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MATH 555: Section H01
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January 27, 2024

MATH 555 Homework 3

Problem 2.30 Show that the function f on $\mathbb{R}$ defined by

$$f(x) = \begin{cases} x^2, & x \geq 0 \\ -x^2, & x < 0 \end{cases}$$

is differentiable on $\mathbb{R}$ but not twice differentiable.

We claim that $f'(x) = 2|x|$. By the power rule, $f'(x) = 2x$ for $x > 0$, and $f'(x) = -2x$ for $x < 0$. For $x = 0$, we can write

$$f'(0) = \lim_{x \to 0} \frac{f(x) - f(0)}{x - 0}.$$

We compute both one-sided limits:

$$\lim_{x \to 0^+} \frac{x^2 - 0^2}{x - 0} = \lim_{x \to 0^+} \frac{x^2}{x} = \lim_{x \to 0^+} x = 0, \quad \lim_{x \to 0^-} \frac{-x^2 + 0^2}{x - 0} = \lim_{x \to 0^-} \frac{-x^2}{x} = \lim_{x \to 0^-} (-x) = 0.$$

Since the limits agree, we have $f'(0) = 0 = 2|0|$. So by definition of absolute value, we have $f'(x) = 2|x|$.

However, $f' = 2|x|$ is not differentiable, as we have shown before that $|x|$ is not differentiable at $x = 0$ (the one-sided limits do not agree). So f is differentiable on $\mathbb{R}$ but not twice differentiable. $\square$

Problem 2.33 Let f be a function that is four times differentiable on an open interval I and let $a \in I$. Let $T(x)$ be the polynomial

$$T(x) = f(a) + f'(a)(x-a) + \frac{f''(a)}{2}(x-a)^2 + \frac{f^{(3)}(a)}{3!}(x-a)^3 + \frac{f^{(4)}(a)}{4!}(x-a)^4,$$

and set

$$g(x) = f(x) - T(x).$$

Prove that

$$g(a) = g'(a) = g''(a) = g^{(3)}(a) = g^{(4)}(a) = 0.$$

We have

$$g(a) = f(a) - T(a) = f(a) - \left[f(a) + f'(a)(0) + \frac{f''(a)}{2}(0)^2 + \frac{f^{(3)}(a)}{3!}(0)^3 + \frac{f^{(4)}(a)}{4!}(0)^4 \right] = f(a) - f(a) = 0,$$

$$g'(a) = f'(a) - T'(a) = f''(a) - \left[f'(a) + f''(a)(0) + \frac{f^{(3)}(a)}{2}(0)^2 + \frac{f^{(4)}(a)}{3!}(0)^3 \right] = f'(a) - f'(a) = 0,$$

$$g''(a) = f''(a) - T''(a) = f'(a) - \left[f''(a) + f^{(3)}(a)(0) + \frac{f^{(4)}(a)}{2}(0)^2 \right] = f''(a) - f''(a) = 0,$$

$$g^{(3)}(a) = f^{(3)}(a) - T^{(3)}(a) = f^{(3)}(a) - \left[f^{(3)}(a) + f^{(4)}(a)(0) \right] = f^{(3)}(a) - f^{(3)}(a) = 0,$$

and

$$g^{(4)}(a) = f^{(4)}(a) - T^{(4)}(a) = f^{(4)}(a) - \left[f^{(4)}(a)\right] = 0.$$

The derivatives were computed by repeatedly applying the power rule. $\square$

Problem 2.34 Let f be five times differentiable on the open interval I and $a, b \in I$ with $a \neq b$. Prove that there is a ξ between a and b such that

$$f(b) = f(a) + f'(a)(b-a) + \frac{f''(a)}{2}(b-a)^2 + \frac{f^{(3)}(a)}{3!}(b-a)^3 + \frac{f^{(4)}(a)}{4!}(b-a)^4 + \frac{f^{(5)}(\xi)}{5!}(b-a)^5.$$

Or in different notation let $T(x)$ be from Problem 2.33, then this is

$$f(b) = T(b) + \frac{f^{(5)}(\xi)}{5!}(b-a)^5.$$

Let $h : I \to \mathbb{R}$ defined by

$$h(x) = f(x) - T(x) - \frac{f(b) - T(b)}{(b-a)^5}(x-a)^5.$$

Then, we have

$$h(b) = f(b) - T(b) - \frac{f(b) - T(b)}{(b-a)^5}(b-a)^5 = h(b) = f(b) - T(b) - [f(b) - T(b)] = 0.$$

