, \vec{e}_n\}$ be the standard basis for $\mathbb{R}^n$, and let $A = [T(\vec{e}_1) \,|\, T(\vec{e}_2) \,|\, \cdots \,|\, T(\vec{e}_n)] \in \operatorname{Mat}_{m\times n}(\mathbb{R})$, where $T(\vec{e}_i)$ is the ith column of A. Show that $T = T_A$: for all $\vec{x} \in \mathbb{R}^n$, we have $T(\vec{x}) = A\vec{x}$.

(Start by writing $\vec{x} = \begin{pmatrix} x_1 \\ \vdots \\ x_n \end{pmatrix} = x_1\vec{e}_1 + \cdots x_n\vec{e}_n$, then use linearity of T.)

(b) Suppose that $T : \mathbb{R}^3 \to \mathbb{R}^3$ is a linear transformation which satisfies

$$T\begin{pmatrix}0\\1\\0\end{pmatrix} = \begin{pmatrix}1\\0\\-1\end{pmatrix}, \quad T\begin{pmatrix}1\\-3\\2\end{pmatrix} = \begin{pmatrix}0\\2\\0\end{pmatrix}, \quad T\begin{pmatrix}2\\-4\\1\end{pmatrix} = \begin{pmatrix}0\\1\\1\end{pmatrix}.$$

Use the result of part (a) to find $A \in \mathrm{Mat}_{3\times 3}(\mathbb{R})$ such that for all $\vec{x} \in \mathbb{R}^3$, we have $T(\vec{x}) = T_A(\vec{x}) = A\vec{x}$.

(c) Suppose that we define $T : \mathbb{R}^n \to \mathbb{R}^n$ for all $\vec{x} \in \mathbb{R}^n$ by $T(\vec{x}) = 3\vec{x}$. Show that T is linear, and find $A \in \mathrm{Mat}_{n\times n}(\mathbb{R})$ such that $T = T_A$.

Note: In class, I stated the following proposition: A function $T : \mathbb{R}^n \to \mathbb{R}^m$ is a linear transformation if and only if there exists $A \in \mathrm{Mat}_{m\times n}(\mathbb{R})$ such that $T = T_A$. I proved the $\Longleftarrow$ implication in class; part (a) proves the $\Longrightarrow$ implication.

Solution.

(a) Let $\vec{x} \in \mathbb{R}^n$, and define the components of A and $\vec{x}$ with

$$A = \begin{pmatrix} a_{11} & a_{12} & \dots & a_{1n} \\ a_{21} & a_{22} & \dots & a_{2n} \\ \vdots & \vdots & \ddots & \vdots \\ a_{m1} & a_{m2} & \dots & a_{mn} \end{pmatrix}, \vec{x} = \begin{pmatrix} x_1 \\ x_2 \\ \vdots \\ x_n \end{pmatrix}.$$

Then, we have

$$\begin{aligned} T(\vec{x}) &= T(x_1\vec{e_1} + x_2\vec{e_2} + \cdots + x_n\vec{e_n}) \\ &= x_1T(\vec{e_1}) + x_2T(\vec{e_2}) + \cdots + x_nT(\vec{e_n}) && \text{(using linearity)} \\ &= x_1\begin{pmatrix} a_{11} \\ a_{21} \\ \vdots \\ a_{m1} \end{pmatrix} + x_2\begin{pmatrix} a_{12} \\ a_{22} \\ \vdots \\ a_{m2} \end{pmatrix} + \cdots + x_n\begin{pmatrix} a_{1n} \\ a_{2n} \\ \vdots \\ a_{mn} \end{pmatrix} \\ &= \begin{pmatrix} x_1a_{11} + x_2a_{12} + \cdots + x_na_{1n} \\ x_1a_{21} + x_2a_{22} + \cdots + x_na_{2n} \\ \vdots \\ x_1a_{m1} + x_2a_{m2} + \cdots + x_na_{mn} \end{pmatrix} \\ &= A\vec{x}. && \text{(definition of matrix multiplication)} \end{aligned}$$

Therefore, we have $T = T_A$, since for all $x \in \mathbb{R}^n$, we have $T(\vec{x}) = A\vec{x}$.

