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SCHC 501 Homework 2

Problem 1 Let $A = \{b, c\}$ and $B = \{2, 3\}$.

(a) Specify the following sets by listing their members.

(b) Classify each statement as true or false.

(i) $(A \times B) \cup (B \times A) = \emptyset$
(ii) $(A \times A) \subseteq (A \times B)$
(iii) $\langle c, c\rangle \subseteq (A \times A)$
(iv) $\{\langle b, 3\rangle, \langle 3, b\rangle\} \subseteq (A \times B) \cup (B \times A)$
(v) $\emptyset \subseteq A \times A$
(vi) $\{\langle b, 2\rangle, \langle c, 3\rangle\}$ is a relation from A to B
(vii) $\{\langle b, b\rangle\}$ is a relation in A.

(c) Consider the following relation from A to $(A \cup B)$:

$$R = \{\langle b, b\rangle, \langle b, 2\rangle, \langle c, 2\rangle, \langle c, 3\rangle\}$$

(i) Specify the domain and range of R.
(ii) Specify the complementary relation R' and the inverse R^{-1}.
(iii) Is $(R')^{-1}$ (the inverse of the complement) equal to $(R^{-1})'$ (the complement of the inverse)?

(a) (i) $A \times B = \{\langle b, 2\rangle, \langle b, 3\rangle, \langle c, 2\rangle, \langle c, 3\rangle\}$
(ii) $B \times A = \{\langle 2, b\rangle, \langle 3, b\rangle, \langle 2, c\rangle, \langle 3, c\rangle\}$
(iii) $A \times A = \{\langle b, b\rangle, \langle b, c\rangle, \langle c, b\rangle, \langle c, c\rangle\}$
(iv) $(A \cup B) \times B = \{\langle b, 2\rangle, \langle c, 2\rangle, \langle 2, 2\rangle, \langle 3, 2\rangle, \langle b, 3\rangle, \langle c, 3\rangle, \langle 2, 3\rangle, \langle 3, 3\rangle\}$
(v) $(A \cap B) \times B = \emptyset$
(vi) $(A - B) \times (B - A) = A \times B = \{\langle b, 2\rangle, \langle b, 3\rangle, \langle c, 2\rangle, \langle c, 3\rangle\}$

(b) (i) **False:** Neither $(A \times B)$ nor $(B \times A)$ is empty, so the union will certainly not be.
(ii) **False:** In particular, we have that $\langle b, b\rangle \in A \times A$ and $\langle b, b\rangle \notin A \times B$.
(iii) **False:** $\langle c, c\rangle$ is not a set, so it cannot be a subset of anything (technically, it is encoded as the set containing c and $\{c\}$, but neither of these elements are in $A \times A$, so the statement is false either way).
(iv) **True:** We have $\langle b, 3\rangle \in A \times B$ and $\langle 3, b\rangle \in B \times A$, so the set containing these elements is the subset of $(A \times B) \cup (B \times A)$.
(v) **True:** Every set has $\emptyset$ as a subset, including $A \times A$.

(vi) **True:** We have that $\{\langle b, 2\rangle, \langle c, 3\rangle\}$ is in $\mathcal{P}(A \times B)$.

(vii) **True:** We have that $\{\langle b, b\rangle\}$ is in $\mathcal{P}(A \times A)$.

(c) (i) The domain of R is $\{b, c\}$, and the range of R is $\{b, 2, 3\}$.

(ii) We have

$$R' = \{\langle b, c\rangle, \langle b, 3\rangle, \langle c, b\rangle, \langle c, c\rangle\}$$

and

$$R^{-1} = \{\langle b, b\rangle, \langle 2, b\rangle, \langle 2, c\rangle, \langle 3, c\rangle\}.$$

(iii) Yes. We prove that $(R')^{-1} = (R^{-1})'$ holds in general by showing that an element is in one set if and only if it is in the other. Let R be a relation from A to B. Then, we have

$$\begin{aligned}
\langle b, a\rangle \in (R')^{-1} &\iff \langle a, b\rangle \in R' && \text{(definition of inverse)}\\
&\iff \langle a, b\rangle \notin R && \text{(definition of complement)}\\
&\iff \langle b, a\rangle \notin R^{-1} && \text{(definition of inverse with contrapositive)}\\
&\iff \langle b, a\rangle \in (R^{-1})'. && \text{(definition of complement)}
\end{aligned}$$

Therefore, $(R')^{-1} = (R^{-1})'$. ☐

Problem 2 Let $A = \{a, b, c\}$ and $B = \{1, 2\}$. How many distinct relations are there from A to B? How many of these are functions from A to B? How many of the functions are onto? one-to-one? Answer the same questions from all relations from B to A.

