Nathan Bickel
MATH 546: Section
Professor: Dr. Vraciu
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MATH 546 Homework 7

The elements of the group D_4 can be represented as:

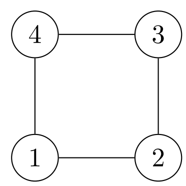
$$e, r, r^2, r^3, t, rt, r^2t, r^3t$$

where r denotes counterclockwise rotation by $\pi/2$, and t denotes reflection about one of the diagonals. We also know that $tr = r^3t$, $r^4 = e$, $t^2 = e$.

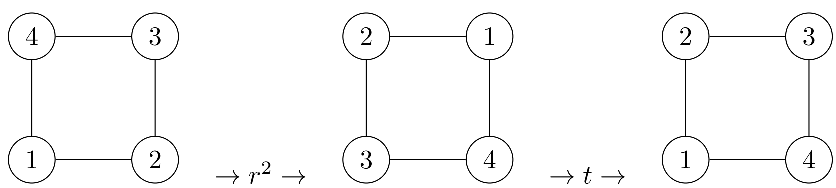
Problem 1 With the notation as above, which one of the elements $e, r, r^2, r^3, t, rt, r^2t, r^3t$ is equal to the reflection about the other diagonal? Justify.

Solution.

Suppose we have labelled our original square as such:



We will assume that the diagonal described above goes from the top left to the bottom right of the triangle. Then, we observe that



reflects the square across the other diagonal. So r^2t is the element that reflects the square across the other diagonal (the same thing happens if the chosen diagonal and other diagonal are exchanged). □

Problem 2 Prove that $Z(D_4) = \{e, r^2\}$.

Solution.

As always, $e \in Z(D_4)$. We also claim $r^2 \in Z(D_4)$. We can check that for each element $a \in D_4$, $a \circ r^2 = r^2 \circ a$:

- $e \circ r^2 = r^2 = r^2 \circ e$
- $r \circ r^2 = r^3 = r^2 \circ r$
- $r^2 \circ r^2 = r^4 = r^2 \circ r^2$
- $r^3 \circ r^2 = r^5 = r^2 \circ r^3$

- $t \circ r^2 = trr = r^3tr = r^3r^3t = r^4r^2t = r^2 \circ t$
- $rt \circ r^2 = rtrr = rr^3tr = r^4tr = tr = r^3t = r^2 \circ rt$
- $r^2t \circ r^2 = r^2trr = r^2r^3tr = r^4rtr = rtr = rr^3t = r^2 \circ r^2t$
- $r^3t \circ r^2 = r^3trr = r^3r^3tr = r^3r^3r^3t = r^4r^2r^3t = r^2 \circ r^3t$

So r^2 commutes with everything in D_4. We next claim that nothing else in $Z(D_4)$. We consider every element:

- r and t do not commute with each other because we have $tr = r^3t$, which is not rt because r^3t is another element. So neither r nor t are in $Z(D_4)$.
- r^3 and r^3t do not commute with each other because
$$r^3t \circ r^3 = r^3trr^2 = r^3r^3tr^2 = r^3r^3trr = r^3r^3r^3tr = r^3r^3r^3r^3t = r^4r^4r^4t = t \neq r^2t = r^4r^2t = r^3 \circ r^3t,$$
so neither r^3 nor r^3t are in $Z(D_4)$.
- rt and r^2t do not commute with each other because
$$rt \circ r^2t = rtrrt = rr^3trt = trt = r^3t^2 = r^3 \neq r = rt^2 = r^2r^3tt = r^2t \circ rt,$$
so neither rt nor r^2t are in $Z(D_4)$.

Thus, only e and r^2 are in $Z(G)$. □

Problem 3 Find all the subgroups of D_4. For each subgroup, list the elements of the subgroup and explain why it is a subgroup. Also explain why there are no other subgroups.

Solution.

By Lagrange's theorem, every subgroup will have order that divides $|D_4|$, so it will have size 1, 2, 4, or 8.

- $\{e\}$ is a subgroup (as always).
- $\{e, r^2\}$ is a subgroup as it is $\langle r^2 \rangle$ ($(r^2)^2 = e$).
- $\{e, t\}$ is a subgroup as it is $\langle t \rangle$ ($t^2 = e$).
- $\{e, rt\}$ is a subgroup as it is $\langle rt \rangle$ ($rtrt = rr^3tt = r^4t^2 = e$).
- $\{e, r^2t\}$ is a subgroup as it is $\langle r^2t \rangle$ ($r^2tr^2t = r^2trrt = r^2r^3trt = r^2r^3r^3tt = r^8t^2 = e$).
- $\{e, r^3t\}$ is a subgroup as it is $\langle r^3t \rangle$ ($r^3tr^3t = r^3trr^2t = r^3r^3tr^2t = r^3r^3r^3trt = r^3r^3r^3r^3tt = r^{12}t^2 = e$).
- $\{e, r, r^2, r^3\}$ is a subgroup as it is $\langle r \rangle$ and $\langle r^3 \rangle$.
- D_4 is a subgroup (as always).

