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Analysis in $\mathbb{R}^n$: Section 1
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Analysis in $\mathbb{R}^n$ Homework 6

Problem 1 Determine with proof whether each function $f : E \to \mathbb{R}$ is uniformly continuous on E:

(a) $f(x) = x^2$ on $E = (-2, 2)$.

(b) $f(x) = \frac{1}{x}$ on $E = (0, +\infty)$.

(c) $f(x) = \frac{1}{x}$ on $E = [1, +\infty)$.

Solution.

(a) We claim f is uniformly continuous over $(-2, 2)$. Let $\varepsilon > 0$, and consider $\delta = \frac{\varepsilon}{4}$. Let $x, y \in (-2, 2)$ such that $|x - y| < \delta$. Since $x < 2$ and $y < 2$, we know that $|x + y| < 4$. So we have

$$|f(x) - f(y)| = |x^2 - y^2| = |(x - y)(x + y)| = |x + y||x - y| < 4|x - y| < 4\delta = 4\frac{\varepsilon}{4} = \varepsilon,$$

and therefore we have $|f(x) - f(y)| < \varepsilon$ so f is uniformly continuous over $(-2, 2)$. $\square$

(b) We claim f is not uniformly continuous over $(0, +\infty)$. Suppose to the contrary that it is. Then, for all $\varepsilon > 0$, there exists some $\delta > 0$ such that for all $x, y \in (0, \infty)$, $|x - y| < \delta \implies \left|\frac{1}{x} - \frac{1}{y}\right| < \varepsilon$. Take $\varepsilon = 1$, $x = \min\{\frac{1}{2}, \delta\}$, and $y = \frac{x}{2}$.

Case 1: $\delta > \frac{1}{2}$. Then, $x = \frac{1}{2}$ and $\frac{1}{4}$, so

$$|x - y| = \left|\frac{1}{2} - \frac{1}{4}\right| = \frac{1}{2} < \delta$$

but

$$\left|\frac{1}{x} - \frac{1}{y}\right| = \left|\frac{1}{1/2} - \frac{1}{1/4}\right| = |2 - 4| = 2 \geq 1 = \varepsilon,$$

a contradiction.

Case 2: $\delta \leq \frac{1}{2}$. Then, $x = \delta$ and $y = \frac{\delta}{2}$, so

$$|x - y| = \left|\delta - \frac{\delta}{2}\right| = \frac{\delta}{2} < \delta$$

but

$$\left|\frac{1}{x} - \frac{1}{y}\right| = \left|\frac{1}{\delta} - \frac{1}{\delta/2}\right| = \left|\frac{1}{\delta} - \frac{2}{\delta}\right| = \left|-\frac{1}{\delta}\right| = \frac{1}{\delta} \geq \frac{1}{1/2} = 2 > 1 = \varepsilon,$$

a contradiction. $\square$

(c) We claim f is uniformly continuous over $[1, +\infty)$. Let $\varepsilon > 0$, and consider $\delta = \varepsilon$. Let $x, y \in [1, +\infty)$ such that $|x - y| < \delta$. Then,

$$\left|\frac{1}{x} - \frac{1}{y}\right| = \left|\frac{y - x}{xy}\right| = \left|\frac{x - y}{xy}\right| = \left|\frac{1}{xy}\right| |x - y| \leq |x - y| < \delta = \varepsilon,$$

where $\left|\frac{1}{xy}\right| |x - y| \leq |x - y|$ is justified because $x \geq 1$ and $y \geq 1$ and thus $xy \geq 1$. $\square$

Problem 2 Let $c \in (a,b)$ and $f : [a,b] \to \mathbb{R}$ be defined by

$$f(x) = \begin{cases} 0 & \text{if } x < c, \\ 1 & \text{if } x \geq c. \end{cases}$$

Prove that f is integrable on $[a,b]$, and find its integral there.

Solution.

