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MATH 574: Section H01
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MATH 574 Homework 1

Collaboration: I discussed some of the problems with Jackson Ginn and Jack Hyatt.

Problem 1 An exam has 5 true/false questions and 5 multiple choice questions. Each multiple choice question has 4 options. In how many different ways can a student answer the exam if:
(a) they leave no answer blank?
(b) they can leave any number of answers blank?
(c) they leave exactly one answer blank?

Solution.

(a) If every question must be answered, then there are 5 problems with 2 choices and 5 problems with 4 choices. Thus, by the product rule, there are $2^5 4^5 = 32768$ ways a student can answer the exam.

(b) If any question can be answered left blank, then the 5 true/false questions can be answered in three ways (true, false, or blank) and the 5 multiple choice questions can be answered in 5 ways (one of the four options or blank). Thus, by the product rule, there are $3^5 5^5 = 759,375$ ways a student can answer the exam.

(c) There are two disjoint cases here. One is that the student answers all the multiple choice questions and all but one of the true/false questions. There are $\binom{5}{4}$ ways to choose the true/false questions to answer, 2 ways to answer each of the 4 true/false questions, and 4 ways to answer each of the 5 multiple choice questions, so by the product rule there are $\binom{5}{4}2^4 4^5 = 81,920$ ways in this case.

The other is that the student answers all the true/false questions and all but one of the multiple choice questions. By similar reasoning, there are $\binom{5}{4}2^5 4^4 = 40,960$ ways in this case. Since there is no overlap between these cases, and they represent all the ways a student can answer the exam and leave one question blank, by the product rule there are $81,920 + 40,960 = 122,880$ ways a student can do this.

Problem 2 A pizza restaurant has the following available toppings: pepperoni, onion, sausage, pineapple, mushroom, black olives, and green olives.
(a) How many different combinations of pizzas are there?
(b) How many different combinations of pizzas are there with at least 2 toppings?
(c) How many different vegetarian pizzas are there?
(d) Now suppose you are allowed to double any of the ingredients (e.g., extra pepperoni). How many different combinations are there?

Solution.

(a) There are 7 toppings, and each one can be either included or not included, which is two options. By the product rule, there are $2^7 = 128$ different combinations of pizza.

(b) There are 8 ways for a pizza to have fewer than two toppings: No toppings, or to have one of the 7 toppings. Since there are 128 possible pizzas, by the subtraction rule, there are $128 - 8 = 120$ combinations of pizza with at least two toppings.

(c) There are 5 vegetarian toppings (pepperoni and sausage are the only meats), so by similar reasoning to part (a), there are $2^5 = 32$ vegetarian pizzas.

(d) There are 7 toppings, each one can be included, included doubly, or not included, which is three options. By the product rule, there are $3^7 = 2,187$ different combinations in this case.

Problem 3 Bit strings
(a) How many bit strings are there of length at most 6, not counting the empty string?
(b) How many bit strings of length 9 are there with exactly two 1s?
(c) How many bit strings of length 9 have at most four 1s?

Solution.

(a) For an arbitrary length $i \in \mathbb{N}$, by the product rule there are 2^i possible bit strings with length i. Thus, by the sum rule, the number of bit strings with integer lengths ranging from 1-6 is

$$\sum_{i=1}^{6} 2^i = 126.$$

(b) There are $\binom{9}{2}$ ways to place two 1s in a bit string of length 9, and since the other 7 bits cannot be 1 they must be 0. Thus, there $\binom{9}{2} = 36$ ways.

(c) We can use similar reasoning as part (b) for $\{0, 1, 3, 4\}$ and then the sum rule. Thus, the number of bit strings with at most four 1s is

$$\sum_{k=0}^{4} \binom{9}{k} = 256.$$

Problem 4 How many bit strings of length 12 contain either 6 consecutive 1s or 6 consecutive 0s?

Solution.

Let B be a sequence $b_1, b_2, ... b_12$ with $b_i \in \{0, 1\}$ for each i. We observe that a string of 6 consecutive characters can start anywhere from b_1 to b_7. We first consider consecutive strings of 6 1s.

Since we only care whether or not at least one consecutive string of 6 1s is present in the bitstring, we can create 7 disjoint sets $O_1, O_2, ... O_7$ where a bistring is in O_i if the first consecutive string of 6 1s in the bitstring starts at b_i. For example, 111111010100 would be in O_1 because the first (and only) consecutive string of 6 1s starts at the first number, and 011111111111 would be in O_2, because the first consecutive string of 6 1s starts at the second number.

With this definition, in O_1, all bitstrings would have the first 6 numbers be 1s, and then there are 2 choices for the last 6 numbers, so there are 2^6 strings by the product rule. For O_2, there are 2 choices for the last 5 numbers. However, the first number must be 0 for all bitstrings, because if it were 1, then the first consecutive string of 6 1s would starting at b_1 rather than b_2, meaning it would belong in O_1. Thus, there are 2^5 strings by the product rule. By the same reasoning as O_2, there are 2^5 bitstrings in $O_3, O_4, ... O_7$. Thus, there are $2^6 + 6(2^5) = 256$ strings in $\bigcup_{i=1}^{7} O_i$.

