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Analysis in $\mathbb{R}^n$: Section 1
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Analysis in $\mathbb{R}^n$ Homework 1

Problem 7 Let $A_1, A_2, A_3, \ldots$ be subsets of a metric space.

(a) If $B_n = \bigcup_{i=1}^{n} A_i$, prove that $\overline{B_n} = \bigcup_{i=1}^{n} \overline{A_i}$, for $n = 1, 2, 3, \ldots$

(b) If $B = \bigcup_{i=1}^{\infty} A_i$, prove that $\overline{B} \supset \bigcup_{i=1}^{\infty} \overline{A_i}$. Show, by an example, that this inclusion can be proper.

Solution.

(a) We will first prove a lemma: if X, Y are subsets of a metric space, then $\overline{X \cup Y} = \overline{X} \cup \overline{Y}$. We have

$$\begin{aligned}\overline{X} \cup \overline{Y} &= (X \cup LP(X)) \cup (Y \cup LP(Y)) \\ &= (X \cup Y) \cup (LP(X) \cup LP(Y)),\end{aligned}$$

so it suffices to show $LP(X) \cup LP(Y) = LP(X \cup Y)$. Let $x \in LP(X \cup Y)$ and suppose toward contradiction that $x \notin LP(X) \cup LP(Y)$. Then, for all $r > 0$, $(X \cup Y) \cap (B_r(x) \backslash \{x\}) \neq \emptyset$. Using the distributive law of sets, we also have $(X \cap B_r(x) \backslash \{x\}) \cup (Y \cap B_r(x) \backslash \{x\}) \neq \emptyset$ for all $r > 0$. But since $x \in LP(X) \cup LP(Y)$, there exists an $r_0 > 0$ such that $X \cap B_{r_0}(x) \backslash \{x\} = \emptyset$ and $Y \cap B_{r_0}(x) \backslash \{x\} = \emptyset$, so $(X \cap B_{r_0}(x) \backslash \{x\}) \cup (Y \cap B_{r_0}(x) \backslash \{x\}) = \emptyset$, a contradiction. So $LP(X \cup Y) \subset LP(X) \cup LP(Y)$, and reverse reasoning can be used to show that $LP(X) \cup LP(Y) \subset LP(X \cup Y)$. Thus, we have $LP(X) = LP(Y)$ and therefore $\overline{X \cup Y} = \overline{X} \cup \overline{Y}$, so the lemma holds.

Next, we will induct on n. Clearly, the base case holds because

$$\overline{B_1} = \overline{\bigcup_{i=1}^{1} A_i} = \overline{A_1} = \bigcup_{i=1}^{1} \overline{A_i}.$$

For the induction step, let $n \in \mathbb{N}$, and assume that $\overline{B_n} = \bigcup_{i=1}^{n} \overline{A_i}$. The same then holds for $n + 1$:

$$\begin{aligned}\overline{B_{n+1}} &= \overline{\bigcup_{i=1}^{n+1} A_i} && \text{(definition of } B_{n+1}\text{)} \\ &= \overline{\bigcup_{i=1}^{n} A_i \cup A_{n+1}} && \text{(splitting union)} \\ &= \overline{B_n \cup A_{n+1}} && \text{(definition of } B_n\text{)} \\ &= \overline{B_n} \cup \overline{A_{n+1}} && \text{(lemma)} \\ &= \bigcup_{i=1}^{n} \overline{A_i} \cup \overline{A_{n+1}} && \text{(induction hypothesis)}\end{aligned}$$

$$= \bigcup_{i=1}^{n+1} \overline{A_i}. \qquad \text{(combining union)}$$

$\square$

(b) Let $x \in \bigcup_{i=1}^{\infty} \overline{A_i}$ and assume toward contradiction that $x \notin \overline{B}$. Since $x \in \bigcup_{i=1}^{\infty} \overline{A_i}$, there exists at least one $i \in \mathbb{N}$ such that $x \in \overline{A_i}$. So by definition, $x \in A_i \cup LP(A_i)$.

- Case 1: x is in A_i but is not in $LP(A_i)$. Then, we have

$$x \in A_i \subset \bigcup_{i=1}^{\infty} A_i \subset \bigcup_{i=1}^{\infty} A_i \cup LP\left(\bigcup_{i=1}^{\infty} A_i\right) = \overline{\bigcup_{i=1}^{\infty} A_i} = \overline{B}.$$

  So $x \in \overline{B}$, a contradiction.

- Case 2: x is in $LP(A)$. Since $x \notin \overline{B}$, $x \notin LP(B)$ and by definition there exists $r > 0$ such that $B \cap B_r(x) \backslash \{x\} = \emptyset$. But since A_i is a subset of B (B is a union of A_i and other sets), we must also have $A \cap B_r(x) \backslash \{x\} = \emptyset$, which contradicts x being a limit point of A.

