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MATH 701 Homework 2

Problem 1.4.1 Prove that $|GL_2(\mathbb{F}_2)| = 6$.

We can easily enumerate the 2×2 matrices over $\{0, 1\}$ (there are $2^{2 \cdot 2} = 16$ of them), and 6 of them have non-zero determinant. In particular, we have

$$GL_2(\mathbb{F}_2) = \left\{ \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix}, \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix}, \begin{pmatrix} 0 & 1 \\ 1 & 1 \end{pmatrix}, \begin{pmatrix} 1 & 0 \\ 1 & 1 \end{pmatrix}, \begin{pmatrix} 1 & 1 \\ 0 & 1 \end{pmatrix}, \begin{pmatrix} 1 & 1 \\ 1 & 0 \end{pmatrix} \right\}.$$

The result also follows as a corollary from Problem 1.4.7.

Problem 1.4.2 Write out all the elements of $GL_2(\mathbb{F}_2)$ and compute the order of each element.

We have:

- $\operatorname{ord} \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix} = 1$,
- $\operatorname{ord} \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix} = 2$,
- $\operatorname{ord} \begin{pmatrix} 0 & 1 \\ 1 & 1 \end{pmatrix} = \infty$,
- $\operatorname{ord} \begin{pmatrix} 1 & 0 \\ 1 & 1 \end{pmatrix} = \infty$,
- $\operatorname{ord} \begin{pmatrix} 1 & 1 \\ 0 & 1 \end{pmatrix} = \infty$,
- $\operatorname{ord} \begin{pmatrix} 1 & 1 \\ 1 & 0 \end{pmatrix} = \infty$.

The first two can be shown easily and directly, and the others can be shown by induction.

Problem 1.4.3 Show that $GL_2(\mathbb{F}_2)$ is non-abelian.

We have

$$\begin{pmatrix} 1 & 0 \\ 1 & 1 \end{pmatrix}, \begin{pmatrix} 1 & 1 \\ 1 & 0 \end{pmatrix} \in GL_2(\mathbb{F}_2),$$

but remembering that $2 \equiv 0 \pmod 2$, we have

$$\begin{pmatrix} 1 & 0 \\ 1 & 1 \end{pmatrix} \begin{pmatrix} 1 & 1 \\ 1 & 0 \end{pmatrix} = \begin{pmatrix} 1 & 1 \\ 0 & 1 \end{pmatrix} \neq \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix} = \begin{pmatrix} 1 & 1 \\ 1 & 0 \end{pmatrix} \begin{pmatrix} 1 & 0 \\ 1 & 1 \end{pmatrix},$$

so $GL_2(\mathbb{F}_2)$ is non-abelian.

Problem 1.4.5 Show that $GL_n(F)$ is a finite group if and only if F has a finite number of elements.

($\Rightarrow$) For all $c \in \mathbb{F}$, we have $\det(cI_n) = c$. Thus, $\{cI_n \mid c \in F - \{0\}\} \subseteq GL_n(F)$, so we have $GL_n(F) \geq |F| - 1$. Thus, if $|F| = \infty$, we also have $|GL_n(F)| = \infty$.

($\Leftarrow$) By Problem 1.4.6, $|GL_n(F)| < |F|^{n^2}$, which is finite if $|F| < \infty$.

Problem 1.4.6 If $|F| = q$ is finite prove that $|GL_n(F)| < q^{n^2}$.

In an $n \times n$ matrix, there are n^2 entries and q choices for each entry, so there are q^{n^2} matrices over F. Not all of them will have determinant 0 (the matrix of all 0s, for example), so $GL_2(F)$ is a proper subset of these matrices. Therefore, $|GL_2(F)| < q^{n^2}$. $\square$

Problem 1.4.7 Let p be a prime. Prove that the order of $GL_2(\mathbb{F}_p)$ is $p^4 - p^3 - p^2 + p$.

A matrix is in $GL_2(\mathbb{F}_p)$ if and only if it is invertible. It is known that a matrix is not invertible if and only if one row is a multiple of the other. For all p^2 matrices where the first row is both zeroes, the first row is always a scalar multiple of the other row. For the other matrices, there are $p^2 - 1$ choices for the first row, and then p choices per row (since p is prime) to make the second row a multiple of the first row. Thus, there are $p^2 + (p^2 - 1)p$ non-invertible matrices. Since there are p^4 total matrices, we have

$$|GL_2(\mathbb{F}_p)| = p^4 - (p^2 + (p^2 - 1)p) = p^4 - (p^2 - p^3 - p) = p^4 - p^3 - p^2 + p.$$

Problem 1.4.8 Show that $GL_n(F)$ is non-abelian for any $n \geq 2$ and any F.