These are the cyclic subgroups (and D_4). We have covered subgroups of size 1 and 8, and since 2 is prime, any subgroup of size 2 must be cyclic. So the only subgroups that could be left are subgroups of size 4. We have found all the cyclic subgroups of size 4 ($\langle r \rangle$), so any other subgroups will not be cyclic.

This will take the form $H = \{e, a, b, ab\}$, where e, a, b are distinct and a and b commute (otherwise, ba would also be in H). Also, we need a, b to have order 2, because the possibilities are 1, 2, and 4. If the order is 1, then they are the identity, and if the order is 4, the subgroup is cyclic. So we will find all the elements a, b in $\{r^2, t, rt, r^2t, r^3t\}$ (which are the order 2 elements) that commute. We will do this with a multiplication table:

$*$	r^2	t	rt	r^2t	r^3t
r^2	e	r^2t	r^3t	t	rt
t	r^2t	e	r^3	r^2	r
rt	r^3t	r	e	r^3	r^2
r^2t	t	r^2	r	e	r^3
r^3t	rt	r^3	r^2	r	e

The distinct elements that commute are

$$\{r^2, t\}, \{r^2, rt\}, \{r^2, r^2t\}, \{r^2, r^3t\}, \{t, r^2t\}, \{rt, r^3t\},$$

so the subgroups of this type are

- $\{e, r^2, t, r^2t\}$ because it is $\langle r^2, t\rangle$.
- $\{e, r^2, rt, r^3t\}$ because it is $\langle r^2, rt\rangle$ and $\langle rt, r^3t\rangle$.
- $\{e, r^2, r^2t, t\}$ because it is $\langle r^2, r^2t\rangle$ and $\langle r^2, t\rangle$.
- $\{e, r^2, r^3t, rt\}$ because it is $\langle r^2, r^3t\rangle$.

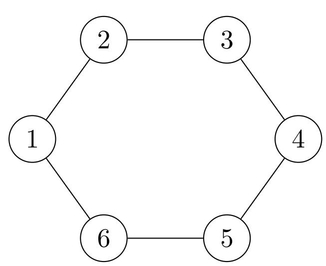
No other subgroups can exist because we have considered every case. In summary, the collection of subgroups of D_4 is

$$\Big\{\{e\}, \{e, r^2\}, \{e, t\}, \{e, rt\}, \{e, r^2t\}, \{e, r^3t\}, \{e, r, r^2, r^3\}, \{e, r^2, t, r^2t\},$$
$$\{e, r^2, rt, r^3t\}, \{e, r^2, r^2t, t\}, \{e, r^2, r^3t, rt\}, \{e, r, r^2, r^3, t, rt, r^2t, r^3t\}\Big\}.$$

Problem 4 Let D_6 be the group of symmetries (rotations and reflections) of a regular hexagon. Using the numbers $1, \ldots, 6$ to label the vertices of the hexagon, write each element of D_6 as a permutation, explaining which elements are rotations and by what angle, and which elements are reflections and about what axis of symmetry.

Solution.

Suppose we have labelled our original hexagon as such:



There are 6 counter-clockwise rotations:

1. By 0: (1)
2. By $\pi/3$: (1 2 3 4 5 6)
3. By $2\pi/3$: (1 3 5)(2 4 6)

4. By π: (1 4)(2 5)(3 6)
5. By $4\pi/3$: (1 5 3)(2 6 4)
6. By $5\pi/3$: (1 6 5 4 3 2)

There are also 6 reflections:

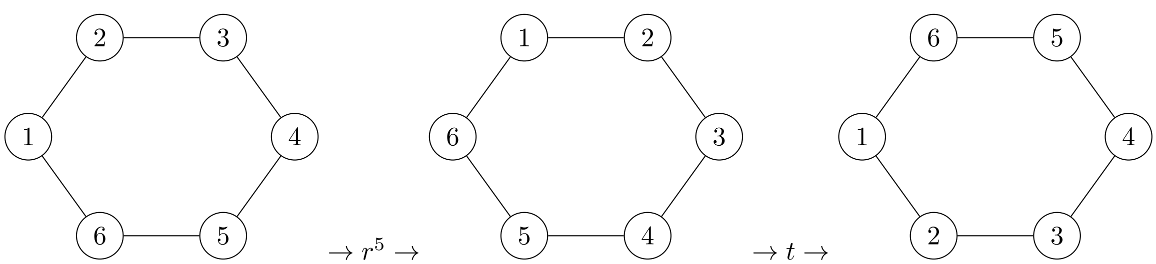
1. Across $\overline{14}$: (2 6)(3 5)
2. Across $\overline{25}$: (1 3)(4 6)
3. Across $\overline{36}$: (1 5)(2 4)
4. Across vertical line: (1 4)(2 3)(5 6)
5. Across top right to bottom left diagonal: (1 6)(2 5)(3 4)
6. Across top left to bottom right diagonal: (1 2)(3 6)(4 5)

