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MATH 300: Section H01
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MATH 300 Homework 8

Problem 1

(a) No: There are 2 elements of $\mathbb{R}$ assigned to each element of $\mathbb{Z}$, so there is no unique element for each element in $\mathbb{Z}$.

(b) Yes: Squaring a number and adding 1 always yields a positive number, so it is always defined on $\mathbb{R}$ to take the square root and it yields a unique number. Thus, every element x of $\mathbb{Z}$ is assigned to $\sqrt{x^2+1}$.

(c) No: 2 is an element of $\mathbb{Z}$ but the function does not assign it to an element of $\mathbb{R}$, because $\frac{1}{2^2-4}$ is not defined.

(d) No: All elements in $\mathbb{Z}$ greater than 5 do not have a value in $\mathbb{R}$ assigned by the function, because this will result in taking the square root of a negative number, an operation not defined on $\mathbb{R}$.

Problem 2

(a) One-To-One, Onto: It is one-to-one because every element in the codomain is assigned to exactly one element in the domain. It is onto because the codomain is equal to the range.

(b) Not One-To-One, Not Onto: It is not one-to-one because $f(a) = f(b) = b$ but $a \neq b$. It is not onto because a is in the range but not the codomain.

(c) Not One-To-One, Not Onto: It is not one-to-one because $f(a) = f(d) = d$ but $a \neq b$. It is not onto because a is in the range but not the codomain.

Problem 3

(a) This is a bijection because each element x in $\mathbb{R}$ is assigned to $-3x+4$, which will be a unique value for all $\mathbb{R}$ and outputs every value in $\mathbb{R}$ at some point (for $c \in \mathbb{R}, f(-\frac{1}{3}c+\frac{4}{3}) = c$). The inverse is $f^{-1}(x) = -\frac{1}{3}x+\frac{4}{3}$.

(b) This is not a bijection, because for all x in the domain, $f(x) = f(-x)$ and thus $f(x)$ is not injective. The inverse is not a function from $\mathbb{R}$ to $\mathbb{R}$ because almost every point in its domain is assigned to either no value or two values rather than one.

(c) This is not a bijection from $\mathbb{R}$ to $\mathbb{R}$ because it is not even a function from $\mathbb{R}$ to $\mathbb{R}$ (it is not defined at $x = -2$, as this would result in divison by 0). The inverse is also not defined from $\mathbb{R}$ to $\mathbb{R}$, because it is not defined at $x = -1$, which would also result in division by 0.

(d) This is a bijection because raising a real to the 5^{th} power results in a unique real, and adding 1 also results in a unique real. Additionally, every real number is output by some element in $\mathbb{R}$ (for $c \in \mathbb{R}$, $f(\sqrt[5]{c-1}) = c$) because 5 is an odd power. The inverse is $f^{-1}(x) = \sqrt[5]{x-1}$.

Problem 4

(a) Yes: At no point do two distinct inputs in $\mathbb{Z}$ produce the same output in $\mathbb{Z}$.

(b) No: For every $n \in \mathbb{Z}$, $f(n) = f(-n)$.

(c) Yes: At no point do two distinct inputs in $\mathbb{Z}$ produce the same output in $\mathbb{Z}$.

(d) No: Every odd $n \in \mathbb{Z}$ will output the same number as the even number 1 greater than it.

Problem 5

(a) Yes: Any $c \in \mathbb{Z}$ can be output by choosing $m = n = c$.

(b) No: There are no m, n such that $(m+n)(m-n) = 2$. The only factors of 2 are 1 and 2, and $(m+n)-(m-n) = 2n$ cannot be odd like $2-1=1$ is.

(c) Yes: Any $c \in \mathbb{Z}$ can be output by choosing $m = c-1$ and $n = 0$.

(d) Yes: Any $c \in \mathbb{Z}$ can be output by choosing $m = 2c$ and $n = c$ if c is non-negative, and $m = c$ and $n = 2c$ if c is negative.

(e) No: Any $c \in \mathbb{Z}$ that is less than -4 cannot be output, because the absolute minimum of $m^2 - 4$ is -4.

