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MATH 576: Section H01
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MATH 576 Homework 3

We will use the notation of equating a position of a game with its set of positions to which the $\mathcal{N}$ext player can legally move. For a set of positions S, by $*S$ we mean the set

$$\{*a : *a \text{ is the nimber associated with some position in } S\}.$$

Problem 1 Write in words the ruleset of the octal game **0.454**. Find the first 10 Nimbers of this game.

The ruleset of the game is as follows. For some $n \in \mathbb{N}$, the game starts with a heap of n tokens. On a player's turn, they must do exactly one of the following:

- Remove one token from the heap and split the remaining tokens into exactly two non-empty heaps.
- Remove two tokens from the heap and, if any tokens remain, split them into exactly two non-empty heaps.
- Remove three tokens from the heap and split the remaining tokens into exactly two non-empty heaps.

We now compute the first 10 nimbers of this game. Let $\underline{n}$ the game with n tokens. Using the notation above and minimum excludents, we compute that:

- $\underline{0}$ has nimber $*0$ since $*\underline{0} = *\varnothing = \varnothing$.
- $\underline{1}$ has nimber $*0$ since $*\underline{1} = *\varnothing = \varnothing$.
- $\underline{2}$ has nimber $*$ since $*\underline{2} = *\{\underline{0}\} = \{*0\}$.
- $\underline{3}$ has nimber $*$ since $*\underline{3} = *\{\underline{1} + \underline{1}\} = \{*0 + *0\} = \{*0\}$.
- $\underline{4}$ has nimber $*2$ since $*\underline{4} = *\{\underline{2} + \underline{1}, \underline{1} + \underline{1}\} = \{* + *0, *0 + *0\} = \{*, *0\} = \{*0, *\}$.
- $\underline{5}$ has nimber $*2$ since

$$\begin{aligned} *\underline{5} &= *\{\underline{3} + \underline{1}, \underline{2} + \underline{2}, \underline{2} + \underline{1}, \underline{1} + \underline{1}\} \\ &= \{* + *0, * + *, * + *0, *0 + *0\} \\ &= \{*, *0, *, *0\} = \{*0, *\}. \end{aligned}$$

- $\underline{6}$ has nimber $*3$ since

$$\begin{aligned} *\underline{6} &= *\{\underline{4} + \underline{1}, \underline{3} + \underline{2}, \underline{3} + \underline{1}, \underline{2} + \underline{2}, \underline{2} + \underline{1}\} \\ &= \{*2 + *0, * + *, * + *0, * + *, * + *0\} \\ &= \{*2, *0, *, *0, *\} = \{*0, *, *2\}. \end{aligned}$$

- $\underline{7}$ has nimber $*4$ since

$$\begin{aligned} *\underline{7} &= *\{\underline{5} + \underline{1}, \underline{4} + \underline{2}, \underline{3} + \underline{3}, \underline{4} + \underline{1}, \underline{3} + \underline{2}, \underline{3} + \underline{1}, \underline{2} + \underline{2}\} \\ &= \{*2 + *0, *2 + *, * + *, *2 + *0, * + *, * + *0, * + *\} \\ &= \{*2, *3, *0, *2, *0, *, *0\} = \{*0, *, *2, *3\}. \end{aligned}$$

- $\underline{8}$ has nimber $*$ since

$$\begin{aligned} *\underline{8} &= *\{\underline{6}+\underline{1}, \underline{5}+\underline{2}, \underline{4}+\underline{3}, \underline{5}+\underline{1}, \underline{4}+\underline{2}, \underline{3}+\underline{3}, \underline{4}+\underline{1}, \underline{3}+\underline{2}\} \\ &= \{*3+*0, *2+*, *2+*, *2+*0, *2+*, *+*, *2+*0, *+*\} \\ &= \{*3, *3, *3, *2, *3, *0, *2, *0\} = \{*0, *2, *3\}. \end{aligned}$$

