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MATH 555 Homework 6

Problem 3.14 Prove that if f is integrable on $[a, b]$ then so is f^2.

Let $\varepsilon > 0$, and let f be integrable on $[a, b]$. We showed in class that this implies that $\max\{f, 0\}$ is integrable, and since $-f$ is integrable by another rule we proved, so is $\max\{-f, 0\}$. Since the sum of integrable functions is integrable,

$$|f| = \max\{f, 0\} + \max\{-f, 0\}$$

is also integrable. Thus, it is bounded, so there exists some $B \neq 0$ such that $|f| \leq B$. Also, by the theorem we proved in class, $|f|$ being integrable implies there are step functions φ, ψ such that

$$\varphi \leq f \leq \psi$$

and

$$\int_a^b (\psi - \varphi)dx < \frac{\varepsilon}{2B}. \tag{1}$$

Let $\varphi' = \max\{0, \varphi\}$, $\psi' = \min\{B, \psi\}$, which are also step functions of $|f|$ since $0 \leq |f| \leq B$, so we have

$$\varphi'^2 \leq f^2 \leq \psi'^2$$

by squaring each part and using that $f^2 = |f|^2$. Then, we can write

$$\begin{aligned}
\int_a^b (\psi'^2 - \varphi'^2)\,dx &= \int_a^b (\psi' + \varphi')(\psi' - \varphi')dx && \text{(factoring)} \\
&\leq \int_a^b (B + B)(\psi' - \varphi')\,dx && (\varphi' \leq \psi' \leq B) \\
&= 2B \int_a^b (\psi' - \varphi')\,dx && \text{(constant integral rule)} \\
&\leq 2B \int_a^b (\psi - \varphi)\,dx && \text{(max/min property)} \\
&< 2B\left(\frac{\varepsilon}{2B}\right) = \varepsilon. && \text{(by 1)}
\end{aligned}$$

So by the theorem we proved in class, this implies that f^2 is integrable. $\square$

Problem 3.15 Prove that if f and g are integrable on $[a, b]$ then so is the product fg.

Let f, g be integrable on $[a, b]$. First, we observe that we can express f, g as a combination of squares with

$$\begin{aligned}
fg &= \frac{4fg}{4} \\
&= \frac{f^2 - f^2 + 4fg + g^2 - g^2}{4} \\
&= \frac{f^2 + 2fg + g^2 - f^2 + 2fg - g^2}{4}
\end{aligned}$$

$$= \frac{f^2 + 2fg + g^2 - (f^2 - 2fg + g^2)}{4}$$
$$= \frac{(f+g)^2 - (f-g)^2}{4}.$$

Since f and g are integrable, so are $f+g$ and $f-g$, and further by Problem 3.14 so are $(f+g)^2$ and $(f-g)^2$. It then follows that $(f+g)^2 - (f-g)^2$ is integrable, and finally so is $\frac{1}{4}\left((f+g)^2 - (f-g)^2\right)$. Since this is equal to fg, we have that fg is integrable on $[a,b]$. □

Problem 4.8 Let u and v continuous on $[a,b]$, differentiable on (a,b), with u' and v' integrable on $[a,b]$. Prove that

$$\int_a^b u(x)v'(x)\,dx = u(x)v(x)\Big|_{x=a}^{b} - \int_a^b u'(x)v(x)\,dx.$$

Let $F(x) = u(x)v(x)$. By the product rule, we have

$$F'(x) = u(x)v'(x) + u'(x)v(x).$$

Since u and v are continuous, they are integrable, so using that u' and v' are integrable and Problem 3.14, $u(x)v'(x)$ and $u'(x)v(x)$ are both integrable, and thus F' is integrable. Also, since u and v are continuous, F is continuous. We can then use the Fundamental Theorem of Calculus to write

$$\begin{aligned} \int_a^b F'(x)\,dx &= F(x)\Big|_{x=a}^{b} \\ \implies \int_a^b \big(u(x)v'(x) + u'(x)v(x)\big)\,dx &= u(x)v(x)\Big|_{x=a}^{b} \\ \implies \int_a^b u(x)v'(x)\,dx + \int_a^b u'(x)v(x)\,dx &= u(x)v(x)\Big|_{x=a}^{b} \qquad \text{(splitting integral)} \\ \implies \int_a^b u(x)v'(x)\,dx &= u(x)v(x)\Big|_{x=a}^{b} - \int_a^b u'(x)v(x)\,dx \end{aligned}$$

as desired. □

Problem 4.9 Let f be $k+1$ times differentiable on an open interval (α,β) and assume that $f^{(k+1)}$ is integrable. Prove that for $a, x \in (\alpha,\beta)$ we have

$$\int_a^x \frac{(x-t)^{k-1}}{(k-1)!} f^{(k)}(t)\,dt = \frac{f^{(k)}(a)}{k!}(x-a)^k + \int_a^x \frac{(x-t)^k}{k!} f^{(k+1)}(t)\,dt.$$

