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MATH 300: Section H01
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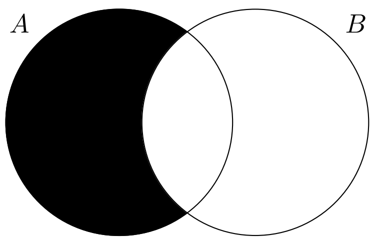
MATH 300 Homework 7

Problem 1

(a) Both $A\backslash B$ and $A\cap\overline{B}$ have the same diagram:



Let A, B be sets.

Assume x is in $A\backslash B$. Then, by definition $x\in A$ and $x\notin B$.

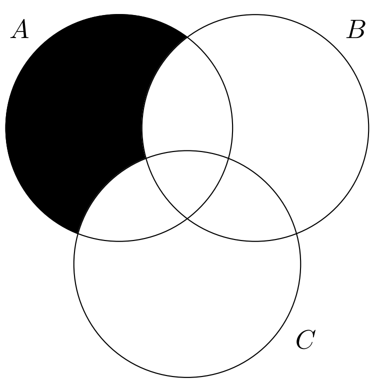
Since $x\in A$ and $x\notin B$ is also the definition of $A\cap\overline{B}$, x is in $A\cap\overline{B}$. Thus, $A\backslash B\subseteq A\cap\overline{B}$.

Then assume x is in $A\cap\overline{B}$. Then, by definition $x\in A$ and $x\notin B$.

Since $x\in A$ and $x\notin B$ is also the definition of $A\backslash B$, x is in $A\backslash B$. Thus, $A\cap\overline{B}\subseteq A\backslash B$.

Therefore, since the sets are subsets of each other, they are equal. $\square$

(b) Both $A\backslash(B\cup C)$ and $(A\backslash B)\cap(A\backslash C)$ have the same diagram:



Let A, B be sets.

Assume x is in $A\backslash(B\cup C)$. Then, by definition, $x\in A\wedge\neg(x\in B\vee x\in C)$.

Using De Morgan's law, this is equivalent to $x\in A\wedge x\notin B\wedge x\notin C$. By the idempotent laws, this is equivalent to $(x\in A\wedge x\notin B)\wedge(x\in A\wedge x\notin C)$.

This is the definition of $(A\backslash B)\cap(A\backslash C)$, so x is in $(A\backslash B)\cap(A\backslash C)$. Thus, $A\backslash(B\cup C)$ is a subset of $(A\backslash B)\cap(A\backslash C)$.

Then assume x is in $(A\backslash B)\cap(A\backslash C)$. Then, by definition, $(x\in A\wedge x\notin B)\wedge(x\in A\wedge x\notin C)$. By the idempotent laws, this is equivalent to $x\in A\wedge x\notin B\wedge x\notin C$.

Using De Morgan's law, this is equivalent to $x \in A \wedge \neg(x \in B \vee x \in C)$. This is the definition of $A \backslash (B \cup C)$, so x is in $A \backslash (B \cup C)$. Thus, $(A \backslash B) \cap (A \backslash C)$ is a subset of $A \backslash (B \cup C)$.

Therefore, since the sets are subsets of each other, they are equal. $\square$

Problem 2

Let A, B be sets.

Assume $A \backslash B = \emptyset$. Then, by definition, there are no elements that are in A but not in B. Equivalently, every element in A is also in B. This is the definition of $A \subseteq B$, so $A \backslash B = \emptyset \Rightarrow A \subseteq B$.

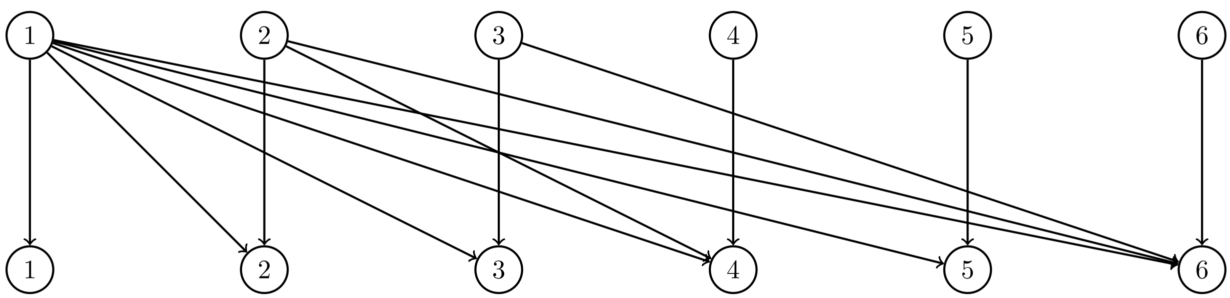
Then, assume $A \backslash B \neq \emptyset$. Then, there is an element in A that is not in B. Thus, by the definition of subset, $A \not\subseteq B$. By the contrapositive, $A \subseteq B \Rightarrow A \backslash B = \emptyset$.

