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MATH 544: Section H01
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MATH 544 Homework 1

Problem 1 Let A, $B \in \text{Mat}_{n\times n}$. The goal of this problem is to show that $\text{rank}(AB) = n$ if and only if $\text{rank}(A) = \text{rank}(B) = n$.
(a) Suppose that $\text{rank}(A) = \text{rank}(B) = n$. Show that $\text{rank}(AB) = n$.
(b) Suppose that $\text{rank}(AB) = n$.
i. First, prove that $\text{rank}(B) = n$.
ii. Show that $\text{rank}(A) = n$.

For (a) and and the first part of (b), think about conditions on a matrix $C \in \text{Mat}_{n\times n}$ which are equivalent to $\text{rank}(C) = \ n$ (C is invertible, $C\vec{x} = \vec{0}$ has only the trivial solution $\vec{x} = \vec{0}$, C is row-equivalent to I_n,...). For the second part of (b), one method is to apply part (a) to the matrices AB and B^{-1}. (This is why the problem asks you to prove $\text{rank}(B) = n$ first, *before* it asks you to prove $\text{rank}(A) = n$.)

Solution.

We have proved in class that the following are equivalent for a matrix $M \in \text{Mat}_{n\times n}$:

1. We have $\text{rank}(M) = n$.
2. M is invertible.
3. There exist elementary matrices $E_1, E_2, \ldots, E_k$ such that $M = E_k \ldots E_2E_1$.
4. $M\vec{x} = \vec{0}$ has only the trivial solution.

We will refer to this hereafter as The Theorem. (This could be construed by some as overly dramatic, but we aren't sure if it is even dramatic enough.)

(a) Since $\text{rank}(A) = \text{rank}(B) = n$, there exist elementary matrices $E_1, E_2, \ldots, E_k$ such that $A = E_k \ldots E_2E_1$ and elementary matrices $E_1', E_2', \ldots, E_{k'}'$ such that $B = E_{k'}' \ldots E_2'E_1'$. So we have

$$AB = (E_k \ldots E_2E_1)\left(E_{k'}' \ldots E_2'E_1'\right),$$

and thus by associativity AB is the product of elementary matrices. So by The Theorem, $\text{rank}(AB) = n$.

(b) *i.* Suppose that $\text{rank}(B) \neq n$. Then, by The Theorem, there exists some $\vec{c} \neq \vec{0}$ such that $B\vec{c} = \vec{0}$. Since $A(\vec{0}) = \vec{0}$, we have $A(B\vec{c}) = \vec{0}$. By associativity, $(AB)\vec{c} = \vec{0}$, so $(AB)\vec{x} = \vec{0}$ has $\vec{c}$ as a non-trivial solution. But then by The Theorem $\text{rank}(AB) \neq n$, a contradiction.

ii. We have that $\text{rank}(AB) = \text{rank}(B) = n$, so B^{-1} exists by The Theorem. Also by The Theorem, we have elementary matrices $E_1, E_2, \ldots, E_k$ such that $AB = E_k \ldots E_2E_1$ and elementary matrices $E_1', E_2', \ldots, E_{k'}'$ such that $B^{-1} = E_{k'}' \ldots E_2'E_1'$. Then, we have

$$A = A\left(BB^{-1}\right) = (AB)B^{-1} = (E_k \ldots E_2E_1)\left(E_{k'}' \ldots E_2'E_1'\right),$$

and thus by associativity A is a product of elementary matrices. So by The Theorem, $\text{rank}(A) = n$.

Problem 2

(a) Give an example of A, $B \in \text{Mat}_{2\times 2}$ such that A and B are invertible, but $A+B$ is not invertible.

(b) Give an example of A, $B \in \text{Mat}_{2\times 2}$ such that neither A nor B is invertible, but $A+B$ is invertible.

Solution.

(a) Let $A = I_2$, $B = -I_2$. Then, $A^{-1} = I_2$ and $B^{-1} = -I_2$, but $A+B$ is not invertible since $\text{rank}(O_{2\times 2}) = 0$.

(b) Let $A = \begin{pmatrix} 1 & 0 \\ 0 & 0 \end{pmatrix}$, $B = \begin{pmatrix} 0 & 0 \\ 0 & 1 \end{pmatrix}$. Then, $\text{rank}(A) = \text{rank}(B) = 1 \leq 2$, so neither A^{-1} nor B^{-1} exist. However, since $I_2^{-1} = I_2$, $A + B = I_2$ is invertible.

Problem 3 Suppose that A, B, and $C \in \text{Mat}_{n\times n}$, that $\text{rank}(A) = n$, and that $AB = AC$. Show that $B = C$.