Let

$$u(t) = f^{(k)}(t),\ v(t) = -\frac{(x-t)^k}{k!}.$$

Then, since f is $k+1$ times differentiable and we have the power and chain rules, we can write

$$u'(t) = f^{(k+1)}(t),\ v'(t) = \frac{(x-t)^{k-1}}{(k-1)!}.$$

Then, since u and v are continuous (their derivatives exist), and u' and v' are integrable from the assumption, from Problem 4.8 we have

$$\int_a^x u(t)v'(t)\,dt = u(t)v(t)\Big|_{t=a}^{x} - \int_a^x u'(t)v(t)\,dt$$

$$\begin{aligned}
&\implies \int_a^x f^{(k)}(t)\frac{(x-t)^{k-1}}{(k-1)!}\,dt = f^{(k)}(t)\frac{-(x-t)^k}{k!}\bigg|_{t=a}^x - \int_a^x f^{(k+1)}(t)\frac{-(x-t)^k}{k!}\,dt \\
&\implies \int_a^x \frac{(x-t)^{k-1}}{(k-1)!}f^{(k)}(t)\,dt = \left[-f^{(k)}(x)\frac{(x-x)^k}{k!} + f^{(k)}(a)\frac{(x-a)^k}{k!}\right] + \int_a^x \frac{(x-t)^k}{k!}f^{(k+1)}(t)\,dt \\
&\implies \int_a^x \frac{(x-t)^{k-1}}{(k-1)!}f^{(k)}(t)\,dt = \frac{f^{(k)}(a)}{k!}(x-a)^k + \int_a^x \frac{(x-t)^k}{k!}f^{(k+1)}(t)\,dt
\end{aligned}$$

as desired. □

Problem 4.10 Let f be $n+1$ times differentiable on (α,β) and assume that $f^{(n+1)}$ is integrable. Then for $a,x \in (\alpha,\beta)$,

$$f(x) = f(a) + f'(a)(x-a) + \frac{f''(a)}{2!}(x-a)^2 + \cdots + \frac{f^{(n)}(a)}{n!}(x-a)^n + R_n(x),$$

where the remainder term $R_n(x)$ is given by

$$R_n(x) = \int_a^x \frac{(x-t)^n}{n!}f^{(n+1)}(t)\,dt.$$

We will induct on n.

Base Case: We will show the $n=0$ case. From the FTC, we can write

$$\int_a^x f'(t)\,dt = f(x) - f(a),$$

so we have

$$\begin{aligned}
f(x) &= f(a) + \int_a^x f'(t)\,dt \\
&= f(a) + \int_a^x \frac{(x-t)^0}{0!}f^{(0+1)}(t)\,dt \\
&= f(a) + R_0(x).
\end{aligned}$$

Induction Step: Let $n \in \mathbb{N}$, $n > 0$, and assume that the claim holds for $n-1$. We will show it holds for n. We observe from Problem 4.10 that we have

$$R_{n-1}(x) = \frac{f^{(n)}(a)}{n!}(x-a)^n + R_n(x). \qquad (\star)$$

From the inductive hypothesis, we have

$$f(x) = f(a) + f'(a)(x-a) + \frac{f''(a)}{2!}(x-a)^2 + \cdots + \frac{f^{(n-1)}(a)}{(n-1)!}(x-a)^n + R_{n-1}(x),$$

and we can then use $(\star)$ to write

$$f(x) = f(a) + f'(a)(x-a) + \frac{f''(a)}{2!}(x-a)^2 + \cdots + \frac{f^{(n-1)}(a)}{(n-1)!}(x-a)^n + \frac{f^{(n)}(a)}{n!}(x-a)^n + R_n(x),$$

so the claim holds for n. □

Problem 4.11 Let the map $x = u(t)$ map the interval $[c,d]$ into the interval $[a,b]$ and assume that $u'(t)$ is integrable on $[c,d]$. Prove that for any continuous function f on $[a,b]$,

$$\int_{u(c)}^{u(d)} f(x)\,dx = \int_c^d f(u(t))u'(t)\,dt.$$

Because u' exists, u is differentiable and therefore continuous. Because the composition of continuous functions is continuous, $f(x)$ is continuous, so it is integrable. Further, since u' is integrable, from Problem 3.15 we have that $f(u(t))u'(t)$ is integrable. So both integrals exist.

Consider F on $[a, b]$ defined by

$$F(s) := \int_a^s f(t)\,dt.$$

Then, from the FTC we have both

$$F'(s) = f(s) \tag{1}$$

and

$$F(u(d)) - F(u(c)) = \int_{u(c)}^{u(d)} f(t)\,dt. \tag{2}$$

Now consider G on $[a, b]$ defined by $F(u(t))$. Then we can use the chain rule and (1) to write

$$G'(t) = F'(u(t))u'(t) = f(u(t))u'(t). \tag{3}$$

So then, we have

$$\begin{aligned}
\int_{u(c)}^{u(d)} f(t)\,dt &= F(u(d)) - F(u(c)) && \text{(by (2))} \\
&= G(d) - G(c) && \text{(definition of } G\text{)} \\
&= \int_c^d G'(t)\,dt && \text{(FTC)} \\
&= \int_c^d f(u(t))u'(t)\,dt && \text{(by (3))}
\end{aligned}$$

as desired. □