Nathan Bickel
MATH 574: Section H01
Professor: Dr. Luo
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MATH 574 Homework 7

Collaboration:

Problem 1 Recall that the Fibonacci numbers satisfy $f_n = f_{n-1} + f_{n-2}$ with initial conditions $f_0 = 0$ and $f_1 = 1$.
(a) Suppose that a sequence $\{b_n\}$ satisfies $b_n = b_{n-1} + b_{n-2}$ with initial conditions $b_0 = 1$ and $b_1 = 2$. Use induction to prove that for all $n \geq 0$, $b_n = f_{n+2}$.
(b) Suppose that a sequence $\{c_n\}$ satisfies $c_n = c_{n-1} + c_{n-2}$ with initial conditions $c_0 = 2$ and $c_1 = 1$. Use induction to prove that for all $n \geq 1$, $c_n = f_{n-1} + f_{n+1}$.

Solution.

(a)

- Let $n = 0$. We have $b_0 = 1 = 0 + 1 = f_{0+2}$.
- Let $n = 1$. We have $b_1 = 2 = 1 + 1 = f_{1+2}$.

So the claim holds for $n \in \{0, 1\}$. Next, let $n \in \mathbb{N}$, $n \geq 1$. Assume that for all k, $0 \leq k \leq n$, $b_k = f_{k+2}$.

Observe from the definition of b_n that $b_{n+1} = b_n + b_{n-1}$. Since we have $0 \leq n \leq n$ and $0 \leq n - 1 \leq n$, we can use the induction hypothesis to write $b_{n+1} = f_{n+2} + f_{n+1}$. By the definition of f_n, we can write

$$b_{n+1} = f_{n+2} + f_{n+1} = f_{(n+1)+2}.$$

So if the claim holds for all integers less than or equal to n, it also holds for $n + 1$. So for all n, we have $b_n = f_{n+2}$.

(b)

- Let $n = 1$. We have $c_1 = 1 = 0 + 1 = f_0 + f_2 = f_{1-1} + f_{1+1}$.
- Let $n = 2$. We have $c_2 = 3 = 1 + 2 = f_1 + f_3 = f_{2-1} + f_{2+1}$.

So the claim holds for $n \in \{1, 2\}$. Next, let $n \in \mathbb{N}$, $n \geq 1$. Assume that for all k, $0 \leq k \leq n$, $c_k = f_{k-1} + f_{k+1}$. Then, we have

$$\begin{aligned}
c_{n+1} &= c_n + c_{n-1} && \text{(recursive definition)} \\
&= (f_{n-1} + f_{n+1}) + (f_{n-2} + f_n) && \text{(IH, since } 0 \leq n \leq n,\ 0 \leq n-1 \leq n\text{)} \\
&= (f_{n-1} + f_{n-2}) + (f_{n+1} + f_n) \\
&= f_n + f_{n+2}. && \text{(recursive Fibonacci definition)}
\end{aligned}$$

So if the claim holds for all integers less than or equal to n, it also holds for $n + 1$. So for all n, we have $c_n = f_{n-1} + f_{n+1}$.

Problem 2 Solve the recurrence relations together with the initial conditions given.
(a) $a_n = 5a_{n-1} - 6a_{n-2}$ for $n \geq 2$, $a_0 = 1$, $a_1 = 0$
(b) $a_n = 4a_{n-2}$ for $n \geq 2$, $a_0 = 0$, $a_1 = 4$
(c) $a_n = 4a_{n-1} - 4a_{n-2}$ for $n \geq 2$, $a_0 = 6, a_1 = 8$

Solution.

(a) We can write the characteristic polynomial $p(\lambda) = \lambda^2 - 5\lambda + 6 = (\lambda - 2)(\lambda - 3)$. This gives us the roots as $\lambda_1 = 2$, $\lambda_2 = 3$. Then, we have $a_n = \alpha_1(2)^n + \alpha_2(3)^n$ for some α_1, α_2. We can find them by solving

$$\begin{cases} a_0 = 1 = \alpha_1(2)^0 + \alpha_2(3)^0 \\ a_1 = 0 = \alpha_1(2)^1 + \alpha_2(3)^1 \end{cases}$$

This gives the solution $\alpha_1 = 3$, $\alpha_2 = -2$. So $a_n = 3(2)^n - 2(3)^n$.

(b) We can write the characteristic polynomial $p(\lambda) = \lambda^2 - 4 = (\lambda + 2)(\lambda - 2)$. This gives us the roots as $\lambda_1 = 2$, $\lambda_2 = -2$. Then, we have $a_n = \alpha_1(2)^n + \alpha_2(-2)^n$ for some α_1, α_2. We can find them by solving

$$\begin{cases} a_0 = 0 = \alpha_1(2)^0 + \alpha_2(-2)^0 \\ a_1 = 4 = \alpha_1(2)^1 + \alpha_2(-2)^1 \end{cases}$$

This gives the solution $\alpha_1 = 1$, $\alpha_2 = -1$. So $a_n = 2^n - (-2)^n$.

