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MATH 575: Section H01
Professor: Dr. Luo
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MATH 575 Homework 10

Collaboration: I discussed some of the problems with Sam, Jack, and Chance.

Problem 1 Use Kempe chains to prove that every planar graph with at most 11 vertices is 4-colorable.

NOTE: Do NOT use the Four Color Theorem.

Solution.

We claim that any planar graph G on at most 11 vertices has a vertex with degree at most 4. We have shown in class that we have

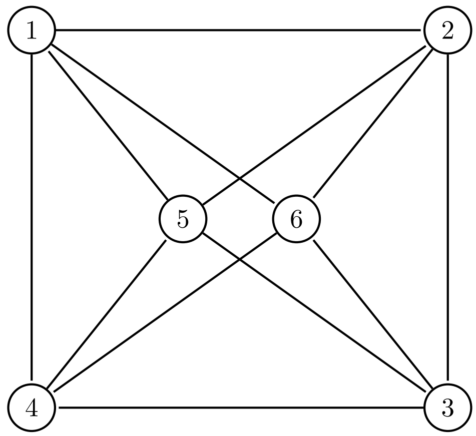
$$\sum_{v \in V(G)} = 2|E(G)| \leq 6n - 12 \implies \overline{d}(G) \leq \frac{6n-12}{n} = 6 - \frac{12}{n},$$

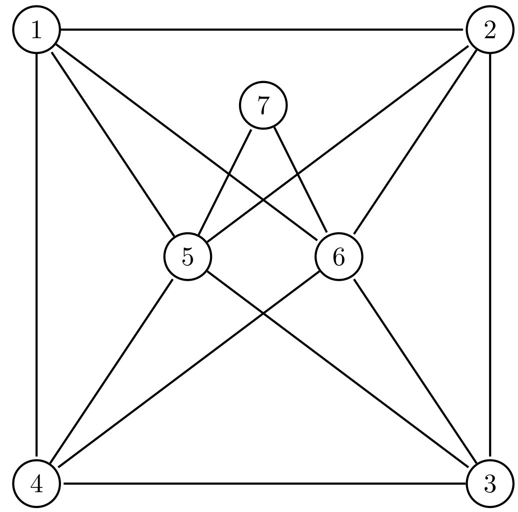
where $\overline{d}(G)$ is the average degree. Since $n < 12$, we have $\overline{d}(G) < 5$ and so we have a vertex with degree at most 4.

We will induct on n. Clearly, any graph on at most 4 vertices is 4-colorable. Let $n \in \mathbb{N}$, $4 < n \leq 11$, and assume that for all $n' < n$, any graph on n' vertices is 4-colorable. Let G be a graph on n vertices, and $v \in V(G)$ be a vertex with $d(v) \leq 4$. Since $G - v$ is 4-colorable by the induction hypothesis, it suffices to properly color v. If $d(v) < 4$, then we can color the neighborhood and v with all different colors, so we will consider $d(v) = 4$.

Let v have neighbors v_1, v_2, v_3, v_4 in clockwise position around v in the plane drawing, and suppose v_i is colored with color i. Let $G_{i,j}$ be the subgraph induced on $G - v$ by the vertices colored i or colored j. If v_1 and v_3 are in separate components in $G_{1,3}$, then we can switch the colors in one of the components so that both v_1 and v_3 have color 1, and then color v with color 3. If they are in the same component, then we have a Kempe chain between v_1 and v_3, and since G is planar, we cannot have also a Kempe chain between v_2 and v_4. In this case, we can switch the color in the one of the components so that both v_2 and v_4 have color 2, and then color v with color 4. Therefore, G is 4-colorable. □

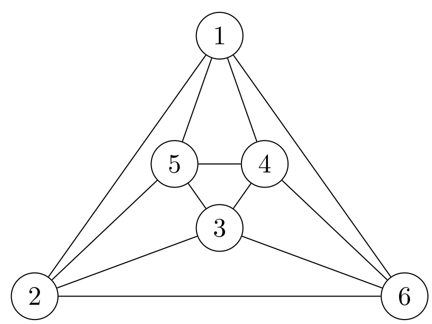
Problem 2 Determine if the following graphs are planar or nonplanar. If it is planar, give a plane drawing. If it is nonplanar, demonstrate the existence of a K_5 or $K_{3,3}$ subdivision.