Therefore, we must have $\overline{B} \supset \bigcup_{i=1}^{\infty} \overline{A_i}$. However, equality does not necessarily hold. For example, consider the collection of circles in $\mathbb{R}^2$ centered at the origin with radii $1/i$:

$$A_i = \left\{(x,y) \in \mathbb{R}^2 : x^2 + y^2 = \frac{1}{i^2}\right\}.$$

Then, $B = \bigcup_{i=1}^{\infty} A_i$ has $(0,0)$ as a limit point, because for any $r > 0$, $B_r\big((0,0)\big)$ will contain (infinitely many) circles in B with radius less than r. However, there is no $i \in \mathbb{N}$ such that A_i contains $(0,0)$ as a limit point, because $B_{i/2}\big((0,0)\big)$ contains no points from A_i. Thus, $(0,0)$ is in B but not in $\bigcup_{i=1}^{\infty} \overline{A_i}$. $\square$

Problem 10 Let X be an infinite set. For $p \in X$ and $q \in X$, define

$$\begin{cases} 1 & (\text{if } p \neq q) \\ 0 & (\text{if } p = q). \end{cases}$$

Prove that this is a metric. Which subsets of the resulting metric space are open? Which are closed?

Solution.

We claim this satisfies all the axioms and thus is a metric:

- 1 and 0 are non-negative, so $d(x,y) \geq 0$ for all $x, y \in X$. Additionally, $d(x,y) = 0$ if and only if $x = y$ by definition.
- Equality and inequality are both symmetric, so if $x = y$, then $y = x$ and $d(x,y) = d(y,x) = 0$. If $x \neq y$, then $y \neq x$ and $d(x,y) = d(y,x) = 1$.
- Let $x, y \in X$. If $x = y$, then $d(x,y) = 0$ so the triangle inequality must hold (there is no way to travel less than 0 distance). If $x \neq y$, then $d(x,y) = 1$. The only way for the distance to be less is if it is 0, which is impossible because $d(x,z) + d(z,y)$ could only equal 0 if $x = z = y$, but x and y are different points.

Let $A \subset X$, $a \in A$, and $r = \frac{1}{2}$. Then, we observe that $A \cap B_r(a) \backslash \{a\} = \emptyset$, because all other points are distance 1 away. So a cannot be a limit point, and thus $LP(A) = \emptyset$. So A is closed because it vacuously contains all its limit points, and also A is open because with the same reasoning A^C is closed. So every subset of X is both open and closed.

Problem 1 Consider $(\mathbb{R}, d)$ with the standard metric, and $A = \left\{\frac{1}{n} : n \in \mathbb{N}\right\}$. Prove that $LP(A) = \{0\}$.

Solution.

We first show $\{0\} \subset LP(A)$. Fix $r > 0$, and consider $n = \left\lceil \frac{1}{r} \right\rceil + 1$. Then, $\frac{1}{n} \in A$ because $n \in \mathbb{N}$, and $\frac{1}{n} < r$. So $A \cap B_r(0) \backslash \{0\} \neq \emptyset$, and thus 0 is a limit point of A.

We next show $LP(A) \subset \{0\}$. Let $x \in (0, 1)$ (clearly no numbers in $(-\infty, 0) \cup [1, \infty]$ are limit points because the open balls extending from the left to 0 or from the right to 1 have no points in A).

Case 1: $x = \frac{1}{n}$ for some $n \in \mathbb{N}$. Consider

$$r = \frac{1}{n} - \frac{1}{n+1}.$$

Then, $A \cap B_r(x) \backslash \{x\}$ has $\frac{1}{n+1}$ on its boundary but nothing closer, so it is empty and x is not a limit point.

Case 2: $x \in \left(\frac{1}{n_1}, \frac{1}{n_2}\right)$ for some $n_1, n_2 \in \mathbb{N}$ with $n_2 - n_1 = 1$. Both $\frac{1}{n_1}$ and $\frac{1}{n_2}$ are candidates for being the closest points in x to A, so consider

$$r = \min\left\{\left|x - \frac{1}{n_1}\right|, \left|x - \frac{1}{n_2}\right|\right\}.$$

Then, $A \cap B_r(x) \backslash \{x\} = \emptyset$, so x is not a limit point.

Therefore, since $\{0\} \subset LP(A)$ and $LP(A) \subset \{0\}$, the sets are equal. □

Problem 2 Let $1 < p < \infty$ and $q \in \mathbb{R}$ be such that $\frac{1}{p} + \frac{1}{q} = 1$ (p and q are said to be conjugate exponents.)

(a) [Young's inequality] Prove that $xy \leq \frac{x^p}{p} + \frac{y^q}{q}$ for all $x, y \geq 0$. (Hint: Fix y, p, q and consider a convenient $f(x)$.)