- $\underline{9}$ has nimber $*$ since

$$\begin{aligned} *\underline{9} &= *\{\underline{7}+\underline{1}, \underline{6}+\underline{2}, \underline{5}+\underline{3}, \underline{4}+\underline{4}, \underline{6}+\underline{1}, \underline{5}+\underline{2}, \underline{4}+\underline{3}, \underline{5}+\underline{1}, \underline{4}+\underline{2}, \underline{3}+\underline{3}\} \\ &= \{*4+*0, *3+*, *2+*, *2+*2, *3+*0, *2+*, *2+*, *2+*0, *2+*, *+*\} \\ &= \{*4, *2, *3, *0, *3, *3, *3, *2, *3, *0\} = \{*0, *2, *3, *4\} \end{aligned}$$

- $\underline{10}$ has number $*6$ since

$$\begin{aligned} *\underline{10} &= *\{\underline{8}+\underline{1}, \underline{7}+\underline{2}, \underline{6}+\underline{3}, \underline{5}+\underline{4}, \underline{7}+\underline{1}, \underline{6}+\underline{2}, \underline{5}+\underline{3}, \underline{4}+\underline{4}, \underline{6}+\underline{1}, \underline{5}+\underline{2}, \underline{4}+\underline{3}\} \\ &= \{*+*0, *4+*, *3+*, *2+*2, *4+*0, *3+*, *2+*, *2+*2, *3+*0, *2+*, *2+*\} \\ &= \{*, *5, *2, *0, *4, *2, *3, *0, *3, *3, *3\} = \{*0, *, *2, *3, *4, *5\}. \end{aligned}$$

Problem 2 Consider the octal game **0.4** (remove one token from a heap; split the remaining tokens into two nonempty heaps). Let $G(n)$ be the Grundy value of the game **0.4** with a starting position of n tokens in a single heap. Let $H(n)$ be the Grundy value of Dawson's Kayles (**0.07**) starting from an initial position of n pins in a row. Prove that for $n \geq 0$, $G(n+1) = H(n)$.

We proceed by induction on n. The claim holds for $n = 0$ holds since the game **0.4** with 1 token and Dawson's Kayles with 0 tokens both have no legal moves and thus both have number $*0$. The claim holds for $n = 1$ for the same reasoning.

Now, let $n \geq 2$, and suppose that for all $n' < n$, we have $G(n'+1) = H(n')$. Observe that by the choice of the octal game, we have

$$G(n+1) = \operatorname{mex}(\{G(a) + G(b) \mid a, b > 0, a + b = n\}). \tag{$\star$}$$

We can then write

$$\begin{aligned} H(n) &= \operatorname{mex}(\{H(a) + H(b) \mid a, b \geq 0, a + b = n - 2\}) && \text{(definition of Dawson's Kayles)} \\ &= \operatorname{mex}(\{G(a+1) + G(b+1) \mid a, b \geq 0, a + b = n - 2\}) && \text{(induction hypothesis)} \\ &= \operatorname{mex}(\{G(k) + G(m) \mid k - 1 \geq 0, m - 1 \geq 0, k - 1 + m - 1 = n - 2\}) && (a = k - 1, b = m - 1) \\ &= \operatorname{mex}(\{G(k) + G(m) \mid k, m \geq 1, k + m = n\}) && \text{(simplifying)} \\ &= \operatorname{mex}(\{G(k) + G(m) \mid k, m > 0, k + m = n\}) && (x \geq 1 \iff x > 0 \text{ for all } x \in \mathbb{Z}) \\ &= G(n+1). && (\text{by } \star) \end{aligned}$$

$\square$

Problem 3 Let $H(n)$ be the Grundy value of Dawson's Kayles (**0.07**) starting from an initial position of n pins in a row. Let $J(n)$ be the Grundy value of Dawson's Chess (**0.137**) on a $3 \times n$ board. Prove that for $n \geq 0$, $H(n+1) = J(n)$.

