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MATH 554: Section H01
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MATH 554 Homework 13

Problem 1

(a) Show that the function
$$f(x) = \frac{x}{1+x^2}$$
is Lipschitz on the closed interval $[-b, b]$ for any $b > 0$.

(b) Use this to prove that if $\langle x_n \rangle_{n=1}^{\infty}$ is a sequence of real numbers with $\lim_{n\to\infty} x_n = L$, then
$$\lim_{n\to\infty} \frac{x_n}{{x_n}^2+1} = \frac{L}{L^2+1}.$$

(a) Let $x, y \in [-b, b]$. Then
$$\begin{aligned}
\big|f(x) - f(y)\big| &= \left|\frac{x}{1+x^2} - \frac{y}{1+y^2}\right| \\
&= \left|\frac{x(1+y^2) - y(1+x^2)}{(1+x^2)(1+y^2)}\right| \\
&= \left|\frac{x + xy^2 - y + x^2y}{x^2y^2 + x^2 + y^2 + 1}\right| \\
&= \left|\frac{xy^2 - x^2y + x - y}{x^2y^2 + x^2 + y^2 + 1}\right| \\
&= \left|\frac{xy(y-x) + x - y}{x^2y^2 + x^2 + y^2 + 1}\right| \\
&= \left|\frac{(1-xy)(y-x)}{x^2y^2 + x^2 + y^2 + 1}\right| && \text{(grouping)} \\
&\leq \left|\frac{(1-xy)(x-y)}{1}\right| && (x^2y^2 + x^2 + y^2 \geq 0) \\
&= |1-xy||x-y| \\
&\leq (|1| + |x||y|)|x-y| && \text{(triangle inequality)} \\
&\leq (1+b^2)|x-y|. && (x, y \leq b \implies |x|^2, |y|^2 \leq b^2)
\end{aligned}$$

So for $M := 1 + b^2$, we have $\big|f(x) - f(y)\big| \leq M|x-y|$. Therefore, $f(x)$ is Lipschitz on $[-b, b]$. $\square$

(b) Since $\langle x_n \rangle$ is convergent, it is bounded, so there exists a $b \in \mathbb{R}^+$ such that $L \in [-b, b]$ and $x_n \in [-b, b]$ for all $n \in \mathbb{N}$. From (a), f is Lipschitz on $[-b, b]$. Let $M := 1 + b^2$, and let $\varepsilon > 0$. Since $\langle x_n \rangle$ converges to L, there exists an N such that for all $n > N$, $|x_n - L| < \frac{\varepsilon}{M}$. Let $n > N$. Then, we have
$$\left|\frac{x_n}{{x_n}^2+1} - \frac{L}{L^2+1}\right| = \big|f(x_n) - f(L)\big|$$

$$
\begin{aligned}
&\leq M|x_n - L| && \text{(Lipschitz)} \\
&< M\left(\frac{\varepsilon}{M}\right) = \varepsilon. && \text{(by choice of } n\text{)}
\end{aligned}
$$

Therefore, by definition

$$\lim_{n\to\infty} \frac{x_n}{{x_n}^2+1} = \frac{L}{L^2+1}.$$

□

Problem 2 Let (A, d_A) and (B, d_B) be metric spaces, and let $E := A \times B$. Define a metric d on E by

$$d\left((a_1, b_1), (a_2, b_2)\right) = d_A(a_1, a_2) + d_B(b_1, b_2).$$

(a) Prove this is a metric on E.

(b) Prove that if A and B are complete, then so is E.

(c) Prove that if A and B are both sequentially compact, then so is E.

(a) We will prove each part of the definition:

- Since d_A and d_B are non-negative everywhere, their sum is also non-negative. Therefore,

  $$d\left((a_1, b_1), (a_2, b_2)\right) \geq 0$$

  for all $(a_1, b_1), (a_2, b_2) \in E$.

