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Linear Algebra: Section 1
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Linear Algebra Homework 4

Problem 1 Find a basis for the space of $n \times n$ symmetric matrices A (i.e. matrices such that $A = A^T$).

Solution.

Define $A_{i,j}$ to be the matrix with i,j-entry and j,i-entry 1 and all other entries 0. Then,

$$\{A_{i,j} : i,j \in \mathbb{N}, 1 \leq i \leq n, i \leq j \leq n\}$$

is a basis for the space of $n \times n$ symmetric matrices. For example, for $n = 3$, the basis is

$$\left\{ \begin{pmatrix} 1 & 0 & 0 \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{pmatrix}, \begin{pmatrix} 0 & 1 & 0 \\ 1 & 0 & 0 \\ 0 & 0 & 0 \end{pmatrix}, \begin{pmatrix} 0 & 0 & 1 \\ 0 & 0 & 0 \\ 1 & 0 & 0 \end{pmatrix}, \begin{pmatrix} 0 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 0 \end{pmatrix}, \begin{pmatrix} 0 & 0 & 0 \\ 0 & 0 & 1 \\ 0 & 1 & 0 \end{pmatrix}, \begin{pmatrix} 0 & 0 & 0 \\ 0 & 0 & 0 \\ 0 & 0 & 1 \end{pmatrix} \right\}.$$

Problem 2 Let A be an $m \times n$ matrix, and let A' be the result of a sequence of elementary row operations on A. Prove that the rows of A' span the same space as the rows of A.

Solution.

It suffices to show that any single row operation does not change the row space (because row operations are performed sequentially, the row space will not change at any step and thus will remain unchanged). Let A be an $m \times n$ matrix with rows $\vec{A_1}, \vec{A_2}, \ldots, \vec{A_m}$, and A' be the result of one elementary row operation performed on A. We would like to show that $\text{Row}(A) = \text{Row}(A')$. There are three types of operations, so we will consider each:

- Case 1: Two rows are swapped. Clearly, the order of the rows does not affect the row space because of the commutative property.
- Case 2: Some row is multiplied by some $c \in \mathbb{R}\backslash\{0\}$. Without loss of generality, suppose it is $\vec{A_1}$. Then, the rows of A' are $c\vec{A_1}, \vec{A_2}, \ldots, \vec{A_m}$, and we have

$$\begin{aligned} & \vec{x} \in \text{Row}(A) \\ \iff & \vec{x} \in \{\vec{A_1}, \vec{A_2}, \ldots, \vec{A_m}\} \\ \iff & \vec{x} = a_1\vec{A_1} + a_2\vec{A_2} + \cdots + a_m\vec{A_m} \text{ for some } a_1, a_2, \ldots, a_m \in \mathbb{R} \\ \iff & \vec{x} = \frac{a_1}{c}c\vec{A_1} + a_2\vec{A_2} + \cdots + a_m\vec{A_m} \text{ for some } a_1, a_2, \ldots, a_m \in \mathbb{R} \\ \iff & \vec{x} = a_1{}'c\vec{A_1} + a_2\vec{A_2} + \cdots + a_m\vec{A_m} \text{ for some } a_1{}', a_2, \ldots, a_m \in \mathbb{R} \qquad (a_1{}' = \tfrac{a_1}{c}) \\ \iff & x \in \{c\vec{A_1}, \vec{A_2}, \ldots, \vec{A_m}\} \\ \iff & \vec{x} \in \text{Row}(A'). \end{aligned}$$

Thus, we have shown a double inclusion, and $\text{Row}(A) = \text{Row}(A')$.

