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MATH 555: Section H01
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# MATH 555 Homework 11

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**Problem 7.8** Let $f_k : [a, b] \to \mathbb{R}$ for $k \in \mathbb{N}$ be continuous functions such that the series

$$F(x) = \sum_{k=0}^{\infty} f_k(x)$$

converges uniformly on $[a, b]$. Prove that $F(x)$ is continuous and

$$\int_a^b F(x)\,dx = \sum_{k=0}^{\infty} \int_a^b f_k(x)\,dx.$$

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Define $\langle F_n \rangle_{n=1}^{\infty}$ by

$$F_n = \sum_{k=0}^{n} f_k(x).$$

Then by definition, $F_n$ converges to $F$ uniformly. Since each $F_n(x)$ is the sum of finitely many continuous functions, each $F_n(x)$ is continuous. Thus, since the uniform limit of continuous functions is continuous, $F(x)$ is continuous. Moreover, since each $F_n$ and $F$ are continuous, they are all Riemann integrable, so we have

$$\begin{aligned}
\int_a^b F(x)\,dx &= \lim_{n\to\infty} \int_a^b F_n(x)\,dx && \text{(Theorem 7.4)} \\
&= \lim_{n\to\infty} \int_a^b \sum_{k=0}^{n} f_k(x)\,dx && \text{(definition of } F_n\text{)} \\
&= \lim_{n\to\infty} \sum_{k=0}^{n} \int_a^b f_k(x)\,dx && \text{(integral property)} \\
&= \sum_{k=0}^{\infty} \int_a^b f_k(x)\,dx. && \text{(infinite sum definition)}
\end{aligned}$$

□

**Problem 7.12** Let $X$ be a compact metric space and for $k \in \mathbb{N}$ let $f_k : X \to \mathbb{R}$ be continuous and assume for each $x \in X$ that $\langle f_k(x) \rangle_{k=1}^{\infty}$ is monotone decreasing (that is $f_{k+1}(x) \le f_k(x)$ for all $k$). Assume there is a continuous function $f : X \to \mathbb{R}$ such that

$$\lim_{k\to\infty} f_k(x) = f(x)$$

for all $x \in X$. Prove that $\lim\limits_{k\to\infty} f_k = f$ uniformly.

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We first note that since $\langle f_k(x) \rangle$ is monotone decreasing for each $x$, we have $f_k(x) \ge f(x)$ for all $x$ and $k$ (this is easy to show with a proof by contradiction), and thus we have

$$\big| f_k(x) - f(x) \big| = f_k(x) - f(x). \tag{1}$$

Let $\varepsilon > 0$, and define for each $k \in \mathbb{N}$

$$U_k := \{x \in X : f_k(x) - f(x) < \varepsilon\}.$$

Let $g : X \to \mathbb{R}$ defined by $g(x) = f_k(x) - f(x)$. Then $g$ is continuous since is the sum of continuous functions. Also, by (1), $g(x) = \big|f_k(x) - f(x)\big|$, so we have $U_k = g^{-1}[(-\varepsilon, \varepsilon)]$. Since the continuous preimage of open sets are open, $U_k$ is open.

We also have that

$$X \subseteq \bigcup_{k=1}^{\infty} U_k.$$

To see this, let $x \in X$. Since $\lim\limits_{k\to\infty} f_k(x) = f(x)$, there exists a $K$ such that for all $k \geq K$,

$$f_k(x) - f(x) = \big|f_k(x) - f(x)\big| < \varepsilon.$$

Thus, by definition we have

$$x \in U_K \subseteq \bigcup_{k=1}^{\infty} U_k.$$

Therefore, $\mathcal{U} = \{U_1, U_2, U_3, \ldots\}$ is an open cover of $X$.

We now note that for all $k \in \mathbb{N}$, we have

$$U_k \subseteq U_{k+1}. \tag{2}$$

To see this, let $x \in U_k$, and observe that

$$\begin{aligned}
f_{k+1}(x) &\leq f_k(x) && \text{(monotonicity assumption)}\\
\implies f_{k+1}(x) - f(x) &\leq f_k(x) - f(x) &&\\
&< \varepsilon && \text{(definition of } x \in U_k)\\
\implies x &\in U_{k+1}. && \text{(definition of } x \in U_{k+1})
\end{aligned}$$

Since $X$ is compact, there exists a finite subcover $\mathcal{U}_0 = \{U_{i_1}, U_{i_2}, \ldots, U_{i_n}\}$ of $X$. Let $N := \max\{i_1, i_2, \ldots, i_n\}$. Then, since $\mathcal{U}_0$ is an open cover of $X$ and we have $U_{i_j} \subseteq U_N$ for all $j$ by (2), we have

$$X \subseteq \bigcup \mathcal{U}_0 \subseteq U_N.$$

Thus, for any $x \in X$, $x \in U_N$,

$$f_N(x) - f(x) < \varepsilon. \tag{3}$$

So for all $n \geq N$, we have

$$\begin{aligned}
\big|f_n(x) - f(x)\big| &= f_n(x) - f(x) && \text{(by (1))}\\
&\leq f_N(x) - f(x) && \text{(monotonicity assumption)}\\
&< \varepsilon. && \text{(by (3))}
\end{aligned}$$

Therefore, since $N$ does not depend on any $x \in X$, $\lim_{k\to\infty} f_k = f$ uniformly. $\square$

**Problem 7.13** Let $\varphi : \mathbb{R} \to \mathbb{R}$ be a Riemann integrable function with

$$\varphi \geq 0 \quad \text{and} \quad \int_{-\infty}^{\infty} \varphi(x)\, dx = 1.$$

Prove that

$$K_n(x) = n\varphi(nx)$$

is a Dirac sequence.