Using the result from Problem 2.34, we can write

$$h(a) = 0 - \frac{(f(b) - T(b))}{(b-a)^5}(0)^5 = 0, \quad h'(a) = 0 - \frac{5(f(b) - T(b))}{(b-a)^5}(0)^4 = 0,$$

$$h''(a) = 0 - \frac{(5)(4)(f(b) - T(b))}{(b-a)^5}(0)^3 = 0, \quad h^{(3)}(a) = 0 - \frac{(5)(4)(3)(f(b) - T(b))}{(b-a)^5}(0)^2 = 0,$$

$$h^{(4)}(a) = 0 - \frac{(5)(4)(3)(2)(f(b) - T(b))}{(b-a)^5}(0)^1 = 0.$$

By the result from last homework, then, there exists a ξ between a and b with $h^{(5)}(\xi) = h^{(4+1)}(\xi) = 0$. We have

$$h^{(5)}(x) = f^{(5)}(x) - T^{(5)}(x) - \frac{5!(f(b) - T(b))}{(b-a)^5}$$

by repeated application of the power rule, and since $T(x)$ is a degree 4 polynomial, $T^{(5)}(x) = 0$ for all x. So we have

$$\begin{aligned}
0 = h^{(5)}(\xi) &= f^{(5)}(\xi) - 0 - \frac{5!(f(b) - T(b))}{(b-a)^5} \\
\implies \frac{5!(f(b) - T(b))}{(b-a)^5} &= f^{(5)}(\xi) \\
\implies f(b) - T(b) &= \frac{f^{(5)}(\xi)(b-a)^5}{5!} \\
\implies f(b) &= T(b) + \frac{f^{(5)}(\xi)}{5!}(b-a)^5
\end{aligned}$$

as desired. $\square$

Problem 2.35 Show that if f is n times differentiable on an open interval I and T_n is its degree n Taylor polynomial at a, then for $0 \leq k \leq n$,

$$T_n^{(k)}(a) = f^{(k)}(a).$$

That is the kth derivatives of T_n and f agree on a for $0 \leq k \leq n$.

We will induct on n.

Base Case: Let $n = 0$. Then

$$T_n^{(n)} = T_n^{(0)}(x) = T_n(x) = \sum_{k=0}^{0} \frac{f^{(0)}(a)}{k!}(x-a)^k = \frac{f(a)}{0!}(x-a)^0 = f(a) = f^{(0)}(a) = f^{(n)}(a),$$

so the claim holds for $n = 0$ since $\{k : 0 \leq k \leq n\} = \{0\}$.

Induction Step: Let $n \in \mathbb{N}$, $n > 0$. Suppose that the claim holds for $n-1$: that is for all $0 \leq k \leq n-1$, we have

$$T_{n-1}^{(k)}(a) = f^{(k)}(a).$$

We note that we can split the series to write

$$T_n(x) = T_{n-1}(x) + \frac{f^{(n)}(a)}{n!}(x-a)^n.$$

Let $k \in \mathbb{Z}$, $0 \leq k \leq n$.

Case 1: $k < n$. Then we have

$$\begin{aligned}
T_n^{(k)}(x) &= T_{n-1}^{(k)} + \frac{d^k}{dx^k}\left[\frac{f^{(n)}(a)}{n!}(x-a)^n\right] \\
&= T_{n-1}^{(k)}(x) + \frac{(n)(n-1)\dots(n-k)f^{(n)}(a)}{n!}(x-a)^{n-k} && \text{(repeated power rule)} \\
&= T_{n-1}^{(k)}(x) + \frac{f^{(n)}(a)}{(n-k-1)!}(x-a)^{n-k} \\
\implies T_n^{(k)}(a) &= T_{n-1}^{(k)}(a) + \frac{f^{(n)}(a)}{(n-k-1)!}(0)^{n-k} && \text{(substitution)} \\
&= T_{n-1}^{(k)}(a) && (k < n \implies n-k \neq 0) \\
&= f^{(k)}(a). && \text{(induction hypothesis)}
\end{aligned}$$

