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Analysis in $\mathbb{R}^n$: Section 1
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Analysis in $\mathbb{R}^n$ Homework 4

Problem 20 Suppose $\{p_n\}$ is a Cauchy sequence in a metric space X, and some subsequence $\{p_{n_l}\}$ converges to a point $p \in X$. Prove that the full sequence $\{p_n\}$ converges to p.

Solution.

Let $\varepsilon > 0$. Because $\{p_n\}$ is Cauchy, there exists some $N_1 \in \mathbb{N}$ such that for all $m, n \geq N_1$, $d(p_m, p_n) < \frac{\varepsilon}{2}$. Also, because $\{p_{nl}\}$ converges to $p \in X$, there exists some $N_2 \in \mathbb{N}$ such that for all $n \geq N_2$, $d(p_n, p) < \frac{\varepsilon}{2}$ if p_n is in the convergent subsequence.

Let $N = \max\{N_1, N_2\}$, $n \in \mathbb{N}$ such that $n \geq N$, and p' be the first point in the convergent subsequence after p_N. Then, we have $d(p_n, p') < \frac{\varepsilon}{2}$ because $N \geq N_1$ and $d(p', p) < \frac{\varepsilon}{2}$ because $N \geq N_2$. So by the triangle inequality, we have

$$d(p_n, p) \leq d(p_n, p') + d(p', p) < \frac{\varepsilon}{2} + \frac{\varepsilon}{2} = \varepsilon.$$

Therefore, all of $\{p_n\}$ converges to p. $\square$

Problem 23 Suppose $\{p_n\}$ and $\{q_n\}$ are Cauchy sequences in a metric space X. Show that the sequence $\{d(p_n, q_n)\}$ converges. Hint: For any m, n,

$$d(p_n, q_n) \leq d(p_n, p_m) + d(p_m, q_m) + d(q_m, q_n);$$

it follows that

$$|d(p_n, q_n) - d(p_m, q_m)|$$

is small if m and n are large.

Solution.

We have defined d as a function from $X \times X$ to $[0, \infty)$, and since $[0, \infty)$ is complete, it suffices to show that $\{d(p_n, q_n)\}$ is Cauchy. Let $\varepsilon > 0$. Because $\{p_n\}$ and $\{q_n\}$ are Cauchy, there exist $N_p, N_q \in \mathbb{N}$ such that for all $m, n \geq N_p$, $d(p_n, p_m) < \frac{\varepsilon}{2}$, and for all $m, n \geq N_q$, $d(q_n, q_m) < \frac{\varepsilon}{2}$.

Let $N = \max\{N_p, N_q\}$, and let $m, n \geq N$. Without loss of generality, assume that $d(p_n, q_n) \geq d(p_m, q_m)$. Then, we can write

$$\begin{aligned}
d(p_n, q_n) &\leq d(p_n, p_m) + d(p_m, q_n) && \text{(triangle inequality)} \\
&\leq d(p_n, p_m) + d(p_m, q_m) + d(q_m, q_n) && \text{(triangle inequality)} \\
\implies d(p_n, q_n) - d(p_m, q_m) &\leq d(p_n, p_m) + d(q_m, q_n) \\
\implies |d(p_n, q_n) - d(p_m, q_m)| &\leq d(p_n, p_m) + d(q_m, q_n) && (d(p_n, q_n) \geq d(p_m, q_m)) \\
&< \frac{\varepsilon}{2} + \frac{\varepsilon}{2} \\
&= \varepsilon.
\end{aligned}$$

Thus, $\{d(p_n, q_n)\}$ is Cauchy in $[0, \infty)$, which implies it converges in $[0, \infty)$. $\square$

Problem 1 Let (X, d) be a discrete metric space. Describe all convergent sequences in X. Describe all Cauchy sequences in X. Is X complete?

Solution.

We have that for any $\varepsilon > 0$, a sequence that converges to $p \in X$ has only finitely many points outside $B_\varepsilon(p)$. Take $\varepsilon = \frac{1}{2}$. Then, since X is discrete and p is 1 unit away from every other element in X, $B_{1/2}(p) = \{p\}$. Therefore, any sequence converging to p is simply the point p itself repeated infinitely many times after some finite number of arbitrary terms.