We can define seven sets $Z_1, Z_2, ... Z_7$ where a bistring is in Z_i if the first consecutive string of 6 0s in the bitstring starts at b_i. By symmetry, there are also 256 strings in $\bigcup_{i=1}^{7} Z_i$. The only strings in $\bigcup_{i=1}^{7} O_i \cap \bigcup_{i=1}^{7} Z_i$ are 111111000000 and 000000111111, because these are the only ways to have both a string of 6 consecutive 1s and 6 consecutive 0s in a bitstring of length 12. So these 2 strings are double-counted, and therefore there are $2\left(2^6 + 6(2^5)\right) - 2 = 510$ bitstrings of length 12 that contain either 6 consecutive 1s or 6 consecutive 0s.

Problem 5 A bowl contains 10 cherry candies and 20 orange candies.
(a) Suppose you are blindfolded. What is the minimum number of candies you must take from the bowl to guarantee 3 candies of the same type?
(b) What is the minimum number of candies you must take from the bowl to guarantee either 4 cherry candies or 7 orange candies?

Solution.

(a) The minimum number is 5 candies. After 4 candies, the requirement will either already be satisfied or you will have two of each candy. Since there are only two options, you are guaranteed on the next pick to choose either a third cherry candy or a third orange candy.

(b) The minimum number is 10 candies. After 9 candies, the requirement will already be satisfied, or you will have chosen 3 cherry candies and 6 orange candies. Since there are only two options, you are guaranteed on the next pick to choose either a 4th cherry candy or a 7th orange candy.

Problem 6 Points in the xy-plane
(a) Let $(x_i, y_i), i = 1, 2, 3, 4, 5$ be a set of five distinct points with integer coordinates in the xy-plane. Prove that the midpoint of the line joining at least one pair of these points has integer coordinates.
(b) Give an example of a set of four points with integer coordinates in the xy-plane such that the midpoints of the lines joining each pair of these points all do not have integer coordinates.

Solution.

(a) Let E be the set of even integers and O be the set of odd integers. Since $E \cup O = \mathbb{Z}$, any point (x, y) in the xy-plane with integer coordinates satisfies exactly one of the following four descriptions:

1. $x \in E, y \in E$
2. $x \in E, y \in O$
3. $x \in O, y \in E$
4. $x \in O, y \in O$

Since there are 5 points, by the PHP at least 2 of the points must have the same description from the list of four options.

Let the points with the same description be (x_1, y_1) and (x_2, y_2). The midpoint will then be $\left(\frac{x_1+x_2}{2}, \frac{y_1+y_2}{2}\right)$. Since the points both satisfy the same description, we know that the x-components are either both even or both odd, and this also holds for the y-components.

We have that the sum of any two odd or any two even integers is even. Therefore, the sum of the x-components and the sum of y-components are both even, so dividing them by 2 yields integers. Thus, the midpoint of (x_1, y_1) and (x_2, y_2) has integer coordinates. $\square$

(b) The set $\{(0,0), (0,1), (1,0), (1,1)\}$ has points with all integer coordinates. The set of the midpoints of the lines connecting the points is $\left\{\left(\frac{1}{2}, 0\right), \left(1, \frac{1}{2}\right), \left(\frac{1}{2}, 1\right), \left(0, \frac{1}{2}\right), \left(\frac{1}{2}, \frac{1}{2}\right)\right\}$, and none of these midpoints have exclusively integer coordinates.

Problem 7 Let $n, m \in \mathbb{N}$. Prove that any sequence of at least $nm + 1$ distinct real numbers must have either an increasing subsequence of length $n + 1$ or a decreasing subsequence of length $m + 1$.

Solution.

Let $m, n \in \mathbb{N}$. Define a sequence S of distinct real numbers with length $l = mn + 1$ where $S = s_1, s_2, ...s_l$. Assume S contains neither an increasing subsequence of length $n + 1$ nor a decreasing subsequence of length $m+1$. Then, the longest increasing subsequence is of length at most n, and the longest decreasing subsequence is of length at most m.

For each a_j in the sequence, we assign a tuple (i_j, d_j) where i_j is the length of the longest increasing subsequence ending in s_j, and s_j is the length of the longest decreasing subsequence ending in s_j.

For all j by our assumption, $i_j \leq n$ and $d_j \leq m$. Since i_j and d_j must be at least 1 for all j (any number is on its own both an increasing and decreasing sequence of length 1), there are at most mn possible tuples by the product rule.

Then, by the PHP, there exist some s_j, s_k in S such that $j \neq k$ and $(i_j, d_j) = (i_k, d_k)$ because there is at least one more number in S than there are possible tuples. WLOG, assume $j < k$.

Case 1: Assume $s_j > s_k$, and let S_1 be the longest decreasing subsequence ending with s_j. By definition, $d_j = |S_1|$. Since s_k follows s_j in S and is less than s_j, by definition $d_k = |S_1| + 1$. But we assumed $d_j = d_k$, so $s_j \not> s_k$ because it leads to a contradiction.

