Nathan Bickel
MATH 554: Section H01
Professor: Dr. Howard
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MATH 554 Take-Home Test 1

Problem 1 Prove that every open ball $B(a, r)$ is open.

Solution.

Let $b \in B(a, r)$. By definition, $d(a, b) < r$, so we have $\rho := r - d(a, b) > 0$. We claim $B(b, \rho) \subseteq B(a, r)$. Let $c \in B(b, \rho)$. Then, we have

$$\begin{aligned} d(a, c) &= d(a, b) + d(b, c) && \text{(triangle inequality)} \\ &< d(a, b) + \rho && (d(b, c) < \rho \text{ because } c \in B(b, \rho)) \\ &= d(a, b) + r - d(a, b) && \text{(definition of } \rho) \\ &= r. \end{aligned}$$

So $d(a, c) < r$, and thus $c \in B(a, r)$ by definition. So $B\left(b, r - d(a, b)\right) \subseteq B(a, r)$ for every $b \in B(a, r)$, and therefore by definition $B(a, r)$ is open. □

Problem 2 Prove that for any $a \in E$ and $r > 0$, the set

$$U = \{x \in E : x \notin \overline{B}(a, r)\} = \{x \in E : d(x, a) > r\}$$

is open.

Solution.

Let $b \in U$. By definition, $d(a, b) > r$, so we have $\rho := d(a, b) - r > 0$. We claim $B(b, \rho) \subseteq U$. Let $c \in B(b, \rho)$. From the triangle inequality, we have

$$d(a, b) \leq d(a, c) + d(c, b) \implies d(a, c) \geq d(a, b) - d(b, c).$$

Then, we have

$$\begin{aligned} d(a, c) &\geq d(a, b) - d(b, c) && \text{(from above)} \\ &> d(a, b) - \rho && (d(b, c) < \rho) \\ &= d(a, b) - (d(a, b) - r) && \text{(by choice of } \rho) \\ &= r. \end{aligned}$$

So $d(a, c) > r$, and thus $c \in U$ by definition. So $B\left(b, d(a, b) - r\right) \subseteq U$ for every $b \in U$, and therefore by definition U is open (it then follows that the complement of U, $\overline{B}(a, r)$, is closed). □

Problem 3 Let $\{U_\alpha : \alpha \in A\}$ be a possibly infinite collection of open subsets of E. Prove that the union

$$U := \bigcup_{\alpha \in A} U_\alpha$$

is open.

Solution.

Let $x \in U$. Then there exists some $\alpha \in A$ such that $x \in U_\alpha$. Since U_α is open, there exists an $r > 0$ such that $B(x,r) \subseteq U_\alpha$. Since $U_\alpha \subseteq U$ by definition, by transitivity we have $B(x,r) \subseteq U$. Therefore, by definition U is open. $\square$

Problem 4 Let $U_1, U_2, \ldots, U_n \subseteq E$ be a finite collection of open subsets of E. Prove that the intersection

$$U = U_1 \cap U_2 \cap \cdots \cap U_n$$

is open.

Solution.

Let $x \in U$. Then $x \in U_i$ for all $i \in \{1, 2, \ldots, n\}$. Since U_i is open for all i, there exist $r_1, r_2, \ldots, r_n \in \mathbb{R}^+$ such that $B(x, r_i) \subseteq U_i$ for all i. Consider $r := \min\{r_1, r_2, \ldots, r_n\}$, which is well-defined since the set is finite. Then $B(x,r) \subseteq B(x, r_i)$ for all i (because any $y \in B(x,r)$ has $d(x,y) < r \leq r_i$). Since $B(x, r_i) \subseteq U_i$ for all i, from transitivity we have $B(x,r) \subseteq U_i$ for all i. By definition of U, then, $B(x,r) \subseteq U$, so by definition U is open. $\square$

Problem 5 This problem shows that the collection of open subsets of $\mathbb{R}$ is not closed under infinite intersections. Let $U_n = (-1/n, 1/n)$ in $\mathbb{R}$, which is open. Show

$$K := \bigcap_{n=1}^{\infty} U_n = \{0\}$$

and therefore the intersection is not open.

Solution.

We will show a double inclusion.

($\subseteq$) Suppose (toward contradiction) that $x \in K$ but $x \neq 0$. Then $x \in (-1/n, 1/n)$ for all $n \in \mathbb{N}$, so $|x| < \frac{1}{n}$. But by Archimedes, there exists an $n \in \mathbb{N}$ such that $\frac{1}{n} < |x|$, a contradiction. So $K \subseteq \{0\}$.

($\supseteq$) If $0 \notin K$, then $|0| \geq \frac{1}{n}$ for some $n \in \mathbb{N}$, so $0 \geq 1$, a contradiction. So $\{0\} \subseteq K$.

Therefore, $K = \{0\}$. $\square$