MATH 552: Section 001
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MATH 552 Homework 9*

Problem C

(a) C is the straight line from $z_0 = -i$ to $z_1 = 1$, so it can be parameterized as $z(t) = ti + t - i$ where $0 \leq t \leq 1$. Thus, $dz = (i+1)dt$.

$$\begin{aligned}
\int_C (2z+1)dz &= \int_0^1 (2[ti + t - i] + 1)(i+1)dt && \text{(parameterizing z)} \\
&= \int_0^1 (4ti + 3 - i) && \text{(simplifying)} \\
&= [2t^2 i + 3t - it]_0^1 \\
&= 2i + 3 - i \\
&= 3 + i
\end{aligned}$$

(b) C is the straight line from $z_0 = -i$ to 0, so it can be parameterized as $z(t) = ti - i$ where $0 \leq t \leq 1$. Thus, $dz = idt$.

$$\begin{aligned}
\int_C (2z+1)dz &= \int_0^1 (2[ti - i] + 1)idt && \text{(parameterizing z)} \\
&= \int_0^1 (-2t + 2 + i) && \text{(simplifying)} \\
&= [-t^2 + 2t + it]_0^1 \\
&= -1 + 2 + i \\
&= 1 + i
\end{aligned}$$

Now C is the straight line to $z_0 = -i$ from 0, so it can be parameterized as $z(t) = -ti$ where $0 \leq t \leq 1$. Thus, $\frac{dz}{dt} = -i$.

$$\begin{aligned}
\int_C (2z+1)dz &= -\int_0^1 (2[-ti] + 1)idt && \text{(parameterizing z)} \\
&= -\int_0^1 (2t + i) && \text{(simplifying)} \\
&= -[t^2 + it]_0^1 \\
&= -1 - i
\end{aligned}$$

(c) C is the circular arc from $z_0 = -i$ to $z_1 = 1$, so it can be parameterized as $z(\theta) = e^{i\theta}$ where $-\frac{\pi}{2} \leq \theta \leq 0$.

Thus, $dz = ie^{i\theta}d\theta$.

$$\begin{aligned}\int_C (2z+1)dz &= \int_{-\frac{\pi}{2}}^{0} (2e^{i\theta}+1)ie^{i\theta}d\theta && \text{(parameterizing z)}\\ &= \int_{-\frac{\pi}{2}}^{0} (2ie^{2i\theta}+ie^{i\theta})d\theta\\ &= [e^{2i\theta}+e^{i\theta}]_{-\pi/2}^{0}\\ &= 3+i\end{aligned}$$

(d) C is C_1+C_2, where C_1 is the straight line from $z_0=-i$ to 0 and C_2 is the straight line from 0 to $z_1=1$. From (b), $\int_{C_1}(2z+1)dz = 1+i$. We can parameterize C_2 as $z(t_2)=t_2$ where $0\le t_2\le 1$. Thus, $dz = dt_2$.

$$\begin{aligned}\int_{C_2}(2z+1)dz &= \int_0^1 (2t+1)dt\\ &= [t^2+t]_0^1\\ &= 1+1=2\end{aligned}$$

Adding the integral over C_1 to the integral over C_2, $\int_C (2z+1)dz = 3+i$.

Problem D

(a) C is the circle wrapping around $|z|=2$ counterclockwise three times, starting at $z=2$. We can parameterize $z(t)$ as $z(t)=2e^{i\theta}$ where $0\le\theta\le 6\pi$. Then, $dz = 2ie^{i\theta}d\theta$.

$$\begin{aligned}\int_C \overline{z}\,dz &= \int_0^{6\pi}(2e^{-i\theta})2ie^{i\theta}d\theta && \text{(using definition of conjugate)}\\ &= \int_0^{6\pi} 4i\,d\theta\\ &= [4i\theta]_0^{6\pi} = 24\pi i\end{aligned}$$

(b) We write $z\overline{z}=|z|^2$, and $\overline{z}=\frac{|z|^2}{z}$ follows. Thus,

$$\int_C \overline{z}\,dz = |z|^2\int_C \frac{1}{z}dz.$$

We know $|z|=2$. Furthermore, since the interior of C that is exterior to the unit circle is analytic everywhere (since the only point that is not analytic, $z=0$, is inside the unit circle), we can define C' to be the unit circle traversed counterclockwise and obtain the same result when integrating as we would along one rotation of C. We know

$$\int_{C'}\frac{1}{z}dz = 2\pi i.$$

Since C traverses the circle three times, we multiply this result by 3 to get $6\pi i$. Therefore,

$$|z|^2\int_C\frac{1}{z}dz = 2^2(6\pi i) = 24\pi i.$$

Problem E

C is the circle wrapping around $|z-i|=4$ counterclockwise once, starting at $z=4+i$. We can parameterize

C as $z(t) = i + 4e^{i\theta}$ where $0 \leq \theta \leq 2\pi$. Then, $dz = 4ie^{i\theta}d\theta$.

$$\begin{aligned}
\int_C \left(\frac{6}{(z-i)^2} + \frac{2}{z-i} + 3(z-i)^2 \right) dz &= \int_0^{2\pi} \left(\frac{6}{4e^{2i\theta}} + \frac{2}{4e^{i\theta}} + 3(4e^{i\theta}) \right) 4ie^{i\theta} d\theta \\
&= \int_0^{2\pi} \left(\frac{6i}{e^{i\theta}} + 2i + 48ie^{2i\theta} \right) d\theta \\
&= -6 \int_0^{2\pi} -ie^{-i\theta} d\theta + 2i \int_0^{2\pi} d\theta + 24 \int_0^{2\pi} 2ie^{2i\theta} d\theta \\
&= -6[e^{-i\theta}]_0^{2\pi} + 2i[\theta]_0^{2\pi} + 24[e^{2i\theta}]_0^{2\pi} \\
&= 6(1-1) + 4\pi i - 0 + 24(1-1) = 4\pi i
\end{aligned}$$

Problem G

According to the Cauchy-Goursat theorem, if a function f is analytic at all points interior to and on a closed contour C, then $\int_C f(z)dz = 0$. Since $P(z)$ is a polynomial, it is analytic everywhere, so the theorem applies to any closed curve defined anywhere on $P(z)$. Since C is closed, the contour integral along it is 0.