Case 2: $k = n$. Then, since $T_{n-1}(x)$ is a degree $n-1$ polynomial, we have $T_{n-1}^{(n)}(x) = 0$ for all x. So

$$\begin{aligned}
T_n^{(n)}(x) &= 0 + \frac{d^n}{dx^n}\left[\frac{f^{(n)}(a)}{n!}(x-a)^n\right] && \text{(splitting sum equation from above)} \\
&= \frac{n!f^{(n)}(a)}{n!}(x-a)^0 && \text{(repeated power rule)} \\
&= f^{(n)}(a).
\end{aligned}$$

So the equation holds for all $0 \leq k \leq n$, and therefore the claim holds for n. $\square$

Problem 2.39 Let $f'' \geq 0$ on an open interval I. Prove that the graph of f is above all its tangent lines. More precisely if $a \in I$, then $f(a) + f'(a)(x-a) \leq f(x)$ for all $x \in I$.

Let $x \in I$. Then from Lagrange, there is a ξ between a and x with

$$f(x) = f(a) + f'(a)(x-a) + \frac{f''(\xi)}{2}(x-a)^2.$$

Since $f''(\xi) \geq 0$ by the assumption, and clearly $\frac{(x-a)^2}{2}$ is non-negative, it follows that

$$f(a) + f'(a)(x-a) \leq f(x).$$

$\square$

Problem 2.40 Let I be an open interval and $f : I \to \mathbb{R}$ be a function. We define f to be convex if and only if for all $x_0, x_1 \in I$ and for all $t \in [0,1]$, the inequality

$$(1-t)f(x_0) + tf(x_1) \geq f\left((1-t)x_0 + tx_1\right)$$

holds. Suppose that f is twice differentiable with $f'' \geq 0$. Prove that f is convex on I.

Let $x_0, x_1 \in I$ and $t \in [0,1]$. Define $x_t = (1-t)x_0 + tx_1$. From Problem 2.39, since $f'' \geq 0$ on I and $x_t \in I$,

$$f(x) \geq f(x_t) + f'(x_t)(x - x_t)$$

holds for all $x \in I$. In particular, then,

$$f(x_0) \geq f(x_t) + f'(x_t)(x_0 - x_t) \text{ and } f(x_1) \geq f(x_t) + f'(x_t)(x_1 - x_t).$$

We can use this to write

$$\begin{aligned}
(1-t)f(x_0) + tf(x_1) &\geq (1-t)\left[f(x_t) + f'(x_t)(x_0 - x_t)\right] + t\left[f(x_t) + f'(x_t)(x_1 - x_t)\right] && \text{(from above)} \\
&= (1-t)\left[f(x_t) + f'(x_t)(x_0 - (1-t)x_0 - tx_1)\right] + t\left[f(x_t) + f'(x_t)(x_1 - (1-t)x_0 - tx_1)\right] \\
&= (1-t)\left[f(x_t) + f'(x_t)(tx_0 - tx_1)\right] + t\left[f(x_t) + f'(x_t)((1-t)x_1 - (1-t)x_0\right] \\
&= (1-t)\left[f(x_t) + tf'(x_t)(x_0 - x_1)\right] + t\left[f(x_t) - (1-t)f'(x_t)(x_0 - x_1)\right] \\
&= (1-t)f(x_t) + tf(x_t) + (1-t)tf'(x_t)(x_0 - x_1) - t(1-t)f'(x_t)(x_0 - x_1) \\
&= f(x_t)(1 - t + t) + 0 \\
&= f(x_t) \\
&= f((1-t)x_0 + tx_1)
\end{aligned}$$

as desired. $\square$

Problem 2.41 Let $\alpha_1, \ldots, \alpha_{n+1} > 0$ with $\alpha_1 + \cdots + \alpha_{n+1} = 1$. Prove that for any real numbers $x_1, \ldots, x_{n+1}$ we have

$$\sum_{k=1}^{n+1} \alpha_k x_k = (1 - \alpha_{n+1}) \sum_{k=1}^{n} \left(\frac{\alpha_k}{1 - \alpha_{n+1}}\right) x_k + \alpha_{n+1} x_{n+1}$$