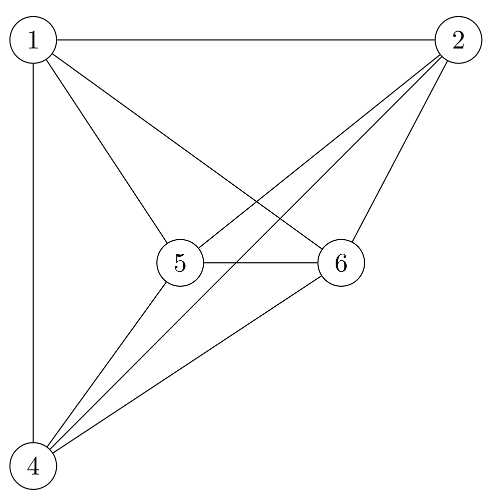


Solution.

The first graph is planar, so we can draw this plane drawing:



The second graph is not planar, because it is a subdivision of a K_5. Consider a K_5 with vertices $1, 2, 4, 5, 6$:

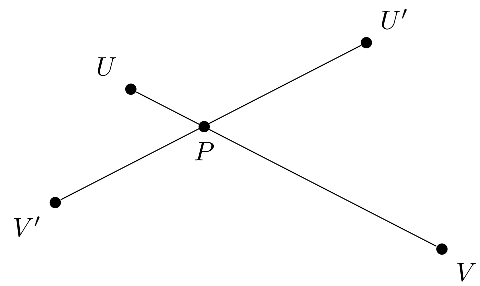


We can see that in the original graph, the edge between 2 and 4 in K_5 is subdivided by 3, and the edge between 5 and 6 in K_5 is subdivided by 7. So the graph is a subdivision of a K_5 and therefore is not planar.

Problem 3 Let X be a set of n points in $\mathbb{R}^2$ such that the (Euclidean) distance between any pair of distinct points in X is at least 1. Prove that there are at most $3n-6$ pairs of points with distance exactly 1.

Solution.

Let G be a graph with $V(G) = X$ and edges between points with Euclidean distance exactly 1, and consider a drawing of G in the Cartesian plane where points are drawn at their coordinates and edges are drawn as straight lines between the vertices. We claim this is a plane drawing of G. To see this, suppose to the contrary that there exist $U, V, U', V' \in V(G)$ with $UV, U'V' \in E(G)$ such that the two edges cross at some point P in the drawing:



By our definition of the edge set, we have that $|\overline{UV}| = |\overline{U'V'}| = 1$. Without loss of generality, assume that $|\overline{UP}| \leq |\overline{VP}|$ and $|\overline{U'P}| \leq |\overline{V'P}|$. Then, we have from the triangle inequality that $|\overline{UU'}| \leq |\overline{UP}| + |\overline{U'P}|$, and since $\overline{UP}$ and $\overline{U'P}$ both have length at most $\frac{1}{2}$, we have $|\overline{UU'}| \leq 1$. If $|\overline{UU'}| < 1$, this is a contradiction, because every pair of points in $V(G)$ are distance at least 1 from each other. If $|\overline{UU'}| = 1$, then $\overline{UV}$ and $\overline{U'V'}$ must be colinear and we would have $|UV'| < 1$, a contradiction.

So no edges in the drawing cross, and thus the drawing is a plane drawing of G. Since G is planar, we have $|E(G)| \leq 3n-6$, and since each edge corresponds to a pair of points with Euclidean distance exactly 1, there are at most $3n-6$ such pairs. □

Problem 4 Recall that a graph is outerplanar if it has a plane drawing with all of its vertices touching the outer face.