Problem 6

(a) $f(n) = (n+1)!$

Assume $c_1, c_2 \in \mathbb{N}$ and $f(c_1) = f(c_2)$. Then, $(c_1+1)! = (c_2+1)!$. The only way to get the same output from a factorial with two inputs is 0! and 1!, and since $c_1, c_2 \in \mathbb{N}$, $c_1 + 1 > 0$ and $c_2 + 1 > 0$. So $c_1 = c_2$, and the function is one-to-one. The function is not onto because many natural numbers, such as 0, are never output (there is no way to multiply non-zero integers, which is what a factorial does, to get 0).

(b) $f(n) = \lfloor \sqrt{n} \rfloor$

The function is not one-to-one because $f(1) = f(2) = 1$ and $1 \neq 2$. The function is surjective because for any $c \in \mathbb{N}$, $f(c^2) = c$.

(c) $f(n) = \begin{cases} n+1 \text{ if } 2|n \\ n-1 \text{ if } 2 \not| n \end{cases}$

Assume $c_1, c_2 \in \mathbb{N}$ and $f(c_1) = f(c_2)$. Since an even output implies an odd input and an odd output implies an even output, c_1 and c_2 must have the same parity. Assume c_1 and c_2 are even. Then $c_1 + 1 = c_2 + 1$, and $c_1 = c_2$. Then assume c_1 and c_2 are not even. Then $c_1 - 1 = c_2 - 1$, and $c_1 = c_2$. Therefore, $f(n)$ is one-to-one. Now, assume $c \in \mathbb{N}$. First, assume it is even. Then, $f(c+1) = c$. Next, assume it is odd. Then, $f(c-1) = c$. So every $c \in \mathbb{N}$ is in the co-domain, and it is onto.

(d) $f(n) = 1$

The function is not one-to-one because $f(1) = f(2) = 1$ but $1 \neq 2$. The function is not onto because there is no c for which $f(c) = 2$ (or any other constant but 1), so it does not cover the whole domain.

Problem 7

(a) $\{1, 4, 9, 16\}$

(b) $\{1, 4, 9\}$

(c) $\{1\}$

(d) $\{1, 4, 9, 16\}$

(e) $\{1, 4, 9\}$

(f) $\{1, 4, 9, 16\}$

(g) $\{y \in \mathbb{R} : 0 \leq y \leq 16\}$

(h) $\{\pm 1, \pm\sqrt{2}, \pm 2\}$

(i) $\{\pm 1, \pm\sqrt{2}, \pm 2, \pm 4\}$

(j) $\{\pm 1, \pm\sqrt{2}, \pm 2\}$

(k) $\{\pm 1, \pm\sqrt{2}, \pm 2, \pm 4\}$

(l) $\{\pm 1, \pm\sqrt{2}, \pm 2\}$

(m) $\{\pm 1, \pm\sqrt{2}, \pm 2, \pm 4\}$

(n) $\{x \in \mathbb{R} : -2 \leq x \leq 2\}$

Problem 8

(a)

$$\begin{aligned}
f(S \cup T) &= \{f(x) : x \in (S \cup T)\} \\
&= \{f(x) : x \in \{r : r \in S \vee r \in T\}\} \\
&= \{f(x) : x \in S \vee x \in T\} \\
&= \{f(x) : x \in S\} \cup \{f(x) : x \in T\} \\
&= f(S) \cup f(T)
\end{aligned}$$

$\square$

(b) Assume $x \in f(S \cap T)$. By definition, $f(S \cap T) = \{f(x) : x \in S \wedge x \in T\}$. So x must be in both S and T, and thus in both $f(S)$ and $f(T)$. This is the definition of intersection, so $f(x)$ is in $f(S) \cap f(T)$ and $f(S \cap T) \subseteq f(S) \cap f(T)$. $\square$

(c)

$$\begin{aligned}
f^{-1}(C \cup D) &= \{a : f(a) \in C \cup D\} \\
&= \{a : f(a) \in \{b : b \in C \vee b \in D\}\} \\
&= \{a : f(a) \in C \vee f(a) \in D\} \\
&= \{a : f(a) \in C\} \cup \{a : f(a) \in D\} \\
&= f^{-1}(C) \cup f^{-1}(D)
\end{aligned}$$

$\square$