Case 2: Assume $s_j < s_k$, and let S_2 be the longest increasing subsequence ending with s_j. By definition, $i_j = |S_2|$. Since s_k follow s_j in S and is greater than s_j, by definition $i_k = |S_1| + 1$. But we assumed $i_j = i_k$, so $s_j \not< s_k$ because it leads to a contradiction.

The only remaining option is that $s_j = s_k$, but we assumed that every number in S is distinct. Thus, our assumption led to a contradiction, and it cannot be true that the longest increasing subsequence is of length at most n and the longest decreasing subsequence is of length at most m. Therefore, S must have either an increasing subsequence of length $n + 1$ or a decreasing subsequence of length $m + 1$. $\square$

Problem 8 Let $n \in \mathbb{N}$. Use Pigeonhole Principle to prove that any set of n consecutive integers contains exactly one integer that is divisible by n.

Solution.

Let S be a set of n consecutive integers. Assume S does not contain exactly one integer that is divisible by n. Then, it either has no integers or more than 1 integer divisible by n.

Case 1: Assume S has more than 1 integer divisible by n. Then, there exist $j, k \in \mathbb{Z}$ such that $j \neq k$ and $jn, kn \in S$. Since S contains all consecutive integers, the $n - 1$ integers between jn and kn are also in S, meaning that $|S| = n - 1 + 2 = n + 1$. But we assumed $|S| = n$, a contradiction.

Case 2: Assume S has no integers divisible by n. Let $s \in S$. Since s cannot have a remainder of 0 when dividing by n or it would be divisible by n, it must have a remainder in $\{1, 2, ...n - 1\}$. Since there are $n - 1$ options, and n integers in S, by the PHP there must be a $t \in S$ such that $s \neq t$ and s and t have the same remainder r.

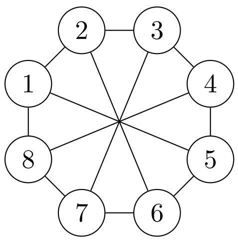
We now use similar reasoning as in case 1. Since s and t both have the remainder r when divided by n, there exist $j, k \in \mathbb{Z}$ such that $j \neq k$ and $jn + r, kn + r \in S$. Since S contains all consecutive integers, the $n - 1$ integers between $jn + r$ and $kn + r$ are also in S, meaning that $|S| = n - 1 + 2 = n + 1$. But we assumed

$|S| = n$, a contradiction.

Therefore, it is not possible for S to have fewer than 1 or more than 1 integer that is a multiple of n, so S must have exactly 1 multiple of n. □

Problem 9 Give a construction of a graph on 8 vertices that contains no K_3 and no $\overline{K_4}$. You do not need to provide a proof, just a picture.

Solution.



Problem 10 Prove that for all integers $s, t \geq 3, R(s,t) \leq R(s-1,t) + R(s,t-1)$.

Solution.

We have for all $n \in \mathbb{Z}$, $n \geq 2$ that $R(n,2) = R(2,n) = n$.

Let $s, t \in \mathbb{Z}$ such that $s \geq 3$ and $t \geq 3$. Assume that $R(s-1,t)$ and $R(s,t-1)$ exist. Then, let G be a graph with $n = R(s-1,t) + R(s,t-1)$. Choose $v \in V(G)$, and let $X = N(v)$ and $Y = \{v' \in V(G) : v' \neq v, v' \notin N(v)\}$. Since there are n vertices, and v is one of them, $|X| + |Y| = n - 1$. By the PHP, we must have $|X| \geq R(s-1,t)$ or $|Y| \geq R(s,t-1)$.

Case 1: Assume $|X| \geq R(s-1,t)$. By definition of a Ramsey number, then, X contains either K_{s-1} or $\overline{K_t}$. First, if X contains a $\overline{K_t}$, then so does G because $X \subseteq G$. Otherwise, because of how we defined X, v is connected to every vertex in X. Therefore, if X contains a K_{s-1}, then G must contain a K_s because there is another vertex (v) in G connected to everything in the K_{s-1}.

Case 2: Assume $|Y| \geq R(s,t-1)$. By definition of a Ramsey number, Y contains either K_s or $\overline{K_{t-1}}$. First, if Y contains a K_s, then so does G because $Y \subseteq G$. Otherwise, because of how we defined Y, v is connected to no vertex in Y. Therefore, if Y contains a $\overline{K_{t-1}}$, then G must contain a $\overline{K_t}$ because there is another vertex (v) in G connected to nothing in the $\overline{K_{t-1}}$.

Thus, if $R(s-1,t)$ and $R(s,t-1)$ exist, a graph with $R(s-1,t) + R(s,t-1)$ vertices is guaranteed to have a K_s or a $\overline{K_t}$. Since $R(s-1,t)$ and $R(s,t-1)$ exist for $s = t = 3$ as shown, they exist for all $s, t \in \mathbb{Z}$ when $s \geq 3$ and $t \geq 3$ and are bounded by $R(s,t) \leq R(s-1,t) + R(s,t-1)$. □