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MATH 544: Section H01
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August 28, 2023

MATH 544 Homework 1

Problem 1.1 Let a and r be real numbers with $r \neq 1$ and $n \geq 2$ an integer. Prove that the sum of the geometric series

$$S = a + ar + ar^2 + \cdots + ar^n$$

is

$$S = \frac{a - ar^{n+1}}{1-r}.$$

We can write

$$rS = r\sum_{k=0}^{n} ar^k = \sum_{k=0}^{n} ar^{k+1} = \sum_{k=1}^{n+1} ar^k.$$

Then, we observe that

$$\begin{aligned}
S - rs &= \sum_{k=0}^{n}(ar^k) - \sum_{k=1}^{n+1}(ar^k) && \text{(from above)} \\
&= \left(a + \sum_{k=1}^{n} ar^k\right) - \left(\sum_{k=1}^{n} ar^k + ar^{n+1}\right) && \text{(splitting sums)} \\
&= a - ar^{n+1} && \text{(distributing/cancelling)} \\
\implies S(1-r) &= a - ar^{n+1} \\
\implies S &= \frac{a - ar^{n+1}}{1-r}. && \text{(given } r \neq 1\text{)}
\end{aligned}$$

□

Problem 1.2 What happens if $r = 1$?

If $r = 1$, then $r^k = 1$ for any $k \in \mathbb{N}$. Thus, we have

$$S = \sum_{k=0}^{n} ar^k = \sum_{k=0}^{n} a = a(n+1).$$

Problem 1.3

(a) Find the sum of $\frac{1}{2} + \frac{1}{4} + \frac{1}{8} + \cdots + \frac{1}{2^n}$.

(b) Find the sum of $P_0(1+r) + P_0(1+r)^2 + \cdots + P_0(1+r)^n$.

(a) We can write

$$\frac{1}{2} + \frac{1}{4} + \frac{1}{8} + \cdots + \frac{1}{2^n} = \sum_{k=1}^{n}\left(\frac{1}{2}\right)^n$$

$$
\begin{aligned}
&= \sum_{k=0}^{n} 1 \left(\frac{1}{2}\right)^{n} - 1 \\
&= \left(\frac{1-(1/2)^{n+1}}{1-(1/2)}\right) - 1 && \text{(sum formula)} \\
&= 2 - 2\left(\frac{1}{2}\right)^{n+1} - 1 \\
&= 1 - \left(\frac{1}{2}\right)^{n} \\
&= 1 - \frac{1}{2^n} = \frac{2^n - 1}{2^n}.
\end{aligned}
$$

(b) Similarly, we have

$$
\begin{aligned}
P_0(1+r) + P_0(1+r)^2 + \cdots + P_0(1+r)^n &= \sum_{k=1}^{n} P_0(1+r)^k \\
&= \sum_{k=0}^{n} P_0(1+r)^k - P_0 \\
&= \frac{P_0 - P_0(1+r)^{n+1}}{1-(1+r)} - P_0 && \text{(sum formula)} \\
&= \frac{P_0(1+r)^{n+1} - P_0}{r} - P_0 \\
&= \frac{P_0(1+r)^{n+1} - P_0(1+r)}{r}.
\end{aligned}
$$

Problem 1.4 Prove that

$$x^n - y^n = (x-y)\left(x^{n-1} + x^{n-2}y + x^{n-3}y^2 + \cdots + xy^{n-2} + y^{n-1}\right)$$

by multiplying out $(x-y)(x^{n-1} + x^{n-2}y + x^{n-3}y^2 + \cdots + xy^{n-2} + y^{n-1})$ and seeing that all but two terms cancel.

