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MATH 544: Section H01
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MATH 544 Homework 2

Problem 1
(a) One possible configuration for a row-echelon matrix in $\mathrm{Mat}_{2\times 3}(\mathbb{R})$ is $\begin{pmatrix} 1 & * & * \\ 0 & 0 & 1 \end{pmatrix}$, where the $*$'s are arbitrary real numbers. Display all possible configurations of 2×3 matrices in row-echelon form. There are seven total. (This requires you to consider all possible positions that 0's, 1's, and $*$'s can take.)
(b) Repeat part (a) for row-echelon matrices in $\mathrm{Mat}_{3\times 2}(\mathbb{R})$.

Solution.

(a)

1. $\begin{pmatrix} 0 & 0 & 0 \\ 0 & 0 & 0 \end{pmatrix}$
2. $\begin{pmatrix} 0 & 0 & 1 \\ 0 & 0 & 0 \end{pmatrix}$
3. $\begin{pmatrix} 0 & 1 & * \\ 0 & 0 & 0 \end{pmatrix}$
4. $\begin{pmatrix} 0 & 1 & * \\ 0 & 0 & 1 \end{pmatrix}$
5. $\begin{pmatrix} 1 & * & * \\ 0 & 0 & 0 \end{pmatrix}$
6. $\begin{pmatrix} 1 & * & * \\ 0 & 0 & 1 \end{pmatrix}$
7. $\begin{pmatrix} 1 & * & * \\ 0 & 1 & * \end{pmatrix}$

(b)

1. $\begin{pmatrix} 0 & 0 \\ 0 & 0 \\ 0 & 0 \end{pmatrix}$
2. $\begin{pmatrix} 0 & 1 \\ 0 & 0 \\ 0 & 0 \end{pmatrix}$
3. $\begin{pmatrix} 1 & * \\ 0 & 0 \\ 0 & 0 \end{pmatrix}$
4. $\begin{pmatrix} 1 & * \\ 0 & 1 \\ 0 & 0 \end{pmatrix}$

Problem 2 Suppose that $A = \begin{pmatrix} 1 & b \\ c & d \end{pmatrix}$ and that $d - bc \neq 0$. Show that A is row-equivalent to I_2.

Solution.

We have

$$\begin{aligned} A &= \begin{pmatrix} 1 & b \\ c & d \end{pmatrix} \\ &\sim \begin{pmatrix} 1 & b \\ 0 & d - cb \end{pmatrix} && (\rho_2 \mapsto \rho_2 - c\rho_1) \end{aligned}$$

$$\sim \begin{pmatrix} 1 & b \\ 0 & 1 \end{pmatrix} \qquad (\rho_2 \mapsto \tfrac{1}{d-cb}\rho_2)$$

$$\sim \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix} = I_2. \qquad (\rho_1 \mapsto \rho_1 - b\rho_2)$$

$\square$

Problem 3 Show that for all $a, b, c \in \mathbb{R}$, the matrices $A = \begin{pmatrix} 2 & 0 & 0 \\ a & -1 & 0 \\ b & c & 3 \end{pmatrix}$ and $B = \begin{pmatrix} 1 & 1 & 2 \\ -2 & 0 & -1 \\ 1 & 3 & 5 \end{pmatrix}$ are **not** row equivalent.

Solution.

We find the reduced row-echelon form of A and B:

$$A = \begin{pmatrix} 2 & 0 & 0 \\ a & -1 & 0 \\ b & c & 3 \end{pmatrix}$$

$$\sim \begin{pmatrix} 1 & 0 & 0 \\ a & -1 & 0 \\ b & c & 3 \end{pmatrix} \qquad (\rho_1 \mapsto \tfrac{1}{2}\rho_1)$$

$$\sim \begin{pmatrix} 1 & 0 & 0 \\ 0 & -1 & 0 \\ b & c & 3 \end{pmatrix} \qquad (\rho_2 \mapsto \rho_2 - a\rho_1)$$

$$\sim \begin{pmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ b & c & 3 \end{pmatrix} \qquad (\rho_2 \mapsto -\rho_2)$$

$$\sim \begin{pmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & c & 3 \end{pmatrix} \qquad (\rho_3 \mapsto \rho_3 - b\rho_1)$$

$$\sim \begin{pmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 3 \end{pmatrix} \qquad (\rho_3 \mapsto \rho_3 - c\rho_2)$$

$$\sim \begin{pmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{pmatrix}. \qquad (\rho_3 \mapsto \tfrac{1}{3}\rho_3)$$

$$B = \begin{pmatrix} 1 & 1 & 2 \\ -2 & 0 & -1 \\ 1 & 3 & 5 \end{pmatrix}$$

$$\sim \begin{pmatrix} 1 & 1 & 2 \\ 0 & 2 & 3 \\ 1 & 3 & 5 \end{pmatrix} \qquad (\rho_2 \mapsto \rho_2 + 2\rho_1)$$

$$\sim \begin{pmatrix} 1 & 1 & 2 \\ 0 & 1 & \frac{3}{2} \\ 1 & 3 & 5 \end{pmatrix} \qquad (\rho_2 \mapsto \tfrac{1}{2}\rho_2)$$

$$\sim \begin{pmatrix} 1 & 1 & 2 \\ 0 & 1 & \frac{3}{2} \\ 0 & 2 & 3 \end{pmatrix} \qquad (\rho_3 \mapsto \rho_3 - \rho_1)$$

$$\sim \begin{pmatrix} 1 & 1 & 2 \\ 0 & 1 & \frac{3}{2} \\ 0 & 0 & 0 \end{pmatrix} \qquad (\rho_3 \mapsto \rho_3 - 2\rho_2)$$

$$\sim \begin{pmatrix} 1 & 0 & \frac{1}{2} \\ 0 & 1 & \frac{3}{2} \\ 0 & 0 & 0 \end{pmatrix}. \qquad (\rho_1 \mapsto \rho_1 - \rho_2)$$

We have shown in class that two matrices are row-equivalent if and only if the reduced row-echelon forms of both matrices are equal. Therefore, since $\operatorname{rref}(A) \neq \operatorname{rref}(B)$, A and B are not row-equivalent. $\square$