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MATH 575: Section H01
Professor: Dr. Luo
March 13, 2023

MATH 575 Homework 7

Collaboration: I discussed some of the problems with Jack. I also discussed with him your request for reciprocal acknowledgement, but will he fulfill it?

Problem 1 Let $k \geq 2$. Suppose G is a k-connected graph with at least $k+1$ vertices, and let $S \subseteq V(G)$ with $|S| = k$. Prove that for every pair of vertices $x, y \in S$, there exists a cycle in G containing x and y that avoids $S - \{x, y\}$.

Let $x, y \in S$. Since G is k-connected, we have from Menger's theorem that there are k internally disjoint x, y-paths in G. Since $|S - \{x, y\}| = k - 2$, at most $k - 2$ of these paths pass through $S - \{x, y\}$ (if not, then by the PHP 2 paths pass through the same vertex, contradicting internal disjointedness). So 2 internally disjoint x, y-paths P_1, P_2 avoid $S - \{x, y\}$. Therefore we can start at x, travel to y along P_1, and travel back to x along P_2 to obtain a cycle in G containing x and y that avoids $S - \{x, y\}$. □

Problem 2 Use Menger's Theorem ($\kappa(x, y) = \lambda(x, y)$ for all nonadjacent x, y) to prove the König–Egerváry Theorem (if G is bipartite, then $\beta(G) = \alpha'(G)$).

Solution.

Let $G = X \cup Y$ be a bipartite graph, and construct a graph G' with two extra vertices x and y, where x is adjacent to every vertex in X and y is adjacent to every vertex in Y. Let S be an x, y-cut in G' with minimum size. Then, we claim S is also a minimum vertex cover in G. If there is an edge $uv \in G$ that is not covered by S, then x, u, v, y is an x, y-path in G', contradicting S being an x, y-cut. Also, if there is a smaller vertex cover S' in G, then S' is a smaller x, y-cut in G', contradicting minimality of S.

Let T be a set of pairwise internally disjoint x, y-paths with maximum size. Then, each path will have 3 edges, since the path needs to pass from x to a vertex in X to a vertex in Y to y. We claim that the set M of middle edges in these paths (the edges passing between between X and Y) is a maximum matching in G. It will be a matching because the vertices in each path are pairwise internally disjoint, so no vertex will be in two edges in the matching. Also, if there is a larger matching M', we can travel from x to each of the edges in M' to y to obtain a larger set of pairwise interally disjoint x, y-paths, contradicting maximality.

So by Menger's theorem, we have $|S| = \kappa(x, y) = \lambda(x, y) = |T| = |M|$. So the size of a minimum vertex cover is equal to the size of a maximum matching. □

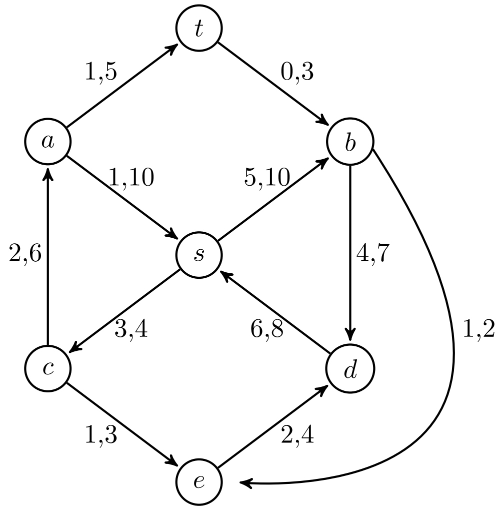
Problem 3 Let D be an s, t-network with no directed path from s to t. Prove that D cannot have a feasible flow with value greater than 0.

Solution.

Let S be the set of vertices v such that D contains a directed path from s to v, and let $T = V(D) - S$. Then, the source is in S, the sink is in T (since there is no directed path from s to t), and S and T partition $V(D)$, so $[S, T]$ is a source/sink cut.

We have $[S, T] = \emptyset$. If not, then there would exist an edge xy with $x \in S$ and $y \in T$. But then we can travel along an s, x-path to x and then to y to obtain an s, y-path, a contradiction since then y should have been in S. So clearly, $\text{cap}(S, T) = 0$, and since $\text{val}(f) \leq \text{cap}(S, T) = 0$ for any feasible f, the value of a maximum feasible flow is 0. $\square$

Problem 4 Consider the following s, t-network with flow f.