(b) Let $x = (x_1, \ldots, x_n), y = (y_1, \ldots, y_n) \in \mathbb{R}^n$ satisfy $\sum_{i=1}^n |x_i|^p = \sum_{i=1}^n |y_i|^q = 1$. Show that

$$\sum_{i=1}^n |x_i y_i| \leq 1.$$

(c) [Hölder's inequality] Prove that for any two elements $x = (x_1, \ldots, x_n), y = (y_1, \ldots, y_n) \in \mathbb{R}^n$,

$$\sum_{i=1}^n |x_i y_i| \leq \left(\sum_{i=1}^n |x_i|^p\right)^{\frac{1}{p}} \left(\sum_{i=1}^n |y_i|^q\right)^{\frac{1}{q}} = \|x\|_p \|y\|_q.$$

(d) [Minkowski's inequality] Prove that for any $x, y \in \mathbb{R}^n$, $\|x + y\|_p \leq \|x\|_p + \|y\|_p$. (Hint: observe that $\sum_{i=1}^n |x_i + y_i|^p \leq \sum_{i=1}^n |x_i + y_i|^{p-1}|x_i| + \sum_{i=1}^n |x_i + y_i|^{p-1}|y_i|$, and apply Minkowski's inequality to the first term to extract an $\|x\|_p$.)

(e) Prove that $(\mathbb{R}^n, d_p)$ is a metric space.

Solution.

(a) Fix $y \in \mathbb{R}_{\geq 0}$. It suffices to show that $f(x) = \frac{x^p}{p} + \frac{y^q}{q} - xy$ is non-negative for all $x \in \mathbb{R}_{\geq 0}$. We compute $f'(x) = x^{p-1} - y$, so $f(x)$ has its critical point at $x = \sqrt[p-1]{y}$. Since x is non-negative and $p > 1$, $f'(x)$ must be increasing at this critical point, so it must be the minimum of $f(x)$. Thus, it suffices to show that the minimum $x = \sqrt[p-1]{y}$ is non-negative. We find that

$$\begin{aligned}
f\left(\sqrt[p-1]{y}\right) &= \frac{\left(\sqrt[p-1]{y}\right)^p}{p} + \frac{y^q}{q} - y\left(\sqrt[p-1]{y}\right) \\
&= \frac{y^{\frac{p}{p-1}}}{p} + \frac{y^q}{q} - y^{1+\frac{1}{p-1}} && \text{(rewriting exponents)} \\
&= \frac{1}{p}\left(y^{\frac{p}{p-1}}\right) + \frac{y^q}{q} - y^{\frac{p}{p-1}} \\
&= -\left(1 - \frac{1}{p}\right)\left(y^{\frac{p}{p-1}}\right) + \frac{y^q}{q} \\
&= -\left(1 - \frac{1}{p}\right)\left(y^{\frac{p}{p-1}}\right) + \frac{y^{\frac{p}{p-1}}}{\frac{1}{1-\frac{1}{p}}} && \left(q = \tfrac{1}{1-\frac{1}{p}} = \tfrac{p}{p-1}\right) \\
&= -\left(1 - \frac{1}{p}\right)\left(y^{\frac{p}{p-1}}\right) + \left(1 - \frac{1}{p}\right)\left(y^{\frac{p}{p-1}}\right) \\
&= 0,
\end{aligned}$$

so $f(x)$ is non-negative everywhere. Therefore, $xy \leq \frac{x^p}{p} + \frac{y^q}{q}$, with equality holding at $x = \sqrt[p-1]{y}$. $\square$

(b) Let $i \in \{1, 2, \ldots, n\}$. From Young's inequality, we have

$$|x_i y_i| \leq \frac{|x_i|^p}{p} + \frac{|y_i|^q}{q} \leq |x_i|^p + |y_i|^p.$$

Since this holds for all i, we have

$$\begin{aligned}
\sum_{i=1}^n |x_i y_i| &\leq \sum_{i=1}^n \frac{|x_i|^p}{p} + \sum_{i=1}^n \frac{|y_i|^q}{q} \\
&= \frac{1}{p}\sum_{i=1}^n |x_i|^p + \frac{1}{q}\sum_{i=1}^n |y_i|^q && \text{(sum property)} \\
&= \frac{1}{p} + \frac{1}{q} && \text{(given that sums equal 1)} \\
&= 1. && \text{(p, q are conjugate pairs)}
\end{aligned}$$