(b) We can solve systems of equations to see that

$$\begin{pmatrix}1\\0\\0\end{pmatrix} = \frac{5}{3}\begin{pmatrix}0\\1\\0\end{pmatrix} - \frac{1}{3}\begin{pmatrix}1\\-3\\2\end{pmatrix} + \frac{2}{3}\begin{pmatrix}2\\-4\\1\end{pmatrix}, \begin{pmatrix}0\\0\\1\end{pmatrix} = \frac{2}{3}\begin{pmatrix}0\\1\\0\end{pmatrix} + \frac{2}{3}\begin{pmatrix}1\\-3\\2\end{pmatrix} - \frac{1}{3}\begin{pmatrix}2\\-4\\1\end{pmatrix}.$$

So we have

$$T\begin{pmatrix}1\\0\\0\end{pmatrix} = T\left(\frac{5}{3}\begin{pmatrix}0\\1\\0\end{pmatrix} - \frac{1}{3}\begin{pmatrix}1\\-3\\2\end{pmatrix} + \frac{2}{3}\begin{pmatrix}2\\-4\\1\end{pmatrix}\right) = \frac{5}{3}\begin{pmatrix}1\\0\\-1\end{pmatrix} - \frac{1}{3}\begin{pmatrix}0\\2\\0\end{pmatrix} + \frac{2}{3}\begin{pmatrix}0\\1\\1\end{pmatrix} = \begin{pmatrix}\frac{5}{3}\\0\\-1\end{pmatrix},$$

$$T\begin{pmatrix}0\\1\\0\end{pmatrix} = \begin{pmatrix}1\\0\\-1\end{pmatrix},$$

$$T\begin{pmatrix}0\\0\\1\end{pmatrix} = T\left(\frac{2}{3}\begin{pmatrix}0\\1\\0\end{pmatrix} + \frac{2}{3}\begin{pmatrix}1\\-3\\2\end{pmatrix} - \frac{1}{3}\begin{pmatrix}2\\-4\\1\end{pmatrix}\right) = \frac{2}{3}\begin{pmatrix}1\\0\\-1\end{pmatrix} + \frac{2}{3}\begin{pmatrix}0\\2\\0\end{pmatrix} - \frac{1}{3}\begin{pmatrix}0\\1\\1\end{pmatrix} = \begin{pmatrix}\frac{2}{3}\\1\\-1\end{pmatrix}.$$

So we now know where our standard basis vectors need to be after the transformation, and thus we have $T(\vec{x}) = T_A(\vec{x}) = A\vec{x}$ for

$$A = \begin{pmatrix}\frac{5}{3} & 1 & \frac{2}{3}\\ 0 & 0 & 1\\ -1 & -1 & -1\end{pmatrix}.$$

(c) Let $c \in \mathbb{R}$ and $\vec{u}, \vec{v} \in \mathbb{R}^n$. Then,

$$T(c\vec{u} + \vec{v}) = 3(c\vec{u} + \vec{v}) = c(3\vec{u}) + 3\vec{v} = cT(\vec{u}) + T(\vec{v}),$$

so T is linear. Let $\{\vec{e_1}, \vec{e_2}, \dots, \vec{e_n}\}$ be the standard basis for $\mathbb{R}^n$. Then, from the procedure in part (a), we have $T = T_A$ for

$$A = \left(3\vec{e_1} \mid 3\vec{e_2} \mid \cdots \mid 3\vec{e_n}\right) = \begin{pmatrix}3 & 0 & 0 & \dots & 0\\ 0 & 3 & 0 & \dots & 0\\ 0 & 0 & 3 & \dots & 0\\ \vdots & \vdots & \vdots & \ddots & \vdots\\ 0 & 0 & 0 & \dots & 3\end{pmatrix}.$$

Problem 5 Suppose that $T : \mathbb{R}_2[x] \to \mathbb{R}_4[x]$ is a linear transformation which satisfies

$$T(2) = 2x^4, \quad T(x-3) = x^3 - 2x, \quad T(x^2 + 2x + 1) = x.$$

Note that $\{2, x-3, x^2+2x+1\}$ is a basis for $\mathbb{R}_2[x]$, a fact that you **do not have to prove**.