Using summation notation, we can write

$$
\begin{aligned}
(x-y)\sum_{k=0}^{n-1} x^{n-1-k}y^k &= x\sum_{k=0}^{n-1} x^{n-1-k}y^k - y\sum_{k=0}^{n-1} x^{n-1-k}y^k \\
&= \sum_{k=0}^{n-1} x^{n-k}y^k - \sum_{k=0}^{n-1} x^{n-1-k}y^{k+1} \\
&= \sum_{k=0}^{n-1} x^{n-k}y^k - \sum_{k=1}^{n} x^{n-k}y^k && \text{(reindexing)} \\
&= \left(x^n + \sum_{k=1}^{n-1} x^{n-k}y^k\right) - \left(\sum_{k=1}^{n-1} x^{n-k}y^k + y^k\right) && \text{(splitting sums)} \\
&= x^n - y^n. && \text{(distributing/cancelling)}
\end{aligned}
$$

□

Problem 1.5 Prove the same statement using a geometric sum.

We can write

$$\begin{aligned}
\sum_{k=0}^{n-1} x^{n-1-k}y^k &= \sum_{k=0}^{n-1} x^{n-1}\left(\frac{y}{x}\right)^k \\
&= \frac{x^{n-1} - x^{n-1}\left(\frac{y}{x}\right)^n}{1-\frac{y}{x}} && \text{(using sum formula)} \\
&= \frac{x^{n-1} - \frac{y^n}{x}}{1-\frac{y}{x}} \\
&= \frac{x^n - y^n}{x-y} && \text{(multiplying by } x/x\text{)} \\
\implies x^n - y^n &= (x-y)\sum_{k=0}^{n-1} x^{n-1-k}y^k.
\end{aligned}$$

□

Problem 1.6 Let $f(x) = c_3x^3 + c_2x^2 + c_1x + c_0$ where c_0,c_1,c_2, and c_3 are constants. Simplify

$$\frac{f(x)-f(a)}{x-a}$$

by showing that $(x-a)$ can be canceled out of the denominator and use this to compute $\lim_{x\to a} \frac{f(x)-f(a)}{x-a}$.

For all $x \neq a$, we have

$$\begin{aligned}
\frac{f(x)-f(a)}{x-a} &= \frac{(c_3x^3 + c_2x^2 + c_1x + c_0) - (c_3a^3 + c_2a^2 + c_1a + c_0)}{x-a} \\
&= \frac{c_3(x^3-a^3)}{x-a} + \frac{c_2(x^2-a^2)}{x-a} + \frac{c_1(x-a)}{x-a} + \frac{c_0-c_0}{x-a} \\
&= \frac{c_3(x^2+xa+a^2)(x-a)}{x-a} + \frac{c_2(x+a)(x-a)}{x-a} + \frac{c_1(x-a)}{x-a} && \text{(factoring)} \\
&= c_3(x^2+xa+a^2) + c_2(x+a) + c_1.
\end{aligned}$$

Evaluating this formula at $x = a$, we find

$$\lim_{x\to a}\frac{f(x)-f(a)}{x-a} = 3c_3a^2 + 2c_2a + c_1,$$

which is the value we expect from calculus for the derivative at $x = a$.

Problem 1.7 Use summation notation to derive a formula for the sum of the series

$$S = \sum_{k=0}^{n-1}(a+kd).$$

We can write

$$\begin{aligned}
2S &= \sum_{k=0}^{n-1}(a+kd) + \sum_{k=0}^{n-1}(a+(n-1-k)d) \\
&= \sum_{k=0}^{n-1}(2a+(n-1)d) \\
&= n(2a+(n-1)d)
\end{aligned}$$

$$\implies S = n\left(\frac{2a+(n-1)d}{2}\right).$$

Problem 1.9 Show that the definition $\binom{n}{k} := \frac{n!}{k!(n-k)!}$ implies $\binom{n}{k} = \binom{n}{n-k}$.