Therefore, since the propositions imply each other, $A \subseteq B \Leftrightarrow A \backslash B = \emptyset$. $\square$

Problem 3

(a) $R = \{(1,1), (1,2), (1,3), (1,4), (1,5), (1,6), (2,2), (2,4), (2,6), (3,3), (3,6), (4,4), (5,5), (6,6)\}$

(b)



Problem 4

(a) No: R is not reflexive, because one cannot be taller than oneself.

(b) Yes: R is reflexive because one is clearly born on the same day as oneself. R is symmetric because if person A is born on the same day as person B, then person B must be born on the same day as person A. R is transitive because if person A is born on the same day as person B and person B is born on the same day as person C, then person A is born on the same day as person C (because they are all born on the same day). Thus, R satisfies all three conditions and is an equivalence relation.

(c) Yes: R is reflexive because one has the same first name as oneself. R is transitive because if person A has the same first name as person B, then person B will have the same first name as person A. R is transitive because if person A has the same first name as person B and person B has the same first name as person C, then person A has the same first name as person C (because they all have the same first name). Thus, R satisfies all three conditions and is an equivalence relation.

(d) No: R is not transitive. For example, one has at least one grandparent in common with one's cousins, but not necessarily grandparents in common with all the cousins of one's cousins.

Problem 5

(a) Let A, B be sets and R be a relation from A to B.

Assume $(a, b) \in R$. Then, $(b, a) \in R^{-1}$. Additionally, $a \in \text{Dom}(R)$, so $(a, a) \in id_{\text{Dom}(R)}$.

Since $R^{-1} \circ R = \{(c,e) : (\exists d)[(c,d) \in R \land (d,e) \in R^{-1}]\}$, (a,a) is in $R^{-1} \circ R$ because choosing $d = b$ yields the pair (a,a).

Therefore, if $(a,a) \in id_{\text{Dom}(R)}$, then $(a,a) \in R^{-1} \circ R$, so $id_{\text{Dom}(R)} \subseteq R^{-1} \circ R$. $\square$

(b) Let A, B be sets and R be a relation from A to B.

Assume $(a,b) \in R$. Then, $(b,a) \in R^{-1}$. Additionally, $b \in \text{Ran}(R)$, so $(b,b) \in id_{\text{Ran}(R)}$.

Since $R \circ R^{-1} = \{(c,e) : (\exists d)[(c,d) \in R^{-1} \land (d,e) \in R]\}$, (b,b) is in $R \circ R^{-1}$ because choosing $d = b$ yields the pair (b,b).

Therefore, if $(b,b) \in id_{\text{Ran}(R)}$, then $(b,b) \in R^{-1} \circ R$, so $id_{\text{Ran}(R)} \subseteq R \circ R^{-1}$. $\square$

Problem 6

The set of numbers that divide 48 is $A = \{-48, -24, -16, -12, -8, -6, -4, -3, -2, -1, 1, 2, 3, 4, 6, 8, 12, 16, 24, 48\}$. To see that the n^{th} term from A divides 48, multiply it by the n^{th} term from the set of integers $A' = \{-1, -2, -3, -4, -6, -8, -12, -16, -24, -48, 48, 24, 16, 12, 8, 6, 4, 3, 2, 1\}$ to get 48. For all other integers z, $\frac{48}{z}$ does not equal an integer, so these are the only factors.

(a) We claim $\text{Dom}(R) = \mathbb{Z}$ and $\text{Ran}(R) = \mathbb{Z}$.

Since R is a relation on $\mathbb{Z}$, by definition $\text{Dom}(R) \subseteq \mathbb{Z}$. Then, assume $x \in \mathbb{Z}$. The pair $(x, x-2)$ is in the relation, because $x - (x-2) = 2$ is even, so x is in $\text{Dom}(R)$ and thus $\mathbb{Z} \subseteq \text{Dom}(R)$. Therefore, $\text{Dom}(R) = \mathbb{Z}$.