- If $(a_1, b_1) = (a_2, b_2)$, then $a_1 = a_2$ and $b_1 = b_2$, so $d_A(a_1, a_2) = d_B(b_1, b_2) = 0$, and thus we have $d\left((a_1, b_1), (a_2, b_2)\right) = 0 + 0 = 0$. Conversely, if we have $d\left((a_1, b_1), (a_2, b_2)\right) = 0$ for some $(a_1, b_1), (a_2, b_2) \in E$, then we must have $d_A(a_1, a_2) = d_B(b_1, b_2) = 0$ (this is the only way to have their sum be zero and both terms be non-negative). So $a_1 = a_2$ and $b_1 = b_2$, and thus $(a_1, b_1) = (a_2, b_2)$. Therefore,

  $$d\left((a_1, b_1), (a_2, b_2)\right) = 0 \iff (a_1, b_1) = (a_2, b_2).$$

- Let $(a_1, b_1), (a_2, b_2) \in E$. Then, we can use the symmetry of d_A and d_B to write

  $$d\left((a_1, b_1), (a_2, b_2)\right) = d_A(a_1, a_2) + d_B(b_1, b_2) = d_A(a_2, a_1) + d_B(b_2, b_1) = d\left((a_2, b_2), (a_1, b_1)\right).$$

- Let $(a_1, b_1), (a_2, b_2), (a_3, b_3) \in E$. Then, we can write

  $$
  \begin{aligned}
  d\left((a_1, b_1), (a_2, b_2)\right) &= d_A(a_1, a_2) + d_B(b_1, b_2) \\
  &\leq d_A(a_1, a_3) + d_A(a_3, a_2) + d_B(b_1, b_3) + d_B(b_3, b_2) && \text{(triangle inequality)} \\
  &= d_A(a_1, a_3) + d_B(b_1, b_3) + d_A(a_3, a_2) + d_B(b_3, b_2) \\
  &= d\left((a_1, b_1), (a_3, b_3)\right) + d\left((a_3, b_3), (a_2, b_2)\right),
  \end{aligned}
  $$

  and therefore the triangle inequality holds.

Therefore, E is a metric space.

(b) Let $\langle p_n \rangle_{n=1}^{\infty}$ be a Cauchy sequence in E. Let $\langle a_n \rangle_{n=1}^{\infty}$ and $\langle b_n \rangle_{n=1}^{\infty}$ be sequences in A and B, respectively, such that for all $n \in \mathbb{N}$, we have $p_n = (a_n, b_n)$. Let $\varepsilon_1 > 0$. Since $\langle p_n \rangle$ is Cauchy, there exists an N_1 such that for all $m, n > N$, we have $d(p_m, p_n) < \varepsilon_1$. Let $m, n > N_1$. Then, since distance metrics are non-negative, we have

$$d_A(a_m, a_n), d_B(b_m, b_n) \leq d_A(a_m, a_n) + d_B(b_m, b_n) = d\left((a_m, b_m), (a_n, b_n)\right) = d(p_m, p_n) < \varepsilon_1.$$

Therefore, $\langle a_n \rangle$ and $\langle b_n \rangle$ are both Cauchy. Since A and B are complete, both sequences are convergent, so we have $a \in A$, $b \in B$ such that

$$\lim_{n \to \infty} a_n = a, \quad \lim_{n \to \infty} b_n = b.$$

We claim that $\langle p_n \rangle$ converges to $p := (a, b)$ (which is in $E = A \times B$). To see this, let $\varepsilon_2 > 0$. Then, there exists an N_2 such that for all $n > N_2$, we have

$$d_A(a_n - a), d_B(b_n - b) < \frac{\varepsilon_2}{2}.$$

Let $n > N_2$. Then, we have

$$d(p_n, p) = d\left((a_n, b_n), (a, b)\right) = d_A(a_n, a) + d_B(b_n, b) < \frac{\varepsilon}{2} + \frac{\varepsilon}{2} = \varepsilon.$$

Thus, by definition $\langle p_n \rangle$ converges to a point in E, and therefore E is complete.

