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MATH 574: Section H01
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MATH 574 Homework 5

Collaboration: I discussed some of the problems with Jackson Ginn, Sam Maloney, and Jack Hyatt.

Problem 1 Let X be a random variable on a sample space S such that $X(s) \geq 0$ for all $s \in S$. Prove that for every number $a > 0$, $P(X \geq a) \leq \frac{E(X)}{a}$. This is called **Markov's inequality**. (Use the formula $E(X) = \sum_{r \in X(S)} rP(X = r)$ and split it into $r < a$ and $r \geq a$.)

Solution.

Let $a \in \mathbb{R}^+$. We define two sets $A_< = \{r \in X(S) : r < a\}$ and $A_\geq = \{r \in X(S) : r \geq a\}$. Since $A_<$ is the set of outcomes less than a and $A_\geq$ is the outcomes greater than or equal to a, $A_<$ and $A_\geq$ are a partition of $X(S)$.

We have that

$$E(X) = \sum_{r \in X(S)} rP(X = r).$$

Since we have a partition, we can rewrite this as

$$E(X) = \sum_{r \in A_<} rP(X = r) + \sum_{r \in A_\geq} rP(X = r).$$

Additionally, given how we defined $A_\geq$, we can write

$$P(X \geq a) = \sum_{r \in A_\geq} P(X = r).$$

Now, assume that $P(X \geq a) > \frac{E(X)}{a}$. This would mean that

$$\begin{aligned}
\sum_{r \in A_\geq} P(X = r) &> \frac{1}{a}\left[\sum_{r \in A_<} rP(X = r) + \sum_{r \in A_\geq} rP(X = r)\right] \\
&\geq \frac{1}{a}\left[\sum_{r \in A_\geq} rP(X = r)\right] && \text{(the first sum is non-negative)} \\
&= \sum_{r \in A_\geq} \frac{r}{a} P(X = r). && \text{(constant rule)}
\end{aligned}$$

However, choose $r \in A_\geq$. We have that $r \geq a$ given how we chose $A_\geq$, so $1 \leq \frac{r}{a}$ and thus we must have $P(X = r) \leq \frac{r}{a}P(X = r)$. Therefore, our conclusion that

$$\sum_{r \in A_\geq} P(X = r) > \sum_{r \in A_\geq} \frac{r}{a} P(X = r)$$

is a contradiction, because every term in the LHS is less than or equal to its corresponding term in the RHS. So we must have $P(X \geq a) \leq \frac{E(X)}{a}$.

Problem 2 A biased coin has probability p of getting heads. Let X be the number of flips it takes to get exactly n heads.
(a) Use the linearity of expectation to prove that $E(X) = n/p$. (Define the random variable X_i to be the number of flips it takes to get the ith heads after getting the $(i-1)$th heads.)
(b) Using part (a), give a double counting proof of the following:

$$\sum_{m=n}^{\infty} m\binom{m-1}{n-1}p^n(1-p)^{m-n} = n/p.$$

Solution.

(a) We define X_i as the number of flips it takes the get the i^{th} heads after getting the $(i-1)^{\text{th}}$ heads (for example, if it takes 5 flips total to get 2 heads and 9 flips total to get 3 heads, $X_3 = 9 - 5 = 4$).

Let $i \in \{1, 2, \ldots, n\}$ and $r \in \mathbb{N}$. We know that X_i follows a geometric distribution, because we are interested in the number of flips until a success and $P(X_i = r) = (1-p)^{r-1}p$. Thus, from class we have that $E(X_i) = \frac{1}{p}$.

We have $X = X_1 + X_2 + \ldots + X_n$, because both count the total flips until n heads, so $E(X) = E(X_1 + X_2 + \ldots + X_n)$. By linearity of expectation, we also have $E(X) = E(X_1) + E(X_2) + \ldots + E(X_n)$. Since $E(X_i) = \frac{1}{p}$ for each i, we have

$$E(X) = \sum_{i=1}^{n} \frac{1}{p} = \frac{n}{p}.$$

(b) Let $m \in \mathbb{N}$, $m \geq n$. We consider the probability of it taking m flips to get n heads. Since the last flip must a heads, there are $\binom{m-1}{n-1}$ ways to arrange the other heads in the sequence of flips. Because there are n heads with probability p and $m - n$ heads with probability $1 - p$, we have $P(X = m) = \binom{m-1}{n-1}p^n(1-p)^{m-n}$.

We have that

$$E(X) = \sum_{r \in X(S)} rP(X = r).$$

$X(S)$ is the subset of the naturals that has least element n, because it must take at least n flips to get n heads. So

$$E(X) = \sum_{m=n}^{\infty} m\binom{m-1}{n-1}p^n(1-p)^{m-n}.$$

Since we showed in (a) that $E(X) = \frac{n}{p}$, we must have

$$\sum_{m=n}^{\infty} m\binom{m-1}{n-1}p^n(1-p)^{m-n} = \frac{n}{p}.$$

Problem 3 A game is played where the player rolls 2 fair 6-sided dice. The player must pay \$1 to play the game. The player wins \$2 if the product of the two dice is an odd number, and \$1 if the sum of the two dice is an odd number.
(a) What is the player's expected net profit for this game?
(b) What is the variance of the player's net profit?