Similarly, a sequence $\{a_n\}$ is Cauchy if and only if for all $\varepsilon > 0$, there exists some $N \in \mathbb{N}$ such that for all $m, n \geq N$, $d(a_m, a_n) < \varepsilon$. If we consider $\varepsilon = \frac{1}{2}$, then a_m and a_n can only be less than distance $\frac{1}{2}$ if $d(a_m, a_n) = 0$, which implies that $a_m = a_n$. So a Cauchy sequence is also a point p repeated infinitely many times after some finite number of arbitrary terms. Therefore, Cauchy sequences always converge in X, and therefore X is complete.

Problem 2 Let $\{x_n\}, \{y_n\}$ be two convergent sequences, and define $\{z_n\}$ by $z_{2n} = x_n$ and $z_{2n-1} = y_n$ for all $n \in \mathbb{N}$. Prove that $\{z_n\}$ is convergent if and only if $\lim\limits_{n\to\infty} x_n = \lim\limits_{n\to\infty} y_n$.

Solution.

We define $x := \lim\limits_{n\to\infty} x_n$ and $y := \lim\limits_{n\to\infty} y_n$.

($\Rightarrow$) We will prove the contrapositive. Suppose that $x \neq y$. Since this implies $d(x, y) > 0$, by definition of convergence, there exists some $N \in \mathbb{N}$ such that for all $n \geq N$, $d(x_n, x) < \frac{d(z,y)}{3}$ and $d(y_n, y) < \frac{d(x,y)}{3}$. Let $n > 2N$. Then, z_n is either in $\{x_n\}$ or in $\{y_n\}$, so we assume without loss of generality that z_n is in $\{x_n\}$ and consequently that z_{n+1} is in $\{y_n\}$. We can then write

$$
\begin{aligned}
& d(x, y) \leq d(x, z_n) + d(z_n, z_{n+1}) + d(z_{n+1}, y) && \text{(triangle inequality)} \\
\implies & d(z_n, z_{n+1}) \geq d(x, y) - d(z_n, x) - d(z_{n+1}, y) \\
& \geq d(x, y) - \frac{d(x, y)}{3} - \frac{d(x, y)}{3} && \text{(convergence definition as discussed)} \\
& \geq \frac{d(x, y)}{3}.
\end{aligned}
$$

Since this is true for arbitrarily large n, $\{z_n\}$ cannot be Cauchy because z_n and z_{n+1} will never be arbitrarily close as n grows. Therefore, $\{z_n\}$ cannot converge.

($\Leftarrow$) Suppose that $x = y$. Then, we claim that $\{z_n\}$ converges to z, where $z = x = y$. Let $\varepsilon > 0$. Since $\{x_n\}$ and $\{y_n\}$ converge, there exists some $N \in \mathbb{N}$ such that for all $n \geq N$, $d(x_n, z) < \varepsilon$ and $d(y_n, z) < \varepsilon$. Let $n > 2N$, and consider z_n. Then, $d(z_n, z) < \varepsilon$ regardless of whether z_n is in $\{x_n\}$ or $\{y_n\}$ by the condition above. Therefore, $\{z_n\}$ converges to z. $\square$

Problem 3 Prove that $(\mathbb{R}^k, d_p)$, where $1 \leq p < \infty$, is a complete metric space.
(You may use without proof that $(\mathbb{R}, |\cdot|)$ and $(\mathbb{R}^k, d_2)$ are complete - a result that will be mentioned on Monday. $(\mathbb{R}^k, d_p)$ is also complete, and the proof is similar to $p < \infty$.)

Solution.