Since $k = n - (n-k)$, it follows that

$$\binom{n}{k} = \frac{n!}{k!(n-k)!} = \frac{n!}{(n-k)!k!} = \frac{n!}{(n-k)!(n-(n-k))!} = \binom{n}{n-k}.$$

Problem 1.10 Prove that the following hold:

$$\begin{aligned}
\binom{n}{0} &= \binom{n}{n} = 1,\\
\binom{n}{1} &= \binom{n}{n-1} = n,\\
\binom{n}{2} &= \binom{n}{n-2} = \frac{n(n-1)}{2},\\
\binom{n}{3} &= \binom{n}{n-3} = \frac{n(n-1)(n-2)}{6}.
\end{aligned}$$

Using the recursive definition of a factorial, we have

$$\begin{aligned}
\binom{n}{0} &= \frac{n!}{(n-0)!0!} = \frac{n!}{n!} = 1,\\
\binom{n}{1} &= \frac{n(n-1)!}{1!(n-1)!} = \frac{n}{1!} = n,\\
\binom{n}{2} &= \frac{n(n-1)(n-2)!}{2!(n-2)!} = \frac{n(n-1)}{2},\\
\binom{n}{3} &= \frac{n(n-1)(n-2)(n-3)!}{3!(n-3)!} = \frac{n(n-1)(n-2)}{6}.
\end{aligned}$$

We showed that $\binom{n}{0} = \binom{n}{n}$ and etc. in problem 9. $\square$

Problem 1.11 Prove that

$$\binom{n}{k} = \frac{n(n-1)\dots(n-k+1)}{k!} = \frac{n^{\underline{k}}}{k!}.$$

From the recursive definition of a factorial, we use cancellation to write

$$\binom{n}{k} = \frac{n(n-1)\dots(n-k+1)(n-k)!}{k!(n-k)!} = \frac{n(n-1)\dots(n-k+1)}{k!} = \frac{n^{\underline{k}}}{k!}.$$

Problem 1.12 Prove that for $1 \le k \le n$ with k, n integers,

$$\binom{n}{k-1} + \binom{n}{k} = \binom{n+1}{k}.$$

Again using recursion, we can write

$$\binom{n}{k-1} + \binom{n}{k} = \frac{n!}{(k-1)!(n-k+1)!} + \frac{n!}{k!(n-k)!}$$

$$
\begin{aligned}
&= \frac{n!}{(k-1)!(n-k+1)(n-k)!} + \frac{n!}{k(k-1)!(n-k)!} \\
&= \frac{kn! + (n-k+1)n!}{k(k-1)!(n-k+1)(n-k)!} \\
&= \frac{kn! + nn! - kn! + n!}{k!(n-k+1)!} \\
&= \frac{(n+1)n!}{k!(n+1-k)!} \\
&= \frac{(n+1)!}{k!(n+1-k)!} \\
&= \binom{n+1}{k}.
\end{aligned}
$$

Problem 1.13 Let k, n be nonnegative integers with $0 \leq k \leq n$. Prove the binomial coefficient $\binom{n}{k}$ is an integer.

We will induct on n.

Base Case: For $n = 0$, we have shown that $\binom{0}{0} = 1$, which is an integer.

Induction Step: Let $n \in \mathbb{Z}$, $n > 0$, and assume that $\binom{n-1}{k}$ is an integer for all $k \in \mathbb{Z}$, $0 \leq k \leq n-1$. Let $k \in \mathbb{Z}$, $0 \leq k \leq n$. If $k \in \{0, n\}$, we have shown that $\binom{n}{0} = \binom{n}{n} = 1$, which is an integer. Otherwise, by the induction hypothesis $\binom{n-1}{k}$ and $\binom{n-1}{k-1}$ are integers. By Pascal's identity, we have $\binom{n}{k} = \binom{n-1}{k} + \binom{n-1}{k-1}$, and thus since $\binom{n}{k}$ is the sum of integers, it is an integer. $\square$

Problem 1.17 Use induction and Pascal's Identity to prove the Binomial Theorem: for $n \in \mathbb{N}$ and $x, y \in \mathbb{R}$,

$$(x+y)^n = \sum_{k=0}^{n} \binom{n}{k} x^{n-k} y^k.$$

We will induct on n.