Solution.

We define X as the profit one makes from a game.

(a) The product of two numbers is odd if both number are odd, or even otherwise. Since each die has probability $\frac{1}{2}$ of being odd, the probability of their product being odd is $\frac{1}{4}$. Additionally, since the sum of two number is odd if their parity is different and even otherwise, the probability of their sum being odd is $\frac{2}{4}$. The only way to get neither of these is if both numbers are even, which happens with probability $\frac{1}{4}$. So we have

$$E(X) = \frac{1}{4}(2-1) + \frac{2}{4}(1-1) + \frac{1}{4}(0-1) = \$0.$$

(b) We have that $V(X) = E((X - E(X))^2)$. Using the same reasoning as (a), we have

$$V(X) = \frac{1}{4}(2-0)^2 + \frac{2}{4}(0-0)^2 + \frac{1}{4}(-1-0)^2 \approx \$^2 0.50.$$

Problem 4 Prove that for an integer $n \geq 1$, $\sum_{k=1}^{n} k = \binom{n+1}{2}$.

Solution.

First, let $n = 1$. We have that

$$\sum_{k=1}^{1}(k) = 1 = \binom{2}{2} = \binom{1+1}{2}.$$

So the claim holds for $n = 1$. Next, let $n \in \mathbb{N}$. Assume that

$$\sum_{k=1}^{n}(k) = \binom{n+1}{2}.$$

We observe that

$$\begin{aligned}
\sum_{k=1}^{n+1}(k) &= \sum_{k=1}^{n}(k) + n + 1 && \text{(splitting sum)} \\
&= \binom{n+1}{2} + n + 1 && \text{(induction hypothesis)} \\
&= \frac{n(n+1)}{2} + n + 1 && \text{(definition of binomial)} \\
&= \frac{n^2 + n + 2n + 2}{2} && \text{(expanding/combining fraction)} \\
&= \frac{(n+2)(n+1)}{2} && \text{(factoring)} \\
&= \binom{n+1+1}{2}. && \text{(definition of binomial)}
\end{aligned}$$

So if the claim holds for n, it also holds for $n+1$. Thus, for all $n \in \mathbb{N}$, we have

$$\sum_{k=1}^{n} k = \binom{n+1}{2}.$$

Problem 5 A random subset of $\{1, \ldots, n\}$ is chosen using the following process: for each element $i \in \{1, \ldots, n\}$ we include i in the subset with probability $1/2$. Let X be the random variable equal to the sum of the elements of the subset. Let Y be the random variable equal to the largest element in the subset.
(a) Compute $E(X)$. (You may find #4 on this homework useful.)
(b) Show that X and Y are not independent.

Solution.

(a) We define X_i as i if i is in the subset, and 0 otherwise. We have that $E(X_i) = \frac{1}{2}(i) + \frac{1}{2}(0) = \frac{i}{2}$. Since X is the sum of the elements in the subset, $X = X_1 + X_2 + \ldots + X_n$. So we have

$$\begin{aligned}
E(X) &= E(X_1 + X_2 + \ldots + X_n) && \text{(since } X = X_1 + X_2 + \ldots + X_n\text{)} \\
&= E(X_1) + E(X_2) + \ldots + E(X_n) && \text{(by linearity of expectation)} \\
&= \frac{1}{2} + \frac{2}{2} + \ldots + \frac{n}{2} && \text{(since } E(X_i) = \tfrac{i}{2}\text{)} \\
&= \frac{1}{2}(1 + 2 + \ldots + n) \\
&= \frac{1}{2}\binom{n+1}{2}. && \text{(by result from \#4)}
\end{aligned}$$

(b) If X and Y are independent, then $P(X = j \text{ and } Y = k) = P(X = j)P(Y = k)$ for all $j \in X(S)$, $k \in Y(S)$. However, choose for example $j = 3$ and $k = 5$. We know that $P(X = 3) > 0$ because $X = 3$ for the subset $\{1, 2\}$ which can occur, and we know that $P(Y = 5) > 0$ because $Y = 5$ for the subset $\{1, 5\}$ which can also occur. So $P(X = 3)P(Y = 5) > 0$. On the other hand, $P(X = 3 \text{ and } Y = 5)$ must be 0, because 5 being in the set would mean the sum would need to be at least 5. So $P(X = 3 \text{ and } Y = 5) = P(X = 3)P(Y = 5)$ cannot hold, and therefore X and Y are not independent.

Problem 6 Let X be a random variable that has geometric distribution with probability of success p. In this question we will show that $V(X) = \frac{1-p}{p^2}$.
(a) For r with $|r| < 1$, prove that

$$\sum_{k=1}^{\infty} k^2 r^{k-1} = \frac{1+r}{(1-r)^3}.$$

You may use results proven in class.
(b) Use part (a) to prove $V(X) = \frac{1-p}{p^2}$.

Solution.