Let $P = \{a, c, b\}$. Then, since f is constant from a to c we have $m_1 = M_1 = 0$, and since f is constant from c to b we have $m_2 = M_2 = 1$. Thus, we have

$$L(f,P) = \sum_{i=1}^{2} m_i(f)\Delta x_i = 0(c-a) + 1(b-c) = \sum_{i=1}^{2} M_i(f)\Delta x_i = U(f,P).$$

So $U(f,P) - L(f,P) = 0 < \varepsilon$ for every $\varepsilon > 0$, so f is integrable on $[a,b]$. In particular, we have

$$\int_a^b f(x)dx = L(f,P) = U(f,P) = b - c.$$

Problem 3 Define $f\colon [0,b] \to \mathbb{R}$ by the formula $f(x) = x$. Show that f is integrable on $[0,b]$ and that $\int_0^b f = \frac{b^2}{2}$. (Hint: consider uniform partitions with intervals of length $\frac{b}{n}$.)

Solution.

We claim f is integrable from 0 to b, and that in particular $\int_0^b f(x)dx = \frac{b^2}{2}$. We have shown in class that to prove this, it suffices to show that for all $\varepsilon > 0$, there exists a partition P such that $U(f,P) - \frac{b^2}{2} < \varepsilon$ and $\frac{b^2}{2} - L(f,P) < \varepsilon$.

Let $P_n = \{x_0, x_1, \ldots, x_n\}$ be a partition with equal subintervals of $[0,b]$. Since each subinterval is equal, we have that $x_i - x_{i-1} = \frac{b}{n}$ for all i, and each $x_i = \frac{i}{n}(b-0) = \frac{ib}{n}$. Also, since f is monotonic and increasing, we have $M_i(f) = x_i$ and $m_i(f) = x_{i-1}$ for all i. Let $\varepsilon > 0$, and choose $n = \frac{b^2}{\varepsilon}$. Then, we have

$$\begin{aligned}
U(f,P_n) - \frac{b^2}{2} &= \sum_{i=1}^{n}(M_i(f)\Delta x_i) - \frac{b^2}{2} && \text{(definition)} \\
&= \sum_{i=1}^{n}[x_i(x_i - x_{i-1})] - \frac{b^2}{2} && \text{(justified above)} \\
&= \sum_{i=1}^{n}\left[\frac{ib}{n}\left(\frac{b}{n}\right)\right] - \frac{b^2}{2} && \text{(justified above)} \\
&= \frac{b^2}{n^2}\sum_{i=1}^{n}(i) - \frac{b^2}{2} && \text{(pulling out constants)} \\
&= \frac{b^2}{n^2}\left[\frac{n(n+1)}{2}\right] - \frac{b^2}{2} && \text{(evaluating sum)} \\
&= \frac{b^2}{2}\left(\frac{n^2+n}{n^2}\right) - \frac{b^2}{2} && \text{(rearranging)} \\
&= \frac{b^2}{2}\left(1 + \frac{1}{n}\right) - \frac{b^2}{2} \\
&= \frac{b^2}{2}\left(\frac{1}{n}\right)
\end{aligned}$$

$$
\begin{aligned}
&= \frac{b^2}{2}\left(\frac{\varepsilon}{b^2}\right) && \text{(we chose } n = \tfrac{b^2}{\varepsilon}\text{)} \\
&= \frac{\varepsilon}{2} < \varepsilon,
\end{aligned}
$$

