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MATH 574: Section H01
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MATH 574 Homework 10

Collaboration:

Problem 1 An integer is called *squarefree* if it is not divisible by the square of a positive integer greater than 1. Find the number of squarefree positive integers less than 100.

Solution.

Let X_n be the set of natural numbers less than 100 divisible by n^2. For any $n \geq 10$, X_n will be empty because n^2 will be at least 100, and a number cannot be divisible by another larger number. Also, we have that $X_9 \subseteq X_3$ (anything divisible by 81 is divisible by 9), $X_8 \subseteq X_4 \subseteq X_2$ (anything divisible by 64 is divisible by 16 is divisible by 4), and $X_6 \subseteq X_2 \cap X_3$ (anything divisible by 36 is divisible by both 4 and 9). So for counting the number of squarefree natural numbers less than 100, we only need to consider X_2, X_3, X_5, and X_7 because if a number is divisible by a relevant square, it will be in one of these sets.

Since there are 99 natural numbers less than 100, we want to determine

$$99 - |X_2 \cup X_3 \cup X_5 \cup X_7|.$$

By the principle of inclusion-exclusion, this is equal to

$$\begin{aligned}99 - (&|X_2| + |X_3| + |X_5| + |X_7| - |X_2 \cap X_3| - |X_2 \cap X_5| - |X_2 \cap X_6| - |X_3 \cap X_5| - |X_3 \cap X_7| - |X_5 \cap X_7| \\ &+ |X_2 \cap X_3 \cap X_5| + |X_2 \cap X_3 \cap X_7| + |X_2 \cap X_5 \cap X_7| + |X_3 \cap X_5 \cap X_7| - |X_2 \cap X_3 \cap X_5 \cap X_7|).\end{aligned}$$

For $X_m \cap X_n$ for coprime integers m, n, we can rewrite this as X_{mn} because x is divisible by m^2 and n^2 if and only if it is divisible by m^2n^2 for any $x \in \mathbb{Z}$. Similarly, we can write

$$X_{a_1} \cap X_{a_2} \cap \ldots \cap X_{a_n} = X_{a_1 a_2 \ldots a_n}.$$

So we can instead write

$$99 - (|X_2| + |X_3| + |X_5| + |X_7| - |X_6| - |X_{10}| - |X_{12}| - |X_{15}| - |X_{21}| - |X_{35}| + |X_{30}| + |X_{42}| + |X_{70}| + |X_{105}| - |X_{210}|).$$

As noted, X_n is empty for $n \geq 10$, so this is simply

$$99 - (|X_2| + |X_3| + |X_5| + |X_7| - |X_6|).$$

As discussed in class, the number of natural numbers less than b divisible by a is $\left\lfloor \frac{b-1}{a} \right\rfloor$, so $X_n = \left\lfloor \frac{99}{n^2} \right\rfloor$ by how we defined this set. So our final number is

$$\begin{aligned}99 - (|X_2| + |X_3| + |X_5| + |X_7| - |X_6|) &= 99 - \left(\left\lfloor \frac{99}{2^2} \right\rfloor + \left\lfloor \frac{99}{3^2} \right\rfloor + \left\lfloor \frac{99}{5^2} \right\rfloor + \left\lfloor \frac{99}{7^2} \right\rfloor - \left\lfloor \frac{99}{6^2} \right\rfloor \right) \\ &= 99 - \left\lfloor \frac{99}{4} \right\rfloor - \left\lfloor \frac{99}{9} \right\rfloor - \left\lfloor \frac{99}{25} \right\rfloor - \left\lfloor \frac{99}{49} \right\rfloor + \left\lfloor \frac{99}{36} \right\rfloor \\ &= 99 - 24 - 11 - 3 - 2 + 2 = 61.\end{aligned}$$

Problem 2 Give a double counting proof of the following: for $n, k \in \mathbb{N}$ with $k \leq n$,

$$\binom{n-1}{k-1} = \sum_{j=0}^{k-1} (-1)^j \binom{k}{j} \binom{n+k-1-j}{k-1-j}.$$

Hint: show that both sides counts the number of ways to place n indistinguishable objects in k distinguishable boxes such that each box gets at least one object.