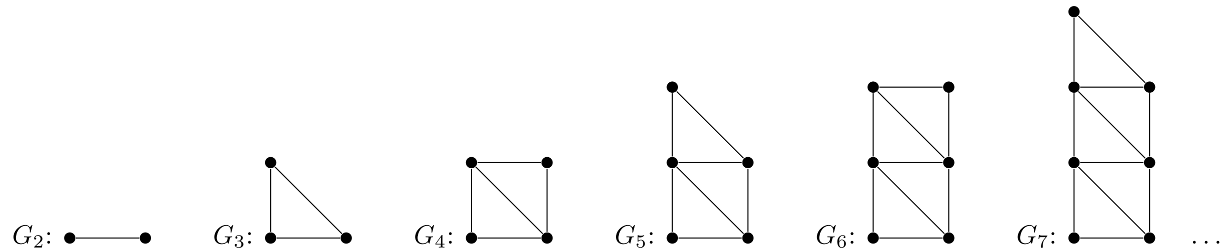
(a) Let $n \geq 2$. Prove that every n-vertex outerplanar graph has at most $2n-3$ edges.

(b) Show that part (a) is best possible for all $n \geq 2$ by iteratively constructing graphs $G_2, G_3, G_4, \ldots$ such that G_n is an n-vertex, outerplanar graph with $2n-3$ edges.

Solution.

(a) We will induct on n. First, let $n = 2$. We can have at most $2(2) - 3 = 1$ edge in any graph on 2 vertices, so the claim holds. Next, let $n \in \mathbb{N}$, $n > 2$, and assume that for all $n' < n$, any outerplanar graph on n' vertices has at most $2n' - 3$ edges. We have proven in class that any outerplanar graph has a vertex with degree at most 2, so we can choose a vertex $v \in V(G)$ with $d(v) \leq 2$. Now, consider $G - v$, which has $n-1$ vertices. By the induction hypothesis, $G - v$ has at most $2(n-1) - 3 = 2n - 5$ edges. Then, since v has at most two neighbors, the number of edges increases by at most 2 when we add it back, so G has at most $2n - 5 + 2 = 2n - 3$ edges. □

(b) We can start with a K_2, and to construct G_n for $n > 2$, connect a vertex to the two endpoints of an edge that was added to make G_{n-1}. Then, since equality holds for K_2, it will continue holding since we add two edges for every vertex. G_n will also be outerplanar, because we are never enclosing vertices already added in an inner face. For example, the first few constructions are shown:



Problem 5 Let G be an n-vertex graph. Suppose for some $t \in \mathbb{N}$ that $d(u) + d(v) \geq n - t$ for every pair of distinct non-adjacent vertices $u, v \in V(G)$. Prove that the vertices of G can be partitioned into at most t pairwise-disjoint paths.

Hint: construct a new graph in which Ore's Theorem can be applied.

Solution.

Construct a graph G' where we add $T = \{v_1, v_2, \ldots, v_t\}$ to G, and connect each of the new vertices to each vertex in G. Then, for all $u \in V(G)$, we will have $d_{G'}(v) - t = d_G(v)$, and thus for every $u, v \in V(G)$,

$$d_{G'}(u) - t + d_{G'}(v) - t = d_G(u) + d_G(v) \geq n - t \implies d_{G'}(u) + d_{G'}(v) \geq n + t.$$

Since G' has $n + t$ vertices, we have from Ore's Theorem that G' has a Hamiltonian cycle. This cycle will contain $v_1, v_2, \ldots, v_t$ and the other vertices will be from G. Let $\{P_{1,2}, P_{2,3}, \ldots, P_{t-1,t}, P_{t,1}\}$ be the set of vertices where $P_{i,j}$ is the path in the Hamiltonian cycle from v_i to v_j, not including the endpoints. This set partitions $V(G)$ into t pairwise-disjoint paths, because every vertex from $V(G)$ is between exactly one pair of vertices from T in the Hamiltonian cycle. $\square$