(c) We can write the characteristic polynomial $p(\lambda) = \lambda^2 - 4\lambda + 4 = (\lambda - 2)^2$. This gives the double root of $\lambda = 2$. Then, we have $a_n = \alpha_1(2)^n + \alpha_2 n(2)^n$ for some α_1, α_2. We can find them by solving

$$\begin{cases} a_0 = 6 = \alpha_1(2)^0 + \alpha_2(0)(2)^0 \\ a_1 = 8 = \alpha_1(2)^1 + \alpha_2(1)(2)^1 \end{cases}$$

This gives the solution $\alpha_1 = 6$, $\alpha_2 = -2$. So $a_n = 6(2)^n - 2n(2)^n$.

Problem 3 Let b_n be the number of bit strings of length n without 2 consecutive 0s. In class, we saw that $\{b_n\}$ satisfies the relation $b_n = b_{n-1} + b_{n-2}$ for $n \geq 2$. Find a solution of this recurrence relation using the initial conditions $b_0 = 1, b_1 = 2$.

Solution.

This is a Fibonacci relation with different initial conditions. Thus, we have from class that

$$b_n = \alpha_1 \left(\frac{1 + \sqrt{5}}{2} \right)^n + \alpha_2 \left(\frac{1 - \sqrt{5}}{2} \right)^n$$

for some α_1, α_2. We can find them by solving

$$\begin{cases} a_0 = 1 = \alpha_1 \left(\frac{1+\sqrt{5}}{2} \right)^0 + \alpha_2 \left(\frac{1-\sqrt{5}}{2} \right)^0 \\ a_1 = 2 = \alpha_1 \left(\frac{1+\sqrt{5}}{2} \right)^1 + \alpha_2 \left(\frac{1-\sqrt{5}}{2} \right)^1 \end{cases}$$

This gives the solution

$$\alpha_1 = \frac{5 + 3\sqrt{5}}{10}, \alpha_2 = \frac{5 - 3\sqrt{5}}{10}.$$

Therefore, we have

$$a_n = \left(\frac{5 + 3\sqrt{5}}{10} \right) \left(\frac{1 + \sqrt{5}}{2} \right)^n + \left(\frac{5 + 3\sqrt{5}}{10} \right) \left(\frac{1 - \sqrt{5}}{2} \right)^n.$$

Problem 4 Find the solution to the recurrence relation $a_n = 2a_{n-1} + a_{n-2} - 2a_{n-3}$ for $n \geq 3$ with initial conditions $a_0 = 3, a_1 = 6, a_2 = 0$.

Solution.

We can write the characteristic polynomial

$$p(\lambda) = \lambda^3 - 2\lambda^2 - \lambda + 2 = \lambda^2(\lambda - 2) - 1(\lambda - 2) = (\lambda - 2)(\lambda - 1)(\lambda + 1).$$

This gives the roots as $\lambda_1 = 2, \lambda_2 = 1, \lambda_3 = -1$. Then, we have $a_n = \alpha_1(2)^n + \alpha_2(1)^n + \alpha_3(-1)^n$ for some $\alpha_1, \alpha_2, \alpha_3$. We can find them by solving

$$\begin{cases} a_0 = 3 = \alpha_1(2)^0 + \alpha_2(1)^0 + \alpha_3(-1)^0 \\ a_1 = 6 = \alpha_1(2)^1 + \alpha_2(1)^1 + \alpha_3(-1)^1 \\ a_2 = 0 = \alpha_1(2)^2 + \alpha_2(1)^2 + \alpha_3(-1)^2 \end{cases}$$

This gives the solution $\alpha_1 = -1$, $\alpha_2 = 6$, $\alpha_3 = -2$. So $a_n = -(2)^n + 6(1)^n - 2(-1)^n$.

Problem 5 Find the solution to the recurrence relation $a_n = 2a_{n-1} - 2a_{n-2}$ for $n \geq 2$ with initial conditions $a_0 = 1$ and $a_1 = 2$. Use your solution to calculate the value of a_{20}. (You will have complex roots. You may use a calculator for a_{20}.)

Solution.

We can write the characteristic polynomial $p(\lambda) = \lambda^2 - 2\lambda + 2$. Using the quadratic formula, this gives the roots as $\lambda_1 = 1 + i, \lambda_2 = 1 - i$. Then, we have $a_n = \alpha_1(1+i)^n + \alpha_2(1-i)^n$ for some α_1, α_2. We can find them by solving

$$\begin{cases} a_0 = 1 = \alpha_1(1+i)^0 + \alpha_2(1-i)^0 \\ a_1 = 2 = \alpha_1(1+i)^1 + \alpha_2(1-i)^1 \end{cases}$$

This gives the solution $\alpha_1 = \frac{1-i}{2}$, $\alpha_2 = \frac{1+i}{2}$. So

$$a_n = \left(\frac{1-i}{2}\right)(1+i)^n + \left(\frac{1+i}{2}\right)(1-i)^n.$$

Using Euler's formula, we can rewrite this in a way more practical to calculate as

$$a_n = \frac{1}{2}\left(\sqrt{2}e^{-i\pi/4}\right)\left(\sqrt{2}e^{i\pi/4}\right)^n + \frac{1}{2}\left(\sqrt{2}e^{i\pi/4}\right)\left(\sqrt{2}e^{-i\pi/4}\right)^n.$$

