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MATH 576: Section H01
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MATH 576 Homework 3

Problem 1 Prove that in Wythoff's game, all of the positions (a, a) for a nonnegative integer a have different nimbers. Furthermore, show that if a, b, c are nonnegative integers with $b \neq c$, then the positions (a, b) and (a, c) have different nimbers.

We prove both claims. First, let $a < b$ and consider the positions (a, a) and (b, b) in Wythoff's game. Let $*m$ be the nimber associated with (a, a) and $*n$ be the nimber associated with (b, b). Since $a < b$, (b, b) has (a, a) as an option, so n is the mex of a set of numbers including m. Thus, we have $n \neq m$ by definition, so (a, a) and (b, b) have different nimbers.

Similarly, if (a, b) and (a, c) are positions with $b < c$, then (a, c) has (a, b) as an option, and so for the same reason as above (a, b) and (a, c) have different nimbers. □

Problem 2 Let $G(a, b)$ be the Grundy value of the position (a, b) in Wythoff's game. Prove that $G(a, b) \leq a + b$.

We proceed by induction on $a + b$. The base case is trivial as $G(0, 0) = 0 \leq 0 + 0$. Now, let $a, b \in \mathbb{N}$ with $a + b > 0$ and suppose that for all $a', b' \in \mathbb{N}$ with $a' + b' < a + b$, we have $G(a', b') \leq a' + b'$.

Suppose (toward contradiction) that $G(a, b) > a + b$. Then by definition of mex, there is an option (a', b') in (a, b) with $G(a', b') = a + b$. Since a player must remove at least one token in each option, we must have $a' + b' < a + b$. But then by the induction hypothesis, we have $a + b = G(a', b') \leq a' + b'$, a contradiction. □

Problem 3 Fix an integer $r > 0$. The game r-Wythoff is the following variation of Wythoff's game: start with two heaps of tokens. On their turn, a player may either: (i) remove any nonzero number of tokens from a single heap; or (ii) remove a tokens from one heap and b tokens from the other heap, where $|a - b| < r$.

By a similar argument as in Theorem 1.1 of Lecture Notes 7, the nth $\mathcal{P}$-position (a_n, b_n) of r-Wythoff is given by the following recursive formula:

$$a_n = \operatorname{mex}\{a_i, b_i : i < n\},$$
$$b_n = a_n + rn.$$

Use this result and Theorem 1.3 in Lecture Notes 7 to prove the following exact formula: the nth $\mathcal{P}$-position of r-Wythoff is given by

$$(a_n, b_n) = (\lfloor n\alpha \rfloor, \lfloor n\beta \rfloor)$$

where

$$\alpha = \frac{1}{2}\left(2 - r + \sqrt{r^2 + 4}\right) \text{ and } \beta = \alpha + r.$$

We proceed by induction on n. The base case is trivial as $(a_0, b_0) = (0, 0) = (\lfloor 0 \cdot \alpha \rfloor, \lfloor 0 \cdot \beta \rfloor)$. Then, let $n > 0$, and suppose that for all $n' < n$, we have $(a_{n'}, b_{n'}) = (\lfloor n'\alpha \rfloor, \lfloor n'\beta \rfloor)$.

We have
$$\beta \geq \alpha = \frac{2-r+\sqrt{r^2+4}}{2} > \frac{2-r+\sqrt{r^2}}{2} = \frac{2-r+r}{2} = 1,$$
and both α and β are irrational (no two perfect squares have difference 4 so $\sqrt{r^2+4}$ is irrational). Moreover, we can write
$$\begin{aligned}
\frac{1}{\alpha}+\frac{1}{\beta} &= \frac{2}{2-r+\sqrt{r^2+4}} + \frac{1}{\alpha+r} \\
&= \frac{2}{2-r+\sqrt{r^2+4}} + \frac{1}{\frac{2-r+\sqrt{r^2+4}}{2}+r} \\
&= \frac{2}{2-r+\sqrt{r^2+4}} + \frac{2}{2-r+\sqrt{r^2+4}+2r} \\
&= \frac{2}{\sqrt{r^2+4}+2-r} + \frac{2}{\sqrt{r^2+4}+2+r} \\
&= \frac{2(\sqrt{r^2+4}+2+r)+2(\sqrt{r^2+4}+2-r)}{(\sqrt{r^2+4}+2+r)(\sqrt{r^2+4}+2-r)} \\
&= \frac{2\sqrt{r^2+4}+4+2r+2\sqrt{r^2+4}+4-2r}{(\sqrt{r^2+4}+2)^2-r^2} && \text{(difference of squares)} \\
&= \frac{2\sqrt{r^2+4}+4+2\sqrt{r^2+4}+4}{r^2+4+4\sqrt{r^2+4}+4-r^2} \\
&= \frac{4\sqrt{r^2+4}+8}{4\sqrt{r^2+4}+8} = 1.
\end{aligned}$$
Therefore, by Theorem 1.3, the sets $\{\lfloor m\alpha \rfloor \mid m \in \mathbb{N}\}$ and $\{\lfloor m\beta \rfloor \mid m \in \mathbb{N}\}$ are complementary. Since $\langle \lfloor m\alpha \rfloor \rangle_{m=1}^{\infty}$ and $\langle \lfloor m\beta \rfloor \rangle_{m=1}^{\infty}$ are strictly increasing sequences and $\alpha < \beta$, it follows using the induction hypothesis that
$$a_n = \operatorname{mex}(\{a_i, b_i \mid i < n\}) = \operatorname{mex}(\{\lfloor i\alpha \rfloor, \lfloor i\beta \rfloor \mid i < n\}) = \lfloor n\alpha \rfloor.$$
Using this, we can write
$$b_n = a_n + rn = \lfloor n\alpha \rfloor + rn = \lfloor n\alpha + nr \rfloor = \lfloor n(\alpha+r) \rfloor = \lfloor n\beta \rfloor.$$
Therefore, $(a_n, b_n) = (\lfloor n\alpha \rfloor, \lfloor n\beta \rfloor)$. ☐

Problem 4 We showed in class that the winning move for the first player in the game of Chomp on an $n \times 2$ board is to remove the square $(n, 2)$ in the top right corner. Prove that if the first player makes any other move on their first turn, then the second player has a winning strategy.

We describe the winning strategy for the second player if the first player removes any square $(a, b) \neq (n, 2)$.

Case 1: $b = 1$. Then on the second player's turn, an $(a-1) \times 2$ board remains, a position which we proved in class a player can win on their turn (by taking the square $(a-1, 2)$).

Case 2: $b = 2$. Then the second player wins by taking the square $(a+1, 1)$ (this square exists since $a < n$ by assumption). Then, on the first player's turn, an $a \times 2$ board remains with square $(a, 2)$ missing, a position which we proved in class a player cannot win on their turn. ☐

Problem 5 Determine the value of the partisan game $\{*0, *, *3, *8, *9 \mid *0, *4, *, *5, *6\}$.

We compute that $2 = \operatorname{mex}(\{0, 1, 3, 8, 9\}) = \operatorname{mex}(\{0, 4, 1, 5, 6\})$, so by Theorem 3.1 in Lecture Notes 9 the value of the game is $*2$. ☐