and

$$\sum_{k=1}^{n} \left(\frac{\alpha_k}{1 - \alpha_{n+1}}\right) = 1.$$

We will show both equalities separately. For the first, we have

$$\begin{aligned}
(1 - \alpha_{n+1}) \sum_{k=1}^{n} \left(\frac{\alpha_k}{1 - \alpha_{n+1}}\right) x_k + \alpha_{n+1} x_{n+1} &= \sum_{k=1}^{n} \left(\frac{(1 - \alpha_{n+1})\alpha_k}{1 - \alpha_{n+1}}\right) x_k + \alpha_{n+1} x_{n+1} && \text{(distributing)} \\
&= \sum_{k=1}^{n} \alpha_k x_k + \alpha_{n+1} x_{n+1} && \text{(cancelling)}
\end{aligned}$$

$$= \sum_{k=1}^{n+1} \alpha_k x_k, \qquad \text{(combining sum)}$$

and for the second we can compute

$$\begin{aligned}
\sum_{k=1}^{n+1} \alpha_k &= 1 && \text{(by assumption)} \\
\implies \sum_{k=1}^{n} \alpha_k + \alpha_{n+1} &= 1 && \text{(splitting sum)} \\
\implies \sum_{k=1}^{n} \alpha_k &= 1 - \alpha_{n+1} && \text{(algebra)} \\
\implies \frac{1}{1-\alpha_{n+1}} \sum_{k=1}^{n} \alpha_k &= 1 && \text{(algebra)} \\
\implies \left(\sum_{k=1}^{n} \frac{\alpha_k}{1-\alpha_{n+1}} \right) &= 1. && \text{(distributing)}
\end{aligned}$$

□

Problem 2.43 Show that $f(x) = x$ and $g(x) = |x|$ are convex on $\mathbb{R}$.

We have that $f''(x) \equiv 0$, so from problem 2.40 f is convex on $\mathbb{R}$. To see that g is convex, let $x, y \in \mathbb{R}$, $t \in [0, 1]$. Then

$$\begin{aligned}
g\left((1-t)x_0 + tx_1\right) &= \left|(1-t)x_0 + tx_1\right| \\
&\leq \left|(1-t)x_0\right| + |tx_1| && \text{(triangle inequality)} \\
&= |1-t|\,|x_0| + |t|\,|x_1| \\
&= (1-t)g(x_0) + tg(x_1), && (0 \leq t \leq 1 \text{ so } t, 1-t \geq 0)
\end{aligned}$$

so g is convex by definition.

Problem 2.44 (Jensen's inequality). If f is convex on the interval I, $x_1, \ldots, x_n \in I$ and $\alpha_1, \ldots, \alpha_n \geq 0$ with $\alpha_1 + \cdots + \alpha_n = 1$, prove that

$$f(\alpha_1 x_1 + \cdots + \alpha_n x_n) \leq \alpha_1 f(x_1) + \cdots + \alpha_n f(x_n).$$

Also, prove that if f is strictly convex, then equality holds if and only if $x_1 = \cdots = x_n$.

Case 1: $x_1 = \cdots = x_n$. Let $x := x_1$. Then, we have

$$\begin{aligned}
f(\alpha_1 x_1 + \cdots + \alpha_n x_n) &= f(\alpha_1 x + \cdots + \alpha_n x) && (x = x_i \text{ for all } i) \\
&= f((\alpha_1 + \cdots + \alpha_n)x) && \text{(distributing)} \\
&= f((1)x) && \text{(sum of } \alpha_i\text{s is 1)} \\
&= f(x) \\
&= (1)f(x) \\
&= (\alpha_1 + \cdots + \alpha_n)f(x) \\
&= \alpha_1 f(x) + \cdots + \alpha_n f(x) && \text{(distributing)}
\end{aligned}$$

$$= \alpha_1 f(x_1) + \cdots + \alpha_n f(x_n).$$

So if f is strictly convex and $x_1 = \cdots = x_n$, then equality for

$$f(\alpha_1 x_1 + \cdots + \alpha_n x_n) \le \alpha_1 f(x_1) + \cdots + \alpha_n f(x_n).$$

holds. It remains to show that the converse of this, as well as the inequality for when f is convex but not necessarily strictly so, hold.

Case 2: $x_i \neq x_{i+1}$ for some $i \in \{1, \ldots, n-1\}$. We will induct on n.

