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MATH 574: Section H01
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September 12, 2022

MATH 574 Homework 3

Collaboration: I discussed some of the questions with Jackson Ginn.

Problem 1 We consider the permutations of $\{1, 2, 3, 4, 5, 6\}$.
(a) Write the first 15 permutations in lexicographical order.
(b) What is the next largest permutation after 265431?
(c) Let $a_1a_2 \ldots a_n$ be a permutation of $\{1, 2, \ldots n\}$. Find a general method for constructing the previous permutation in lexicographical order of $a_1a_2 \ldots a_n$.
(d) Use part (c) to find the permutation before 265431.

Solution.
(a)

1. 123456
2. 123465
3. 123546
4. 123564
5. 123645
6. 123654
7. 124356
8. 124365
9. 124536
10. 124563
11. 124635
12. 124653
13. 125346
14. 125364
15. 125436

(b) Using the algorithm we discussed in class, the next largest permutation is 312456.
(c)

1. Find the greatest i such that $a_i > a_j$ for some $j \in \{i + 1, \ldots, n\}$.
2. Switch a_i with the greatest a_j such that $j \in \{i + 1, \ldots, n\}$ and $a_i > a_j$, so a_i is reassigned the old value of a_j. We know at least one j in the set will satisfy this based on how we chose i.
3. Reorder $a_{i+1} \ldots a_n$ in descending order, so $a_{i+1} > \cdots > a_n$.

(d) We begin with the permutation 265431, so $n = 6$ because there are 6 numbers.

1. The greatest i such that $a_i > a_j$ for some $j \in \{i + 1, \ldots, n\}$ is 5, because $a_5 = 3 > 1 = a_6$, and 6 is in $\{5 + 1, \ldots, 6\}$.

2. The greatest j for which $j \in \{5+1, \ldots, 6\}$ and $a_5 > a_j$ is $j = 6$ (clearly, since there is only one element in the set). So we switch a_i and a_j and obtain 265413.

3. We now reorder $a_{5+1} \ldots a_6$ in descending order. Since there is only one term, it is clearly in descending order, so we are done. The permutation before 265431 is 265413.

Problem 2 Write the next largest subset of size 6 of $\{1, \ldots 9\}$ for each of the following subsets.
(a) $\{2, 3, 4, 5, 6, 7\}$
(b) $\{3, 4, 6, 7, 8, 9\}$
(c) $\{1, 2, 4, 7, 8, 9\}$

Solution.

We use the algorithm we discussed in class.

(a) $\{2, 3, 4, 5, 6, 8\}$

(b) $\{3, 5, 6, 7, 8, 9\}$

(c) $\{1, 2, 5, 6, 7, 8\}$

Problem 3 200 people are entered into a drawing where each person has an equal chance to win. What's the probability that Alice, Bob, and Charlie win the first, second, and third place prize respectively if
(a) no one can win more than one prize?
(b) winning more than one prize is allowed?

Solution.

(a) There are 200 choices for the first prize, 199 for the second prize, and 198 for the third prize (assuming first, then second, then third is chosen). Thus, there are $200 \times 199 \times 198 = 7,880,400$ ways to award the prizes, and one way to award Alice, Bob, and Charlie the first, second, and third prizes respectively. Since each outcome is equally likely, the probability of this is $\frac{1}{7,880,400}$.

(b) There are 200 choices for each prize, so there are $200^3 = 8,000,000$ ways to award the three prizes. By the same reasoning as (a), the probability is $\frac{1}{8,000,000}$.

Problem 4 You are dealt a 5-card hand from a standard 52-card deck at random.

- Let A be the event that your hand contains at least 2 7's.
- Let B be the event that your hand contains a four of a kind.
- Let C be the event that your hand contains a flush (including royal flush and straight flush).

(a) Compute the probabilities $p(A)$, $p(B)$, and $p(C)$.
(b) Compute $p(B|A)$. Are A and B independent?
(c) Compute $p(A|C)$. Are A and C independent?

Solution.