(a)

$$\begin{aligned}
\sum_{k=1}^{\infty} k^2 r^{k-1} &= \sum_{k=1}^{\infty} (k+1-1)kr^{k-1} && (k = k+1-1) \\
&= \sum_{k=1}^{\infty} (k+1)kr^{k-1} - kr^{k-1} && \text{(distributing)} \\
&= \sum_{k=1}^{\infty} (k+1)kr^{k-1} - \sum_{k=1}^{\infty} kr^{k-1} && \text{(splitting sum)} \\
&= \sum_{k=1}^{\infty} \frac{d^2}{dr^2}\left[r^{k+1}\right] - \sum_{k=1}^{\infty} \frac{d}{dr}\left[r^k\right] && \text{(re-expressing)} \\
&= \frac{d^2}{dr^2}\left[\sum_{k=1}^{\infty} r^{k+1}\right] - \frac{d}{dr}\left[\sum_{k=1}^{\infty} r^k\right] && \text{(sum rule of derivatives)} \\
&= \frac{d^2}{dr^2}\left[\sum_{k=0}^{\infty} (r^k) - r^1 - r^0\right] - \frac{d}{dr}\left[\sum_{k=0}^{\infty} (r^k) - r^0\right] && \text{(adjusting bounds)} \\
&= \frac{d^2}{dr^2}\left[\frac{1}{1-r} - r - 1\right] - \frac{d}{dr}\left[\frac{1}{1-r} - 1\right] && \text{(evaluating sums/simplifying)}
\end{aligned}$$

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\begin{aligned}
&= \frac{d^2}{dr^2}\left[\frac{1}{1-r}\right] - \frac{d^2}{dr^2}[r] - \frac{d^2}{dr^2}[1] - \frac{d}{dr}\left[\frac{1}{1-r}\right] + \frac{d}{dr}[1] && \text{(sum rule)} \\
&= \frac{2}{(1-r)^3} - \frac{1}{(1-r)^2} && \text{(evaluating derivatives)} \\
&= \frac{1+r}{(1-r)^3} && \text{(combining sum)}
\end{aligned}
$$

(b)

$$
\begin{aligned}
V(X) &= E(X^2) - E(X)^2 && \text{(from class)} \\
&= \sum_{r \in X^2(S)} \Big(rP(X^2 = r)\Big) - \left(\frac{1}{p}\right)^2 && \text{(by definition)} \\
&= \sum_{r \in X(S)} \Big(r^2 P(X = r)\Big) - \frac{1}{p^2} \\
&= \sum_{r=1}^{\infty} \Big(r^2(1-p)^{r-1}p\Big) - \frac{1}{p^2} && \text{(probability for geometric distribution)} \\
&= p\sum_{r=1}^{\infty} \Big(r^2(1-p)^{r-1}\Big) - \frac{1}{p^2} \\
&= p\frac{1+(1-p)}{(1-(1-p))^3} - \frac{1}{p^2} && \text{(using result from (a))} \\
&= \frac{p(2-p)}{p^3} - \frac{1}{p^2} \\
&= \frac{2-p}{p^2} - \frac{1}{p^2} \\
&= \frac{1-p}{p^2}.
\end{aligned}
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Problem 7 Suppose that the number of cans of soda pop filled in a day at a bottling plant is a random variable with an expected value of 10,000 and a variance of 1000.
(a) Use Markov's inequality (#1 on this homework) to obtain an upper bound on the probability that the plant will fill more than 11,000 cans on a particular day.
(b) Use Chebyshev's inequality to obtain a lower bound on the probability that the plant will fill between 9000 and 11,000 cans on a particular day.

Solution.

(a) From Markov's inequality, we have that for every number $a > 0$, $P(X \geq a) \leq \frac{E(X)}{a}$. Letting $a = 11,000$, we have

$$P(X \geq 11,000) \leq \frac{E(X)}{11,000} = \frac{10,000}{11,000} = \frac{10}{11}.$$

(b) We have from class that for any $c \in \mathbb{R}^+$ and random variable X with $V(X) = \sigma^2$, we have

$$P(|X - E(X)| \leq c\sigma) \geq 1 - \frac{1}{c^2}.$$

We have $\sigma = \sqrt{1000}$, so $c = \sqrt{1000}$ yields

$$P(|X - 10,000| \leq 1000) \geq 1 - \frac{1}{1000}.$$

Therefore,

$$P(9000 \leq X \leq 11,000) \geq \frac{999}{1000}.$$

Problem 8 A biased coin has probability $p = .99$ for heads. Suppose we flip the coin 1000 times. Use Chebyshev's formula to give an upper bound for the probability that we get heads at most 900 times.

Solution.

Let X be the number of heads from the 1000 flips. Since X is a binomial distribution, $E(X) = 1000(0.99) = 990$ and $V(X) = 1000(0.99)(1 - 0.99) = 9.9$. We have from Chebyshev's Inequality that

$$P(|X - E(X)| \geq k) \leq \frac{V(X)}{k^2}.$$

So if we choose $k = 90$,

$$P(|X - 990| \geq 90) \leq \frac{9.9}{90^2}.$$

Since X cannot exceed 1000, we have

$$P(X \leq 900) \leq \frac{11}{9000}.$$