Since R is a relation on $\mathbb{Z}$, by definition $\text{Ran}(R) \subseteq \mathbb{Z}$. Then, assume $y \in \mathbb{Z}$. The pair $(y+2, y)$ is in the relation, because $y + 2 - y = 2$ is even, so y is in $\text{Ran}(R)$ and thus $\mathbb{Z} \subseteq \text{Ran}(R)$. Therefore, $\text{Ran}(R) = \mathbb{Z}$. $\square$

(b) We claim $\text{Dom}(S) = \text{Ran}(S) = A$ (see definition of A above).

First, assume $a \in A$. Then, $\left(a, \frac{48}{a}\right)$ and $\left(\frac{48}{a}, a\right)$ are both in S because $\frac{48 \times a}{a} = \frac{a \times 48}{a} = 48$. Thus, $A \subseteq \text{Dom}(S)$ and $A \subseteq \text{Ran}(S)$.

Then, assume $a \in \text{Dom}(S)$. The only way to multiply two integers to obtain 48 is if the first number is a factor of 48, which is how A was defined, so $\text{Dom}(S) \subseteq A$.

Finally, assume $a \in \text{Ran}(S)$. The only way to multiply two integers to obtain 48 is if the second number is a factor of 48, which is how A was defined, so $\text{Ran}(S) \subseteq A$.

Thus, both equalities are verified, because the sets are subsets of each other. $\square$

(c) $R \circ S = \{(a,c) \in \mathbb{Z} \times \mathbb{Z} : (\exists b)[(a,b) \in S \land (b,c) \in R]\}$

Since $ab = 48$ as $(a,b) \in S$, $b = \frac{48}{a}$. Thus, $R \circ S = \{(a,c) \in \mathbb{Z} \times \mathbb{Z} : \left(a, \frac{48}{a}\right) \in S \land \left(\frac{48}{a}, c\right) \in R \land \frac{48}{a} \in \mathbb{Z}\}$. As $\left(a, \frac{48}{a}\right) \in S$ is defined on $\mathbb{Z} \times \mathbb{Z}$, the only way for $\frac{48}{a}$ to be an integer is if $a|48$. Also, as $\left(\frac{48}{a}, c\right) \in R$, $2|\left(\frac{48}{a} - c\right)$.

Therefore, $R \circ S = \{(a,c) \in \mathbb{Z} \times \mathbb{Z} : a|48 \land 2|\left(\frac{48}{a} - c\right)\}$. $\square$

(d) We claim $\text{Dom}(R \circ S) = A$ and $\text{Ran}(R \circ S) = \mathbb{Z}$. The first equality follows directly from the definition of $R \circ S$ at the end of (c), because $a|48 \Leftrightarrow a \in A$ since A is the factors of 48.

Assume $c \in \text{Ran}(R \circ S)$. Then, $2|(\frac{48}{a} - c)$, so there exists a $k \in \mathbb{Z}$ such that $2k = \frac{48}{a} - c$. So $c = \frac{48}{a} - 2k$, and since it the difference of 2 integers, $c \in \mathbb{Z}$. Thus, $\text{Ran}(R \circ S) \subseteq \mathbb{Z}$.

Then, assume $c \in \mathbb{Z}$.

Case 1: Choose $a = 48$, as it is in $\text{Dom}(R \circ S)$. Then, $2|\left(\frac{48}{48} - c\right) \equiv 2|(1-c)$. Any odd c will satisfy this proposition, because the difference of two odds is even and 1 is odd.

Case 2: Choose $a = 1$, as it is in $\text{Dom}(R \circ S)$. Then, $2|\left(\frac{48}{1} - c\right) \equiv 2|(48-c)$. Any even c will satisfy this proposition, because the difference of two evens is even and 48 is even. Thus, any odd or even c is in $\text{Ran}(R \circ S)$, so $\mathbb{Z} \subseteq \text{Ran}(R \circ S)$.