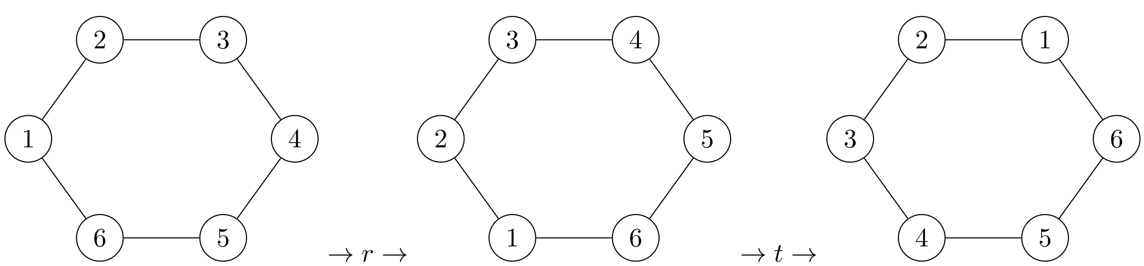
D_6 is the group generated by these twelve elements, with the operation being function composition.

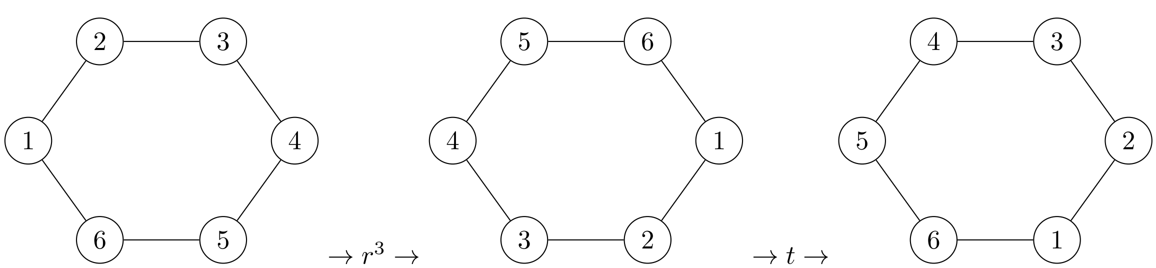
Problem 5 Let r denote the rotation of the regular hexagon by an angle of $\pi/3$, and let t denote the reflection about one of the diagonals. Prove that each of the elements of D_6 that you found in problem 4 can be obtained as $r^i t^j$ with $i \in \{0, 1, 2, 3, 4, 5\}$ and $j \in \{0, 1\}$.

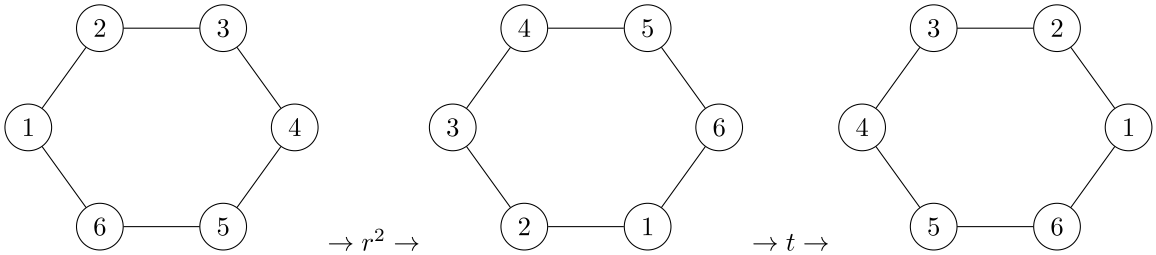
Solution.

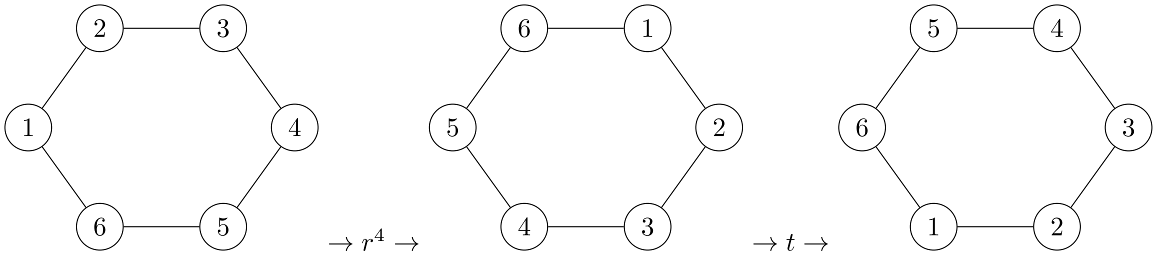
Clearly, all the rotations can be expressed as $r^k t^0$, since they are repeated iterations of one rotation. If we let t be the reflection across the diagonal from top left to bottom right, then reflection 6 from above is $r^0 t^1$. It suffices to check the other reflections, in order from above:

1. 

2. 

3. 

4. 

5. 

In all the above, we were able to get the desired reflection with appropriate rotation and reflecting across the one diagonal. Thus, all the elements from number 4 are represented. $\square$