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Analysis in $\mathbb{R}^n$: Section 1
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Analysis in $\mathbb{R}^n$ Homework 2

Problem 9 Let E° denote the set of all interior points of a set E. [See Definition 2.18(e); E° is called the *interior* of E.]

(a) Prove that E° is always open.

(b) Prove that E is open if and only if $E^\circ = E$.

(c) If $G \subset E$ and G is open, prove that $G \subset E^\circ$.

Solution.

(a) We have shown in class that E° is open if and only if for every $p \in E^\circ$, there exists an $r > 0$ such that $B_r(p) \subset E^\circ$. Let $p \in E^\circ$. Since p is an interior point, by definition there exists a neighborhood N_p of p such that $N_p \subset E$. Let $x \in N$. We have showed in class that since N_p is an open ball, it is open in E, so there is a neighborhood N_x of x in E. Thus, x is an interior point in E so $x \in E^\circ$. Therefore, $N_p \subset E^\circ$ and E° is open.

(b) ($\Leftarrow$) Suppose $E = E^\circ$. From **(a)**, E° is open, so E is as well since $E = E^\circ$.

($\Rightarrow$) Suppose E is open. We have $E^\circ \subset E$ by definition, so it suffices to show $E \subset E^\circ$. Let $p \in E$. Since E is open, from the theorem in (a) there exists an $r > 0$ such that $B_r(p) \subset E$. Then, there is a neighborhood N of p in E, so p is an interior point and thus $p \in E^\circ$. So $E^\circ \subset E$ and $E = E^\circ$.

(c) Suppose $G \subset E$ is open and let $p \in G$. Since G is open, there exists an $r > 0$ such that $B_r(p) \subset G \subset E$, so $B_r(p)$ must lie in E meaning p is an interior point of E. Since E° is the set of interior points, $p \in E^\circ$ and therefore $G \subset E^\circ$.

Problem 14 Give an example of an open cover of the segment $(0, 1)$ which has no finite sub-cover.

Solution.

Consider the set

$$C = \left\{ \left(x - \frac{x}{2}, x + \frac{x}{2} \right) : x \in (0,1) \right\} = \left\{ \left(\frac{x}{2}, \frac{3x}{2} \right) : x \in (0,1) \right\}.$$

This is an open cover for $(0, 1)$: every set in C is open because it is an open interval, and every $x \in (0,1)$ is in $\bigcup_{I \in C} I$ because x is in the interval $\left(\frac{x}{2}, \frac{3x}{2}\right)$. However, we claim there is no finite sub-cover $C_n \subset C$ that covers $(0,1)$. Suppose toward contradiction that $C_n = \{I_1, I_2, \ldots, I_n\} \subset C$ is a cover for $(0,1)$. Let $X_n = \{x_1, x_2, \ldots, x_n\}$ be the center points of the intervals, where $I_i = \left(\frac{x_i}{2}, \frac{3x_i}{2}\right)$ for $1 \le i \le n$. Let x be the least element in X_n (which is well defined because X_n is finite). Then, $\left(\frac{x}{2}, \frac{3x}{2}\right)$ is covered by C_n, but $(0, \frac{x}{2}]$ is not because x is the least element. This is a contradiction because we assumed C_n covered $(0,1)$. $\square$

Problem 1 Let (X, d) be a metric space and $A \subset X$. Define $\mathcal{U} := \{U \subset A : U \text{ is open}\}$. Prove that $A^\circ = \bigcup_{U \in \mathcal{U}} U$.

Solution.

($\subset$) From 9(a), we know that $A^\circ \subset A$ is open, so $A^\circ \in \mathcal{U}$. Thus, we clearly have $A^\circ \subset \bigcup_{U \in \mathcal{U}} U$.

($\supset$) Let $x \in \bigcup_{U \in \mathcal{U}} U$. Then, $x \in U$ for some open set $U \subset A$. By the theorem from 9(a), there exists an $r > 0$ such that $B_r(x) \subset U$. Since $U \subset A$, $B_r(x) \subset A$ and thus there is a neighborhood N of x lying in A. So by definition, x is an interior point and thus is in A°, and therefore $\bigcup_{U \in \mathcal{U}} U \subset A^\circ$. $\square$

Problem 2 Prove that every open set in $(\mathbb{R}, |\cdot|)$ is a disjoint union of open intervals. I.e. if $U \subset \mathbb{R}$ is open, there exist open intervals $\{I_\alpha\}$ such that $U = \bigcup_\alpha I_\alpha$, and for all $\alpha \neq \alpha'$, $I_\alpha \cap I_{\alpha'} = \emptyset$.

Solution.

Let U be an open set in $(\mathbb{R}, |\cdot|)$. Since U is open, it contains no isolated points in $\mathbb{R}$ (if there were an isolated point, then it would be a limit point in U^c but not in U^c, meaning U^c would not be closed). Thus, every point in U must be in some interval in $\mathbb{R}$, and so there exists some set of intervals in $\mathbb{R}$ whose union equals U. Let I_α be such a set with each interval maximal (if it were any wider, it would contain points not in U). Then, every interval in $\{I_\alpha\}$ must be open. If it is not, then there must be some interval $I \in \{I_\alpha\}$ that includes an endpoint x. But then x is a limit point of $U^c = \mathbb{R} \backslash U$ but $x \notin U^c$ (because it is in I), so U^c is not closed, contradicting the openness of U. Additionally, the sets in I_α must be pairwise disjoint: if this is not the case, then we have $I_1 = (x_1, x_2), I_2 = (x_3, x_4) \in \{I_\alpha\}$ with $x_2 > x_3$ and $I_1 \cap I_2 \neq \emptyset$. But then (x_1, x_4) is a wider interval that contains only points in U, contradicting maximality. $\square$