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MATH 554: Section H01
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MATH 554 Homework 13

Problem 1 Give an ε, δ proof that $f(x) = x^3 - x$ is continuous at at all points a.

Let $a \in \mathbb{R}$, and let $\varepsilon > 0$. We will show that there exists a $\delta > 0$ such that for all $x \in \mathbb{R}$, we have

$$|x - a| < \delta \implies \big|f(x) - f(a)\big| < \varepsilon.$$

Consider

$$\delta := \min\left\{1, \frac{\varepsilon}{3|a|^2 + 3|a| + 2}\right\},$$

and let $x \in \mathbb{R}$ such that $|x - a| < \delta$. Then, from the triangle inequality we have

$$|x| = |x - a + a| \leq |a| + \delta \leq |a| + 1,$$

which we can use to write

$$\begin{aligned}
|f(x) - f(a)| &= \Big|(x^3 - x) - (a^3 - a)\Big| \\
&= \Big|x^3 - a^3 + a - x|\Big| \\
&\leq \Big|x^3 - a^3\Big| + \big|-(x - a)\big| && \text{(triangle inequality)} \\
&= |x - a|\Big|x^2 + ax + a^2\Big| + |x - a| && \text{(factoring)} \\
&= |x - a|\left(\Big|x^2 + ax + a^2\Big| + 1\right) \\
&\leq |x - a|\left(|x|^2 + |a||x| + |a|^2 + 1\right) && \text{(triangle inequality)} \\
&\leq |x - a|\left((|a| + 1)^2 + |a|(|a| + 1) + |a|^2 + 1\right) && \text{($|x| \leq |a| + 1$, shown above)} \\
&= |x - a|\left(|a|^2 + 2|a| + 1 + |a|^2 + |a| + |a|^2 + 1\right) && \text{(expanding)} \\
&= |x - a|\left(3|a|^2 + 3|a| + 2\right) \\
&< \delta\left(3|a|^2 + 3|a| + 2\right) && \text{($|x - a| < \delta$)} \\
&\leq \left(\frac{\varepsilon}{3|a|^2 + 3|a| + 2}\right)\left(3|a|^2 + 3|a| + 2\right) = \varepsilon.
\end{aligned}$$

Therefore, f is continuous at all points $a \in \mathbb{R}$. □

Problem 2 Give an ε, δ proof that $f(x) = \sqrt{|x|}$ is continuous at $x = 0$.

Let $\varepsilon > 0$. We will show that there exists $\delta > 0$ such that for all $x \in \mathbb{R}$, we have

$$|x - 0| < \delta \implies \big|f(x) - f(0)\big| < \varepsilon.$$

Consider $\delta := \varepsilon$. Let $x \in \mathbb{R}$ such that $|x - 0| < \delta$ (so $|x| < \delta$). Then, we can write

$$\big|f(x) - f(0)\big|^2 = \Big|\sqrt{|x|} - \sqrt{|0|}\Big|^2$$

$$
\begin{aligned}
&= \left|\sqrt{|x|}\right|^2 \\
&= \left(\sqrt{|x|}\right)^2 && \text{(absolute value property)} \\
&= |x| && \text{(square root property)} \\
&< \delta = \varepsilon^2. && \text{(by assumption/definition)}
\end{aligned}
$$

So $\left|f(x) - f(0)\right|^2 < \varepsilon^2$. It follows that $\left|f(x) - f(0)\right| < \varepsilon$, and therefore $f(x)$ is continuous at $x = 0$. $\square$

Problem 3 Give an ε, δ proof that $f(x) = \frac{x}{1+x}$ is continuous at any point $a \neq -1$.

Let $a \in \mathbb{R}\backslash\{-1\}$, and let $\varepsilon > 0$. We will show that there exists a $\delta > 0$ such that for all $x \in \mathbb{R}$, we have

$$|x - a| < \delta \implies \left|f(x) - f(a)\right| < \varepsilon.$$

Consider

$$\delta := \min\left\{\frac{|a+1|}{2}, \frac{|a+1|^2\varepsilon}{2}\right\},$$

and let $x \in \mathbb{R}$ such that $|x - a| < \delta$. Then, from the reverse triangle inequality we have

$$
\begin{aligned}
|x+1| &= \left|x - a - (-a-1)\right| \\
&\geq \left||x-a| - |a+1|\right| && \text{(reverse triangle inequality)} \\
&\geq |a+1| - |x-a| \\
&\geq |a+1| - \frac{|a+1|}{2} && \left(|x-a| < \delta \leq \tfrac{|a+1|}{2}\right) \\
&= \frac{|a+1|}{2},
\end{aligned}
$$

which we can use to write

$$
\begin{aligned}
\left|f(x) - f(a)\right| &= \left|\frac{x}{1+x} - \frac{a}{1+a}\right| \\
&= \left|\frac{x(a+1) - a(x+1)}{(x+1)(a+1)}\right| \\
&= \left|\frac{xa + x - ax - a}{(x+1)(a+1)}\right| \\
&= \frac{|x-a|}{|x+1||a+1|} \\
&\leq \frac{|x-a|}{\left(\frac{|a+1|}{2}\right)\left(|a+1|\right)} && \text{(from above)} \\
&= \frac{2|x-a|}{|a+1|^2} \\
&< \frac{2\delta}{|a+1|^2} \\
&\leq \left(\frac{2}{|a+1|^2}\right)\left(\frac{|a+1|^2\varepsilon}{2}\right) = \varepsilon.
\end{aligned}
$$

Therefore, f is continuous at all points $a \neq -1$. $\square$

Problem 4 Let $g : \mathbb{R} \to \mathbb{R}$ given by

$$g(x) = \begin{cases} 1 & \text{if } x \in \mathbb{Q}, \\ 0 & \text{if } x \notin \mathbb{Q} \end{cases}.$$

Show that g is not continuous at any point.