- Case 3: Some row has been added to another row. Without loss of generality, suppose $\vec{A_2}$ has been added to $\vec{A_1}$. Then, the rows of A' are $\vec{A_1}+\vec{A_2}, \vec{A_2}, \dots, \vec{A_m}$, and we have

$$\begin{aligned}
&\vec{x} \in \text{Row}(A) \\
\iff &\vec{x} \in \{\vec{A_1}, \vec{A_2}, \dots, \vec{A_m}\} \\
\iff &\vec{x} = a_1\vec{A_1} + a_2\vec{A_2} + \cdots + a_m\vec{A_m} \text{ for some } a_1, a_2, \dots, a_m \in \mathbb{R} \\
\iff &\vec{x} = (a_1\vec{A_2} - a_1\vec{A_2}) + a_1\vec{A_1} + a_2\vec{A_2} + \cdots + a_m\vec{A_m} \text{ for some } a_1, a_2, \dots, a_m \in \mathbb{R} && \text{(adding 0)} \\
\iff &\vec{x} = a_1\vec{A_1} + a_1\vec{A_2} + a_2\vec{A_2} - a_1\vec{A_2} + \cdots + a_m\vec{A_m} \text{ for some } a_1, a_2, \dots, a_m \in \mathbb{R} && \text{(commutative)} \\
\iff &\vec{x} = a_1(\vec{A_1} + \vec{A_2}) + (a_2 - a_1)\vec{A_2} + \cdots + a_m\vec{A_m} \text{ for some } a_1, a_2, \dots, a_m \in \mathbb{R} \\
\iff &\vec{x} = a_1(\vec{A_1} + \vec{A_2}) + a_2'\vec{A_2} + \cdots + a_m\vec{A_m} \text{ for some } a_1, a_2', \dots, a_m \in \mathbb{R} && (a_2' = a_2 - a_1) \\
\iff &\vec{x} \in \{\vec{A_1}+\vec{A_2}, \vec{A_2}, \dots, \vec{A_m}\} \\
\iff &\vec{x} \in \text{Row}(A').
\end{aligned}$$

Thus, we have shown a double inclusion, and $\text{Row}(A) = \text{Row}(A')$.

Therefore, elementary row operations do not change the row space of a matrix. $\square$

Problem 3 Find a basis for the space of solutions in $\mathbb{R}^n$ of the equation

$$x_1 + x_2 + \cdots + nx_n = 0.$$

Solution.

We can write a matrix

$$A = \begin{pmatrix} 1 & 1 & \dots & 1 & n \end{pmatrix}.$$

Since this is in reduced row-echelon form, we have

$$\text{Null}(A) = \text{Span}\left\{ \begin{pmatrix} -1 \\ 1 \\ 0 \\ \vdots \\ 0 \end{pmatrix}, \begin{pmatrix} -1 \\ 0 \\ 1 \\ \vdots \\ 0 \end{pmatrix}, \dots, \begin{pmatrix} -n \\ 0 \\ 0 \\ \vdots \\ 1 \end{pmatrix} \right\}.$$

More specifically, we can define a basis for the space of solutions as

$$\{\vec{x_i} \in \mathbb{R}^n : i \in \mathbb{N}, 2 \le i \le n\},$$

where for $2 \le i \le n-1$, $\vec{x_i}$ has first entry -1 and ith entry 1, and $\vec{x_n}$ has first entry $-n$ and nth entry 1.

Problem 4

(a) Determine the basechange matrix in $\mathbb{R}^2$ when the old basis is standard basis $B = (e_1, e_2)$ and the new basis is $B' = (e_1 + e_2, e_1 - e_2)$.

(b) Determine the basechange matrix when the old basis is the standard basis B in $\mathbb{R}^n$ and the new basis is $B = (e_n, e_{n-1}, \dots, e_1)$.

Solution.

(a) The basechange matrix is
$$\begin{pmatrix} 1 & 1 \\ 1 & -1 \end{pmatrix}.$$

(b) The basechange matrix is a matrix with zeroes everywhere except for ones on the off-diagonal. For example, for $n = 3$ the basechange matrix is
$$\begin{pmatrix} 0 & 0 & 1 \\ 0 & 1 & 0 \\ 1 & 0 & 0 \end{pmatrix}.$$

Problem 5 Prove that a set $(\vec{v_1}, \dots, \vec{v_n})$ of vectors in $\mathbb{R}^n$ is a basis if and only if the matrix obtained by assembling the coordinate vectors of v_i is invertible.

Solution.

