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MATH 544: Section H01
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MATH 544 Homework 1

Problem 1 Show that subtraction of matrices in $\mathrm{Mat}_{2\times 3}$ is neither commutative nor associative.

Solution.

If subtraction in $\mathrm{Mat}_{2\times 3}$ is commutative, then for all $A, B \in \mathrm{Mat}_{2\times 3}$, $A - B = B - A$. If subtraction is associative, then for all $A, B, C \in \mathrm{Mat}_{2\times 3}$, $(A - B) - C = A - (B - C)$. But suppose that we have

$$A = C = \begin{pmatrix} 1 & 1 & 1 \\ 1 & 1 & 1 \end{pmatrix}, B = \begin{pmatrix} 0 & 0 & 0 \\ 0 & 0 & 0 \end{pmatrix}.$$

Then,

$$A - B = \begin{pmatrix} 1 & 1 & 1 \\ 1 & 1 & 1 \end{pmatrix} \neq \begin{pmatrix} -1 & -1 & -1 \\ -1 & -1 & -1 \end{pmatrix} = B - A,$$

and

$$(A - B) - C = \begin{pmatrix} 1 & 1 & 1 \\ 1 & 1 & 1 \end{pmatrix} - \begin{pmatrix} 1 & 1 & 1 \\ 1 & 1 & 1 \end{pmatrix} = \begin{pmatrix} 0 & 0 & 0 \\ 0 & 0 & 0 \end{pmatrix} \neq$$
$$\begin{pmatrix} 2 & 2 & 2 \\ 2 & 2 & 2 \end{pmatrix} = \begin{pmatrix} 1 & 1 & 1 \\ 1 & 1 & 1 \end{pmatrix} - \begin{pmatrix} -1 & -1 & -1 \\ -1 & -1 & -1 \end{pmatrix} = A - (B - C).$$

So subtraction in $\mathrm{Mat}_{2\times 3}$ is neither commutative nor associative, because the definitions do not hold for our choice of A, B, and C. $\square$

Problem 2 Let $A \in \mathrm{Mat}_{m\times n}$, and let $B \in \mathrm{Mat}_{n\times p}$. Suppose that B has a column of zeros. Show that AB has a column of zeroes.

Solution.

Let A_i denote the ith row of A, and let B_j denote the jth column of B. Since B has a column of zeros, we have some $j \in \{1, 2, \ldots, p\}$ such that $(B)_{kj} = 0$ for all $k \in \{1, 2, \ldots, n\}$.

We claim that the jth column of AB will be a column of zeros. To see this, let $i \in \{1, 2, \ldots, m\}$ be an arbitrary row index of AB. Then, we have

$$(AB)_{ij} = \sum_{k=1}^{n} (A)_{ik}(B)_{kj}.$$

Since $(B)_{kj} = 0$ for all k, each product in the sum is 0, and thus $(AB)_{ij} = 0$ for all rows. So AB has a column of zeros. $\square$

Problem 3 Let $A, B \in \mathrm{Mat}_{2\times 2}$ with $A \neq O_{2\times 2}$. Suppose that $A^2 = AB$. Prove or give a counterexample to the following statement: $A = B$.

Solution.

Suppose we have the matrices

$$A = \begin{pmatrix} 0 & 1 \\ 0 & 1 \end{pmatrix}, B = \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix}.$$

Then, $A^2 = \begin{pmatrix} 0 & 1 \\ 0 & 1 \end{pmatrix} = AB$ but $A \neq B$, so the implication is not true in general. $\square$

Problem 4 Suppose that $A, B \in \mathrm{Mat}_{n\times n}$ are symmetric. Show that AB is symmetric if and only if A and B commute.

Solution.

We note that since A and B are symmetric, we have $A = A^T$ and $B = B^T$.

First, assume AB is symmetric. Then,

$$\begin{aligned} AB &= (AB)^T && \text{(definition of symmetric)} \\ &= B^T A^T && \text{(from class)} \\ &= BA. && (B = B^T,\ A = A^T) \end{aligned}$$

So AB being symmetric implies that A and B commute since $AB = BA$. Next, assume A and B commute. Then,

$$\begin{aligned} AB &= BA && \text{(definition of commuting)} \\ &= B^T A^T && (B = B^T,\ A = A^T) \\ &= (AB)^T. && \text{(from class)} \end{aligned}$$

So A and B commuting implies that AB is symmetric since $AB = (AB)^T$. Therefore, the statements are equivalent. $\square$

Problem 5
(a) Find a matrix $A \in \mathrm{Mat}_{2\times 2}$ such that $A \neq O_{2\times 2}$ but $A^2 = O_{2\times 2}$.
(b) Find a matrix $A \in \mathrm{Mat}_{3\times 3}$ such that $A^2 \neq O_{3\times 3}$ but $A^3 = O_{3\times 3}$.
(c) Let $n \geq 1$ be an integer. Make a conjecture about matrices $A \in \mathrm{Mat}_{n\times n}$ such that $A^{n-1} \neq O_{n\times n}$ by $A^n = O_{n\times n}$.

Solution.

(a) $A = \begin{pmatrix} 0 & 1 \\ 0 & 0 \end{pmatrix}$.

(b) $A = \begin{pmatrix} 0 & 1 & 1 \\ 0 & 0 & 1 \\ 0 & 0 & 0 \end{pmatrix}$.

(c) I conjecture that any matrix $A \in \mathrm{Mat}_{n\times n}$ of the form

$$(A)_{ij} = \begin{cases} 1 & \text{if } j \geq i \\ 0 & \text{else} \end{cases}$$

will satisfy $A^{n-1} \neq O_{n\times n}$ and $A^n = O_{n\times n}$.