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To show that $\langle K_n \rangle_{n=1}^{\infty}$ is a Dirac sequence, we need to show that

(a) $K_n \geq 0$ for all $n$,

(b) For all $n$,

$$\int_{-\infty}^{\infty} K_n(x)\,dx = 1,$$

(c) For all $\delta > 0$,

$$\lim_{n\to\infty} \int_{|x|\geq\delta} K_n(x)\,dx = 0.$$

We will show each in order.

(a) Since $\varphi \geq 0$, $n\varphi(nx) \geq 0$ since non-negative numbers are closed under multiplication.

(b) We have

$$\begin{aligned}
\int_{-\infty}^{\infty} K_n(x)\,dx &= \int_{-\infty}^{\infty} n\varphi(nx)\,dx \\
&= \int_{-\infty\cdot n}^{\infty\cdot n} \varphi(u)\,du && (u = nx,\ du = n\,dx) \\
&= \int_{-\infty}^{\infty} \varphi(x)\,dx = 1. && \text{(by assumption)}
\end{aligned}$$

(c) Let $\delta > 0$. Then, we have

$$\begin{aligned}
\lim_{n\to\infty} \int_{|x|\geq\delta} K_n(x)\,dx &= \lim_{n\to\infty} \int_{|x|\geq\delta} n\varphi(nx)\,dx \\
&= \lim_{n\to\infty} \int_{|u|\geq n\delta} \varphi(u)\,du && (u = nx,\ du = n\,dx) \\
&= \lim_{n\to\infty} \left[\int_{-\infty}^{\infty} \varphi(x)\,dx - \int_{-n\delta}^{n\delta} \varphi(x)\,dx\right] && \text{(by definition of integration limit)} \\
&= \int_{-\infty}^{\infty} \varphi(x)\,dx - \lim_{n\to\infty} \int_{-n\delta}^{n\delta} \varphi(x)\,dx && \text{(first integral is constant)} \\
&= \int_{-\infty}^{\infty} \varphi(x)\,dx - \int_{-\infty}^{\infty} \varphi(x)\,dx && \text{(definition of improper integral)} \\
&= 0.
\end{aligned}$$

So $\langle K_n \rangle$ is a Dirac sequence. □

**Problem 7.14** Let $f$ be a bounded continuous function on $\mathbb{R}$ and $\langle K_n \rangle_{n=1}^{\infty}$ be a Dirac sequence. Let

$$f_n(x) = \int_{-\infty}^{\infty} f(x-y)K_n(y)\,dy.$$

Prove that $\lim\limits_{n\to\infty} f_n(x) = f(x)$ pointwise.

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Let $x \in \mathbb{R}$, and let $\varepsilon > 0$. Since $f$ is continuous, there exists a $\delta > 0$ such that for all $y \in \mathbb{R}$,

$$|y - x| < \delta \implies \big|f(y) - f(x)\big| < \frac{\varepsilon}{2}. \tag{1}$$

Also, since $f$ is bounded, there exists a $B \in \mathbb{R}$ such that

$$|f| < B. \tag{2}$$

Finally, since $\langle K_n \rangle$ is a Dirac sequence, there exists an $N \in \mathbb{N}$ such that for all $n \geq N$,

$$\int_{|y| \geq \delta} K_n(y)\,dy < \frac{\varepsilon}{4B}. \tag{3}$$

Let $n \geq N$. With the above, we can write

$$\begin{aligned}
\big|f_n(x) - f(x)\big| &= \left|\int_{-\infty}^{\infty} f(x-y)K_n(y)\,dy - f(x)\right| \\
&= \left|\int_{-\infty}^{\infty} f(x-y)K_n(y)\,dy - \int_{-\infty}^{\infty} f(x)K_n(y)\,dy\right| && \text{(multiplying } f(x) \text{ by 1)} \\
&= \left|\int_{-\infty}^{\infty} \big[f(x-y) - f(x)\big]\,K_n(y)\,dy\right| && \text{(combining integrals)} \\
&= \left|\int_{|y|<\delta} \big[f(x-y) - f(x)\big]\,K_n(y)\,dy + \int_{|y|\geq\delta} \big[f(x-y) - f(x)\big]\,K_n(y)\,dy\right| \\
&\leq \int_{|y|<\delta} \big|f(x-y) - f(x)\big|\,K_n(y)\,dy + \int_{|y|<\delta} \big|f(x-y) - f(x)\big|\,K_n(y)\,dy && \text{(integral property)} \\
&< \int_{|y|<\delta} \left(\frac{\varepsilon}{2}\right) K_n(y)\,dy + \int_{|y|\geq\delta} \big|f(x-y) - f(x)\big|\,K_n(y)\,dy && \text{(by (1): } |(x-y)-x| = |y| < \delta) \\
&= \frac{\varepsilon}{2}\int_{|y|<\delta} K_n(y)\,dy + \int_{|y|\geq\delta} \big|f(x-y) - f(x)\big|\,K_n(y)\,dy \\
&\leq \frac{\varepsilon}{2} + \int_{|y|\geq\delta} \big|f(x-y) - f(x)\big|\,K_n(y)\,dy && \text{(first integral} \leq 1 \text{ since } K_n \text{ Dirac)} \\
&\leq \frac{\varepsilon}{2} + \int_{|y|\geq\delta} \Big[\big|f(x-y)\big| + \big|f(x)\big|\Big]\,K_n(y)\,dy && \text{(triangle inequality)} \\
&\leq \frac{\varepsilon}{2} + \int_{|y|\geq\delta} (B+B)K_n(y)\,dy && \text{(by (2))} \\
&= \frac{\varepsilon}{2} + 2B\int_{|y|\geq\delta} K_n(y)\,dy \\
&< \frac{\varepsilon}{2} + 2B\left(\frac{\varepsilon}{4B}\right) && \text{(by (3))} \\
&= \frac{\varepsilon}{2} + \frac{\varepsilon}{2} = \varepsilon.
\end{aligned}$$