(a) Verify that f is feasible.

(b) Use the Ford-Fulkerson algorithm to find a maximum flow of the network.

Prove that your final flow is maximum by constructing a minimum cut.

Solution.

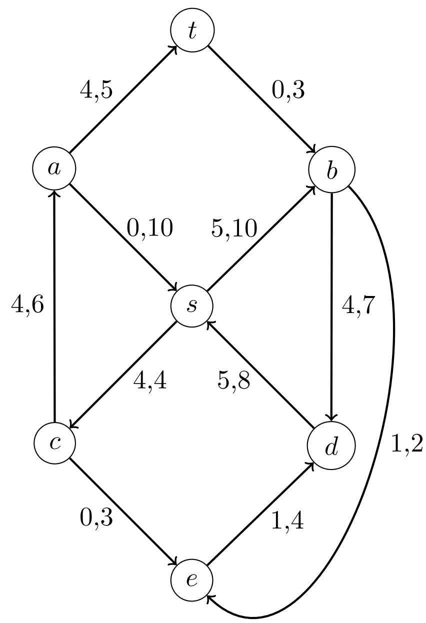
(a) First, it is clear that for all e in the edge set, $0 \leq f(e) \leq c(e)$. Next, we note that we have:

- $f^+(a) - f^-(a) = (1 + 1) - (2) = 0$
- $f^+(b) - f^-(b) = (4 + 1) - (5 + 0) = 0$
- $f^+(c) - f^-(c) = (2 + 1) - (3) = 0$
- $f^+(d) - f^-(d) = (6) - (2 + 4) = 0$
- $f^+(e) - f^-(e) = (2) - (1 + 1) = 0.$

So conversation of flow is conserved for all vertices that are not the sink or source. Thus, f is feasible.

(b) We iteratively find f-augmenting paths until we obtain the flow below:

- s, a, t
- s, c, a, t
- s, d, e, c, a, t



Let $S = \{s, b, d, e\}$ and $T = \{a, c, t\}$. Then, S and T partition the vertices, and $[S, T]$ is a source/sink cut. Since the slack in each edge in $[S, T]$ is 0, we have found a maximum flow.

Problem 5 A warehouse stores 3 different chemicals A, B, and C. Tomorrow, 4 trucks will arrive to transport the barrels of chemicals to another location. Due to safety concerns there are some restrictions for their transportation.

i. Chemical A can only be transported in Truck #1 or Truck #2. No truck can carry more than 2 barrels of Chemical A.

ii. Chemical B can only be transported in Truck #2 or Truck #3. No truck can carry more than 2 barrels of Chemical B.

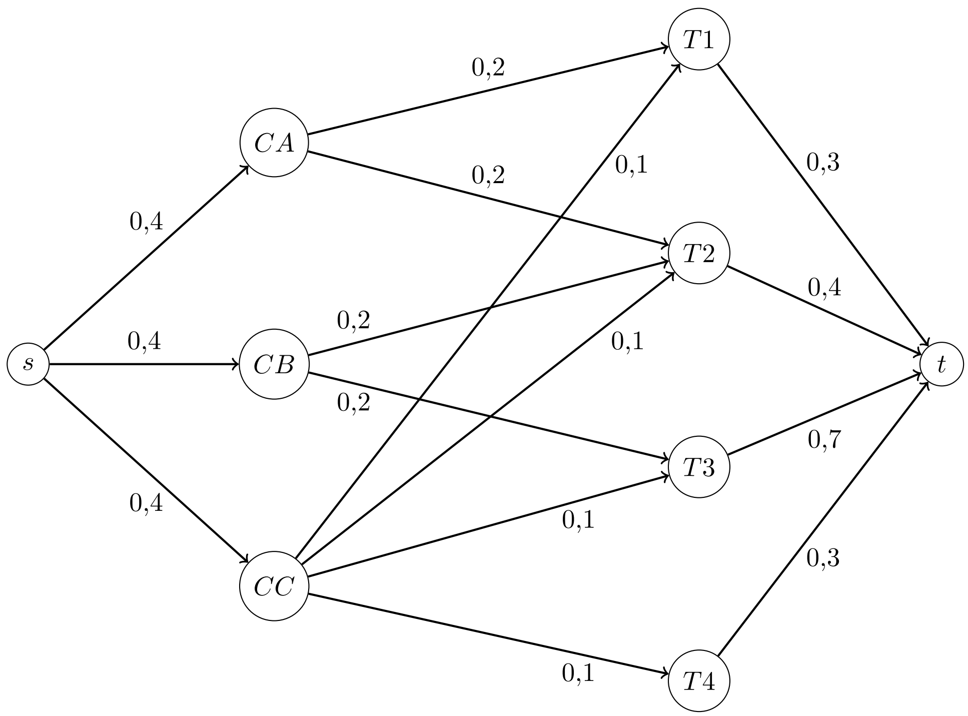
iii. Chemical C can be transported in any truck, but no truck can carry more than 1 barrel of Chemical C.