$\square$

(c) Consider the unit vectors $\frac{x}{\|x\|_p}$ and $\frac{y}{\|y\|_p}$. Then, we observe that

$$\begin{aligned}
&\frac{1}{\|x\|_p \|y\|_q} \sum_{i=1}^n |x_i y_i| \\
&= \sum_{i=1}^n \left| \frac{x_i}{\|x\|_p} \frac{y_i}{\|y\|_q} \right| && \text{(sum property)} \\
&\leq 1 && \text{(from part \textbf{(b)})} \\
&= \left(\sum_{i=1}^n \left(\frac{|x_i|}{\|x\|_p} \right)^p \right)^{1/p} \left(\sum_{i=1}^n \left(\frac{|y_i|}{\|y\|_q} \right)^q \right)^{1/q} && \text{(unit vectors have length 1)}
\end{aligned}$$

$$= \left(\left(\frac{1}{\|x\|_p} \right)^p \sum_{i=1}^n |x_i|^p \right)^{1/p} \left(\left(\frac{1}{\|y\|_q} \right)^q \sum_{i=1}^n |y_i|^q \right)^{1/q} \qquad \text{(sum property)}$$

$$= \frac{1}{\|x\|_p \|y\|_q} \left(\sum_{i=1}^n |x_i|^p \right)^{\frac{1}{p}} \left(\sum_{i=1}^n |y_i|^q \right)^{\frac{1}{q}}, \qquad \text{(exponent property)}$$

which after multiplying by $\|x\|_p \|y\|_q$ implies that

$$\sum_{i=1}^n |x_i y_i| \le \left(\sum_{i=1}^n |x_i|^p \right)^{\frac{1}{p}} \left(\sum_{i=1}^n |y_i|^q \right)^{\frac{1}{q}} = \|x\|_p \|y\|_q.$$

□

(d) We observe that

$$\begin{aligned}
\sum_{i=1}^n |x_i + y_i|^p &= \sum_{i=1}^n |x_i + y_i| |x_i + y_i|^{p-1} \\
&\le \sum_{i=1}^n (|x_i| + |y_i|) |x_i + y_i|^{p-1} && \text{(triangle inequality)} \\
&= \sum_{i=1}^n |x_i| |x_i + y_i|^{p-1} + \sum_{i=1}^n |y_i| |x_i + y_i|^{p-1} && \text{(sum property)} \\
&= \sum_{i=1}^n |x_i (x_i + y_i)^{p-1}| + \sum_{i=1}^n |y_i (x_i + y_i)^{p-1}| && \text{(absolute value property)} \\
&\le \left(\sum_{i=1}^n |x_i|^p \right)^{\frac{1}{p}} \left(\sum_{i=1}^n |x_i + y_i|^{\frac{p(p-1)}{p-1}} \right)^{\frac{p-1}{p}} \\
&\quad + \left(\sum_{i=1}^n |y_i|^p \right)^{\frac{1}{p}} \left(\sum_{i=1}^n |x_i + y_i|^{\frac{p(p-1)}{p-1}} \right)^{\frac{p-1}{p}} && \text{(Hölder's inequality)} \\
&= \left[\left(\sum_{i=1}^n |x_i|^p \right)^{\frac{1}{p}} + \left(\sum_{i=1}^n |y_i|^p \right)^{\frac{1}{p}} \right] \left(\sum_{i=1}^n |x_i + y_i|^p \right)^{\frac{p-1}{p}} && \text{(redistributing)}
\end{aligned}$$

$$\implies \left(\sum_{i=1}^n |x_i + y_i|^p \right)^{\frac{p}{p}} \le \left[\left(\sum_{i=1}^n |x_i|^p \right)^{\frac{1}{p}} + \left(\sum_{i=1}^n |y_i|^p \right)^{\frac{1}{p}} \right] \left(\sum_{i=1}^n |x_i + y_i|^p \right)^{\frac{p-1}{p}}$$

$$\implies \left(\sum_{i=1}^n |x_i + y_i|^p \right)^{\frac{1}{p}} \le \left(\sum_{i=1}^n |x_i|^p \right)^{\frac{1}{p}} + \left(\sum_{i=1}^n |y_i|^p \right)^{\frac{1}{p}}$$

$$\implies \|x + y\|_p \le \|x\|_p + \|y\|_p. \qquad \text{(p-norm definition)}$$

□

(e) We claim this satisfies all the axioms and thus is a metric:

- Every distance is a sum of absolute values and thus is non-negative. Additionally, $d_p(x, y) = 0$ if $x = y$ (all the components will be the same and thus the distance is a sum of 0's) and only if $x = y$ (if two components differ, it will contribute to the sum and make it positive).

- Since the sum involves absolute values, we have $d_p(x, y) = d_p(y, x)$ for all $x, y \in \mathbb{R}^n$ because the order of subtraction has no affect.
- From **(d)**, we have $\|x + y\|_p \leq \|x\|_p \|y\|_p$ for all $x, y \in \mathbb{R}^n$.

□