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MATH 574: Section H01
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MATH 574 Homework 4

Collaboration: I discussed some of the problems with Jackson Ginn, Sam Maloney, Miriam Rozin, and Emma Devine.

Problem 1 Give a double counting proof of the following: For all integers $n \geq 1$,

$$\sum_{k=1}^{n} k\binom{n}{k} = n2^{n-1}.$$

Solution.

We have $n2^{n-1}$, which can represent the number of ways to choose, from a group of n people, a committee with a leader and size ranging from 1 to n. This is because there are n choices for the leader, leaving $n-1$ people that can serve alongside the leader. As we have shown, then, there are 2^{n-1} possible subsets of these remaining people, so by the product rule there are $n2^{n-1}$ possible such committees.

Since the size of these committees can range from 1 to n, we can also consider each size k individually. There are $\binom{n}{k}$ ways to choose a committee of k people from the group of n, and then in that group k choices for who the leader will be. So by the product rule, there are $k\binom{n}{k}$ ways to choose a committee of size k, and thus the total number of committees is

$$\sum_{k=1}^{n} k\binom{n}{k}.$$

Since these two expressions count the same situation, they must be equal.

Problem 2 A 7-digit phone number (with digits in 0-9) is selected randomly. What is the probability that the phone number has strictly increasing digits?

Solution.

For a phone number to have strictly increasing digits, each digit must be included at most once and there is only one way to order them such that they are in increasing order. Thus, we can define a phone number using a bit string $a_0a_1 \ldots a_9$ where $\{i : a_i = 1\}$ is ordered in increasing order (for example, the bit string 1011011011 would map to the phone number 0235689). Since we are interested in phone numbers of length 7, there are $\binom{10}{7}$ possible such bit string/phone number pairs. Since by the product rule there are 10^7 possible phone numbers, the probability of a random phone number having strictly increasing digits is

$$\frac{\binom{10}{7}}{10^7} = \frac{120}{10^7} = \frac{3}{250,000}.$$

Problem 3 Find the probability that a randomly generated bit string of length 10 does not contain a 0 if bits are independent and if
(a) a 0 bit and a 1 bit are equally likely.

(b) the probability that a bit is a 1 is 0.6.
(c) the probability that the ith bit is a 1 is $(\frac{1}{2})^i$ for $i = 1, 2, 3, \ldots, 10$.

Solution.

Since the bits are independent, we can multiply the probability of success (getting a 1, since this is the only way to not get a 0) for each bit.

(a) The probability of each bit being 1 is $\frac{1}{2}$, so the probability of getting no zeroes in the bit string is

$$\left(\frac{1}{2}\right)^{10} = \frac{1}{1024}.$$

(b) Similarly, since the probability of a 1 has changed to $\frac{3}{5}$, the probability of getting all 1s is

$$\left(\frac{3}{5}\right)^{10} = \frac{59,049}{9,765,625}.$$

(c) Similarly, we multiply the probability of getting a 1 for each bit. So the probability of getting all 1s is

$$\left(\frac{1}{2}\right)^1 \left(\frac{1}{2}\right)^2 \cdots \left(\frac{1}{2}\right)^{10} = \left(\frac{1}{2}\right)^{1+2+\ldots+10} = \frac{1}{2^{55}}.$$

Problem 4 A biased coin has probability for heads 3/4 and tails 1/4. A player flips the coin until a total of 3 tails show up. For $n \geq 1$, what is the probability that the player stops after n flips?

Solution.

Since the last flip must be the third heads, there are $\binom{n-1}{3-1}$ ways to arrange the flips. Thus, the probability the player stops after n flips is

$$\binom{n-1}{2}\left(\frac{3}{4}\right)^{n-3}\left(\frac{1}{4}\right)^3.$$

Problem 5 Find each of the following probabilities when n independent Bernoulli trials are carried out with probability of success p.
(a) the probability of no failures
(b) the probability of at least one failure
(c) the probability of at most one failure
(d) the probability of at least two failures

Solution.

We use a binomial distribution for each part, which takes the form

$$\binom{n}{k} p^n (1-p)^{n-k}$$

for the probability of getting k successes from n Bernoulli trials with a probability p.