(a) Find $T(1)$, $T(x)$, and $T(x^2)$.

(b) Let $p(x) = a_2x^2 + a_1x + a_0 \in \mathbb{R}_2[x]$. Use part (a) to find a formula for $T(p(x)) \in \mathbb{R}_4[x]$.

Solution.

(a) We have

$$T(1) = T\left(\frac{2}{2}\right) = \frac{1}{2}(2x^4) = x^4,$$
$$T(x) = T(x-3+3) = T(x-3) + T(3) = x^3 - 2x + 3x^4 = 3x^4 + x^3 - 2x,$$

$$T(x^2) = T(x^2 + 2x + 1 - 2x - 1) = T(x^2 + 2x + 1) - 2T(x) - T(1) = -7x^4 - 2x^3 + 5x.$$

(b) We have

$$\begin{aligned}
T(p(x)) &= T(a_2x^2 + a_1x + a_0) \\
&= a_2T(x^2) + a_1T(x) + a_0T(1) && \text{(using linearity)} \\
&= a_2(-7x^4 - 2x^3 + 5x) + a_1(3x^4 + x^3 - 2x) + a_0x^4. && \text{(substituting from (a))} \\
&= (-7a_2 + 3a_1 + a_0)x^4 + (-2a_2 + a_1)x^3 + (5a_2 - 2a_1)x.
\end{aligned}$$

Problem 6 Let $M = \begin{pmatrix} 1 & 2 \\ 0 & 3 \end{pmatrix} \in \text{Mat}_{2\times 2}(\mathbb{R})$, and define the function $T : \text{Mat}_{2\times 2}(\mathbb{R}) \to \text{Mat}_{2\times 2}(\mathbb{R})$ for all $A \in \text{Mat}_{2\times 2}(\mathbb{R})$ by $T(A) = AM - MA$.

(a) Show that T is a linear transformation. (Use the definition.)

(b) Compute $\text{rank}(T)$, and find a basis for $\text{Im}(T)$, the image of T.

(c) Compute $\text{nullity}(T)$, and find a basis for

$$\ker(T) = \{A \in \text{Mat}_{2\times 2}(\mathbb{R}) \mid T(A) = O_{2\times 2}\}.$$

Solution.

(a) Let $c \in \mathbb{R}$, $A, B \in \text{Mat}_{2\times 2}(\mathbb{R})$. Then,

$$\begin{aligned}
T(cA + B) &= (cA + B)M - M(cA + B) && \text{(definition)} \\
&= cAM + BM - cMA - MB \\
&= c(AM - MA) + (BM - MB) \\
&= cT(A) + T(B), && \text{(definition)}
\end{aligned}$$

so T is linear.

(b) We have shown that

$$B = \left\{ \begin{pmatrix} 1 & 0 \\ 0 & 0 \end{pmatrix}, \begin{pmatrix} 0 & 1 \\ 0 & 0 \end{pmatrix}, \begin{pmatrix} 0 & 0 \\ 1 & 0 \end{pmatrix}, \begin{pmatrix} 0 & 0 \\ 0 & 1 \end{pmatrix} \right\}$$

is a basis for $\text{Mat}_{2\times 2}(\mathbb{R})$, so we know that

$$B' = \left\{ T\begin{pmatrix} 1 & 0 \\ 0 & 0 \end{pmatrix}, T\begin{pmatrix} 0 & 1 \\ 0 & 0 \end{pmatrix}, T\begin{pmatrix} 0 & 0 \\ 1 & 0 \end{pmatrix}, T\begin{pmatrix} 0 & 0 \\ 0 & 1 \end{pmatrix} \right\} = \left\{ \begin{pmatrix} 0 & 2 \\ 0 & 0 \end{pmatrix}, \begin{pmatrix} 0 & 2 \\ 0 & 0 \end{pmatrix}, \begin{pmatrix} -2 & 0 \\ -2 & 2 \end{pmatrix}, \begin{pmatrix} 0 & -2 \\ 0 & 0 \end{pmatrix} \right\}$$