Therefore, $\lim\limits_{n\to\infty} f_n(x) = f(x)$ pointwise. □

**Problem 7.15** Let $f$ be a function on $\mathbb{R}$ that is both bounded and uniformly continuous and let $\langle K_n \rangle_{n=1}^{\infty}$ be a Dirac sequence. Define

$$f_n(x) = \int_{-\infty}^{\infty} f(x-y)K_n(y)\,dy.$$

Prove that $\lim\limits_{n\to\infty} f_n(x) = f(x)$ uniformly.

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Let $\varepsilon > 0$. Since $f$ is uniformly continuous, there exists a $\delta > 0$ such that for all $x, y \in \mathbb{R}$,

$$|y - x| < \delta \implies \big|f(y) - f(x)\big| < \frac{\varepsilon}{2}. \tag{1}$$

Also, since $f$ is bounded, there exists a $B \in \mathbb{R}$ such that

$$|f| < B. \tag{2}$$

Finally, since $\langle K_n \rangle$ is a Dirac sequence, there exists an $N \in \mathbb{N}$ such that for all $n \geq N$,

$$\int_{|y| \geq \delta} K_n(y)\, dy < \frac{\varepsilon}{4B}. \tag{3}$$

Let $n \geq N$, and let $x \in \mathbb{R}$. With the above, we can write

$$\begin{aligned}
\big|f_n(x) - f(x)\big| &= \left| \int_{-\infty}^{\infty} f(x-y) K_n(y)\, dy - f(x) \right| \\
&= \left| \int_{-\infty}^{\infty} f(x-y) K_n(y)\, dy - \int_{-\infty}^{\infty} f(x) K_n(y)\, dy \right| && \text{(multiplying } f(x) \text{ by 1)} \\
&= \left| \int_{-\infty}^{\infty} \big[f(x-y) - f(x)\big] K_n(y)\, dy \right| && \text{(combining integrals)} \\
&= \left| \int_{|y|<\delta} \big[f(x-y) - f(x)\big] K_n(y)\, dy + \int_{|y|\geq\delta} \big[f(x-y) - f(x)\big] K_n(y)\, dy \right| \\
&\leq \int_{|y|<\delta} \big|f(x-y) - f(x)\big| K_n(y)\, dy + \int_{|y|<\delta} \big|f(x-y) - f(x)\big| K_n(y)\, dy && \text{(integral property)} \\
&< \int_{|y|<\delta} \left(\frac{\varepsilon}{2}\right) K_n(y)\, dy + \int_{|y|\geq\delta} \big|f(x-y) - f(x)\big| K_n(y)\, dy && \text{(by (1): } |(x-y) - x| = |y| < \delta) \\
&= \frac{\varepsilon}{2} \int_{|y|<\delta} K_n(y)\, dy + \int_{|y|\geq\delta} \big|f(x-y) - f(x)\big| K_n(y)\, dy \\
&\leq \frac{\varepsilon}{2} + \int_{|y|\geq\delta} \big|f(x-y) - f(x)\big| K_n(y)\, dy && \text{(first integral} \leq 1 \text{ since } K_n \text{ Dirac)} \\
&\leq \frac{\varepsilon}{2} + \int_{|y|\geq\delta} \Big[\big|f(x-y)\big| + \big|f(x)\big|\Big] K_n(y)\, dy && \text{(triangle inequality)} \\
&\leq \frac{\varepsilon}{2} + \int_{|y|\geq\delta} (B+B) K_n(y)\, dy && \text{(by (2))} \\
&= \frac{\varepsilon}{2} + 2B \int_{|y|\geq\delta} K_n(y)\, dy \\
&< \frac{\varepsilon}{2} + 2B \left(\frac{\varepsilon}{4B}\right) && \text{(by (3))} \\
&= \frac{\varepsilon}{2} + \frac{\varepsilon}{2} = \varepsilon.
\end{aligned}$$

Therefore, $\lim_{n\to\infty} f_n(x) = f(x)$ uniformly. □

**Problem 7.16** Let $f$ be a continuous function such that for some interval $[\alpha, \beta]$ we have $f(x) = 0$ for all $x \notin [\alpha, \beta]$. Prove that $f$ is bounded and uniformly continuous.

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Since $[\alpha, \beta]$ is compact and $f$ is continuous, $f\big[[\alpha, \beta]\big]$ is compact, so it is bounded. Thus, there exists a $B \in \mathbb{R}$ such that $|f(x)| \leq B$ for all $x \in [\alpha, \beta]$. Since $|f(x)| = 0 \leq B$ for all $x \notin [\alpha, \beta]$, we have $|f(x)| \leq B$ for all $x \in \mathbb{R}$. Thus, $f$ is bounded.