Base Case: Let $n = 2$ (Case 1 covers $n = 1$). Since f is convex, we have

$$f(\alpha_1 x_1 + \alpha_2 x_2) \le \alpha_1 f(x_1) + \alpha_2 f(x_2)$$

by definition. Also, the Case 2 assumption forces $x_1 < x_2$, and since f is strictly convex,

$$f(\alpha_1 x_1 + \alpha_2 x_2) < \alpha_1 f(x_1) + \alpha_2 f(x_2)$$

follows by definition.

Induction Step: Let $n \in \mathbb{N}$, $n \ge 2$. Suppose that if we have $x_1', \ldots, x_n' \in I$ and $\alpha_1', \ldots, \alpha_n' > 0$ with $\alpha_1' + \cdots + \alpha_n' = 1$, then

1. If f is convex, then $f(\alpha_1' x_1' + \cdots + \alpha_n' x_n') \le \alpha_1' f(x_1') + \cdots + \alpha_n' f(x_n')$.
2. If f is strictly convex, then this inequality is strict.

We will show both hold for $n+1$ as well. Suppose we have the givens above for $n+1$ (so $x_i \neq x_{i+1}$ for some $i \in \{1, \ldots, n\}$).

1. Suppose f is convex. Consider

$$\alpha_1' := \frac{\alpha_1}{1 - \alpha_{n+1}}, \alpha_2' := \frac{\alpha_2}{1 - \alpha_{n+1}}, \ldots, \alpha_n' := \frac{\alpha_n}{1 - \alpha_{n+1}}.$$

   We showed in Problem 2.41 that $\alpha_1' + \cdots + \alpha_n' = 1$. Also, consider $\alpha := 1 - \alpha_{n+1}, \beta := \alpha_{n+1}$, which clearly satisfy $\alpha + \beta = 1$. Then, we can write

$$\begin{aligned}
f(\alpha_1 x_1 + \cdots + \alpha_{n+1} x_{n+1}) &= f\left(\sum_{k=1}^{n+1} \alpha_k x_k\right) && (1)\\
&= f\left((1 - \alpha_{n+1}) \sum_{k=1}^{n} \left(\frac{\alpha_k}{1 - \alpha_{n+1}}\right) x_k + \alpha_{n+1} x_{n+1}\right) && \text{(by Problem 2.41)}\\
&= f\left(\alpha \sum_{k=1}^{n} \alpha_k' x_k + \beta x_{n+1}\right) && \text{(will use convexity definition)}\\
&\le \alpha f\left(\sum_{k=1}^{n} \alpha_k' x_k\right) + \beta f(x_{n+1}) && (2)\\
&= \alpha f(\alpha_1' x_1 + \cdots + \alpha_n' x_n) + \beta f(x_{n+1}) && \text{(will use induction hypothesis)}\\
&\le \alpha \left(\alpha_1' f(x_1) + \cdots + \alpha_n' f(x_n)\right) + \beta f(x_{n+1}) && (3)
\end{aligned}$$

$$= (1 - \alpha_{n+1}) \left(\frac{\alpha_1 f(x_1)}{1 - \alpha_{n+1}} + \cdots + \frac{\alpha_n f(x_n)}{1 - \alpha_{n+1}} \right) + \alpha_{n+1} f(x_{n+1}) \tag{4}$$

$$= \alpha_1 f(x_1) + \cdots + \alpha_{n+1} f(x_{n+1}). \tag{5}$$

So the inequality holds for $n+1$.

2. Suppose f is strictly convex. We can use the same calculations as above to show that the inequality is strict if f is strictly convex.

   Case 2.1: $x_i \neq x_{i+1}$ for some $i \in \{1, \dots, n-1\}$. Then, equation (3) from above can be written as a strict inequality by part 2 of the induction hypothesis.

   Case 2.2: $x_n \neq x_{n+1}$. Since Case 2.1 does not hold, we can define $x := x_1 = \cdots = x_n$ and write

   $$f\left(\alpha \sum_{k=1}^{n} \alpha'_k x_k + \beta x_{n+1}\right) = f\left(\alpha x \sum_{k=1}^{n} \alpha'_k + \beta x_{n+1}\right) = f(\alpha x + \beta x_{n+1}),$$

   with the second equality coming from Problem 2.41. Since $x \neq x_{n+1}$, equation (2) from above can then be written as a strict inequality by definition of strict convexity.

So both statements hold for $n+1$, and therefore by induction the claim holds for all $n \in \mathbb{N}$. $\square$