Solution.

Suppose we have children $c_1, c_2, \ldots, c_k$ and we want to count the number of ways to distribute n indistinguishable cookies to them where each child gets at least one cookie.

One way to do this is to first give each child one cookie, which would leave $n-k$ cookies to be distributed. Then, as we've proved before, there are $\binom{n-k+k-1}{k-1} = \binom{n-1}{k-1}$ ways to distribute these remaining cookies.

Another way to do this is to count the number of ways to distribute the cookies where at least one child doesn't get a cookie, and then subtract this from the number of ways to distribute the cookies with no restrictions. Let X_i be the event that child c_i doesn't get any cookies. Then, we are looking for

$$\binom{n+k-1}{k-1} - |X_1 \cup X_2 \cup \ldots \cup X_k|.$$

By the principle of inclusion-exclusion (and distributing the negative), this is equal to

$$\binom{n+k-1}{k-1} - \sum_{1 \leq i \leq k} |X_i| + \sum_{1 \leq i \leq j \leq k} |X_i \cap X_j| - \sum_{1 \leq i \leq j \leq \ell \leq k} |X_i \cap X_j \cap X_\ell| + \ldots + (-1)^k |X_1 \cap X_2 \cap \ldots \cap X_k|.$$

We note that the last term must be 0, because we cannot distribute cookies in a such a way that all k children get no cookies.

To evaluate the rest of the sum, let $i_1, i_2, \ldots, i_j \in \{1, 2, \ldots, k\}$. Then, $|X_{i_1} \cap X_{i_2} \cap \ldots \cap X_{i_j}|$ is the number of ways to distribute n cookies such that children $c_{i_1}, c_{i_2}, \ldots, c_{i_j}$ get no cookies. So there are $k-j$ children left to distribute the n cookies to, which can be done in $\binom{n+(k-j)-1}{(k-j)-1} = \binom{n+k-1-j}{k-1-j}$ ways. Since there are $\binom{k}{j}$ ways to choose j children to not give cookies to, we can write our expression as

$$\binom{n+k-1}{k-1} - \binom{k}{1}\binom{n+k-1-1}{k-1-1} + \binom{k}{2}\binom{n-k-1-2}{k-1-2} - \binom{k}{3}\binom{n+k-1-3}{k-1-3}$$
$$+ \ldots + (-1)^j \binom{k}{j}\binom{n+k-1-j}{k-1-j} + \ldots + (-1)^{k-1}\binom{k}{k-1}\binom{n+k-1-(k-1)}{k-1-(k-1)} + 0.$$

Multiplying $\binom{n+k-1}{k-1}$ by $\binom{k}{0} = 1$, we can write this sum as

$$\sum_{j=0}^{k-1} (-1)^j \binom{k}{j} \binom{n+k-1-j}{k-1-j}.$$

Therefore, since

$$\binom{n-1}{k-1} \text{ and } \sum_{j=0}^{k-1} (-1)^j \binom{k}{j} \binom{n+k-1-j}{k-1-j}$$

both count the number of ways to give k children n cookies where each child gets at least one cookie, they are equal. $\square$

Problem 3 We want to tile a $1 \times m$ row using 1×1 colored tiles. Suppose we have n different colors to work with, and tiles of the same color are indistinguishable.

(a) For $m \geq n$, use Inclusion/Exclusion to determine the number of different ways to tile the $1 \times m$ row such that each color is used at least once.
(b) What can you say about part (a) when $m = n$? What should your summation simplify to?

Solution.