Let $\underline{n}$ be the game of Dawson's Kayles with n tokens, and let $\overline{n}$ be the game of Dawson's Chess with n tokens. We proceed by induction on n. For the base case, we verify that:

- For $n = 0$, we have
$$*\underline{0} = *\underline{1} = *\overline{0} = *\varnothing = \varnothing,$$
so $\underline{0}, \underline{1}, \overline{0}$ all have nimber $*0$ and thus $H(1) = J(0)$.
- For $n = 1$, we have
$$H(2) = \text{mex}(*\underline{2}) = \text{mex}(*\{\underline{0}\}) = \text{mex}(\{*0\}) = \text{mex}(*\{\overline{0}\}) = \text{mex}(*\overline{1}) = J(1).$$
- For $n = 2$, we have
$$H(3) = \text{mex}(*\underline{3}) = \text{mex}(*\{\underline{1}\}) = \text{mex}(\{*0\}) = \text{mex}(*\{\overline{0}\}) = \text{mex}(*\overline{2}) = J(2).$$

Now, let $n \geq 3$, and suppose that for all $n' < n$, we have $H(n'+1) = J(n')$. From Problem 2, we have
$$H(n+1) = \{H(a) + H(b) \mid a, b \geq 0, a + b = n - 1\}. \tag{$\star$}$$
We can then write
$$\begin{aligned}
J(n) &= \text{mex}(\{J(a) + J(b) \mid a, b \geq 0, a + b = n - 3\} \cup \{J(n-2)\}) && \text{(by rules of } \mathbf{0.137}\text{)} \\
&= \text{mex}(\{H(a+1) + H(b+1) \mid a, b \geq 0, a + b = n - 3\} \cup \{H(n-1)\}) && \text{(induction hypothesis)} \\
&= \text{mex}(\{H(k) + H(m) \mid k - 1 \geq 0, m - 1 \geq 0, k - 1 + m - 1 = n - 3\} \cup \{H(n-1)\}) \\
&= \text{mex}(\{H(k) + H(m) \mid k, m \geq 1, k + m = n - 1\} \cup \{H(n-1)\}) \\
&= \text{mex}(\{H(k) + H(m) \mid k, m \geq 1, k + m = n - 1\} \cup \{H(n-1) + H(0)\}) && \text{(since } H(0) = *0\text{)} \\
&= \text{mex}(\{H(k) + H(m) \mid k, m \geq 0, k + m = n - 1\}) && \text{(combining sets)} \\
&= H(n+1). && \text{(by } \star\text{)}
\end{aligned}$$
□

Problem 4 Let a and b be positive integers. Prove that the Subtraction-$\{a, b\}$ game is periodic with period $a + b$ and preperiod 0.

Without loss of generality, suppose $a \leq b$. Let $\underline{n}$ be the Subtraction-$\{a, b\}$ game with n tokens, and let $G(n)$ be the Grundy value of $\underline{n}$. Note that $\underline{n}$ always has at most two options, so $G(n) \in \{*0, *, *2\}$ for all $n \geq 0$. It suffices to show that for all $n \geq 0$, $G(n) = G(n + a + b)$.

We first prove a lemma: for all $n \geq 0$, we claim we have
$$G(n) = *0 \iff G(n + a + b) = *0. \tag{$\star$}$$
($\Rightarrow$) Observe that for all $n \geq 0$, if $\underline{n}$ is a $\mathcal{P}$-position, then $\underline{n + a + b}$ is a $\mathcal{P}$-position: if the $\mathcal{N}$ext player takes a tokens, the $\mathcal{P}$revious player can take b tokens to reduce the heap to $\underline{n} \in \mathcal{P}$, and if the $\mathcal{N}$ext player takes b tokens the $\mathcal{P}$revious player can take a tokens to reduce the heap to $\underline{n} \in \mathcal{P}$. Since all $\mathcal{P}$-positions have nimber $*0$, we have that for all $n \geq 0$, we have $G(n) = *0 \implies G(n + a + b) = *0$.

($\Leftarrow$) Suppose that $\underline{n + a + b} \in \mathcal{P}$ but $\underline{n} \in \mathcal{N}$. Since $\underline{n + a + b} = \{\underline{n + b}, \underline{n + a}\} \in \mathcal{P}$, we have that $\underline{n + b}, \underline{n + a} \in \mathcal{N}$. Since $\underline{n + b} = \{\underline{n + b - a}, \underline{n}\} \in \mathcal{N}$ and $\underline{n} \in \mathcal{N}$, we have $\underline{n + b - a} = \{\underline{n + b - 2a}, \underline{n - a}\} \in \mathcal{P}$. Thus, $\underline{n - a} \in \mathcal{N}$. Since $\underline{n + a} = \{\underline{n}, \underline{n + a - b}\} \in \mathcal{N}$ and $\underline{n} \in \mathcal{N}$, we have $\underline{n + a - b} = \{\underline{n - b}, \underline{n + a - 2b}\}$. Thus, $\underline{n - b} \in \mathcal{N}$. But then $\underline{n} = \{\underline{n - a}, \underline{n - b}\}$ has only options in $\mathcal{N}$ and thus $\underline{n} \in \mathcal{P}$, a contradiction.