Simplifying, we have

$$a_n = 2^{(n-1)/2}\left(e^{(n+1)i\pi/4} + e^{-(n-1)i\pi/4}\right) = 2^{(n+1)/2}\cosh\left(\frac{(n-1)i\pi}{4}\right).$$

Therefore, we have

$$a_{20} = 2^{21/2}\cosh\left(\frac{19i\pi}{4}\right) = -1024.$$

Problem 6 A model for the number of lobsters caught per year is based on the assumption that the number of lobsters caught in a year is the average of the number caught in the two previous years.
(a) Find a recurrence relation for $\{L_n\}$, where L_n is the number of lobsters caught in year n, under the assumption for this model.

(b) Find L_n if $4,000$ lobsters were caught in year 1 and $10,000$ were caught in year 2.
(c) What is the long-term behavior of L_n? That is, what is $\lim_{n\to\infty} L_n$?

Solution.

(a) $L_n = \frac{1}{2}L_{n-1} + \frac{1}{2}L_{n-2}$

(b) We can write the characteristic polynomial

$$p(\lambda) = \lambda^2 - \frac{1}{2}\lambda - \frac{1}{2} = \frac{1}{2}(2\lambda+1)(\lambda-1).$$

This gives us the roots as $\lambda_1 = 1$, $\lambda_2 = -\frac{1}{2}$. Then, we have $a_n = \alpha_1(1)^n + \alpha_2\left(-\frac{1}{2}\right)^n$ for some α_1, α_2. We can find them by solving

$$\begin{cases} a_1 = 4000 = \alpha_1(1)^1 + \alpha_2\left(-\frac{1}{2}\right)^1 \\ a_2 = 10000 = \alpha_1(1)^2 + \alpha_2\left(-\frac{1}{2}\right)^2 \end{cases}$$

This gives the solution $\alpha_1 = 8000$, $\alpha_2 = 8000$. So $a_n = 8000(1)^n + 8000\left(-\frac{1}{2}\right)^n$.

(c)

$$\begin{aligned}
&\lim_{n\to\infty}\left[8000(1)^n + 8000\left(-\frac{1}{2}\right)^n\right] \\
&= \lim_{n\to\infty}(8000) + \lim_{n\to\infty}\left[\frac{8000(-1)^n}{2^n}\right] \\
&= 8000 + 0 = 8000. \qquad \text{(2^n dominates the numerator as its magnitude never exceeds 8000)}
\end{aligned}$$

Problem 7 Let a_n be the number of ways a $2\times n$ rectangular chessboard can be tiled using 1×2 and 2×2 pieces.
(a) Determine a_1 and a_2.
(b) Find a recurrence relation for $\{a_n\}$.
(c) Find a solution of the recurrence relation in part (b) using the initial conditions in part (a).

Solution.

(a) Since the only way to tile a 2×1 chessboard is with a 2×1 piece, $a_1 = 1$. With a 2×2 chessboard, we can tile it with two vertical 2×1 pieces, two horizontal 2×1 pieces, or 1 2×2 piece. So $a_2 = 3$.

(b) Every tiling of a $2\times n$ chessboard, $n\geq 2$ satisfies exactly one of these three cases:

Case 1: The rightmost file is a vertical 2×1 piece. Then, there are a_{n-1} ways to tile the files to the left since there are $n-1$ such files.

Case 2: The rightmost 2 files are tiled by two horizontal 2×1 pieces. Then, there are a_{n-2} ways to tile the files to the left since there are $n-2$ such files.

Case 3: The rightmost 2 files are tiled by a 2×2 piece. Then, there are a_{n-2} ways to tile the files to the left since there are $n-2$ such files.

Therefore, for a $2\times n$ chessboard, there are $a_{n-1} + a_{n-2} + a_{n-2}$ ways to tile it. So $a_n = a_{n-1} + 2a_{n-2}$.

(c) We can write the characteristic polynomial $p(\lambda) = \lambda^2 - \lambda - 2 = (\lambda-2)(\lambda+1)$. This gives us the roots as $\lambda_1 = 2$, $\lambda_2 = -1$. Then, we have $a_n = \alpha_1(2)^n + \alpha_2(-1)^n$ for some α_1, α_2. We can find them by solving

$$\begin{cases} a_1 = 1 = \alpha_1(2)^1 + \alpha_2(-1)^1 \\ a_2 = 3 = \alpha_1(2)^2 + \alpha_2(-1)^2 \end{cases}$$

This gives the solution $\alpha_1 = \frac{2}{3}$, $\alpha_2 = \frac{1}{3}$. So $a_n = \left(\frac{2}{3}\right)(2)^n + \left(\frac{1}{3}\right)(-1)^n$.