Let $\{x_n\}$ be Cauchy in $(\mathbb{R}^k, d_p)$ and let $\varepsilon > 0$. Then, there exists $N \in \mathbb{N}$ such that for all $m, n \geq N$,

$$d_p(x_m, x_n) = \sqrt[p]{|x_{m1} - x_{n1}|^p + |x_{m2} - x_{n2}|^p + \cdots + |x_{mk} - x_{nk}|^p} < \varepsilon,$$

where x_{ni} represents the ith coordinate of x_n. We can consider each sequence of coordinates individually: let $i \in \{1, 2, \dots, k\}$, and let $\{x_{ni}\}$ represent the sequence in $\mathbb{R}$ of ith coordinates of $\{x_n\}$.

Let $m, n \geq N$. Since $d_p(x_m, x_n) < \varepsilon$, we also have $d_p(x_{mi}, x_{ni}) < \varepsilon$ (we are considering distance in only one dimension which must be less than or equal to the distance in all k dimensions). Thus, $\{x_{ni}\}$ is Cauchy in $(\mathbb{R}, d_p)$. Since d_p is equivalent to $|\cdot|$ in $\mathbb{R}$, $\{x_{ni}\}$ is Cauchy in $(\mathbb{R}, |\cdot|)$. Thus, because we have that $(\mathbb{R}, |\cdot|)$ is complete, $\{x_{ni}\}$ converges to some $s_i \in \mathbb{R}$. Because it converges, there exists some N_i such that for all $n_i \geq N_i$, $|x_{n_i} - s_i| < \frac{\varepsilon^p}{k}$.

We claim that $\{x_n\}$ converges to $s = (s_1, s_2, \dots, s_k)$. Let $N = \max\{N_1, N_2, \dots, N_k\}$, and $n \geq N$. From the result above, we have $|x_n - s_i| < \frac{\varepsilon^p}{k}$, and since $|x_n - s_i|$ will eventually be less than 1, we have $|x_n - s_i|^p \leq |x_n - s_i| < \frac{\varepsilon^p}{k}$. So we can write

$$|x_{n1} - s_1|^p + |x_{n2} - s_2|^p + \dots + |x_{nk} - s_k|^p < \frac{\varepsilon^p}{k} + \frac{\varepsilon^p}{k} + \dots + \frac{\varepsilon^p}{k} = \varepsilon^p,$$

which implies that

$$d_p(x_n, s) = \sqrt[p]{|x_{n1} - s_1|^p + |x_{n2} - s_2|^p + \dots + |x_{nk} - s_k|^p} < \varepsilon.$$

Therefore, $\{x_n\}$ converges to s. $\square$

Problem 4 Let K be a compact set in $\mathbb{R}^k$, $a \in \mathbb{R}^k$ and

$$\alpha = \inf\{\|a - x\|_2 \ : \ x \in K\}.$$

(a) Prove that there exists a sequence $\{x_n\}$ in $\mathbb{R}^k$ such that $\{\|a - x_n\|_2\}$ converges to α.

(b) Show that there exists $b \in K$ such that

$$\|a - b\|_2 = \inf\{\|a - x\|_2 \ : \ x \in K\}.$$

Solution.

(a) Let $D = \{\|a - x\|_2 : x \in K\}$. Then, $D \subset \mathbb{R}$. We have proven in class that since $\alpha = \inf D$, we have that for all $\varepsilon > 0$, there exists a $d \in D$ such that $\alpha \leq d < \alpha + \varepsilon$. Thus, we can choose an arbitrary $d \in D$, and define $\{x_n\}$ by letting x_n be some $x \in K$ with $\|a - x\|_2 = \alpha + \frac{d}{n}$. We know $\alpha + \frac{d}{n} \in D$ because of the property proved about α, so this sequence is well-defined and $\{\|a - x_n\|_2\}$ converges to α because $\frac{d}{n}$ converges to 0.

(b) Because $\{x_n\}$ in (a) converges, there exists a $b \in \mathbb{R}^k$ such that $\|a - b\|_2 = \alpha$. Since there exists a sequence in K that converges to such a b, b is a limit point of K. Since K is a compact set in $\mathbb{R}^k$, it is closed, which means it includes all of its limit points. Therefore, we must have $b \in K$. $\square$