Thus, the lemma holds. We observe that for all $0 \leq n < a$, we have $G(n) = 0$ since $\underline{n} = \varnothing$. So for all $n < a$, we have $G(n) = *0 = G(n + a + b)$ from the lemma.

Now, let $n \geq a$, and suppose that for all $n' < n$, we have $G(n') = G(n' + a + b)$. We show that $G(n) = G(n + a + b)$.

Case 1: $n < b$. Then $G(n) = \operatorname{mex}(\{G(n-a)\})$, so $G(n) \in \{*0, *\}$.

Case 1.1: $G(n) = *0$. Then $G(n + a + b) = *0$ by the lemma.

Case 1.2: $G(n) = *$. Since $G(n) = \operatorname{mex}\{G(n-a)\} = *$, we must have $G(n-a) = *0$, and thus by the lemma we have $G(n+b) = *0$. Also, we have $G(n+a) = \operatorname{mex}\{G(n), G(n+a-b)\}$, and since $G(n) = *$ we have $G(n+a) \neq *$. So $\{G(n+b), G(n+a)\}$ contains $*0$ but not $*$, and thus

$$G(n + a + b) = \operatorname{mex}\{G(n + b), G(n + a)\} = * = G(n).$$

So in both cases, $G(n) = G(n + a + b)$.

Case 2: $n \geq b$. We have

$$\begin{aligned}
G(n) &= \operatorname{mex}(\{G(n-a), G(n-b)\}) && \text{(definition)}\\
&= \operatorname{mex}(\{G((n-a)+(a+b)), G((n-b)+(a+b))\}) && \text{(induction hypothesis)}\\
&= \operatorname{mex}(\{G((n+a+b)-a), G((n+a+b)-b)\}) && \text{(rearranging)}\\
&= G(n+a+b). && \text{(definition)}
\end{aligned}$$

Therefore, the Subtraction-$\{a, b\}$ game is periodic with period $a + b$ and preperiod 0.

Problem 5 Lasker's Nim is the following variation of Nim: on their turn, a player may remove any number of tokens from any one heap, or split a heap of tokens into two nonempty heaps (in octal game notation, this is **4.333**...). Compute the first 10 Nimbers of Lasker's Nim.

Let $\underline{n}$ be the game of Lasker's Nim. We now compute the first 10 nimbers of this game. Using minimum excludents, we compute that:

- $\underline{0}$ has nimber $*0$ since $*\underline{0} = *\varnothing = \varnothing$.
- $\underline{1}$ has nimber $*$ since $*\underline{1} = *\{\underline{0}\} = \{*0\}$.
- $\underline{2}$ has nimber $*2$ since $*\underline{2} = *\{\underline{0}, \underline{1}, \underline{1} + \underline{1}\} = \{*0, *, * + *\} = \{*0, *, *0\} = \{*0, *\}$.
- $\underline{3}$ has nimber $*4$ since $*\underline{3} = *\{\underline{0}, \underline{1}, \underline{2}, \underline{2} + \underline{1}\} = \{*0, *, *2, *2 + *1\} = \{*0, *, *2, *3\}$.
- $\underline{4}$ has nimber $*3$ since

$$\begin{aligned}
*\underline{4} &= *\{\underline{0}, \underline{1}, \underline{2}, \underline{3}, \underline{3} + \underline{1}, \underline{2} + \underline{2}\}\\
&= \{*0, *, *2, *4, *4 + *, *2 + *2\}\\
&= \{*0, *, *2, *5, *0\}\\
&= \{*0, *, *2, *5\}.
\end{aligned}$$

