MATH 552: Section 001
Professor: Dr. Miller
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MATH 552 Homework 9ˆ

Problem 47.2+ Let C denote the line segment from $z = i$ to $z = 1$, and show that

$$\left| \int_C \frac{dz}{z^4} \right| \leq 4\sqrt{2}$$

without evaluating the integral.

Solution.

By the theorem, $\left|\int_C \frac{dz}{z^4}\right| \leq ML$ where M is the maximum taken on by $\left|\frac{1}{z^4}\right|$ on the curve and L is the length of the curve.

We write $z = re^{i\theta}$, so $z^{-4} = r^{-4}e^{-4i\theta}$. Then, $|z^{-4}|$ is r^{-4}.

On the line segment, r takes on a maximum of 1 at i and 1, and a minimum of $\frac{1}{\sqrt{2}}$ when it is halfway along the curve. Since we are taking the reciprocal, $|z^{-4}|$ takes its maximum when r takes its minimum. Thus, the maximum of $|z^{-4}|$ is $\left(\frac{1}{\sqrt{2}}\right)^{-4} = 4$.

The length of the curve is $|i - 1| = \sqrt{2}$. Thus, $M = 4$ and $L = \sqrt{2}$, so

$$\left| \int_C \frac{dz}{z^4} \right| \leq ML = 4\sqrt{2}.$$

Problem 47.5+ Let C_R be the circle $|z| = R$ ($R > 1$), described in the counterclockwise direction. Show that

$$\left| \int_{C_R} \frac{\operatorname{Log} z}{z^2}\ dz \right| < 2\pi\left(\frac{\pi + \ln R}{R}\right),$$

and then use l'Hospital's rule to show that the value of this integral tends to zero as R tends to infinity.

Solution.

$$\begin{aligned}
\left|\int_C \frac{\operatorname{Log} z}{z^2}dz\right| &= \left|\int_{-\pi}^{\pi} \frac{\operatorname{Log}(Re^{i\theta})}{R^2e^{2i\theta}}Rie^{i\theta}d\theta\right| && \text{(parameterizing } C\text{)}\\
&= \left|\int_{-\pi}^{\pi} \frac{\operatorname{Log}(Re^{i\theta})}{Re^{i\theta}}i\ d\theta\right| \\
&\leq \int_{-\pi}^{\pi} \left|\frac{\operatorname{Log}(Re^{i\theta})}{Re^{i\theta}}\right| |i|\ d\theta && \text{(using lemma in 4.47)}\\
&= \int_{-\pi}^{\pi} \left|\frac{\ln R + i\theta}{Re^{i\theta}}\right| d\theta && (|i| = 1)\\
&= \frac{1}{R}\int_{-\pi}^{\pi} \frac{|\ln R + i\theta|}{|e^{i\theta}|}d\theta && \text{(rearranging)}\\
&= \frac{1}{R}\int_{-\pi}^{\pi} |\ln R + i\theta| d\theta && (|e^{i\theta}| = 1 \text{ by Euler's formula})\\
&\leq \frac{1}{R}\int_{-\pi}^{\pi} [|\ln R| + |i\theta|] d\theta && \text{(triangle inequality)}\\
&< \frac{1}{R}\int_{-\pi}^{\pi} (\ln R + \pi) d\theta && (|i\theta| \text{ never exceeds } \pi \text{ but is sometimes less})\\
&= \frac{\pi + \ln R}{R}\int_{-\pi}^{\pi} d\theta && \text{(pulling out constant)}\\
&= \frac{\pi + \ln R}{R}[\theta]_{-\pi}^{\pi}\\
&= 2\pi\left(\frac{\pi + \ln R}{R}\right)
\end{aligned}$$

The inequality is shown. Since

$$\lim_{R\to\infty} 2\pi(\pi + \ln R) = \infty \text{ and } \lim_{R\to\infty} R = \infty,$$

by L'Hôpital's rule:

$$\begin{aligned}
\lim_{R\to\infty} \frac{2\pi(\pi + \ln R)}{R} &= \lim_{R\to\infty} \frac{\frac{d}{dR}[2\pi(\pi + \ln R)]}{\frac{d}{dR}[R]}\\
&= \lim_{R\to\infty} \frac{2\pi/R}{1}\\
&= \lim_{R\to\infty} \frac{2\pi}{R} = 0.
\end{aligned}$$

Problem F

(a) Observe that $\left|\frac{1}{z^2-1}\right|$ takes on its maximum value on C at $z = \pm 3$. This is because z is being squared, so the closer z is to an axis on the circle, the less the denominator is. Additionally, since it is being subtracted by 1, the intersections of the curve and real axis will minimize the denominator with $(\pm 3)^2 - 1 = 8$.

The maximum value the function takes on is thus $\frac{1}{8}$, and the length of C is 6π (since the diameter is 6). Therefore, because of the theorem that states $\left|\int_C f(z)dz\right| \leq ML$,

$$\left|\int_C \frac{dz}{z^2 - 1}\right| \leq \frac{3\pi}{4}.$$

(b) This does not necessarily imply that the integral is 0, because the interior of C is not analytic everywhere as the function is not even defined at $z = \pm i$. We can only use the Cauchy-Goursat theorem if the entire interior of the curve is analytic.

Problem H

The function is analytic everywhere except $z = 3i$. Since this is outside C, every point inside and on C is analytic and therefore the Cauchy-Goursat theorem applies. As C is a closed curve, the contour integral evalutes to 0.