Let $\varepsilon > 0$. Since $[\alpha, \beta]$ is compact and $f$ is continuous, $f$ is uniformly continuous on $[\alpha, \beta]$. So there exists a $\delta > 0$ such that for all $x_1, x_2 \in [\alpha, \beta]$,

$$|x_1 - x_2| < \delta \implies \left|f(x_1) - f(x_2)\right| < \varepsilon.$$

Now, let $x_1, x_2 \in \mathbb{R}$ such that $|x_1 - x_2| < \delta$.

Case 1: $x_1, x_2 \in [\alpha, \beta]$. Then, the above $\delta$ clearly satisfies

$$|x_1 - x_2| < \delta \implies \left|f(x_1) - f(x_2)\right| < \varepsilon.$$

Case 2: $x_1, x_2 \notin [\alpha, \beta]$. Then, $\left|f(x_1) - f(x_2)\right| = |0 - 0| = 0 < \varepsilon$, so

$$|x_1 - x_2| < \delta \implies \left|f(x_1) - f(x_2)\right| < \varepsilon$$

holds vacuously for any $\delta$.

Case 3: One of $x_1$ or $x_2$ is in $[\alpha, \beta]$ and the other is not. Without loss of generality, suppose $x_1 \in [\alpha, \beta]$ and $x_2 \notin [\alpha, \beta]$. We note that we have $f(\alpha) = f(\beta) = 0$, since $f$ is continuous and each has a one-sided limit equal to 0. If $x_1 \leq x_2$, then $\beta$ is between $x_1$ and $x_2$, so we have

$$\begin{aligned}
\left|f(x_2) - f(x_1)\right| &= \left|f(x_2) - f(\beta) + f(\beta) - f(x_1)\right| \\
&\leq \left|f(x_2) - f(\beta)\right| + \left|f(\beta) - f(x_1)\right| \\
&< \varepsilon + |0 + 0| && (|x_2 - B| < \delta) \\
&= \varepsilon.
\end{aligned}$$

Similarly if $x_1 > x_2$, then $\alpha$ is between $x_1$ and $x_2$, so we have

$$\begin{aligned}
\left|f(x_1) - f(x_2)\right| &= \left|f(x_1) - f(\alpha) + f(\alpha) - f(x_2)\right| \\
&\leq \left|f(x_1) - f(\alpha)\right| + \left|f(\alpha) - f(x_2)\right| \\
&< \varepsilon + |0 + 0| && (|x_2 - B| < \delta) \\
&= \varepsilon.
\end{aligned}$$

Therefore, in all cases, we have that for all $x_1, x_2 \in \mathbb{R}$,

$$|x_1 - x_2| < \delta \implies \left|f(x_1) - f(x_2)\right| < \varepsilon.$$

So $f$ is uniformly continuous. □

**Problem 7.17** Let $f$ be bounded and continuous on $\mathbb{R}$ and let $\langle K_n \rangle_{n=1}^{\infty}$ be a Dirac sequence and

$$f_n(x) = \int_{-\infty}^{\infty} f(x - y) K_n(y)\, dy.$$

Prove that $f_n$ can be rewritten as

$$f_n(x) = \int_{-\infty}^{\infty} f(y) K_n(x - y)\, dy.$$

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We can use the substitution $z = x - y$. Then, $y = x - z$, and $dz = -dy$. So we have

$$\begin{aligned}
f_n(x) &= \int_{-\infty}^{\infty} f(x-y)K_n(y)\,dy \\
&= -\int_{-\infty}^{\infty} -f(x-y)K_n(y)\,dy \\
&= -\int_{x-(-\infty)}^{x-\infty} f(z)K_n(x-z)\,dz && \text{(substitution)} \\
&= \int_{x-\infty}^{x+\infty} f(z)K_n(x-z)\,dz && \text{(flipping integral)} \\
&= \int_{-\infty}^{\infty} f(y)K_n(x-y)\,dy. && \text{(changing variable/evaluating bounds)}
\end{aligned}$$

□

**Problem 7.18** Find

$$\int_{-1}^{1}(1-x^2)^n\,dx.$$

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Let $n \in \mathbb{N}$, and let

$$I_n := \int_{-1}^{1}(1-x^2)^n\,dx.$$

Define

$$I(m,n) := \int_{-1}^{1}(1-x)^m(1+x)^n\,dx.$$

This is useful as we have

$$I_n = \int_{-1}^{1}(1-x^2)^n\,dx = \int_{-1}^{1}[(1-x)(1+x)]^n\,dx = \int_{-1}^{1}(1-x)^n(1+x)^n\,dx = I(n,n).$$

We will first prove three lemmas.

*Lemma 1:* For all $m,n \geq 0$, we have

$$I(m,n) = \frac{m}{n+1}I(m-1,n+1).$$

Proof: We can use integration by parts with $u=(1-x)^m$, $dv=(1+x)^n\,dx$ to write

$$\begin{aligned}
I(m,n) &= \int_{-1}^{1}(1-x)^m(1+x)^n\,dx \\
&= \frac{(1-x)^m(1+x)^{n+1}}{n+1}\bigg|_{x=-1}^{1} - \int_{-1}^{1}\frac{(1+x)^{n+1}(-m)(1-x)^{m-1}}{n+1}\,dx && \text{(from described } u,v) \\
&= \frac{0^m2^{n+1}}{n+1} - \frac{2^m0^{n+1}}{n+1} + \int_{-1}^{1}\frac{m(1-x)^{m-1}(1+x)^{n+1}}{n+1}\,dx \\
&= \frac{m}{n+1}\int_{-1}^{1}(1-x)^{m-1}(1+x)^{n+1}\,dx \\
&= \frac{m}{n+1}I(m-1,n+1)
\end{aligned}$$

as desired. □

*Lemma 2:* For all $n \geq 0$, we have

$$I(0,n) = \frac{2^{n+1}}{n+1}.$$

Proof: This is straightfoward with $u$-substitution by choosing $u = 1 + x$. We have

$$I(0, n) = \int_{-1}^{1} (1 + x)^n \, dx = \int_{0}^{2} u^n \, du = \frac{u^{n+1}}{n+1} = \frac{2^{n+1}}{n+1}.$$

