Nathan Bickel
Analysis in $\mathbb{R}^n$: Section 1
Professor: Shabani
August 1, 2023

Analysis in $\mathbb{R}^n$ Homework 7

Problem 1 Let $R \subset \mathbb{R}^n$ be a rectangle and $f : R \to \mathbb{R}$ continuous. Prove that f is integrable on R.

Solution.

Since R is a rectangle, it is closed and bounded in $\mathbb{R}^n$ and thus compact. Thus, since f is continuous, we have shown in class that it is also uniformly continuous. So there exists a $\delta > 0$ such that for all $\varepsilon > 0$ with the standard metric in $\mathbb{R}^n$,

$$d(x, y) < \delta \implies |f(x) - f(y)| < \frac{\varepsilon}{2\operatorname{Vol}(R)}.$$

Let P_k be a uniform partition of R, with each dimension subdivided into k intervals, and $\mathcal{P}_k$ be the sub-rectangles of P_k. Then, we have $|\mathcal{P}_k| = k^n$, and thus $\operatorname{Vol}(S) = \frac{\operatorname{Vol}(R)}{k^n}$ for all $S \in \mathcal{P}_k$. Additionally, $|f(x) - f(y)| < \frac{\varepsilon}{2\operatorname{Vol}(R)}$ for all $x, y \in S$, so we have $M_S(f) - m_S(f) \leq \frac{\varepsilon}{2\operatorname{Vol}(R)} < \frac{\varepsilon}{\operatorname{Vol}(R)}$.

Choose a $k \in \mathbb{N}$ such that any two points x, y in a sub-rectangle of P_k satisfy $d(x, y) < \delta$ (clearly, this can always be achieved for a sufficiently high k). Let $\varepsilon > 0$. Then,

$$\begin{aligned}
U(f, P) - L(f, P) &= \sum_{S \in \mathcal{P}_k} M_S(f)\operatorname{Vol}(S) - \sum_{S \in \mathcal{P}_k} m_S(f)\operatorname{Vol}(S) \\
&= \sum_{S \in \mathcal{P}_k} (M_S(f) - m_S(f))\operatorname{Vol}(S) \\
&< \sum_{S \in \mathcal{P}_k} \frac{\varepsilon}{\operatorname{Vol}(R)} \frac{\operatorname{Vol}(R)}{k^n} && \text{(justified above)} \\
&= \frac{\varepsilon}{k^n} \sum_{S \in \mathcal{P}_k} 1 && \text{(pulling out constants)} \\
&= \frac{\varepsilon}{k^n} k^n = \varepsilon. && (k^n \text{ subrectangles})
\end{aligned}$$

Therefore, f is integrable on R by the theorem we proved in class. $\square$

Problem 2 Let $R \subset \mathbb{R}^n$ be a rectangle.

(a) Prove that any rectangle $S \subset R$ is measurable.

(b) Prove that any finite subset of R is measurable.

(c) Suppose that $A, B \subset R$ are measurable. Prove that $A \cap B$ and $A \cup B$ are measurable too, and that

$$\operatorname{Vol}(A \cup B) = \operatorname{Vol}(A) + \operatorname{Vol}(B) - \operatorname{Vol}(A \cap B).$$

(d) Deduce that if $A_1, \ldots, A_m \subset R$ are measurable, then $\bigcup_{j=1}^{m} A_j$ and $\bigcap_{j=1}^{m} A_j$ are measurable.

Solution.

(a) Let

$$[a'_1, b'_1] \times [a'_2, b'_2] \times \cdots \times [a'_n, b'_n] = S \subset R = [a_1, b_1] \times [a_2, b_2] \times \cdots \times [a_n, b_n]$$

be a rectangle. Then,

$$P = \{a_1, a'_1, b'_1, b_1\} \times \{a_2, a'_2, b'_2, b_2\} \times \cdots \times \{a_n, a'_n, b'_n, b_n\}$$

is a partition of R where $S \in \mathcal{P}$, and there is a rectangle on either side of S in each dimension with other rectangles filling in the rest of R. We can then write

$$\begin{aligned} U(\mathbb{1}_S, P) - L(\mathbb{1}_S, P) &= \sum_{T \in \mathcal{P}} M_T \operatorname{Vol}(T) - \sum_{T \in \mathcal{P}} m_T \operatorname{Vol}(T) \\ &= M_S \operatorname{Vol}(S) - m_s \operatorname{Vol}(S) && (\mathbb{1}_S \text{ is 0 outside } S) \\ &= 1 \operatorname{Vol}(S) - 1 \operatorname{Vol}(S) = 0. && (\mathbb{1}_S \text{ is constant across } S) \end{aligned}$$

So $U(\mathbb{1}_S, P) - L(\mathbb{1}_S, P) < \varepsilon$ for all $\varepsilon > 0$, and thus $\mathbb{1}_S$ is integrable on R. Therefore, S is measurable.