(a)

- A: For a hand to contain s 7's, there are $\binom{4}{s}$ ways to choose which 7s are picked and $\binom{48}{5-s}$ to choose which other cards are picked. Since there are $\binom{52}{5}$ ways to choose a hand, the probability of our hand containing at least 2 7s is

$$\sum_{s=2}^{4} \frac{\binom{4}{s}\binom{48}{5-s}}{\binom{52}{5}} = \frac{2257}{54145}.$$

- B: There are 13 choices for which "kind" there will be four of, and 48 choices for the other card. Thus, the probability of our hand containing a four of a kind is

$$\frac{(13)(48)}{\binom{52}{5}} = \frac{1}{4165}.$$

- C: We first choose a card. Then, there is a $\frac{12}{51}$ chance the next card will have the same suit, and then $\frac{11}{50}$, and so on. So the probability of our hand containing a flush is

$$(1)\left(\frac{12}{51}\right)\left(\frac{11}{50}\right)\left(\frac{10}{49}\right)\left(\frac{9}{48}\right) = \frac{33}{16660}.$$

(b) Since $P(A \cap B)$ requires 4 7s and one other card, there are 48 options for this out of $\binom{52}{5}$. So

$$P(B|A) = \frac{P(A \cap B)}{P(A)} = \frac{48/\binom{52}{5}}{108336/\binom{52}{5}} = \frac{1}{2257}.$$

Therefore, A and B are not independent, because $P(B) = \frac{1}{4165} \neq \frac{1}{2257} = P(B|A)$.

(c) If C happens, then there can be at most one 7 in the hand because there is only one 7 of each suit in the deck. Therefore, $p(A|C) = 0$, so A and C are not independent because $p(A|C) = 0 \neq \frac{2257}{54145} = p(A)$.

Problem 5 Prove that if E and F are events, then

$$p(E \cap F) \geq p(E) + p(F) - 1.$$

When does equality hold? That is, what conditions on E and F imply that $p(E \cap F) = p(E) + p(F) - 1$?

Solution.

We have from class that

$$P(E \cup F) = P(E) + P(F) - P(E \cap F).$$

Rearranging,

$$P(E \cap F) = P(E) + P(F) - P(E \cup F).$$

We claim that $p(E \cap F) \geq p(E) + p(F) - 1$. Assume to the contrary that $p(E \cap F) < p(E) + p(F) - 1$.

$$\begin{aligned}
&p(E \cap F) < p(E) + p(F) - 1 \\
&\implies P(E) + P(F) - P(E \cup F) < p(E) + p(F) - 1 && \text{(substituting for } P(E \cap F)) \\
&\implies -P(E \cup F) < -1 \\
&\implies P(E \cup F) > 1.
\end{aligned}$$

But this is a contradiction, because probabilities can never exceed 1. Therefore, we must have

$$p(E \cap F) \geq p(E) + p(F) - 1.$$

Equality holds when $P(E \cup F) = 1$, as can be seen by substituting 1 for $P(E \cup F)$ in the equation for $P(E \cap F)$ above.

Problem 6 Prove that for all $n \geq 2$, if $E_1, E_2, \ldots, E_n$ are n events then

$$p(E_1 \cap E_2 \cap \cdots \cap E_n) \geq p(E_1) + p(E_2) + \cdots + p(E_n) - (n-1).$$

Solution.

First, let $n = 2$. From problem (5), we have that $p(E \cap F) \geq p(E) + p(F) - 1$. Therefore, we also have $p(E_1 \cap E_2) \geq p(E_1) + p(E_2) - (2-1)$, so the claim holds for $n = 2$.

Then, let $n \in \mathbb{N}$ such that $n \geq 2$. Assume that

$$p\left(\bigcap_{i=1}^{n} E_i\right) \geq \sum_{i=1}^{n} (p(E_i)) - (n-1).$$

We claim that

$$p\left(\bigcap_{i=1}^{n+1} E_i\right) \geq \sum_{i=1}^{n+1} (p(E_i)) - (n+1-1).$$

Assume this is not true. Then,

$$\begin{aligned}
& p\left(\bigcap_{i=1}^{n+1} E_i\right) < \sum_{i=1}^{n+1} (p(E_i)) - (n+1-1) \\
\implies & p\left(\bigcap_{i=1}^{n} (E_i) \cap E_{n+1}\right) < \sum_{i=1}^{n} (p(E_i)) + p(E_{n+1}) - (n-1) - 1 && \text{(splitting intersection and sum)} \\
\implies & p\left(\bigcap_{i=1}^{n} (E_i) \cap E_{n+1}\right) - p(E_{n+1}) + 1 < \sum_{i=1}^{n} (p(E_i)) - (n-1) && \text{(rearranging inequality)} \\
\implies & p\left(\bigcap_{i=1}^{n} (E_i) \cap E_{n+1}\right) - p(E_{n+1}) + 1 < p\left(\bigcap_{i=1}^{n} E_i\right) && \text{(induction hypothesis)} \\
\implies & p\left(\bigcap_{i=1}^{n} (E_i) \cap E_{n+1}\right) < p\left(\bigcap_{i=1}^{n} E_i\right) + p(E_{n+1}) - 1. && \text{(rearranging inequality)}
\end{aligned}$$