$\square$

*Lemma 3:* For all $1 \leq k \leq n$, we have

$$I(n, n) = \prod_{i=1}^{k} \left( \frac{n - (i - 1)}{n + i} \right) I(n - k, n + k).$$

Proof: We will use induction on $k$. The base case $k = 1$ follows directly from Lemma 1 as

$$I(n, n) = \frac{n}{n+1} I(n - 1, n + 1) = \frac{n - (1 - 1)}{n + 1} I(n - 1, n + 1).$$

Now, let $1 < k \leq n$ and assume that the claim holds for $k - 1$. Then, we can use Lemma 1 to write

$$\begin{aligned}
I(n, n) &= \prod_{i=1}^{k-1} \left( \frac{n - (i - 1)}{n + i} \right) I(n - k + 1, n + k - 1) && \text{(IH)} \\
&= \prod_{i=1}^{k-1} \left( \frac{n - (i - 1)}{n + i} \right) \left( \frac{n - k + 1}{n + k - 1 + 1} \right) I(n - k + 1 - 1, n + k - 1 + 1) && \text{(Lemma 1)} \\
&= \prod_{i=1}^{k-1} \left( \frac{n - (i - 1)}{n + i} \right) \left( \frac{n - (k - 1)}{n + k} \right) I(n - k, n + k) && \text{(simplifying)} \\
&= \prod_{i=1}^{k} \left( \frac{n - (i - 1)}{n + i} \right) I(n - k, n + k). && \text{(combining product)}
\end{aligned}$$

So by induction, the lemma holds for all $1 \leq k \leq n$. $\square$

We can now compute $I_n$ by writing

$$\begin{aligned}
I(n, n) &= \prod_{i=1}^{n} \left( \frac{n - (i - 1)}{n + i} \right) I(n - n, n + n) && \text{(Lemma 3 with } k = n\text{)} \\
&= \frac{(n)(n + 1) \dots (1)}{(n + 1)(n + 2) \dots (2n)} I(0, 2n) \\
&= \frac{(n!)^2}{(2n)!} I(0, 2n) \\
&= \frac{(n!)^2}{(2n)!} \left( \frac{2^{2n+1}}{2n + 1} \right) && \text{(Lemma 2)} \\
&= \frac{2^{2n+1} (n!)^2}{(2n + 1)!}.
\end{aligned}$$

Therefore,

$$I_n := \int_{-1}^{1} (1 - x^2)^n \, dx = \frac{2^{2n+1} (n!)^2}{(2n + 1)!}.$$

$\square$

**Problem 7.19** Let $f : [\alpha, \beta] \to \mathbb{R}$ be a continuous function where $f(x) = 0$ for $x \notin [\alpha, \beta]$. Define $F : [0, 1] \to \mathbb{R}$ to be the function

$$F(x) := f(\alpha + (\beta - \alpha)x)$$

and let $P_n : [0,1] \to \mathbb{R}$ be polynomials such that $P_n \to F$ uniformly and set

$$p_n(x) = P_n\left(\frac{x-\alpha}{\beta-\alpha}\right).$$

Prove that each $p_n$ is a polynomial and $p_n \to f$ uniformly.

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Let

$$g(x) := \frac{x-\alpha}{\beta-\alpha} = \left(\frac{1}{\beta-\alpha}\right)x - \frac{\alpha}{\beta-\alpha},$$

which is a polynomial. Then $p_n(x) = P_n(g(x))$, and since the composition of two polynomials is a polynomial, each $p_n(x)$ is a polynomial.

We now show that $p_n \to f$ uniformly. We first note that for all $x \in [\alpha, \beta]$, we have

$$F\left(\frac{x-\alpha}{\beta-\alpha}\right) = f\left(\alpha + (\beta-\alpha)\left(\frac{x-\alpha}{\beta-\alpha}\right)\right) = f(\alpha + x - \alpha) = f(x).$$

Since $P_n \to F$ uniformly, there exists an $N$ such that for all $n \geq N$ and for all $x \in [0,1]$,

$$|P_n(x) - F(x)| < \varepsilon.$$

Let $n \geq N$. Then, for all $x \in [\alpha, \beta]$, we have

$$\left|p_n(x) - f(x)\right| = \left|P_n\left(\frac{x-\alpha}{\beta-\alpha}\right) - F\left(\frac{x-\alpha}{\beta-\alpha}\right)\right| < \varepsilon,$$

since

$$\frac{x-\alpha}{\beta-\alpha} \in [0,1].$$

So by definition, $p_n \to f$ uniformly. □

**Problem 7.20** Let $f : [a,b] \to \mathbb{R}$ be continuous. Prove that there is a sequence of polynomials $p_n : [a,b] \to \mathbb{R}$ with $p_n \to f$ uniformly.