spans $\text{Im}(T)$. We create a matrix A where the columns of A come from the components of the matrices in B': each column is one matrix, going from left to right and top to bottom in the matrix. Then, we use Gauss-Jordan elimination to conclude that

$$A = \begin{pmatrix} 0 & 0 & -2 & 0 \\ 2 & 2 & 0 & -2 \\ 0 & 0 & -2 & 0 \\ 0 & 0 & 2 & 0 \end{pmatrix} \sim \begin{pmatrix} 1 & 1 & 0 & -1 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \end{pmatrix},$$

which is in reduced row-echelon form. So the first and third columns in A have leading ones, and thus we should choose the first and third matrices that appear in the way we have written B' to be a basis. So

$$\beta = \left\{ \begin{pmatrix} 0 & 2 \\ 0 & 0 \end{pmatrix}, \begin{pmatrix} -2 & 0 \\ -2 & 2 \end{pmatrix} \right\}$$

is a basis for $\mathrm{Im}(T)$, and therefore we have $\mathrm{rank}(T) = \dim(\mathrm{Im}(T)) = |\beta| = 2$.

(c) Let

$$M = \begin{pmatrix} m_{11} & m_{12} \\ m_{21} & m_{22} \end{pmatrix} \in \mathrm{Mat}_{2\times 2}, \vec{x} = \begin{pmatrix} m_{11} \\ m_{12} \\ m_{21} \\ m_{22} \end{pmatrix}.$$

By the way we have defined A, we have that $A\vec{x} = \vec{0} \iff T(M) = O_{2\times 2}$, so it suffices to find a basis for $\mathrm{Null}(A)$. From the reduced row-echelon form we computed, we conclude that

$$B = \left\{ \begin{pmatrix} -1 \\ 1 \\ 0 \\ 0 \end{pmatrix}, \begin{pmatrix} 1 \\ 0 \\ 0 \\ 1 \end{pmatrix} \right\}$$

is a basis for $\mathrm{Null}(A) = \ker(T)$. Therefore, $\mathrm{nullity}(T) = \dim(\ker(T)) = |B| = 2$.

Problem 8 Suppose that

$$\vec{u}_1 = \begin{pmatrix} 1 \\ -1 \end{pmatrix}, \quad \vec{u}_2 = \begin{pmatrix} 2 \\ -1 \end{pmatrix}, \quad \vec{u}_3 = \begin{pmatrix} -3 \\ 2 \end{pmatrix}, \quad \vec{v}_1 = \begin{pmatrix} 1 \\ 0 \end{pmatrix}, \quad \vec{v}_2 = \begin{pmatrix} 0 \\ 1 \end{pmatrix}, \quad \vec{v}_3 = \begin{pmatrix} 1 \\ 1 \end{pmatrix}.$$

True or **False**: There exists a linear transformation $T : \mathbb{R}^2 \to \mathbb{R}^2$ such that, for $i = 1, 2, 3$, we have $T(\vec{u}_i) = \vec{v}_i$. Explain your reasoning.

Solution.

Suppose that some such T does exist. Then, we have

$$\begin{aligned} \begin{pmatrix} 1 \\ 1 \end{pmatrix} &= T\begin{pmatrix} -3 \\ 2 \end{pmatrix} && \text{(hypothesis)} \\ &= T\left(-\begin{pmatrix} 1 \\ -1 \end{pmatrix} - \begin{pmatrix} 2 \\ -1 \end{pmatrix}\right) && \text{(rewriting vector)} \\ &= -T\begin{pmatrix} 1 \\ -1 \end{pmatrix} - T\begin{pmatrix} 2 \\ -1 \end{pmatrix} && \text{(linearity)} \\ &= -\begin{pmatrix} 1 \\ 0 \end{pmatrix} - \begin{pmatrix} 0 \\ 1 \end{pmatrix} \\ &= \begin{pmatrix} -1 \\ -1 \end{pmatrix}, \end{aligned}$$

a contradiction. So the statement is false.