(b) Let $S \subset R$ be finite. Let P_k be a uniform partition of R, with each dimension subdivided into k intervals, and $\mathcal{P}_k$ be the sub-rectangles of P_k. Then, we have $|\mathcal{P}_k| = k^n$, and thus $\operatorname{Vol}(S) = \frac{\operatorname{Vol}(R)}{k^n}$ for all $S \in \mathcal{P}_k$. Let $\varepsilon > 0$, let k be such that $k^n \geq \frac{|S| \operatorname{Vol}(R)}{\varepsilon}$, and let $\mathcal{S} \subset \mathcal{P}$ be the subrectangles in $\mathcal{P}$ that contains points in S (so $|\mathcal{S}| \leq |S|$). Then, we have

$$\begin{aligned} U(\mathbb{1}_S, P) - L(\mathbb{1}_S, P) &= \sum_{T \in \mathcal{P}} M_T \operatorname{Vol}(T) - \sum_{T \in \mathcal{P}} m_T \operatorname{Vol}(T) \\ &= \sum_{T \in \mathcal{S}} M_T \operatorname{Vol}(T) - \sum_{T \in \mathcal{S}} m_T \operatorname{Vol}(T) && (\mathbb{1}_S \text{ is constant outside } \mathcal{S}) \\ &= \sum_{T \in \mathcal{S}} \operatorname{Vol}(T) && (M_T = 1 \text{ and } m_T = 0) \\ &= \sum_{T \in \mathcal{S}} \frac{\operatorname{Vol}(R)}{k^n} \\ &\leq |S| \frac{\operatorname{Vol}(R)}{k^n} && (|\mathcal{S}| < |S|) \\ &\leq \frac{|S| \operatorname{Vol}(R)}{\frac{|S| \operatorname{Vol}(R)}{\varepsilon}} = \varepsilon. && (\text{by choice of } k) \end{aligned}$$

So for all $\varepsilon > 0$, there exists a partition P such that $U(\mathbb{1}_S, P) - L(\mathbb{1}_S, P) < \varepsilon$.

(c) We first consider $A \cap B$. We can write $\mathbb{1}_{A \cap B} = \mathbb{1}_A \mathbb{1}_B$, because this product will be 1 if and only if $\mathbb{1}_A(x) = \mathbb{1}_B(x) = 1$ for all $x \in R$. This means x is in both A and B, and thus is in $A \cap B$, which is how $\mathbb{1}_{A \cap B}$ is defined. Since $\mathbb{1}_A$ and $\mathbb{1}_B$ are both integrable in R, we know that $\mathbb{1}_{A \cap B} = \mathbb{1}_A \mathbb{1}_B$ is also integrable, and therefore $A \cap B$ is measurable.

Now, we consider $A \cup B$. We can write $\mathbb{1}_{A \cup B} = \mathbb{1}_A + \mathbb{1}_B - \mathbb{1}_{A \cap B}$, because $\mathbb{1}_{A \cup B} = \mathbb{1}_A(x) + \mathbb{1}_B(x)$ will be 1 for all $x \in A \cup B$ except for $A \cap B$, where it will be 2. Thus, we can subtract $\mathbb{1}_{A \cap B}$ to make the function 1 everywhere on $A \cup B$. Then, since this is the sum of integrable functions, it is integrable. Moreover, by the properties we proved in class, we have

$$\operatorname{Vol}(A \cup B) = \int_R (\mathbb{1}_A + \mathbb{1}_B - \mathbb{1}_{A \cap B}) = \int_R \mathbb{1}_A + \int_R \mathbb{1}_B - \int_R \mathbb{1}_{A \cap B} = \operatorname{Vol}(A) + \operatorname{Vol}(B) - \operatorname{Vol}(A \cap B).$$

(d) We will induct on m. First, let $m = 1$. Then, we are given that $A_1 = \bigcap_{j=1}^{1} A_j = \bigcup_{j=1}^{1} A_j$ is measurable.