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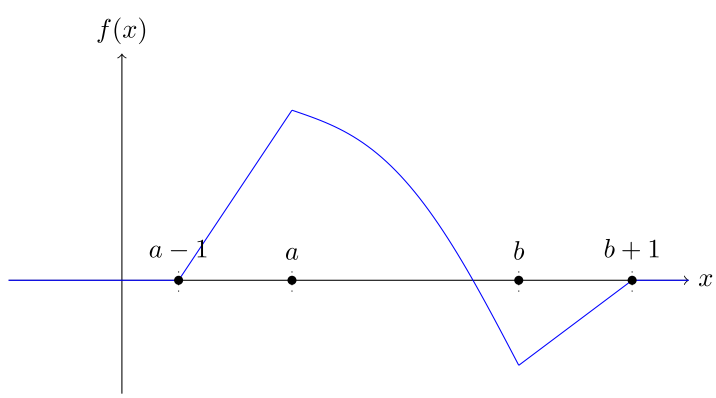
We can extend $f$'s domain from $[a,b]$ to $\mathbb{R}$ by redefining $f$ as

$$f(x) := \begin{cases} 0, & x < a-1; \\ (x-(a-1))f(a), & a-1 \leq x < a; \\ f(x), & a \leq x \leq b; \\ ((b+1)-x)f(b), & b < x \leq b+1; \\ 0, & b+1 < x. \end{cases}$$

An example is shown (drawn by ChatGPT):



From the picture, $f$ is clearly continuous. Let $\alpha := a - 1$ and $\beta := b + 1$. Then, $f(x) = 0$ for all $x \notin [\alpha, \beta]$.

Define $F : [0,1] \to \mathbb{R}$ by

$$F(x) := f(\alpha + (\beta - \alpha)x),$$

define the Dirac sequence

$$K_n(x) := \begin{cases} c_n(1-x^2)^n, & |x| \le 1; \\ 0, & |x| > 1. \end{cases}$$

with

$$c_n := \frac{1}{\int_{-1}^{1} (1-x^2)^n \, dx},$$

and define $P_n : [0,1] \to \mathbb{R}$ by

$$P_n(x) = \int_{-1}^{1} K_n(x-y) F(y) \, dy.$$

Then by Proposition 7.21 in the notes, since $F(x) = 0$ for all $x \notin [0,1]$, we have that $P_n \to F$ uniformly and that each $P_n$ restricted to $[0,1]$ is a polynomial.

We can now directly apply Problem 7.19 to conclude that

$$p_n(x) = P_n\left(\frac{x-\alpha}{\beta-\alpha}\right)$$

is a sequence of polynomials from $[a,b]$ to $\mathbb{R}$ with $p_n \to f$ uniformly. □

**Problem 7.21** Let $f : [a,b] \to \mathbb{R}$ be continuous and assume that

$$\int_a^b f(x) x^n \, dx = 0$$

for all $n \in \mathbb{N}$. Sow that $f(x) = 0$ for all $x \in [a,b]$.

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We claim that for any polynomial $p(x)$, we have

$$\int_a^b f(x) p(x) \, dx = 0. \tag{1}$$

To see this, let

$$p(x) = \sum_{k=0}^{m} a_k x^k$$

be a polynomial of degree $m$. Then, we have

$$\begin{aligned} \int_a^b f(x)p(x) &= \int_a^b f(x) \sum_{k=0}^{m} a_k x^k \, dx \\ &= \sum_{n=0}^{m} \int_a^b f(x) a_n x^n \, dx && \text{(integral property)} \\ &= \sum_{k=0}^{m} a_n \int_a^b f(x) x^n \, dx && \text{(distributive property)} \\ &= \sum_{k=0}^{m} a_n (0) && \text{(by assumption)} \\ &= 0. \end{aligned}$$

By Problem 7.20, there exists a sequence of polynomials $p_n$ that converge to $f$ uniformly. Thus, we have $\lim_{n\to\infty} p_n = f$ uniformly, and since $f$ is continuous, we can multiply by $f$ to write

$$\lim_{n\to\infty} f p_n = f^2. \tag{2}$$

So we have

$$\begin{aligned}
\int_a^b [f(x)]^2 \, dx &= \lim_{n\to\infty} \int_a^b f(x) p_n(x)\, dx && \text{(combining (2) with Theorem 7.4)} \\
&= \lim_{n\to\infty} (0) && \text{(from (1))} \\
&= 0.
\end{aligned}$$

Since $[f(x)]^2$ is non-negative,

$$\int_a^b [f(x)]^2 \, dx = 0$$

is possible only if $f(x) = 0$ for all $x \in [a, b]$. □

**Problem 7.22** Let $f, g : [a, b] \to \mathbb{R}$ be continuous functions such that

$$\int_a^b f(x) x^n \, dx = \int_a^b g(x) x^n \, dx$$

for all $n \in \mathbb{N}$. Show that $f(x) = g(x)$ for all $x \in [a, b]$.

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Let $h : [a, b] \to \mathbb{R}$ defined by $h(x) = f(x) - g(x)$. Since $f$ and $g$ are continuous, $h$ is continuous. Then, we have

$$\int_a^b h(x) x^n \, dx = \int_a^b [f(x) - g(x)] x^n \, dx = \int_a^b f(x) x^n \, dx - \int_a^b g(x) x^n \, dx = 0$$

for all $n \in \mathbb{N}$, so by Problem 7.21, $h(x) = 0$ for all $x \in [a, b]$. Therefore,

$$0 = h(x) = f(x) - g(x) \implies f(x) = g(x)$$

for all $x \in [a, b]$. □

**Problem 7.23** For the rest of the problems, let $f : \mathbb{R} \to \mathbb{R}$ be a function such that for some $b > 0$ we have $f(x) = 0$ for all $x$ with $|x| \geq b$, $f$ is Riemann integrable on $[-b, b]$, and there is a constant $B$ such that $\left|f(x)\right| \leq B$ for all $x$.

