MATH 552: Section 001
Professor: Dr. Miller
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MATH 552 Homework 11ˆ

Problem 5 The function

$$f(z) = \frac{-1}{(z-1)(z-2)} = \frac{1}{z-1} - \frac{1}{z-2},$$

which has two singular points $z = 1$ and $z = 2$, is analytic in the domains (Fig. 84)

$$D_1 : |z| < 1, \quad D_2 : 1 < |z| < 2, \quad D_3 : 2 < |z| < \infty.$$

Find the series representation in powers of z for $f(z)$ in each of those domains.

Solution.

For $|z| < 1$:

$$\begin{aligned}
f(z) &= \frac{1}{z-1} - \frac{1}{z-2} \\
&= \frac{1}{z-1} - \frac{1}{z-2}\left(\frac{1/2}{1/2}\right) \\
&= \frac{1}{z-1} - \frac{1}{2}\left(\frac{1}{\frac{z}{2}-1}\right) \\
&= \frac{1}{2}\left(\frac{1}{1-\frac{z}{2}}\right) - \frac{1}{1-z} && \text{(writing in form of power series: } \left|\tfrac{z}{2}\right| < |z| < 1) \\
&= \frac{1}{2}\left(1 + \left(\frac{z}{2}\right) + \left(\frac{z}{2}\right)^2 + \left(\frac{z}{2}\right)^3 + \ldots\right) - (1 + z + z^2 + z^3 + \ldots) \\
&= \left(\frac{1}{2} - 1\right) + \left(\frac{z}{4} - z\right) + \left(\frac{z^2}{8} - z^2\right) + \left(\frac{z^3}{16} - z^3\right) + \ldots
\end{aligned}$$

For $1 < |z| < 2$:

$$\begin{aligned}
f(z) &= \frac{1}{z-1} - \frac{1}{z-2} \\
&= \frac{1}{z-1}\left(\frac{1/z}{1/z}\right) - \frac{1}{z-2}\left(\frac{1/2}{1/2}\right) \\
&= \frac{1}{z}\left(\frac{1}{1-\frac{1}{z}}\right) + \frac{1}{2}\left(\frac{1}{1-\frac{z}{2}}\right) && \text{(writing in form of power series: } 1 < |z| < 2 \Rightarrow \left|\tfrac{z}{2}\right| < 1 \wedge \left|\tfrac{1}{z}\right| < 1) \\
&= \frac{1}{z}\left(1 + \left(\frac{1}{z}\right) + \left(\frac{1}{z}\right)^2 + \left(\frac{1}{z}\right)^3 + \ldots\right) + \frac{1}{2}\left(1 + \left(\frac{z}{2}\right) + \left(\frac{z}{2}\right)^2 + \left(\frac{z}{2}\right)^3 + \ldots\right) \\
&= \left(\frac{1}{2} + \frac{z}{4} + \frac{z^2}{8} + \frac{z^3}{16} + \ldots\right) + \left(\frac{1}{z} + \frac{1}{z^2} + \frac{1}{z^3} + \frac{1}{z^4} + \ldots\right)
\end{aligned}$$

For $2 < |z| < \infty$:

$$\begin{aligned}
f(z) &= \frac{1}{z-1} - \frac{1}{z-2} \\
&= \frac{1}{z-1}\left(\frac{1/z}{1/z}\right) - \frac{1}{z-2}\left(\frac{1/z}{1/z}\right) \\
&= \frac{1}{z}\left(\frac{1}{1-\frac{1}{z}}\right) - \frac{1}{z}\left(\frac{1}{1-\frac{2}{z}}\right) && \text{(writing in form of power series: } |z|>2 \Rightarrow \left|\tfrac{1}{z}\right| < \left|\tfrac{2}{z}\right| < 1) \\
&= \frac{1}{z}\left(1 + \left(\frac{1}{z}\right) + \left(\frac{1}{z}\right)^2 + \left(\frac{1}{z}\right)^3 + \ldots\right) - \frac{1}{z}\left(1 + \left(\frac{2}{z}\right) + \left(\frac{2}{z}\right)^2 + \left(\frac{2}{z}\right)^3 + ..\right) \\
&= \left(\frac{1}{z} + \frac{1}{z^2} + \frac{1}{z^3} + \frac{1}{z^4} + \ldots\right) - \left(\frac{1}{z} + \frac{2}{z^2} + \frac{4}{z^3} + \frac{8}{z^4} + \ldots\right) \\
&= -\frac{1}{z^2} - \frac{3}{z^3} - \frac{7}{z^4} + \ldots
\end{aligned}$$

Problem 6 Show that when $0 < |z-1| < 2$,

$$\frac{z}{(z-1)(z-3)} = -3\sum_{n=0}^{\infty} \frac{(z-1)^n}{2^{n+2}} - \frac{1}{2(z-1)}.$$

Solution.

$$\begin{aligned}
\frac{z}{(z-1)(z-3)} &= -\frac{1}{2(z-1)} + \frac{3}{2(z-3)} && \text{(using partial fractions)} \\
&= -\frac{1}{2(z-1)} + \frac{3}{2(z-1-2)} \\
&= -\frac{1}{2(z-1)} - \frac{3}{4-2(z-1)}\left(\frac{1/4}{1/4}\right) \\
&= -\frac{1}{2(z-1)} - \frac{3}{4}\left(\frac{1}{1-\frac{z-1}{2}}\right) && \text{(writing in form of power series: } |z-1|<2 \Rightarrow \left|\tfrac{z-1}{2}\right| < 1) \\
&= -\frac{1}{2(z-1)} - \frac{3}{4}\sum_{i+0}^{\infty}\left(\frac{z-1}{2}\right)^n \\
&= -\frac{1}{2(z-1)} - 3\sum_{i=0}^{\infty}\frac{(z-1)^n}{4(2)^n} \\
&= -3\sum_{i=0}^{\infty}\frac{(z-1)^n}{2^{n+2}} - \frac{1}{2(z-1)}
\end{aligned}$$