($\Rightarrow$) We will prove the contrapositive. Suppose $A = \left(\vec{v_1} \mid \vec{v_2} \mid \dots \mid \vec{v_n}\right)$ is not invertible. We have shown this is equivalent to $\operatorname{rank}(A) < n$, and since $\operatorname{rank}(A) + \operatorname{nullity}(A) = n$, we must have $\operatorname{nullity}(A) \geq 1$. This means that a basis for $\operatorname{Null}(A)$ has a non-zero $\vec{x} \in \mathbb{R}^n$ such that $A\vec{x} = \vec{0}$. Then, $\vec{x}$ is a non-trivial solution to the dependency equation for the set of vectors, and therefore the set is not linearly independent. Thus, it is not a basis.

($\Leftarrow$) Suppose $A = \left(\vec{v_1} \mid \vec{v_2} \mid \dots \mid \vec{v_n}\right)$ is invertible. Then, similar reasoning as above applies: we have $\operatorname{rank}(A) = n$ and $\operatorname{nullity}(A) = 0$, so $\operatorname{Null}(A) = \{\vec{0}\}$. Thus, the dependency equation of the set has only the trivial solution, so it is linearly independent. Since the set has n vectors, this implies that it is a basis. □

Problem 6 Let A be an $m \times n$ matrix. Use the dimension formula to prove that the space of solutions to the system $AX = 0$ has dimension at least $n - m$.

Solution.

Case 1: $n \leq m$. Since we know dimension is always non-negative, we have $\operatorname{nullity}(A) \geq 0 \geq n - m$.

Case 2: $m < n$. The rank of a matrix is never greater than the number of columns or greater than the number of rows, so we have $\operatorname{rank}(A) \leq m$. The dimension formula states that $\operatorname{rank}(A) = n - \operatorname{nullity}(A)$, so we have $n - \operatorname{nullity}(A) \leq m$ which we can rearrange to $\operatorname{nullity}(A) \geq n - m$. □

Problem 7 Let $W_1, W_2, \dots, W_k$ be subspaces of V. We define the sum of the subspaces $W_1, W_2, \dots, W_k$ as
$$W_1 + W_2 + \cdots + W_k = \{v \in V : v = w_1 + w_2 + \cdots + w_k \text{ with } w_i \in W_i \text{ for each } i\}.$$

(a) Prove that $W_1 + W_2 + \cdots + W_k$ is a subspace of V.

(b) Prove that $\dim(W_1 + \cdots + W_k) \leq \dim(W_1) + \cdots + \dim(W_k)$.

Solution.

(a) We have shown that it suffices to show that some $W \subset V$ is a subspace if $c\vec{u} + \vec{v} \in W$ for all $\vec{u}, \vec{v} \in W$, $c \in \mathbb{R}$, so we will take this approach. Let $\vec{u}, \vec{v} \in W_1 + W_2 + \cdots + W_k$ and $c \in \mathbb{R}$. By definition, $\vec{u} = w_1 + w_2 + \cdots + w_k$ and $\vec{v} = w_1{}' + w_2{}' + \cdots + w_k{}'$ for some $w_i, w_i{}' \in W_i$ for $1 \leq i \leq k$. Then, we have

$$c\vec{u} + \vec{v} = c(w_1 + w_2 + \cdots + w_k) + (w_1{}' + w_2{}' + \cdots + w_k{}') = (cw_1 + w_1{}') + (cw_2 + w_2{}') + \cdots + (cw_k + w_k{}'),$$

and since each of these $cw_i + w_i{}' \in W_i$ for all $1 \leq i \leq n$, $c\vec{u} + \vec{v}$ is in $W_1 + W_2 + \cdots + W_k$. Therefore, $W_1 + W_2 + \cdots + W_k$ is a subspace.