We have that $|A \times B| = |A||B| = 2 \cdot 3 = 6$. The number of relations from A to B is equal to $\mathcal{P}(A \times B) = 2^{|A \times B|} = 2^6$, so there are 64 distinct relations from A to B. Since a function must assign exactly one value to each element of A, there are 2 choices in B for each of the three elements in A, so there are $2^3 = 8$ functions. The only functions that are not onto are the functions that send all elements in A to one element in B, so since there are two elements in B, there are $8 - 2 = 6$ onto functions. None of the functions are one-to-one by the pigeonhole principle, so there are 0 one-to-one functions.

The number of relations from B to A is also 64, since switching the order of the pairs doesn't change the number of relations. By the same reasoning as above, there are $3^2 = 9$ functions from B to A. None of the functions from B to A are onto since there are not enough elements in B to "cover" all elements in B. The functions that are not one-to-one are the functions that map both 1 and 2 to one of the elements in A, so since there are three elements in A, there are $9 - 3 = 6$ one-to-one functions.

Problem 3 Let

$$R_1 = \{\langle 1, 1\rangle, \langle 2, 1\rangle, \langle 3, 4\rangle, \langle 2, 2\rangle, \langle 3, 3\rangle, \langle 4, 4\rangle, \langle 4, 1\rangle\}$$

$$R_2 = \{\langle 3, 4\rangle, \langle 1, 2\rangle, \langle 1, 4\rangle, \langle 2, 3\rangle, \langle 2, 4\rangle, \langle 1, 3\rangle\}$$

(both relations in A, where $A = \{1, 2, 3, 4\}$).

(a) Form the composites $R_2 \circ R_1$ and $R_1 \circ R_2$. Are they equal?

(b) Show that $R_1^{-1} \circ R_1 \neq id_A$ and that $R_2^{-1} \circ R_2 \not\subseteq id_A$.

(a) We have

$$R_2 \circ R_1 = \{\langle 1,2\rangle, \langle 1,3\rangle, \langle 1,4\rangle, \langle 2,2\rangle, \langle 2,3\rangle, \langle 2,4\rangle, \langle 3,4\rangle, \langle 4,2\rangle, \langle 4,3\rangle, \langle 4,4\rangle\}$$

and

$$R_1 \circ R_2 = \{\langle 1,1\rangle, \langle 1,2\rangle, \langle 1,3\rangle, \langle 1,4\rangle, \langle 2,1\rangle, \langle 2,3\rangle, \langle 2,4\rangle, \langle 3,1\rangle, \langle 3,4\rangle\}.$$

They are not equal: in particular, $\langle 1,1\rangle \in R_1 \circ R_2$ but $\langle 1,1\rangle \notin R_2 \circ R_1$.

(b) Since $\langle 1,2\rangle \in R_1^{-1}$ and $\langle 1,1\rangle \in R_1$, we have that $\langle 1,2\rangle \in R_1^{-1} \circ R_1$, and since $\langle 1,2\rangle \notin id_A$, the sets are not equal. Similarly, since $\langle 4,3\rangle \in R_2^{-1}$ and $\langle 1,4\rangle \in R_2$, we have $\langle 1,3\rangle \in R_2^{-1} \circ R_2$, and since $\langle 1,3\rangle \notin id_A$, the sets are not equal.

Problem 4 For the functions F and G in Figure 2-3:

(a) Show that $(G \circ F)^{-1} = F^{-1} \circ G^{-1}$.

(b) Show that the corresponding equation holds for relations R and S in Figure 2-6.

We will prove the statement in general, and then both parts will follow as corollaries. Let A, B, C be sets, let R be a relation from A to B, and let S be a relation from B to C. We will prove that $(S \circ R)^{-1} = R^{-1} \circ S^{-1}$. Let $a \in A, c \in C$ such that $\langle c, a\rangle \in (S \circ R)^{-1}$. This is true if and only if $\langle a, c\rangle \in S \circ R$, which is true if and only if there exists some $b \in B$ such that $\langle a, b\rangle \in R$ and $\langle b, c\rangle \in S$. This is equivalent to there existing some $b \in B$ such that $\langle c, b\rangle \in S^{-1}$ and $b, a \in R^{-1}$, which is true if and only if $\langle c, a\rangle \in R^{-1} \circ S^{-1}$. Therefore, equality of the sets holds and parts **(a)** and **(b)** follow directly. $\square$