and similarly,

$$
\begin{aligned}
\frac{b^2}{2} - L(f, P_n) &= \frac{b^2}{2} - \sum_{i=1}^{n} m_i(f)\Delta x_i \\
&= \frac{b^2}{2} - \sum_{i=1}^{n} x_{i-1}(x_i - x_{i-1})\Delta x_i \\
&= \frac{b^2}{2} - \sum_{i=1}^{n} \frac{(i-1)b}{n}\left(\frac{b}{n}\right) \\
&= \frac{b^2}{2} - \frac{b^2}{n^2}\sum_{i=1}^{n}(i-1) \\
&= \frac{b^2}{2} - \frac{b^2}{n^2}\left[\sum_{i=1}^{n} i - \sum_{i=1}^{n} 1\right] && \text{(splitting sum)} \\
&= \frac{b^2}{2} - \frac{b^2}{n^2}\left[\frac{n(n+1)}{2} - n\right] \\
&= \frac{b^2}{2} - \frac{b^2}{n^2}\left(\frac{n^2 - n}{2}\right) \\
&= \frac{b^2}{2} - \frac{b^2}{2}\left(\frac{n^2 - n}{n^2}\right) \\
&= \frac{b^2}{2} - \frac{b^2}{2}\left(1 - \frac{1}{n}\right) \\
&= \frac{b^2}{2}\left(\frac{1}{n}\right) \\
&= \frac{b^2}{2}\left(\frac{\varepsilon}{b^2}\right) \\
&= \frac{\varepsilon}{2} < \varepsilon.
\end{aligned}
$$

Therefore, f is integrable on $[0, b]$, and we have $\int_0^b f(x)dx = \frac{b^2}{2}$. $\square$

Problem 4 Let $f : [a, b] \longrightarrow \mathbb{R}$ be a bounded function. Let P_n be a partition of $[a, b]$ into n equal subintervals. Prove that if $\lim\limits_{n\longrightarrow\infty} U(f, P_n) = \lim\limits_{n\longrightarrow\infty} L(f, P_n) = L$, then f is integrable on $[a, b]$ and $\int_a^b f = L$.

Solution.

Suppose we have $\lim\limits_{n\to\infty} U(f, P_n) = \lim\limits_{n\to\infty} L(f, P_n) = L$. Let $\varepsilon > 0$. Then, there exists some $N \in \mathbb{N}$ such for all $n \geq N$, we have $|U(f, P_n) - L| < \varepsilon$ and $|L(f, P_n) - L| < \varepsilon$. Since, as n grows, $U(f, P_n)$ is monotonic and decreasing and $L(f, P_n)$ is monotonic and increasing, this means that $U(f, P_N) - L < \varepsilon$ and $L - L(f, P_N) < \varepsilon$. We have shown in class that this implies that f is integrable over $[a, b]$ and $\int_a^b f(x)dx = L$. $\square$

Problem 5

(a) Let $x_0 \in [a, b]$ be fixed, and $f, g : [a, b] \longrightarrow \mathbb{R}$ be bounded functions such that $f(x) = g(x),\ \forall x \neq x_0$. Prove that if g is integrable, then f is also integrable and $\int_a^b f = \int_a^b g$.

(b) **(Bonus)** Let $f, g : [a,b] \longrightarrow \mathbb{R}$ be bounded functions such that $f(x) = g(x)$ for all but finitely many $x \in [a,b]$. Prove that if g is integrable, then f is also integrable and $\int_a^b f = \int_a^b g$.

Solution.

(a) Suppose g is integrable over $[a,b]$ with $\int_a^b g = \Omega$. It suffices to show that $h(x) := f(x) - g(x)$ is integrable with $\int_a^b h = 0$. Then, $f(x)$ will be integrable with $\int_a^b f = \Omega$. We have shown in class that this is true if for every $\varepsilon > 0$, we have a partition P such that $U(h,P) < \varepsilon$ and $-L(h,P) < \varepsilon$.

Since f and g differ only at $x = x_0$, h is zero on $[a,b]$ except at x_0, where $|h(x)| = y_0$ where $y_0 := |f(x_0) - g(x_0)|$. Let $P_n = \{z_0, z_1, \ldots, z_n\}$ be a uniform partition of $[a,b]$. Since it is uniform, we have that $\Delta z_i := z_i - z_{i-1} = \frac{b-a}{n}$ for all i.