(c) Let $\langle p_n \rangle_{n=1}^{\infty}$ be a sequence in E. Let $\langle a_n \rangle_{n=1}^{\infty}$ and $\langle b_n \rangle_{n=1}^{\infty}$ be sequences in A and B, respectively, such that for all $n \in \mathbb{N}$, we have $p_n = (a_n, b_n)$. Since A is sequentially compact, there exists a subsequence $\langle a_{n_k} \rangle_{k=1}^{\infty}$ of $\langle a_n \rangle$ that converges to some point $a \in A$. Since B is sequentially compact, there exists a subsequence $\langle b_{n_{k_j}} \rangle_{j=1}^{\infty}$ of $\langle b_{n_k} \rangle$ that converges to some point $b \in B$. Then $\langle a_{n_{k_j}} \rangle$ converges since it is a subsequence of a convergent sequence.

Consider $\langle p_{n_{k_j}} \rangle_{j=1}^{\infty}$. Using the same argument from part (b), we can conclude that since we have

$$\lim_{j \to \infty} a_{n_{k_j}} = a, \quad \lim_{j \to \infty} b_{n_{k_j}} = b,$$

we also have

$$\lim_{j \to \infty} p_{n_{k_j}} = \lim_{j \to \infty} \left(a_{n_{k_j}}, b_{n_{k_j}}\right) = (a, b).$$

Since $(a, b) \in E$, we have that any sequence has a convergent subsequence, and therefore E is sequentially compact. $\square$

Problem 3 Let (E, d) and (E', d') be metric spaces and let $f : E \to E'$ satisfy

$$d'\left(f(p), f(q)\right) \leq M d(p, q)$$

for some constant $M > 0$. Prove that if $V \subset E'$ is an open set than

$$U := f^{-1}[V] = \{p \in E : f(p) \in V\}$$

is also open.

Let $p \in U$. Then, $f(p) \in V$, and since V is open, there exists an $r > 0$ such that $B(f(p), r) \subseteq V$. Consider $\delta := \frac{r}{M}$. We claim $B(p, \delta) \subseteq U$. To see this, let $x \in B(p, \delta)$. Since f is Lipschitz, we have that

$$d'\left(f(p), f(x)\right) \leq M d(p, x) < M\delta = M\left(\frac{r}{M}\right) = r.$$

Therefore, we have $f(x) \in B(f(p), r)$. Since $B(f(p), r) \subseteq V$, we have $f(x) \in V$, and thus $x \in U$. So $B(p, \delta) \subseteq U$, and since $\delta > 0$ and p was an arbitrary point in U, we have that U is open. $\square$

Problem 4 Let $\mathbb{R}^2$ have its usual metric

$$d\left((x_1, y_1), (x_2, y_2)\right) = \sqrt{(x_1 - x_2)^2 + (y_1 - y_2)^2}.$$

In different notation if $\mathbf{x_1} = (x_1, y_1)$ and $\mathbf{x_2} = (x_2, y_2)$ then

$$d(\mathbf{x_1}, \mathbf{x_2}) = \|x_1 - x_2\|.$$

(a) Let $\mathbf{a}, \mathbf{b} \in \mathbb{R}^2$. Prove that the map $f : \mathbb{R} \to \mathbb{R}^2$ defined by

$$f(t) = (1 - t)\mathbf{a} + t\mathbf{b}$$

is Lipschitz.

(b) For $\mathbf{a}, \mathbf{b} \in \mathbb{R}^2$ define the segment with endpoints $\mathbf{a}$ and $\mathbf{b}$ as

$$[\mathbf{a}, \mathbf{b}] = \{(1 - t)\mathbf{a} + t\mathbf{b} : 0 \le t \le 1\}.$$

Prove that $[\mathbf{a}, \mathbf{b}]$ is connected.