Therefore, the second inequality is verified because they are subsets of each other. $\square$

(e) $R \circ S = \{(a,c) \in \mathbb{Z} \times \mathbb{Z} : (\exists b)[(a,b) \in R \wedge (b,c) \in S]\}$

Since $bc = 48$ as $(b,c) \in S$, $b = \frac{48}{c}$. Thus, $R \circ S = \{(a,c) \in \mathbb{Z} \times \mathbb{Z} : \left(a, \frac{48}{c}\right) \in R \wedge \left(\frac{48}{c}, c\right) \in S\}$. As $\left(\frac{48}{c}, c\right) \in S$ is defined on $\mathbb{Z} \times \mathbb{Z}$, the only way for $\frac{48}{c}$ to be an integer is if $c|48$. Also, as $\left(a, \frac{48}{c}\right) \in R$, $2|\left(a - \frac{48}{c}\right)$.

Therefore, $R \circ S = \{(a,c) \in \mathbb{Z} \times \mathbb{Z} : 2|\left(a - \frac{48}{c}\right) \wedge c|48\}$. $\square$

(f) We claim $\text{Dom}(S \circ R) = \mathbb{Z}$ and $\text{Ran}(S \circ R) = A$. The second equality follows directly from the definition of $S \circ R$ at the end of (e), because $c|48 \Leftrightarrow c \in A$ since A is the factors of 48.

Assume $a \in \text{Dom}(S \circ R)$. Then, $2|(a - \frac{48}{c})$, so there exists a $k \in \mathbb{Z}$ such that $2k = a - \frac{48}{c}$. So $a = \frac{48}{a} + 2k$, and since it the sum of 2 integers, $a \in \mathbb{Z}$. Thus, $\text{Dom}(S \circ R) \subseteq \mathbb{Z}$.

Then, assume $a \in \mathbb{Z}$.

Case 1: Choose $c = 48$, as it is in $\text{Ran}(S \circ R)$. Then, $2|\left(a - \frac{48}{48}\right) \equiv 2|(a-1)$. Any odd a will satisfy this proposition, because the difference of two odds is even and 1 is odd.

Case 2: Choose $c = 1$, as it is in $\text{Ran}(S \circ R)$. Then, $2|\left(a - \frac{48}{1}\right) \equiv 2|(a-48)$. Any even a will satisfy this proposition, because the difference of two evens is even and 48 is even. Thus, any odd or even a is in $\text{Dom}(S \circ R)$, so $\mathbb{Z} \subseteq \text{Ran}(S \circ R)$.

Therefore, the second inequality is verified because they are subsets of each other. $\square$

(g) We claim $R \circ R = R$.

By definition, $R \circ R = \{(a,c) \in \mathbb{Z} \times \mathbb{Z} : (\exists b)[(a,b) \in R \wedge (b,c) \in R]\}$. Assume $(a,c) \in R \circ R$. Then, there is a b such that $2|(a-b)$, so $2m = a - b$ for some integer m. Rearranging, $b = a - 2m$. It is also stipulated that $2|(b-c)$ and thus $2n = b - c$ for some n, so $b = 2n + c$. Equating the values of b, $a - 2m = 2n + c \Rightarrow a - c = 2(m+n)$. Since $m + n$ is an integer as it is the sum of two integers, $2|(a-c)$.

Since R is defined as $\{(x,y) \in \mathbb{Z} \times \mathbb{Z} : 2|(x-y)\}$, $R \circ R = R$ because for any pair $(a,c) \in R \circ R$, $2|(a-c)$. $\square$

(h) We claim $S \circ S = id_A$.

By definition, $S \circ S = \{(a,c) \in \mathbb{Z} \times \mathbb{Z} : (\exists b)[(a,b) \in S \wedge (b,c) \in S]\}$. Assume $(a,c) \in S \circ S$. Then, $b = \frac{48}{a}$ since $ab = 48$. Also, $b = \frac{48}{c}$ since $bc = 48$. So, $\frac{48}{a} = \frac{48}{c}$, and thus $a = c$ if $(a,c) \in S \circ S$.

Since $b = \frac{48}{a} = \frac{48}{c}$, a and c must be factors of 48. This is how A is defined, so $a \in A \wedge c \in A$. Thus, $S \circ S = \{(a,a) \in \mathbb{Z} \times \mathbb{Z} : a \in A\}$. This is the definition of id_A. $\square$

(i) We claim both are true.