Let $\varepsilon > 0$ and $n = \frac{2y_0}{\varepsilon}(b-a)$. We know that x_0 will lie in one of these intervals, so let i_0 be the $i \in \{1, 2, \ldots, n\}$ such that $x_0 \in [z_{i-1}, z_i]$ (if x_0 is on the border of two intervals, it is not difficult to choose a larger n where this does not happen). Then, we have

$$\begin{aligned}
U(h,P_n) &= \sum_{i=1}^{n} M_i(h)\Delta z_i \\
&= M_{i_0}(h)\Delta z_{i_0} && (h \text{ is zero everywhere except } x_0) \\
&= M_{i_0}(h)\left(\frac{b-a}{n}\right) && (\text{explained above}) \\
&= \sup\left\{h(x) \mid x \in [z_{i_0-1}, z_{i_0}]\right\}\left(\frac{b-a}{n}\right) && (\text{definition}) \\
&\le y_0\left(\frac{b-a}{n}\right) && (\text{function never exceeds } h_0) \\
&= \frac{y_0(b-a)}{\frac{2y_0(b-a)}{\varepsilon}} && (\text{substituting our choice of } n) \\
&= \frac{\varepsilon}{2} < \varepsilon,
\end{aligned}$$

and similarly

$$\begin{aligned}
L(h,P_n) &= -\sum_{i=1}^{n} m_i(h)\Delta z_i \\
&= m_{i_0}(h)\Delta z_{i_0} \\
&= m_{i_0}(h)\left(\frac{b-a}{n}\right) \\
&= \inf\left\{h(x) \mid x \in [z_{i_0-1}, z_{i_0}]\right\}\left(\frac{b-a}{n}\right) \\
&\ge -y_0\left(\frac{b-a}{n}\right) \\
&= \frac{-y_0(b-a)}{\frac{2y_0(b-a)}{\varepsilon}} \\
&= -\frac{\varepsilon}{2} \\
\implies -L(h,P_n) &\le \frac{\varepsilon}{2} < \varepsilon.
\end{aligned}$$

Therefore, we have

$$\int_a^b h = 0 \implies \int_a^b f - g = 0 \implies \int_a^b f - \int_a^b g = 0 \implies \int_a^b f = \int_a^b g.$$

□

(b) We will use a similar strategy as in **(a)**. Suppose g is integrable over $[a, b]$ with $\int_a^b g = \Omega$, and $f(x) = g(x)$ for all $x \in [a, b] \backslash \{x_0, x_1, \ldots, x_k\}$. We will show $h(x) := f(x) - g(x)$ is integrable with $\int_a^b h = 0$ by showing that for every $\varepsilon > 0$, we have a partition P such that $U(h, P) < \varepsilon$ and $-L(h, P) < \varepsilon$.

Define $y_j := |f(x_j) - g(x_j)|$ for all $j \in \{1, 2, \ldots, k\}$. Then, h is zero everywhere on $[a, b]$ except at finitely many points, where $|h(x_j)| = y_j$. Let $y := \max\{y_0, y_1, \ldots, y_k\}$. Let $P_n = \{z_0, z_1, \ldots, z_n\}$ be a uniform partition of $[a, b]$. Then, $\Delta z_i := z_i - z_{i-1} = \frac{b-a}{n}$ for all $i \in \{1, 2, \ldots, n\}$.