Moreover, each truck has their own carrying capacity: Truck #1 can carry at most 3 total barrels; Truck #2 can carry at most 4 total barrrels; Truck #3 can carry at most at most 7 total barrels; and Truck #4 can carry at most 3 total barrels. Suppose the warehouse currently has 4 barrels of each chemical in storage (12 total barrels).

Find the maximum total number of barrels that can be shipped using the 4 trucks. Verify that your answer is maximum.

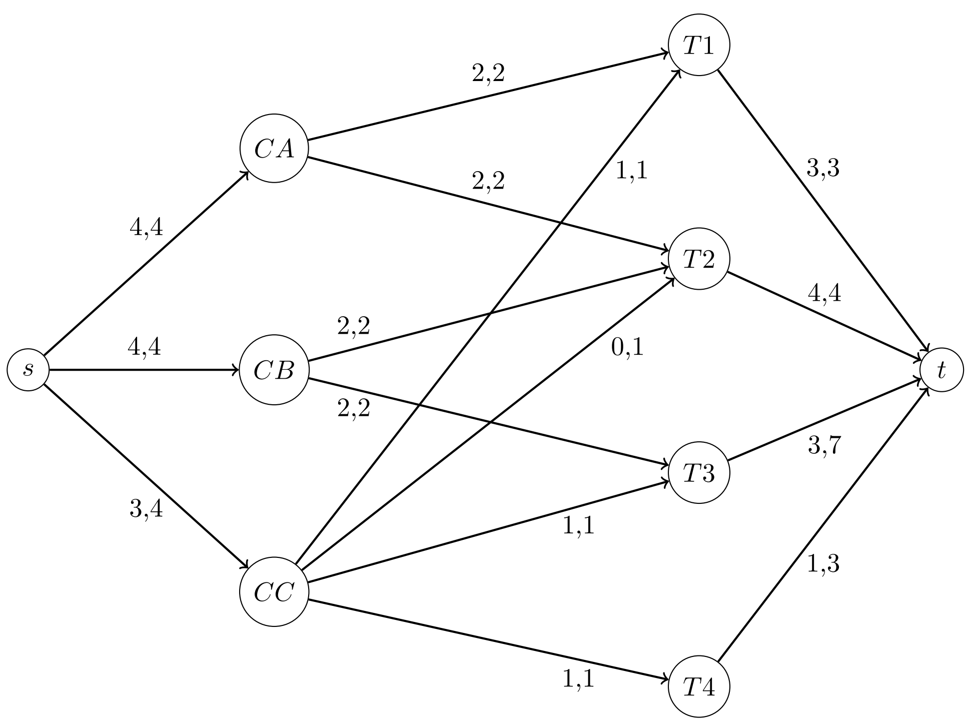
Solution.

We construct a network flow f that represents this situation. Let s be the warehouse, t be the location to which the barrels are being shipped, CA be chemical A and so on, and $T1$ be truck 1 and so on. We start with zero flow, and use capacities that match the problem's constraints.



We now augment the flow using the following f-augmenting paths:

- $s, CA, T1, t$
- $s, CA, T2, t$
- $s, CB, T2, t$
- $s, CB, T3, t$
- $s, CC, T1, t$
- $s, CC, T3, t$
- $s, CC, T4, t$



Now, let $S = \{s, CC, T2\}$ and $T = \{CA, CB, T1, T3, T4, t\}$. This is a source/sink cut because S and T partition the vertex set with $s \in S$, $t \in T$. Since the slack in each edge in $[S, T]$ is 0, we have found a maximum flow. Therefore, since $\text{val}(f) = 3 + 4 + 3 + 1 = 11$, the maximum total number of barrels that can be shipped using the 4 trucks is 11. $\square$