Assume $(x,y) \in S$. Since $xy = 48$, $yx = 48$, so $(y,x) \in S^{-1}$. Thus, $S \subseteq S^{-1}$. Then, assume $(x,y) \in S^{-1}$. Since $yx = 48$, $xy = 48$, so $(y,x) \in S$. Thus, $S^{-1} \subseteq S$. Therefore, since the sets are subsets of each other, $S = S^{-1}$.

Assume $(x,y) \in R$. Then, $2|(x-y)$, so there exists a $k \in \mathbb{Z}$ such that $2k = x - y$. Negating this, $y - x = -2k$. Since k is an integer, so is $-k$, so $2|(y-x)$ and $(y,x) \in S^{-1}$. Thus, $R \subseteq R^{-1}$. Then, assume $(x,y) \in R$.

Then, $2|(x-y)$, so there exists a $k \in \mathbb{Z}$ such that $2k = x - y$ and consequentially $-2k = y - x$. Then, $2|(y-x)$ and $(y,x) \in S$. Thus, $R^{-1} \subseteq R$. Therefore, since the sets are subsets of each other, $R = R^{-1}$. $\square$

Problem 7

(a) $R^{-1} = \{(1,1),(1,2),(3,3),(7,4),(4,7),(3,7)\}$

(b) $\text{Dom}(R) = \{1,2,3,4,7\}$

$\text{Ran}(R) = \{1,3,4,7\}$

(c) $R \circ R = \{(1,1),(2,1),(3,3),(4,4),(4,3),(7,7),(7,3)\}$

$\text{Dom}(R \circ R) = \{1,2,3,4,7\}$

$\text{Ran}(R \circ R) = \{1,3,4,7\}$

(d) $R^{-1} \circ R = \{(1,1),(3,3),(7,7),(4,4),(4,3),(3,4)\}$

(e) $R \circ R^{-1} = \{(1,1),(1,2),(2,1),(2,2),(3,3),(3,7),(4,4),(7,7),(7,3)\}$

Problem 8

(a) Let A be a set and S be a relation on A.

Assume $(a,c) \in S \circ S$. By definition, $S \circ S = \{(a,c) \in A \times A : (\exists b)[(a,b) \in S \wedge (b,c) \in S]\}$. Since S is transitive, $(a,b) \in S \wedge (b,c) \in S \Rightarrow (a,c) \in S$ by definition, so $(a,c) \in S$. Therefore, $(S \circ S) \subseteq S$. $\square$

(b) Assume $S^{-1} = S$. Then, by definition of set equality, any $(a,b) \in S$ means $(a,b) \in S^{-1}$, because $S \subseteq S^{-1}$. Additionally, any $(b,a) \in S^{-1}$ means $(b,a) \in S$ because $S^{-1} \subseteq S$. By the definition of inverse, if (a,b) is in S then (b,a) is in S^{-1}, and since $S^{-1} \subseteq S$, $(b,a) \in S$. This is the definition of a relation being symmetric, so $S^{-1} = S \Rightarrow S$ is symmetric.

Then, assume S is symmetric. Assume $(a,b) \in S$. By definition of being symmetric, $(b,a) \in S$, and by definition of inverse $(a,b) \in S^{-1}$. So $S \subseteq S^{-1}$, since every element in S is in S^{-1}. Finally, assume $(b,a) \in S^{-1}$. Then, $(a,b) \in S$, and by definition of being symmetric $(b,a) \in S$. So $S^{-1} \subseteq S$, since every element in S^{-1} is in S. Thus, $S^{-1} = S \Leftarrow S$ is symmetric.

Therefore, $S^{-1} = S \Leftrightarrow S$ is symmetric. $\square$

Problem 9

Let S be a relation from A to B and R be a relation from B to C.

$$\begin{aligned}
(R \circ S)^{-1} &= \{(c,a) : (\exists b)[(a,b) \in S \wedge (b,c) \in R]\} && \text{(by definition)} \\
&= \{(c,a) : (\exists b)[(b,c) \in R \wedge (a,b) \in S]\} && \text{(commutative property)} \\
&= \{(a,c) : (\exists b)[(b,a) \in R \wedge (c,b) \in S]\} && \text{(renaming variables)} \\
&= \{(a,c) : (\exists b)[(a,b) \in R^{-1} \wedge (b,c) \in S^{-1}]\} && \text{(definition of inverse)} \\
&= S^{-1} \circ R^{-1} && \text{(definition of composition)}
\end{aligned}$$

Thus, the sets are equal because they are defined in the same way. $\square$