(a) This is the same as the probability of n successes:

$$\binom{n}{n} p^n (1-p)^{n-n} = p^n.$$

(b) There will either be no failures or at least 1 failure, so this is the complemement of the probability from (a):

$$1 - p^n.$$

(c) This is the same as the probability of n or $n-1$ successes:

$$\binom{n}{n} p^n (1-p)^{n-n} = p^n + \binom{n}{n-1} p^{n-1}(1-p)^{n-(n-1)} = p^n + np^{n-1}(1-p).$$

(d) There will either be at most one failure or at least two failures, so this is the complement of the probability from (c):

$$1 - p^n - np^{n-1}(1-p).$$

Problem 6 In the Erdős–Renyi random graph model, a random graph $G(n,p)$ is generated as folllows: the vertex set of $G(n,p)$ is the set of n vertices $\{1, 2, \ldots, n\}$. For each pair of distinct vertices i and j, we add the edge ij to the graph with probability p (independently of the other edges).
(a) If $p > 0$, how large is the sample space of this experiment? That is, how many possible graphs $G(n,p)$ can be generated from this process?
(b) For a nonnegative integer k, what is the probability that the resulting graph $G(n,p)$ contains exactly k edges?
(c) For a nonnegative integer k, what is the probability that a given vertex has exactly k neighbors?

Solution.

(a) We can represent the vertices an adjacency matrix M where $M_{i,j} = 1$ represents a connection between vertex i and j. Since vertices cannot connect to themselves and our graph is not directed, this will be a symmetric matrix with 0 diagonals. So there are $1 + 2 + \ldots + n - 1 = \frac{n(n-1)}{2}$ possible edges, and since an edge can either exist or not, the sample space has size

$$2^{\frac{n(n-1)}{2}}.$$

(b) Since there are $\frac{n(n-1)}{2}$ possible edges, by the binomial distribution the probability of k edges is

$$\binom{\frac{n(n-1)}{2}}{k} p^k (1-p)^{\frac{n(n-1)}{2} - k}.$$

(c) Since each vertex has $n-1$ possible neighbors, and each is a neighbor with probability p, the probability of k neighbors is

$$\binom{n-1}{k} p^k (1-p)^{n-1-k}.$$

Problem 7 In a lottery, a set of 6 numbers are chosen randomly from $\{1, \ldots, 50\}$ (without repeats, and the order of the numbers do not matter). A person must pay \$1 to play the lottery. They select 6 numbers and win if and only if they get all 6 numbers correct.
(a) Suppose the prize for winning is 1 million dollars. What is the expected net profit for a player?
(b) Suppose that the prize of the lottery increases by 1 million dollars at the end of every week if there is no winner. A player refuses to enter if their expected net profit is negative. How long must the player wait before they can play?

Solution.

(a) The probability of winning is $1/\binom{50}{6}$, since there is only one winning group out of all possible. So the expected value is

$$(0-1)\left(1-\frac{1}{\binom{50}{6}}\right)+(1,000,000-1)\left(\frac{1}{\binom{50}{6}}\right)=-\frac{148907}{158907}\approx -\$0.94.$$

(b) We can multiply $1,000,000$ by i to get the payout for Week i. Then, we solve

$$(0-1)\left(1-\frac{1}{\binom{50}{6}}\right)+(1,000,000i-1)\left(\frac{1}{\binom{50}{6}}\right)=0$$

and find $i=\frac{158907}{10000}\approx 15.9$, so the player will have to wait until the 16^{th} week before they can play.

Problem 8 You play a game of Plinko where a ball is placed at the top node and falls with gravity along the lines drawn. Each time the ball hits a peg (denoted by the nodes), the ball will either fall to the left or right with probability 1/2 each. In the end, the player wins a prize depending on which bin the ball lands in.

(a) How many total paths are there from the top of the Plinko board to the bins?

(b) For $i=1,2,3,4,5,6$, determine the probability that the ball lands in Bin i.

(c) Suppose that the prize for landing in either Bin 1 or 6 is \$5 and the prize for landing in either Bin 2 or 5 is \$1. However if the ball lands in either Bin 3 or 4 then the player must pay \$1. What is the expected payout of the Plinko game?

(a) There are 5 levels the ball must fall down. At each level, it has 2 choices (left or right) of how to fall to the next level, so there are $2^5=32$ possible paths.

(b) To get to box i, we must make $i-1$ right turns and $6-i$ left turns. We can model this as a bit string of length 5, and thus for each box i there are $\binom{5}{i-1}$ ways to place the right turns. For example, in box 1, we must take $1-1=0$ right turns and $6-1=5$ left turns, and so there is $\binom{5}{1-1}=1$ possible path.

Since there are 32 possible paths, the probabilities that the ball lands in Bins 1, 2, 3, 4, 5, and 6 are $\frac{1}{32}$, $\frac{5}{32}$, $\frac{5}{16}$, $\frac{5}{16}$, $\frac{5}{32}$, and $\frac{1}{32}$ respectively.

(c) Using the expected value formula, we have

$$\left(\frac{1}{32}\right)(5)+\left(\frac{5}{32}\right)(1)+\left(\frac{5}{16}\right)(-1)+\left(\frac{5}{16}\right)(-1)+\left(\frac{5}{32}\right)(1)+\left(\frac{1}{32}\right)(5)=0.$$

So the expected payout is neither positive nor negative.