Let $\varepsilon > 0$ and $n = \frac{2yk}{\varepsilon}(b - a)$. We know that for each $j \in \{1, 2, \ldots, k\}$, x_j will lie in one of these intervals, so let i_j be the $i \in \{1, 2, \ldots, n\}$ such that $x_j \in [z_{i-1}, z_i]$ (if any x_j is on the border of two intervals, it is not difficult to choose a larger n where this does not happen). Then, we have

$$\begin{aligned}
U(h, P_n) &= \sum_{i=1}^{n} M_i(h)\Delta z_i \\
&\leq M_{i_0}(h)\Delta z_{i_0} + M_{i_1}(h)\Delta z_{i_1} + \cdots + M_{i_k}(h)\Delta z_{i_k} \\
&= (M_{i_0} + M_{i_1} + \cdots + M_{i_k})\left(\frac{b-a}{n}\right) && \text{(duplicates may cause overcounting)} \\
&\leq (y_0 + y_1 + \cdots + y_k)\left(\frac{b-a}{n}\right) && \text{(each interval never exceeds } y_j\text{)} \\
&\leq (y + y + \cdots + y)\left(\frac{b-a}{n}\right) && (y_j \leq y \text{ for all } j) \\
&= \frac{yk(b-a)}{n} \\
&= \frac{yk(b-a)}{\frac{2yk(b-a)}{\varepsilon}} \\
&= \frac{\varepsilon}{2} < \varepsilon,
\end{aligned}$$

and similarly

$$\begin{aligned}
L(h, P_n) &= \sum_{i=1}^{n} m_i(h)\Delta z_i \\
&\geq m_{i_0}(h)\Delta z_{i_0} + m_{i_1}(h)\Delta z_{i_1} + \cdots + m_{i_k}(h)\Delta z_{i_k} \\
&= (m_{i_0} + m_{i_1} + \cdots + m_{i_k})\left(\frac{b-a}{n}\right) \\
&\geq (-y_0 - y_1 - \cdots - y_k)\left(\frac{b-a}{n}\right) \\
&\geq (-y - y - \cdots - y)\left(\frac{b-a}{n}\right) \\
&= \frac{-yk}{(b-a)}n \\
&= \frac{-yk(b-a)}{\frac{2yk(b-a)}{\varepsilon}} \\
&= -\frac{\varepsilon}{2}
\end{aligned}$$

$$\implies -L(h, P_n) \le \frac{\varepsilon}{2} < \varepsilon.$$

Therefore, we have

$$\int_a^b h = 0 \implies \int_a^b f - g = 0 \implies \int_a^b f - \int_a^b g = 0 \implies \int_a^b f = \int_a^b g.$$

□

Problem 6 Let $A \subset \mathbb{R}$ be a non-empty set and let $f : A \to \mathbb{R}$ be a bounded function. Show that

$$\sup\{f(x) \mid x \in A\} - \inf\{f(x) \mid x \in A\} = \sup\{f(x) - f(y) \mid x, y \in A\}.$$

Solution.

For brevity, we define

$$M := \sup\{f(x) \mid x \in A\}, m := \{f(x) \mid x \in A\}, \Delta = \sup\{f(x) - f(y) \mid x, y \in A\}.$$

Suppose (toward contradiction) that $M - m \neq \Delta$.

Case 1: $M - m < \Delta$. Then there exists $\varepsilon > 0$ such that $\Delta = M - m + \varepsilon$. Then, by our theorem from class there exists some $z \in \{f(x) - f(y) \mid x, y \in A\}$ such that $z \ge M - m + \frac{\varepsilon}{2}$. So there exists $x, y \in A$ such that $z = f(x) - f(y)$. Since $f(y) \in f(A)$, we have $f(y) \ge m$. So we have $f(x) \ge M + \frac{\varepsilon}{2}$, a contradiction because M is the supremum of $f(A)$.

Case 2: $M - m > \Delta$. Then there exists $\varepsilon > 0$ such that $\Delta = M - m - \varepsilon$. By the theorem from class, there exists $x \in A$ such that $f(x) \ge M - \frac{\varepsilon}{4}$ and $y \in f(A)$ such that $f(y) \le m + \frac{\varepsilon}{4}$. But then $f(x) - f(y) \ge M - m - \frac{\varepsilon}{2}$, which is greater than Δ, a contradiction because Δ is the supremum of $\{f(x) - f(y) \mid x, y \in A\}$. □