Prove that if $\langle K_n \rangle_{n=1}^{\infty}$ is a Dirac sequence and

$$f_n(x) = \int_{-\infty}^{\infty} K_n(y) f(x - y)\, dy = \int_{-\infty}^{\infty} K_n(x - y) f(y)\, dy,$$

then at any point $x$ where $f$ is continuous

$$\lim_{n\to\infty} f_n(x) = f(x).$$

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Let $x \in \mathbb{R}$, and let $\varepsilon > 0$. Since $f$ is continuous at $x$, there exists a $\delta > 0$ such that for all $y \in \mathbb{R}$,

$$|y - x| < \delta \implies \left|f(y) - f(x)\right| < \frac{\varepsilon}{2}. \tag{1}$$

Also, since $\langle K_n \rangle$ is a Dirac sequence, there exists an $N \in \mathbb{N}$ such that for all $n \geq N$,

$$\int_{|y|\geq\delta} K_n(y)\,dy < \frac{\varepsilon}{4B}. \tag{3}$$

Let $n \geq N$. With the above, we can write

$$\begin{aligned}
\big|f_n(x) - f(x)\big| &= \left|\int_{-\infty}^{\infty} f(x-y)K_n(y)\,dy - f(x)\right| \\
&= \left|\int_{-\infty}^{\infty} f(x-y)K_n(y)\,dy - \int_{-\infty}^{\infty} f(x)K_n(y)\,dy\right| && \text{(multiplying } f(x) \text{ by 1)} \\
&= \left|\int_{-\infty}^{\infty} \big[f(x-y) - f(x)\big]\, K_n(y)\,dy\right| && \text{(combining integrals)} \\
&= \left|\int_{|y|<\delta} \big[f(x-y) - f(x)\big]\, K_n(y)\,dy + \int_{|y|\geq\delta} \big[f(x-y) - f(x)\big]\, K_n(y)\,dy\right| \\
&\leq \int_{|y|<\delta} \big|f(x-y) - f(x)\big|\, K_n(y)\,dy + \int_{|y|<\delta} \big|f(x-y) - f(x)\big|\, K_n(y)\,dy && \text{(integral property)} \\
&< \int_{|y|<\delta} \left(\frac{\varepsilon}{2}\right) K_n(y)\,dy + \int_{|y|\geq\delta} \big|f(x-y) - f(x)\big|\, K_n(y)\,dy && \text{(by (1): } |(x-y)-x| = |y| < \delta) \\
&= \frac{\varepsilon}{2}\int_{|y|<\delta} K_n(y)\,dy + \int_{|y|\geq\delta} \big|f(x-y) - f(x)\big|\, K_n(y)\,dy \\
&\leq \frac{\varepsilon}{2} + \int_{|y|\geq\delta} \big|f(x-y) - f(x)\big|\, K_n(y)\,dy && \text{(first integral} \leq 1 \text{ since } K_n \text{ Dirac)} \\
&\leq \frac{\varepsilon}{2} + \int_{|y|\geq\delta} \Big[\big|f(x-y)\big| + \big|f(x)\big|\Big]\, K_n(y)\,dy && \text{(triangle inequality)} \\
&\leq \frac{\varepsilon}{2} + \int_{|y|\geq\delta} (B+B)K_n(y)\,dy && \text{(by (2))} \\
&= \frac{\varepsilon}{2} + 2B\int_{|y|\geq\delta} K_n(y)\,dy \\
&< \frac{\varepsilon}{2} + 2B\left(\frac{\varepsilon}{4B}\right) && \text{(by (3))} \\
&= \frac{\varepsilon}{2} + \frac{\varepsilon}{2} = \varepsilon.
\end{aligned}$$

Therefore, $\lim_{n\to\infty} f_n(x) = f(x)$. □

**Problem 7.24** Let $\langle K_n \rangle_{n=1}^{\infty}$ be a differentiable Dirac sequence. Prove that for each $n \in \mathbb{N}$,

$$f_n(x) = \int_{-\infty}^{\infty} K_n(x-y)f(y)\,dy$$

is differentiable and

$$f_n'(x) = \int_{-\infty}^{\infty} K_n'(x-y)f(y)\,dy.$$

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Let $n \in \mathbb{N}$ and $\varepsilon > 0$. We want to show that

$$\lim_{h\to 0} \frac{f_n(x+h) - f_n(x)}{h} = \int_{-\infty}^{\infty} K_n'(x-y)f(y)\,dy.$$

This means that we want to find a $\delta > 0$ such that for all $h \in \mathbb{R}$,

$$|h| < \delta \implies \left| \frac{f_n(x+h) - f_n(x)}{h} - \int_{-\infty}^{\infty} K_n'(x-y)f(y)\,dy \right| < \varepsilon.$$

Since $\langle K_n \rangle$ is a differentiable Dirac sequence, there exists a $\delta > 0$ such that for all $h \in \mathbb{R}$ and for all $x \in \mathbb{R}$,

$$|h| < \delta \implies \left| \frac{K_n(x+h) - K_n(x)}{h} - K_n'(x) \right| < \frac{\varepsilon}{2Bb}. \tag{$*$}$$

