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Analysis in $\mathbb{R}^n$: Section 1
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# Analysis in $\mathbb{R}^n$ Homework 3

**Problem 20** Are closures and interiors of connected sets always connected? (Look at subsets of $\mathbb{R}^2$.)

Solution.

No. For example, consider $B_1, \overline{B_1{}'} \in \mathbb{R}^2$ centered at $(1,0)$ and $(-1,0)$ respectively. We showed in class that $E = B_1 \cup \overline{B_1{}'}$ is connected, but we claim $E^\circ$, the interior of $E$, is disconnected. To see this, consider $A = B_1$, $B = B_1{}'$ centered at $(1,0)$ and $(-1,0)$ respectively. Then, $E^\circ = A \cup B$, because $E^\circ$ is the interior of both balls. Clearly, both $A$ and $B$ are nonempty, and we have $\overline{A} \cap B = A \cap \overline{B} = \emptyset$, so $E^\circ$ is disconnected. So $E$ is a connected set that does not have both a connected interior and a connected closure. $\square$

**Problem 1** Prove that the intersection of compact sets is compact.

Solution.

We have shown the following in class: Let $(X,d)$ be a metric space, $F \subset K \subset X$, and $K$ a compact set. Then:

(a) $K$ is closed;

(b) $F$ is compact if and only if it is closed.

Consider a collection $C$ of compact sets. By (a), every set in $F$ is closed. Therefore, $F = \bigcap\limits_{K \in C} K$ is closed (the intersection of closed sets is closed). Since $F$ is a subset of a compact set, by (b) $F$ is also compact. $\square$

**Problem 2** Let $(X,d)$ be a metric space and $A \subset X$. Prove that

$$\overline{A} = \{x \in X \ : \ \forall r > 0, A \cap B_r(x) \neq \emptyset\}.$$

Solution.

We can write

$$\begin{aligned}
\overline{A} &= A \cup LP(A) && \text{(definition of closure)}\\
&= \{x \in X : x \in A \text{ or } x \in LP(A)\} && \text{(definition of union)}\\
&= \{x \in X : A \cap \{x\} \neq \emptyset \text{ or } \forall r > 0, A \cap B_r(x)\backslash\{x\} \neq \emptyset\} && \text{(definition of LP)}\\
&= \{x \in X : \forall r > 0, (A \cap \{x\}) \cup (A \cap B_r(x)\backslash\{x\}) \neq \emptyset\} && \text{(definition of union)}\\
&= \{x \in X : \forall r > 0, A \cap (\{x\} \cup B_r(x)\backslash\{x\}) \neq \emptyset\} && \text{(distributive property)}\\
&= \{x \in X : \forall r > 0, A \cap B_r(x) \neq \emptyset\}.
\end{aligned}$$

$\square$

**Problem 3** Let $S \subset \mathbb{R}$. Prove that if $\sup S$ exists, then it is an element of $\overline{S}$.

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Solution.

Suppose that $\alpha = \sup S$ exists. Then, we have shown in class that for all $r > 0$, there exists $s \in S$ such that $\alpha - r < s \leq \alpha < \alpha + r$. Since $B_r(\alpha) = (\alpha - r, \alpha + r)$ in $\mathbb{R}$, this means that for all $r > 0$, $S \cap B_r(\alpha) \backslash \{\alpha\}$ contains an element and thus is non-empty. Thus, $\alpha$ is a limit point of $S$ and therefore $\alpha \in \overline{S}$. □

**Problem 4** Let $A, B$ be two non-empty, bounded sets of real numbers. Prove the followings:

(a) $\sup(A + B) = \sup A + \sup B$, where $A + B = \{a + b \ : \ a \in A, b \in B\}$;

(b) $\sup(-A) = -\inf A$, where $-A = \{-a \ : \ a \in A\}$.

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Solution.

(a) Since $A$ and $B$ are bounded sets, $A + B$ will also be bounded and thus $\sup(A + B)$ exists. Suppose $\alpha = \sup(A + B) \neq \sup A + \sup B$.

Case 1: $\alpha < \sup A + \sup B$. Then, there exists some $\varepsilon > 0$ such that $\alpha = \sup A + \sup B - \varepsilon$. We have shown in class that there exists some $a \in A$ such that $a \in \left(\sup A - \frac{\varepsilon}{2}, \sup A\right]$ and some $b \in B$ such that $b \in \left(\sup B - \frac{\varepsilon}{2}, \sup B\right]$. We have $a + b \in A + B$, but then

$$\alpha = \sup A + \sup B - \varepsilon = \left(\sup A - \frac{\varepsilon}{2}\right) + \left(\sup B - \frac{\varepsilon}{2}\right) < a + b \in A + B,$$

contradicting $\alpha$ being an upper bound for $A + B$.

Case 2: $\alpha > \sup A + \sup B$. Then, there exists some $\varepsilon > 0$ such that $\alpha = \sup A + \sup B + \varepsilon$. We have shown in class that there exists some $p \in A + B$ such that $p \in (\sup A + \sup B, \sup A + \sup B + \varepsilon]$. Since $p \in A + B$, there exist some $a \in A$, $b \in B$ such that $p = a + b$. We must have $a \leq \sup a$ and $b \leq \sup b$ by definition, but then $p \leq \sup a + \sup b$, contradicting the interval above.

Therefore, $\sup(A + B) = \sup A + \sup B$. □

(b) Let $\beta = \inf A$ and $\alpha = \sup(-A)$. Suppose $\alpha \neq -\beta$.

Case 1: $\alpha < -\beta$. Then $\alpha = -\beta - \varepsilon$ for some $\varepsilon > 0$. We have shown in class that there exists some $a \in A$ such that $a \in [\beta, \beta + \varepsilon)$. But then $-a \in (-\beta - \varepsilon, -\beta]$, so $-a > -\beta - \varepsilon = \alpha = \sup(-A)$, a contradiction.

Case 2: $\alpha > -\beta$. Then $\alpha = -\beta + \varepsilon$ for some $\varepsilon > 0$. We have shown in class that there exists some $a \in -A$ such that $a \in (-\beta, -\beta + \varepsilon]$, so it follows that $-a \in A$ and $-a \in [\beta - \varepsilon, \beta)$. But then $-a < \beta = \inf(A)$, a contradiction.

Therefore, $\sup -A = -\inf A$. □

**Problem 5** Let $(X, d)$ be a metric space, $p \in X$, and $\{p_n\}$ a sequence in $X$ such that $d(p_n, p) < 1/n$ for all $n \in \mathbb{N}$. Prove that $\{p_n\}$ converges to $p$.

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Solution.

We say $\{p_n\}$ converges to $p$ if, for all $\varepsilon > 0$, there exists $N \in \mathbb{N}$ such that for all $n \geq N$, $d(p_n, p) < \varepsilon$. Let $\varepsilon > 0$, and consider $N = \frac{1}{\lfloor \varepsilon \rfloor}$. Then, any $n \geq N$ will satisfy

$$d(p_n, p) < \frac{1}{n} \leq \frac{1}{N} = \lfloor \varepsilon \rfloor < \varepsilon,$$

so $\{p_n\}$ converges to $p$. $\square$