(a) We have colors $c_1, c_2, \ldots, c_n$, and we want to count the number of ways to tile a $1 \times m$ row using every color. Let X_i be the event that no tiles of color c_i are used. Then, there are

$$n^m - |X_1 \cup X_2 \cup \ldots \cup X_n|$$

such tilings, because n^m is the number of all tilings by the product rule and the union is all the ways to tile the row while not using one or more of the colors. By the principle of inclusion-exclusion (and distributing the negative), this is equal to

$$n^m - \sum_{1 \leq i \leq n} |X_i| + \sum_{1 \leq i \leq j \leq n} |X_i \cap X_j| - \sum_{1 \leq i \leq j \leq k \leq n} |X_i \cap X_j \cap X_k| + \ldots + (-1)^n |X_1 \cap X_2 \cap \ldots \cap X_n|.$$

To evaluate the sum, let $i_1, i_2, \ldots, i_j \in \{1, 2, \ldots, k\}$. Then, $|X_{i_1} \cap X_{i_2} \cap \ldots \cap X_{i_j}|$ is the number of ways to tile the row while not using colors $c_{i_1}, c_{i_2}, \ldots, c_{i_j}$. So there are $n - j$ colors left that can be used, which by the product rule can be done in $(n-j)^m$ ways. Since there are $\binom{k}{j}$ ways to choose j children to not give cookies to, we can write our expression as

$$n^m - \binom{n}{1}(n-1)^m + \binom{n}{2}(n-2)^m - \binom{n}{3}(n-3)^m + \ldots + (-1)^k \binom{n}{k}(n-k)^m + \ldots + (-1)^n \binom{n}{n}(n-n)^m.$$

Since we can write $n^m = \binom{n}{0}(n-0)^m$, the number of tilings using every color is

$$\sum_{k=0}^{n} (-1)^k \binom{n}{k}(n-k)^m.$$

(b) In this case, the sum should simplify to $n!$ because we will need to use exactly one tile of each color, so there are n choices for the first tile, $n-1$ for the second, $n-2$ for the third, and so on until there are 2 choices for the $(n-1)^{\text{th}}$ and 1 choice for the n^{th}. By the product rule, this is $n(n-1)(n-2)\ldots(2)(1) = n!$.

Problem 4 Consider the following relations defined on the set $\{a, b, c\}$. For each relation, determine whether it is symmetric, reflexive, transitive. If a property does not hold, give a reason why.
(a) $R_1 = \{(a,b), (a,c), (c,c), (b,b), (c,b), (b,c)\}$.
(b) $R_2 = \{(a,a), (b,b), (a,b), (b,a)\}$.

Solution.

(a)

- R_1 is not symmetric because $(a,b) \in R_1$ but $(b,a) \notin R_1$.
- R_1 is not reflexive because $(a,a) \notin R_1$.
- R_1 is transitive because for every $x, y, z \in \{a, b, c\}$, $(x,y), (y,z) \in R_1 \implies (x,z) \in R_1$.

(b)

- R_2 is symmetric because for every $x, y \in \{a, b, c\}$, $(x, y) \in R_2 \implies (y, x) \in R_2$.
- R_2 is not reflexive because $(c, c) \notin R_2$.
- R_2 is transitive because for every $x, y, z \in \{a, b, c\}$, $(x, y), (y, z) \in R_1 \implies (x, z) \in R_2$.

Problem 5 Let A be the set of all people in the world. Consider the following relations defined on the set $\{a, b, c\}$. For each relation, determine whether it is symmetric, reflexive, transitive. If a property does not hold, give a reason why.
(a) R_1 defined on A as xR_1y if person x and person y are born in the same year.
(b) R_2 defined on A as xR_2y if the heights of person x and person y are within two inches of each other.
(c) R_3 defined on A as xR_3y if person x has met person y.

Solution.

(a)

- R_1 is symmetric because if person A is born in the same year as person B, then person B is born in the same year as person A.
- R_1 is reflexive because every person is born in the same year as oneself.
- R_1 is transitive because if person A is born in the same year as person B and person B is born in the same year as person C then person A is born in the same year as person C (they are all born in the same year).