Base Case: For $n = 0$, we have

$$(x+y)^0 = 1 = \binom{0}{0} x^0 y^0 = \sum_{k=0}^{0} \binom{0}{k} x^{0-k} y^k.$$

Induction Step: Let $n \in \mathbb{N}$, $n > 0$, and assume

$$(x+y)^{n-1} = \sum_{k=0}^{n-1} x^{n-1-k} y^k.$$

Then, we can write

$$
\begin{aligned}
(x+y)^n &= (x+y)(x+y)^{n-1} && (n > 0) \\
&= (x+y) \sum_{k=0}^{n-1} \binom{n-1}{k} x^{n-1-k} y^k && \text{(induction hypothesis)} \\
&= \sum_{k=0}^{n-1} \binom{n-1}{k} x^{n-k} y^k + \sum_{k=0}^{n-1} \binom{n-1}{k} x^{n-1-k} y^{k+1} && \text{(distributing)} \\
&= \sum_{k=0}^{n-1} \binom{n-1}{k} x^{n-k} y^k + \sum_{k=1}^{n} \binom{n-1}{k-1} x^{n-1-(k-1)} y^{(k-1)+1} && \text{(reindexing)}
\end{aligned}
$$

$$= \binom{n-1}{0} x^n y^0 + \sum_{k=1}^{n-1} \binom{n-1}{k} x^{n-k} y^k + \sum_{k=1}^{n-1} \binom{n-1}{k-1} x^{n-k} y^k + \binom{n-1}{n-1} x^0 y^n \qquad \text{(splitting sums)}$$

$$= x^n + \sum_{k=1}^{n-1} \left[\binom{n-1}{k} + \binom{n-1}{k-1} \right] x^{n-k} y^k + y^n \qquad \text{(combining sums)}$$

$$= \binom{n}{0} x^n y^0 + \sum_{k=1}^{n-1} \binom{n}{k} x^{n-k} y^k + \binom{n}{n} x^0 y^n \qquad \text{(Pascal's identity)}$$

$$= \sum_{k=0}^{n} \binom{n}{k} x^{n-k} y^k. \qquad \text{(combining sum)}$$

$\square$

Problem 1 Show that

$$x^3 = x^{\underline{3}} + 3x^{\underline{2}} + x^{\underline{1}}$$

and use this to find a formula for

$$\sum_{k=1}^{n} k^3.$$

We can write

$$\begin{aligned}
x^{\underline{3}} + 3x^{\underline{2}} + x^{\underline{1}} &= x(x-1)(x-2) + 3x(x-1) + x \\
&= x[(x-1)(x-2) + 3(x-1) + 1] \\
&= x(x^2 - 3x + 2 + 3x - 3 + 1) \\
&= x(x^2) && \text{(cancelling)} \\
&= x^3.
\end{aligned}$$

Using this, we can compute

$$\begin{aligned}
\sum_{k=1}^{n} k^3 &= \sum_{k=1}^{n} k^{\underline{3}} + 3k^{\underline{2}} + k^{\underline{1}} \\
&= \left[\frac{k^{\underline{4}}}{4} + k^{\underline{3}} + \frac{k^{\underline{2}}}{2} \right]_1^{n+1} && \text{(fundamental theorem of sums)} \\
&= \frac{(n+1)(n)(n-1)(n-2)}{4} + (n+1)(n)(n-1) + \frac{(n+1)n}{2} && \text{(all lower bounds are 0)} \\
&= n(n+1) \left[\frac{(n-1)(n-2)}{4} + n - 1 + \frac{1}{2} \right] && \text{(factoring)} \\
&= n(n+1) \left(\frac{n^2 - 3n + 2 + 4n - 4 + 2}{4} \right) && \text{(combining fractions)} \\
&= n(n+1) \left(\frac{n^2 + n}{4} \right) && \text{(cancelling)} \\
&= \frac{n^2(n+1)^2}{4}.
\end{aligned}$$