Next, let $m \in \mathbb{N}$, and assume that $\bigcap_{j=1}^{m-1} A_j$ and $\bigcup_{j=1}^{m-1} A_j$ are measurable. Then, $\bigcap_{j=1}^{m} A_j = \bigcap_{j=1}^{m-1} A_j \cap A_m$

is measurable by **(c)**, and similarly so is $\bigcup_{j=1}^{m} A_j = \bigcup_{j=1}^{m-1} A_j \cup A_m$. Thus, the statement is true for any $m \in \mathbb{N}$. $\square$

Problem 3 Let $R \subset \mathbb{R}^n$ be a rectangle and $A \subset R$ a measurable set. Prove that $\mathrm{Vol}(A) = 0$ if and only if for every $\varepsilon > 0$ there exist finitely many rectangles $R_1, \dots, R_m \subset R$ such that $A \subset \bigcup_{j=1}^{m} R_j$ and $\sum_{j=1}^{m} \mathrm{Vol}(R_j) < \varepsilon$.

Solution.

($\Longrightarrow$) Suppose $\mathrm{Vol}(A) = \int_R \mathbb{1}_A = 0$. Then, we have shown in class that for all $\varepsilon > 0$, there exists a partition P of R such that $U(\mathbb{1}_A, P) < \varepsilon$. Since $\mathbb{1}_A \leq 1$ by definition, each rectangle $S \in \mathcal{P}$ that contains a point in A will have $M_S(\mathbb{1}_A) = 1$ and all other rectangles $S \in \mathcal{P}$ will have $M_S(\mathbb{1}_A) = 0$. So we have $\sum_{S \in \mathcal{P}} \mathrm{Vol}(S) < \varepsilon$, and since the partition is finite, there exist finitely many rectangles $R_1, \dots, R_m \subset R$ such that $A \subset \bigcup_{j=1}^{m} R_j$ and $\sum_{j=1}^{m} \mathrm{Vol}(R_j) < \varepsilon$.

($\Longleftarrow$) Let $\varepsilon > 0$ and suppose there exist finitely many rectangles $R_1, \dots, R_m \subset R$ such that

$$A \subset \bigcup_{j=1}^{m} R_j \text{ and } \sum_{j=1}^{m} \mathrm{Vol}(R_j) < \varepsilon.$$

Since A is covered by the rectangles, we have

$$\mathrm{Vol}(A) \leq \mathrm{Vol}\left(\bigcup_{j=1}^{m} R_j \right) \leq \sum_{j=1}^{m} \mathrm{Vol}(R_j) < \varepsilon.$$

Since this is true for every positive ε, we must have $\mathrm{Vol}(A) = 0$. $\square$

Problem 4 Let $R \subset \mathbb{R}^n$ be a rectangle, $A \subset R$ a measurable set with $\mathrm{Vol}(A) = 0$, and $f : R \to \mathbb{R}$ a bounded function that is integrable in R. Prove that

$$\int_A f = 0.$$

Solution.

Assume that $\mathrm{Vol}(A) = \int_R \mathbb{1}_A = 0$. Then, from the theorem in class there exists a partition P such that for all $\varepsilon > 0$, $U(\mathbb{1}_A, P) < \frac{\varepsilon}{M}$ and $-L(\mathbb{1}_A, P) < \frac{\varepsilon}{m}$, where M is the supremum of f over R and m is the infinum. Then, we have

$$\begin{aligned}
&\frac{\varepsilon}{M} > U(\mathbb{1}_A, P) \\
\Longrightarrow\ &\varepsilon > MU(\mathbb{1}_A, P) \\
&= U(M\mathbb{1}_A, P) && \text{(sum property)} \\
&\geq U(f\mathbb{1}_A, P) && (M \geq f \text{ by definition})
\end{aligned}$$

and similarly

$$\begin{aligned} \frac{\varepsilon}{m} &> -L(\mathbb{1}_A, P) \\ \implies \varepsilon &> -mL(\mathbb{1}_A, P) \\ &= -L(m\mathbb{1}_A, P) \\ &\geq -L(f\mathbb{1}_A, P). && (m \leq f \implies -m \geq -f) \end{aligned}$$

We have shown in class that this implies that $\int_A f = 0$. □