Solution.

We have shown that since $\text{rank}(A) = n$, A^{-1} exists. So we have

$$\begin{aligned} AB &= AC && \text{(from assumption)} \\ (A^{-1}A)B &= (A^{-1}A)C \\ B &= C. && (A^{-1}A = I_n) \end{aligned}$$

Problem 4

(a) Let $A = \begin{pmatrix} 1 & 2 & 3 & 4 \\ 0 & 2 & 3 & 4 \\ 0 & 0 & 3 & 4 \\ 0 & 0 & 0 & 4 \end{pmatrix}$. Compute the inverse of A, if it exists.

(b) A matrix $A \in \text{Mat}_{n\times n}$ is **upper-triangular** if and only if for all $i > j$, we have $(A)_{ij} = 0$: every entry below the main diagonal is zero. Prove that an upper-triangular matrix $A \in \text{Mat}_{n\times n}$ is invertible if and only if every entry on its main diagonal is non-zero: for all $1 \leq i \leq n$, we have $(A)_{ii} \neq 0$.

Solution.

(a) We compute the reduced row-echelon form of $(A|I_4)$:

$$\begin{aligned} (A|I_4) &= \left(\begin{array}{cccc|cccc} 1 & 2 & 3 & 4 & 1 & 0 & 0 & 0 \\ 0 & 2 & 3 & 4 & 0 & 1 & 0 & 0 \\ 0 & 0 & 3 & 4 & 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 4 & 0 & 0 & 0 & 1 \end{array}\right) \\ &\sim \left(\begin{array}{cccc|cccc} 1 & 2 & 3 & 4 & 1 & 0 & 0 & 0 \\ 0 & 2 & 3 & 4 & 0 & 1 & 0 & 0 \\ 0 & 0 & 3 & 0 & 0 & 0 & 1 & -1 \\ 0 & 0 & 0 & 4 & 0 & 0 & 0 & 1 \end{array}\right) && (\rho_3 \mapsto \rho_3 - \rho_4) \\ &\sim \left(\begin{array}{cccc|cccc} 1 & 2 & 3 & 4 & 1 & 0 & 0 & 0 \\ 0 & 2 & 3 & 0 & 0 & 1 & 0 & -1 \\ 0 & 0 & 3 & 0 & 0 & 0 & 1 & -1 \\ 0 & 0 & 0 & 4 & 0 & 0 & 0 & 1 \end{array}\right) && (\rho_2 \mapsto \rho_2 - \rho_4) \end{aligned}$$

$$\sim \left(\begin{array}{cccc|cccc} 1 & 2 & 3 & 0 & 1 & 0 & 0 & -1 \\ 0 & 2 & 3 & 0 & 0 & 1 & 0 & -1 \\ 0 & 0 & 3 & 0 & 0 & 0 & 1 & -1 \\ 0 & 0 & 0 & 4 & 0 & 0 & 0 & 1 \end{array}\right) \qquad (\rho_1 \mapsto \rho_1 - \rho_4)$$

$$\sim \left(\begin{array}{cccc|cccc} 1 & 2 & 3 & 0 & 1 & 0 & 0 & -1 \\ 0 & 2 & 0 & 0 & 0 & 1 & -1 & 0 \\ 0 & 0 & 3 & 0 & 0 & 0 & 1 & -1 \\ 0 & 0 & 0 & 4 & 0 & 0 & 0 & 1 \end{array}\right) \qquad (\rho_2 \mapsto \rho_2 - \rho_3)$$

$$\sim \left(\begin{array}{cccc|cccc} 1 & 2 & 0 & 0 & 1 & 0 & -1 & 0 \\ 0 & 2 & 0 & 0 & 0 & 1 & -1 & 0 \\ 0 & 0 & 3 & 0 & 0 & 0 & 1 & -1 \\ 0 & 0 & 0 & 4 & 0 & 0 & 0 & 1 \end{array}\right) \qquad (\rho_1 \mapsto \rho_1 - \rho_3)$$

$$\sim \left(\begin{array}{cccc|cccc} 1 & 0 & 0 & 0 & 1 & -1 & 0 & 0 \\ 0 & 2 & 0 & 0 & 0 & 1 & -1 & 0 \\ 0 & 0 & 3 & 0 & 0 & 0 & 1 & -1 \\ 0 & 0 & 0 & 4 & 0 & 0 & 0 & 1 \end{array}\right) \qquad (\rho_1 \mapsto \rho_1 - \rho_2)$$