We will expand the matrices from Problem 1.4.3. Let F be a field, 0 be the additive identity of F, 1 be the multiplicative identity of F, and $2 := 1 + 1$. Let

$$A = \left(\begin{array}{cc|c} 1 & 0 & \multirow{2}{*}{$O_{2\times(n-2)}$} \\ 1 & 1 & \\ \hline \multicolumn{2}{c|}{O_{(n-2)\times 2}} & I_{n-2} \end{array}\right), \quad B = \left(\begin{array}{cc|c} 1 & 1 & \multirow{2}{*}{$O_{2\times(n-2)}$} \\ 1 & 0 & \\ \hline \multicolumn{2}{c|}{O_{(n-2)\times 2}} & I_{n-2} \end{array}\right).$$

Then we have $A, B \in GL_n(F)$, and

$$AB = \left(\begin{array}{cc|c} 1 & 1 & \multirow{2}{*}{$O_{2\times(n-2)}$} \\ 2 & 1 & \\ \hline \multicolumn{2}{c|}{O_{(n-2)\times 2}} & I_{n-2} \end{array}\right) \neq \left(\begin{array}{cc|c} 2 & 1 & \multirow{2}{*}{$O_{2\times(n-2)}$} \\ 1 & 0 & \\ \hline \multicolumn{2}{c|}{O_{(n-2)\times 2}} & I_{n-2} \end{array}\right) = BA.$$

Therefore, $GL_n(F)$ is not abelian.

Problem 1.4.10 Let $G = \left\{ \begin{pmatrix} a & b \\ 0 & c \end{pmatrix} \middle| \, a, b, c \in \mathbb{R}, a \neq 0, c \neq 0 \right\}$.

(a) Compute the product of $\begin{pmatrix} a_1 & b_1 \\ 0 & c_1 \end{pmatrix}$ and $\begin{pmatrix} a_2 & b_2 \\ 0 & c_2 \end{pmatrix}$ to show that G is closed under matrix multiplication.

(b) Find the matrix inverse of $\begin{pmatrix} a & b \\ 0 & c \end{pmatrix}$ and deduce that G is closed under inverses.

(c) Deduce that G is a subgroup of $GL_2(\mathbb{R})$.

(d) Prove that the set of elements of G whose two diagonal entries are equal (i.e., $a = c$) is also a subgroup of $GL_2(\mathbb{R})$.

(a) We have

$$\begin{pmatrix} a_1 & b_1 \\ 0 & c_1 \end{pmatrix}\begin{pmatrix} a_2 & b_2 \\ 0 & c_2 \end{pmatrix} = \begin{pmatrix} a_1a_2 & a_1a_2 + b_1c_2 \\ 0 & c_1c_2 \end{pmatrix}.$$

Since none of a_1, a_2, c_1, c_2 are 0, neither are a_1a_2 nor c_1c_2. Thus, the matrix product is in G.

(b) We obtain (from the closed form of the inverse of a 2×2 matrix with non-zero determinant) that the inverse is

$$\begin{pmatrix} \frac{1}{a} & -\frac{b}{ac} \\ 0 & \frac{1}{c} \end{pmatrix}.$$

Since a and c are non-zero, so are $\frac{1}{a}$ and $\frac{1}{c}$, so the inverse is in G.

(c) Since $I_2 \in G$, we have that G has the identity, and matrix multiplication is known to be associative. We also have $G \subseteq GL_2(\mathbb{R})$ as the determinant is $ac - 0 = ac$, which is non-zero since a and c are non-zero. Therefore, $G \leq GL_2(\mathbb{R})$.

(d) Let

$$\begin{pmatrix} a_1 & b_1 \\ 0 & a_1 \end{pmatrix}, \begin{pmatrix} a_2 & b_2 \\ 0 & a_2 \end{pmatrix} \in G.$$

Then, we have

$$\begin{pmatrix} a_1 & b_1 \\ 0 & a_1 \end{pmatrix}\begin{pmatrix} a_2 & b_2 \\ 0 & a_2 \end{pmatrix}^{-1} = \begin{pmatrix} a_1 & b_1 \\ 0 & a_1 \end{pmatrix}\begin{pmatrix} \frac{1}{a_2} & -\frac{b_2}{a_2^2} \\ 0 & \frac{1}{a_2} \end{pmatrix} = \begin{pmatrix} \frac{a_1}{a_2} & \frac{b_1a_2 - a_1b_2}{a_2^2} \\ 0 & \frac{a_1}{a_2} \end{pmatrix} \in H.$$

Since I_2 is an element of G whose two diagonal entries are equal, the set is non-empty. Therefore, it is a subgroup of G, and by transitivity a subgroup of $GL_2(\mathbb{R})$.

Problem 1.5.1 Compute the order of each of the elements in Q_8.

We have:

- $\text{ord}(1) = 1$
- $\text{ord}(-1) = 2$
- $\text{ord}(i) = 4$
- $\text{ord}(-i) = 4$
- $\text{ord}(j) = 4$
- $\text{ord}(-j) = 4$
- $\text{ord}(k) = 4$
- $\text{ord}(-k) = 4$.

Each of these can be shown directly.