But again, we know from problem (5) that this cannot be true, because $p(E \cap F) \geq p(E) + p(F) - 1$ is true for

$$E = \bigcap_{i=1}^{n} (E_i) \text{ and } F = E_{n+1}.$$

So our claim must be true for $n+1$ if it is for n, and therefore for any $n \geq 2$ we have

$$p(E_1 \cap E_2 \cap \cdots \cap E_n) \geq p(E_1) + p(E_2) + \cdots + p(E_n) - (n-1).$$

Problem 7 Prove that for all $n \geq 2$, if $E_1, E_2, \ldots, E_n$ are n events from a finite sample space, then

$$p(E_1 \cup E_2 \cup \cdots \cup E_n) \leq p(E_1) + p(E_2) + \cdots + p(E_n).$$

Solution.

First, let $n = 2$. As show in problem 5, $P(E \cup F) = P(E) + P(F) - P(E \cap F)$. Thus,

$$P(E_1 \cup E_2) + P(E_1 \cap E_2) \leq P(E_1) + P(E_2),$$

and since $P(E_1 \cap E_2)$ is non-negative, the claim holds for $n = 2$.

Next, let $n \in \mathbb{N}$ such that $n \geq 2$. Assume that

$$p\left(\bigcup_{i=0}^{n} E_i\right) \leq \sum_{i=0}^{n} p(E_i).$$

We observe that:

$$\begin{aligned}
p\left(\bigcup_{i=0}^{n+1} E_i\right) &= p\left(\bigcup_{i=0}^{n} E_i \cup E_{n+1}\right) && \text{(splitting union)} \\
&\leq p\left(\bigcup_{i=0}^{n} E_i\right) + p\left(E_{n+1}\right) && \text{(as shown in base case for } n = 2\text{)} \\
&\leq \sum_{i=0}^{n} (p(E_i)) + p(E_{n+1}) && \text{(induction hypothesis)} \\
&= \sum_{i=0}^{n+1} (p(E_i)). && \text{(combining sum)}
\end{aligned}$$

So if the claim holds for n, it also holds for $n + 1$. Therefore, for all $n \geq 2$, we have

$$p(E_1 \cup E_2 \cup \cdots \cup E_n) \leq p(E_1) + p(E_2) + \cdots + p(E_n).$$

Problem 8 A bit string of length 12 is generated at random. Let A be the event that the string contains six consecutive 1s. Let B be the event that the first bit is 0. Determine $p(A)$, $p(B)$, $p(A \cap B)$, $p(A|B)$ and $p(B|A)$. Compare the quantities $p(B)$ with $p(B|A)$ and $p(A)$ with $p(A|B)$ and briefly interpret your results.

Solution.

- From Homework 1, problem (4), we have that there are 2^8 bit strings of length 12 with 6 consecutive 1s. Since there are 2^{12} possible bit strings of length 12, $p(A) = \frac{2^8}{2^{12}} = \frac{1}{16}$.
- There are 2^{11} bit strings of length 12 starting with 0 (since the first bit has only one choice), so $p(B) = \frac{2^{11}}{2^{12}} = \frac{1}{2}$.
- We take the same approach as in Homework 1, problem (4), but we treat the bit string as a bit string of length 11 since we can ignore the leading 0. So there are $2^5 + 5(2^4)$ possible bit strings, because the first class has 5 free bits and the second through sixth classes have 4 free bits. Since there are 2^{12} possible bit strings, $P(A \cap B) = \frac{2^5+5(2^4)}{2^{12}} = \frac{7}{256}$.
- We have that $p(A|B) = \frac{p(A \cap B)}{p(B)}$, so $p(A|B) = \frac{7/256}{1/2} = \frac{7}{128}$.
- We have that $p(B|A) = \frac{p(A \cap B)}{p(A)}$, so $p(B|A) = \frac{7/256}{1/16} = \frac{7}{16}$.

Since $p(A) = \frac{1}{16} \neq \frac{7}{128} = p(A|B)$ and $p(B) = \frac{1}{2} \neq \frac{7}{16} = p(B|A)$, A and B are not independent.