$$\sim \left(\begin{array}{cccc|cccc} 1 & 0 & 0 & 0 & 1 & -1 & 0 & 0 \\ 0 & 1 & 0 & 0 & 0 & \frac{1}{2} & -\frac{1}{2} & 0 \\ 0 & 0 & 3 & 0 & 0 & 0 & 1 & -1 \\ 0 & 0 & 0 & 4 & 0 & 0 & 0 & 1 \end{array}\right) \qquad (\rho_2 \mapsto \tfrac{1}{2}\rho_2)$$

$$\sim \left(\begin{array}{cccc|cccc} 1 & 0 & 0 & 0 & 1 & -1 & 0 & 0 \\ 0 & 1 & 0 & 0 & 0 & \frac{1}{2} & -\frac{1}{2} & 0 \\ 0 & 0 & 1 & 0 & 0 & 0 & \frac{1}{3} & -\frac{1}{3} \\ 0 & 0 & 0 & 4 & 0 & 0 & 0 & 1 \end{array}\right) \qquad (\rho_3 \mapsto \tfrac{1}{3}\rho_3)$$

$$\sim \left(\begin{array}{cccc|cccc} 1 & 0 & 0 & 0 & 1 & -1 & 0 & 0 \\ 0 & 1 & 0 & 0 & 0 & \frac{1}{2} & -\frac{1}{2} & 0 \\ 0 & 0 & 1 & 0 & 0 & 0 & \frac{1}{3} & -\frac{1}{3} \\ 0 & 0 & 0 & 1 & 0 & 0 & 0 & \frac{1}{4} \end{array}\right). \qquad (\rho_3 \mapsto \tfrac{1}{4}\rho_4)$$

We have shown in class that since the left part has been reduced to the identity matrix, the inverse will be the right part and $A^{-1} = \begin{pmatrix} 1 & -1 & 0 & 0 \\ 0 & \frac{1}{2} & -\frac{1}{2} & 0 \\ 0 & 0 & \frac{1}{3} & -\frac{1}{3} \\ 0 & 0 & 0 & \frac{1}{4} \end{pmatrix}$.

(b) Let $A' \in \mathrm{Mat}_{n\times n}$ be a matrix. For each row i in A, find the first non-zero element in the row and then let row i in A' be row i in A divided by the reciprocal of this element. For example, if

$$A = \begin{pmatrix} 3 & 6 \\ 0 & 2 \end{pmatrix}, \text{ then } A' = \begin{pmatrix} 1 & 2 \\ 0 & 1 \end{pmatrix}.$$

We note that A' is row-equivalent to A and that A' will be in row-echelon form; thus, the number of leading ones in A' will be the rank of A.

($\Rightarrow$) We will prove the contrapositive. Assume there is a zero entry on the main diagonal. Then, all the following leading ones in A' will be shifted over from the diagonal by at least one, since row-echelon form is in a staircase. So there can be at most $n-1$ leading ones in A', and thus $\operatorname{rank}(A) < n$. So by The Theorem, A is not invertible.

($\Leftarrow$) Assume that every entry on the main diagonal is non-zero. Then, every entry on the main diagonal of A' will be a leading one, and thus $\operatorname{rank}(A) = n$. So by The Theorem, A is invertible. $\square$

Problem 5 Suppose that $A \in \mathrm{Mat}_{n\times n}$ satisfies $A^{10} = O_{n\times n}$. Prove that $I_n - A$ is invertible. (*Hint*: As a warm-up, suppose instead that $A^2 = O_{n\times n}$.)

Suppose that $A^{10} = O_{n\times n}$, and let $\vec{c}$ be a solution of $(I_n - A)\vec{x} = \vec{0}$. Then,

$$\begin{aligned}
(I_n - A)\vec{c} &= \vec{0} \\
\implies I_n\vec{c} - A\vec{c} &= \vec{0} && \text{(distributive property)} \\
\implies \vec{c} - A\vec{c} &= \vec{0} && \text{(identity property)} \\
\implies \vec{c} &= A\vec{c} \\
&= A(A\vec{c}) && \text{(replacing since } \vec{c} = A\vec{c}\text{)} \\
&= A^2\vec{c} && \text{(associative property)} \\
&\dots \\
&= A^{10}\vec{c} && \text{(applying above two steps repeatedly)} \\
&= O_{n\times n}\vec{c} && (A^{10} = O_{n\times n}) \\
&= \vec{0}. && (O_{n\times n} \text{ times any vector is } \vec{0})
\end{aligned}$$

So $(I_n - A)\vec{x} = \vec{0}$ has only the trivial solution $\vec{c} = \vec{0}$, and thus by The Theorem $I_n - A$ is invertible. $\square$