(b)

- R_2 is symmetric because if person A's height is within 2 inches of person B's, then person B's height will be within 2 inches of person A's as only the direction and not magnitude will change.
- R_2 is reflexive because one has the same highest as oneself (which is obviously a difference less than 2 inches).
- R_2 is not transitive. Suppose Alice is 58.5 inches tall, Bob is 60 inches tall, and Charlie is 61.5 inches tall. Then Alice's height is within two inches of Bob's height and Bob's height is within two inches of Charlie's height, so both pairs are in the relation. However, Alice's height is not within 2 inches of Charlie's height because $|61.5 - 58.5| = 3 > 2$, so this pair is not in the relation which could not be the case if R_2 were transitive.

(c)

- R_3 is symmetric because if person A has met person B, person B has met person A (assuming person B was paying attention?)
- It may make sense to say R_3 isn't reflexive. If this relation is being used to make a social network graph, having self-edges would be unnecessarily complicated, and it would be odd to say one has met oneself. However, if one can meet oneself, then R_3 is reflexive.
- R_3 is not transitive. If Bob is popular, then it is very possible (in reality, very probable) that he has met two people that have never met each other.

Problem 6 State whether each of the following relations is an symmetric, reflexive, transitive. If a property does not hold, give a reason why.
(a) R_1 defined on $\mathbb{R}$ as xR_1y if and only if $xy \geq 0$.
(b) R_2 defined on $\mathbb{R}$ as xR_2y if and only if $xy > 0$.
(c) R_3 defined on $\mathbb{R}$ as xR_3y if and only if $|x - y| < 1$.

Solution.

(a)

- R_1 is symmetric. If $xy \geq 0$, $yx \geq 0$.
- R_1 is reflexive because no real number times itself is negative.
- R_1 is not transitive: $(1, 0), (0, -1) \in R_1$ because $(1)(0) \geq 0$ and $(0)(-1) \geq 0$, but $(1, -1) \notin R_1$ because $(1)(-1) \ngeq 0$.

(b)

- R_2 is symmetric. If $xy > 0$, $yx > 0$.
- R_2 is not reflexive: $(0, 0) \notin R_2$ because $(0)(0) \not> 0$.
- R_2 is transitive. If $(x, y) \in R_2$, then x, y have the same sign, so if $(x, y), (y, z) \in R_2$, then x, z must have the same sign and thus are in R_2.

(c)

- R_3 is symmetric because $|x - y| = | - (x - y)| = |y - x|$.
- R_3 is reflexive because $|x - x| = 0 < 1$.
- R_3 is not transitive because $(0, 0.75), (0.75, 1.5) \in R_3$ because $|0 - 0.75| = |0.75 - 1.5| = 0.75 < 1$ but $(0, 1.5) \notin R_3$ because $|0 - 1.5| = 1.5 \not< 1$.

Problem 7 Let R_1 be a relation on the set A and R_2 be a relation on the set B. Let R be the relation defined on $A \times B$ such that $(a, b)R(a', b')$ if and only if aR_1a' and bR_2b'. Prove that if R_1 and R_2 are equivalence relations then R is also an equivalence relation.

Solution.

Assume that R_1 and R_2 are equivalance relations. Thus, they are both symmetric, reflexive, and transitive.

First, let $a, a' \in A$ and $b, b' \in B$ and assume $(a, b)R(a', b')$. Then, aR_1a' and bR_2b' by definition. Since R_1 and R_2 are symmetric, we also have $a'R_1a$ and $b'R_2b$. Then, $(a', b')R(a, b)$ by definition. So R is symmetric.

Next, let $a \in A$ and $b \in B$ and assume $(a, b) \in A \times B$. Since R_1 and R_2 are reflexive, aR_1a and bR_2b. Then, $(a, b)R(a, b)$ by definition. So R is reflexive.

Finally, let $a, a', a'' \in A$ and $b, b', b'' \in B$, and assume $(a, b)R(a', b')$ and $(a', b')R(a'', b'')$. Then, aR_1a' and $a'R_1a''$, so since R_1 is transitive aR_1a''. Similarly, bR_2b' and $b'R_2b''$, so since R_2 is transitive bR_2b''. Then, $(a, b)R(a'', b'')$ by definition. So R is transitive.

Since R is symmetric, reflexive, and transitive, it is an equivalence relation.