(b) Let $B_1, B_2, \ldots, B_k$ be bases for $W_1, W_2, \ldots, W_k$ respectively, and consider $S = B_1 \cup B_2 \cup \cdots \cup B_k$. Let $\vec{x} \in W_1 + W_2 + \cdots + W_k$. Then, $\vec{x} = \vec{w_1} + \vec{w_2} + \cdots + \vec{w_k}$ for some $\vec{w_i} \in W_i$. Since each $\vec{w_i}$ can be expressed as a linear combination of vectors in B_i, $\vec{x}$ can be expressed as a linear combination of vectors in S. Therefore, S spans $W_1 + W_2 + \cdots + W_k$, so we have

$$\dim(W_1 + W_2 + \cdots + W_k) \leq |S| \leq |B_1| + |B_2| + \cdots + |B_k| = \dim(B_1) + \dim(B_2) + \cdots + \dim(B_k).$$

$\square$

Problem 8 Determine the dimensions of the kernel and the image of the linear operator on $\mathbb{R}^n$ defined by $T(x_1, \ldots, x_n) = (x_1 + x_n, x_2 + x_{n-1}, \ldots, x_n + 1)$.

Solution.

We can write T as left-multiplication by a matrix A, where A has all entries 0 except for the i,j-entries where $i = j$ or where $i + j = n + 1$. For example, for $n = 3$ we have

$$T\begin{pmatrix} x_1 \\ x_2 \\ x_3 \end{pmatrix} = \begin{pmatrix} 1 & 0 & 1 \\ 0 & 1 & 0 \\ 1 & 0 & 1 \end{pmatrix}\begin{pmatrix} x_1 \\ x_2 \\ x_3 \end{pmatrix} = \begin{pmatrix} x_1 + x_3 \\ x_2 + x_2 \\ x_3 + x_1 \end{pmatrix},$$

and for $n = 4$ we have

$$T\begin{pmatrix} x_1 \\ x_2 \\ x_3 \\ x_4 \end{pmatrix} = \begin{pmatrix} 1 & 0 & 0 & 1 \\ 0 & 1 & 1 & 0 \\ 0 & 1 & 1 & 0 \\ 1 & 0 & 0 & 1 \end{pmatrix}\begin{pmatrix} x_1 \\ x_2 \\ x_3 \\ x_4 \end{pmatrix} = \begin{pmatrix} x_1 + x_4 \\ x_2 + x_3 \\ x_3 + x_2 \\ x_4 + x_1 \end{pmatrix}.$$

To find the rank of A, we can reduce A to reduced row-echelon form and count the number of leading ones. Since the matrix is vertically symmetric (row i is the same as row $n+1-i$), these bottom rows will become all zeros in the row-echelon form and the other rows will have leading ones. So $\dim(\text{Im}(T)) = \text{rank}(A) = \lceil \frac{n}{2} \rceil$, and thus by the dimension formula $\dim(\text{Ker}(T)) = \text{nullity}(A) = \lfloor \frac{n}{2} \rfloor$.

Problem 9 Let $T : V \to V$ be a linear operator. Prove that if W_1 and W_2 are T-invariant subspaces of V, then so are $W_1 + W_2$ and $W_1 \cap W_2$.

Solution.

We will first show that $W_1 + W_2$ is T-invariant. Let $\vec{w} \in W_1 + W_2$, which means there exist $\vec{w_1} \in W_1$, $\vec{w_2} \in W_2$ such that $\vec{w} = \vec{w_1} + \vec{w_2}$. By linearity, we have $T(\vec{w}) = T(\vec{w_1} + \vec{w_2}) = T(\vec{w_1}) + T(\vec{w_2})$, and since W_1 and W_2 are T-invariant, we have $T(\vec{w_1}) \in W_1$ and $T(\vec{w_2}) \in W_2$. Therefore, by definition

$$T(\vec{w}) = T(\vec{w_1}) + T(\vec{w_2}) \in W_1 + W_2.$$

We will now show that $W_1 \cap W_2$ is T-invariant. Let $\vec{w} \in W_1 \cap W_2$. Then, since $\vec{w} \in W_1$ we also have $T(\vec{w}) \in W_1$ by the T-invariance of W_1, and similarly $T(\vec{w}) \in W_2$. Therefore, by definition

$$T(\vec{w}) \in W_1 \cap W_2.$$

$\square$