- $\underline{5}$ has nimber $*5$ since

$$\begin{aligned}
*\underline{5} &= *\{\underline{0}, \underline{1}, \underline{2}, \underline{3}, \underline{4}, \underline{4} + \underline{1}, \underline{3} + \underline{2}\}\\
&= \{*0, *, *2, *4, *3, *3 + *, *4 + *2\}\\
&= \{*0, *, *2, *4, *3, *2, *6\}\\
&= \{*0, *, *2, *3, *4, *6\}.
\end{aligned}$$

- $\underline{6}$ has nimber $*6$ since

$$\begin{aligned}
*\underline{6} &= *\{\underline{0}, \underline{1}, \underline{2}, \underline{3}, \underline{4}, \underline{5}, \underline{5} + \underline{1}, \underline{4} + \underline{2}, \underline{3} + \underline{3}\} \\
&= \{*0, *, *2, *4, *3, *5, *5 + *, *3 + *2, *4 + *4\} \\
&= \{*0, *, *2, *4, *3, *5, *4, *, *0\} \\
&= \{*0, *, *2, *3, *4, *5\}.
\end{aligned}$$

- $\underline{7}$ has nimber $*8$ since

$$\begin{aligned}
*\underline{7} &= \{\underline{0}, \underline{1}, \underline{2}, \underline{3}, \underline{4}, \underline{5}, \underline{6}, \underline{6} + \underline{1}, \underline{5} + \underline{2}, \underline{4} + \underline{3}\} \\
&= \{*0, *, *2, *4, *3, *5, *6, *7, *7, *7\} \\
&= \{*0, *, *2, *3, *4, *5, *6, *7\}
\end{aligned}$$

- $\underline{8}$ has nimber $*7$ since

$$\begin{aligned}
*\underline{8} &= *\{\underline{0}, \underline{1}, \underline{2}, \underline{3}, \underline{4}, \underline{5}, \underline{6}, \underline{7}, \underline{7} + \underline{1}, \underline{6} + \underline{2}, \underline{5} + \underline{3}, \underline{4} + \underline{4}\} \\
&= \{*0, *, *2, *4, *3, *5, *6, *8, *8 + *, *6 + *2, *5 + *4, *3 + *3\} \\
&= \{*0, *, *2, *4, *3, *5, *6, *8, *9, *4, *, *0\} \\
&= \{*0, *, *2, *3, *4, *5, *6, *8, *9\}.
\end{aligned}$$

- $\underline{9}$ has number $*9$ since

$$\begin{aligned}
*\underline{9} &= *\{\underline{0}, \underline{1}, \underline{2}, \underline{3}, \underline{4}, \underline{5}, \underline{6}, \underline{7}, \underline{8}, \underline{8} + \underline{1}, \underline{7} + \underline{2}, \underline{6} + \underline{3}, \underline{5} + \underline{4}\} \\
&= \{*0, *, *2, *4, *3, *5, *6, *8, *7, *7 + *, *8 + *2, *6 + *4, *5 + *3\} \\
&= \{*0, *, *2, *4, *3, *5, *6, *8, *7, *6, *10, *2, *6\} \\
&= \{*0, *, *2, *3, *4, *5, *6, *7, *8, *10\}
\end{aligned}$$

- $\underline{1}0$ has number $*10$ since

$$\begin{aligned}
*\underline{10} &= *\{\underline{0}, \underline{1}, \underline{2}, \underline{3}, \underline{4}, \underline{5}, \underline{6}, \underline{7}, \underline{8}, \underline{9}, \underline{9} + \underline{1}, \underline{8} + \underline{2}, \underline{7} + \underline{3}, \underline{6} + \underline{4}, \underline{5} + \underline{5}\} \\
&= \{*0, *, *2, *4, *3, *5, *6, *8, *7, *9, *9 + *, *7 + *2, *8 + *4, *6 + *3, *5 + *5\} \\
&= \{*0, *, *2, *4, *3, *5, *6, *8, *7, *9, *8, *5, *12, *5, *0\} \\
&= \{*0, *, *2, *3, *4, *5, *6, *7, *8, *9, *12\}.
\end{aligned}$$