Let $h$ so that $|h| < \delta$. Then, we have

$$\begin{aligned}
&\left| \frac{f_n(x+h) - f_n(x)}{h} - \int_{-\infty}^{\infty} K_n'(x-y)f(y)\,dy \right| \\
=&\left| \frac{1}{h}\int_{-\infty}^{\infty} K_n(x+h-y)f(y)\,dy - \frac{1}{h}\int_{-\infty}^{\infty} K_n(x-y)f(y)\,dy - \int_{-\infty}^{\infty} K_n'(x-y)f(y)\,dy \right| && \text{(definition of } f_n\text{)} \\
=&\left| \int_{-\infty}^{\infty} \left( \frac{K_n(x+h-y)f(y) - K_n(x-y)f(y)}{h} - K_n'(x-y)f(y) \right) dy \right| && \text{(combining integrals)} \\
=&\left| \int_{-\infty}^{\infty} \left( \frac{K_n(x-y+h) - K_n(x-y)}{h} - K_n'(x-y) \right) f(y)\,dy \right| && \text{(rearranging)} \\
<&\left| \int_{-\infty}^{\infty} \left( \frac{\varepsilon}{B} \right) f(y)\,dy \right| && \text{(applying } (*) \text{ since } x-y \in \mathbb{R}\text{)} \\
\leq&\frac{\varepsilon}{2Bb}\int_{-\infty}^{\infty} \left|f(y)\right|dy && \text{(integeral property)} \\
=&\frac{\varepsilon}{2Bb}\int_{-b}^{b} \left|f(y)\right|dy && \text{(since } f \equiv 0 \text{ for } x \notin [-b,b]\text{)} \\
\leq&\frac{\varepsilon}{2Bb}\int_{-b}^{b} B\,dy && \text{(definition of } B\text{)} \\
=&\frac{\varepsilon}{2Bb}(B(b-(-b)) && \text{(evaluating integral)} \\
=&\frac{\varepsilon 2Bb}{2Bb} = \varepsilon.
\end{aligned}$$

Thus, the limit we wanted to show holds, and therefore we have that $f_n(x)$ is differentiable with

$$f_n'(x) = \int_{-\infty}^{\infty} K_n'(x-y)f(y)\,dy.$$

□

**Problem 7.25** Assume that $f$ is differentiable with $f'$ uniformly continuous and let $\langle K_n \rangle_{n=1}^{\infty}$ be a differentiable Dirac sequence. Prove that the derivative of

$$f_n(x) = \int_{-\infty}^{\infty} K_n(x-y)f(y)\,dy$$

can be written as

$$f_n'(x) = \int_{-\infty}^{\infty} K_n(x-y)f'(y)\,dy.$$

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We can write

$$
\begin{aligned}
f_n'(x) &= \int_{-\infty}^{\infty} K_n'(x-y)f(y)\,dy && \text{(from Problem 7.24)} \\
&= \int_{-b}^{b} K_n'(x-y)f(y)\,dy && (f(y)=0 \text{ for all } x \notin [-b,b]) \\
&= f(y)K_n(x-y)\Big|_{y=-b}^{b} - \int_{-b}^{b} -K_n(x-y)f'(y)\,dy && (\text{by parts with } u=f(y),\ v=-K_n(x-y)) \\
&= 0 - 0 + \int_{-b}^{b} K_n(x-y)f'(y)\,dy && (f(b)=f(-b)=0 \text{ by assumption}) \\
&= \int_{-\infty}^{\infty} K_n(x-y)f'(y)\,dy. && (f'(y)=0 \text{ for all } x \notin [-b,b] \text{ since } f(y)=0)
\end{aligned}
$$

□

**Problem 7.26** Assume that $f$ is differentiable with $f'$ uniformly continuous and let $\langle K_n \rangle_{n=1}^{\infty}$ be a differentiable Dirac sequence. Prove that if

$$f_n(x) = \int_{\infty}^{\infty} K_n(x-y)f(y)\,dy,$$

then the limit $\lim_{n\to\infty} f_n' = f'$ holds uniformly.

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Let $\varepsilon > 0$. Since $f'$ is uniformly continuous, there exists a $\delta > 0$ such that for all $x, y \in \mathbb{R}$,

$$|x-y| < \delta \implies \left|f'(x) - f'(y)\right| < \varepsilon.$$

Since $\langle K_n \rangle$ is a Dirac sequence, there exists an $N$ such that for all $n \geq N$,

$$\int_{|x|\geq\delta} K_n(x)\,dx = 0.$$

Let $n \geq N$. Then, we have

$$
\begin{aligned}
\left|f_n'(x) - f'(x)\right| &= \left|\int_{-\infty}^{\infty} K_n(x-y)f'(y)\,dy - f'(x)\right| && \text{(from Problem 7.25)} \\
&= \left|\int_{-\infty}^{\infty} K_n(x-y)f'(y)\,dy - \int_{-\infty}^{\infty} K_n(x-y)f'(x)\,dy\right| \\
&= \left|\int_{-\infty}^{\infty} K_n(x-y)[f'(y) - f'(x)]\,dy\right| \\
&= \left|\int_{-\infty}^{\infty} K_n(u)[f'(x-u) - f'(x)]\,du\right| && (\text{substitution with } u = x-y \text{ as in Problem 7.17}) \\
&= \left|\int_{-\delta}^{\delta} K_n(u)[f'(x-u) - f'(x)]\,du\right| && (\text{by choice of } \delta \text{ and } N) \\
&< \left|\int_{-\delta}^{\delta} K_n(u)\varepsilon\,du\right| && (\text{since } |(x-u)-x| < \delta) \\
&= \varepsilon. && \text{(integral is 1 by definition)}
\end{aligned}
$$

Therefore,

$$\lim_{n\to\infty} f_n' = f'$$

uniformly. □