Let $a \in \mathbb{R}$ be arbitrary. We will show g is not continuous at a. Consider $\varepsilon := \frac{1}{2}$, and let $\delta > 0$ be arbitrary.

Case 1: $a \in \mathbb{Q}$. By a property of rational numbers we have proved, there exists some $x \notin \mathbb{Q}$ with $a < x < a+\delta$. But then $|x-a| < \delta$ and $\big|f(x) - f(a)\big| = |0-1| = 1 \geq \frac{1}{2}$, so in this case f is not continuous at a.

Case 2: $a \notin \mathbb{Q}$. Similarly, there exists some $x \in \mathbb{Q}$ with $a < x < a+\delta$. But then $|x-a| < \delta$ and $\big|f(x) - f(a)\big| = |1-0| = 1 \geq \frac{1}{2}$, so in this case also f is not continuous at a.

Therefore, f is continuous nowhere. □

Problem 5 Define the functions $f, g : \mathbb{R}^2 \to \mathbb{R}$ by $f(x,y) = x$ and $g(x,y) = y$. Show that f and g are continuous.

Let $\mathbf{a_0} = (x_0, y_0) \in \mathbb{R}^2$, and let $\varepsilon > 0$. We will show there exists a $\delta > 0$ such that for all $\mathbf{a} \in \mathbb{R}^2$, we have

$$\|\mathbf{a} - \mathbf{a_0}\| < \delta \implies \big|f(\mathbf{a}) - f(\mathbf{a_0})\big| < \varepsilon.$$

Consider $\delta := \varepsilon$, and let $\mathbf{a} = (x,y) \in \mathbb{R}^2$ such that $\|\mathbf{a} - \mathbf{a_0}\| < \delta$. Then, we have

$$\begin{aligned}
\big|f(\mathbf{a}) - f(\mathbf{a_0})\big| &= |x - x_0| \\
&= \sqrt{|x-x_0|^2} \\
&\leq \sqrt{|x-x_0|^2 + |y-y_0|^2} \\
&= \|\mathbf{a} - \mathbf{a_0}\| \\
&< \delta = \varepsilon.
\end{aligned}$$

So f is continuous at all points $a_0 \in \mathbb{R}^2$. The same argument holds for g by switching the roles of x and y.□

Problem 6

(a) Show that the function
$$f(x) = \begin{cases} 0, & x \leq 0; \\ x\cos\big(1/x\big), & x > 0. \end{cases}$$
is continuous at $x = 0$.

(b) Show that the function
$$g(x) = \begin{cases} 0, & x \leq 0; \\ \cos\big(1/x\big), & x > 0. \end{cases}$$
is not continuous at $x = 0$.

(a) Let $\varepsilon > 0$. We will show that there exists $\delta > 0$ such that for all $x \in \mathbb{R}$, we have

$$|x - 0| < \delta \implies \big|f(x) - f(0)\big| < \varepsilon.$$

Consider $\delta := \varepsilon$, and let $x \in \mathbb{R}$ such that $|x-0| < \delta$ (so $|x| < \delta$). Then, we have

$$\big|f(x) - f(0)\big| = \big|f(x)\big|$$

$$
\begin{aligned}
&\leq \left|x\cos\bigl(1/x\bigr)\right| && \text{(if } x \leq 0\text{, then } f(x) = 0 \leq x\cos\bigl(1/x\bigr)) \\
&= |x|\left|\cos\bigl(1/x\bigr)\right| \\
&\leq |x| && (\cos(\theta) \leq 1 \text{ for all } \theta \in \mathbb{R}) \\
&< \delta = \varepsilon.
\end{aligned}
$$

Therefore, f is continuous at $x = 0$.

(b) Consider $\varepsilon = \frac{1}{2}$, and let $\delta \in \mathbb{R}^+$ be arbitrary. By the Archimedean property, there exists some $n \in \mathbb{N}$ such that $\frac{1}{n} < 2\pi\delta$.

Consider $x = \frac{1}{2\pi n}$. Then we have $\left|0 - \frac{1}{2\pi n}\right| = \frac{1}{2\pi n} < \delta$ but

$$\bigl|g(x) - g(0)\bigr| = \bigl|g(x)\bigr| = \left|\cos\left(1/\frac{1}{2\pi n}\right)\right| = \bigl|\cos(2\pi n)\bigr| = 1 \geq \frac{1}{2} = \varepsilon.$$

Therefore, g is not continuous at $x = 0$.

Problem 7 Let (E, d) and (E', d') be metric spaces and assume that f is continuous at the point p_0. Let $\langle p_n \rangle_{n=1}^{\infty}$ be a sequence in E with $\lim\limits_{n\to\infty} p_n = p_0$. Prove that

$$\lim_{n\to\infty} f(p_n) = f(p_0).$$

Let $\varepsilon > 0$. Since f is continuous at p_0, there exists some $\delta > 0$ such that for all $p \in E$, we have

$$d(p, p_0) < \delta \implies d'(f(p), f(p_0)).$$

Since $\langle p_n \rangle$ converges to p_0, there exists some N such that for all $n > N$, $d(p_n, p) < \delta$. Let $n > N$. Then, by the continuity assumption, since we have $d(p_n, p) < \delta$, we also have

$$d'(f(p_n), f(p_0)) < \varepsilon.$$

Therefore, $\langle f(p_n) \rangle_{n=1}^{\infty}$ converges to $f(p_0)$ by definition. $\square$