(a) Let $x, y \in \mathbb{R}$. Then, we can write

$$\begin{aligned}
\|f(x) - f(y)\| &= \left\| \left[(1 - x)\mathbf{a} + x\mathbf{b}\right] - \left[(1 - y)\mathbf{a} + y\mathbf{b}\right] \right\| \\
&= \|(1 - x)\mathbf{a} - (1 - y)\mathbf{a} + x\mathbf{b} - y\mathbf{b}\| \\
&= \|(y - x)\mathbf{a} + (x - y)\mathbf{b}\| \\
&\le \|(y - x)\mathbf{a}\| + \|(x - y)\mathbf{b}\| && \text{(triangle inequality)} \\
&= \|\mathbf{a}\||y - x| + \|\mathbf{b}\||x - y| \\
&= \left(\|\mathbf{a}\| + \|\mathbf{b}\|\right)|x - y|. && \text{(factoring/using } |y - x| = |x - y|\text{)}
\end{aligned}$$

Thus, for $M := \|\mathbf{a}\| + \|\mathbf{b}\|$, we have $\|f(x) - f(y)\| \le M|x - y|$, and therefore f is Lipschitz. $\square$

(b) Suppose (toward contradiction) that $[\mathbf{a}, \mathbf{b}]$ is disconnected. Then there exists a disconnection $[\mathbf{a}, \mathbf{b}] = A \cup B$. Using this disconnection, we can define

$$A_0 = \{t \in [0, 1] : (1 - t)\mathbf{a} + t\mathbf{b} \in A\}, \quad B_0 = \{t \in [0, 1] : (1 - t)\mathbf{a} + t\mathbf{b} \in B\}.$$

We claim that $A_0 \cup B_0$ is a disconnection of the interval $[0, 1]$. To see this, we observe that using the f from above, we can express the definition as

$$[\mathbf{a}, \mathbf{b}] = \{f(t) : t \in [0, 1]\}.$$

Then, we have

$$A_0 = \{t \in [0, 1] : f(t) \in A\} = f^{-1}[A], \quad B_0 = \{t \in [0, 1] : f(t) \in B\} = f^{-1}[B].$$

Since A, B are open by definition of disconnection, from problem 3 we have that $A_0 = f^{-1}[A]$ and $B_0 = f^{-1}[B]$ are also open.

Since $A \cap B = \varnothing$, we also have $A_0 \cap B_0 = \varnothing$: if there were some element $t \in A_0 \cap B_0$, then $f(t) \in A$ and $f(t) \in B$, contradicting $A \cap B = \varnothing$. Since $A \cup B = [\mathbf{a}, \mathbf{b}]$, we have $A_0 \cup B_0 = [0,1]$: if there were some element $t \in [0,1] \backslash A_0 \cup B_0$, then $f(t) \notin A$ and $f(t) \notin B$, contradicting $A \cup B = [\mathbf{a}, \mathbf{b}]$. Finally, we have $A_0 \neq \varnothing$ because there is some $a \in A$ and thus there exists t so $a = f(t) \in A_0$, and similarly $B_0 \neq \varnothing$. Therefore, $A_0 \cup B_0$ is a disconnection of $[0,1]$, a contradiction because intervals are connected. $\square$

Problem 5 Let $\mathcal{F}$ be a collection of subsets of the set E. Then $\mathcal{F}$ has the finite intersection property if and only if every finite subset $\mathcal{F}_0 = \{F_1, F_2, \ldots, F_n\}$ of $\mathcal{F}$ has nonempty intersection. That is

$$\bigcap \mathcal{F}_0 = F_1 \cap F_2 \cap \cdots \cap F_n \neq \varnothing.$$

Let E be a metric space.

(a) Prove that if E is compact, then any collection, $\mathcal{F}$, of closed subsets of E with the finite intersection property has nonempty intersection. That is if $\mathcal{F}$ has the finite intersection property, then

$$\bigcap \mathcal{F} \neq \varnothing.$$

(b) Prove that if every collection of closed sets, $\mathcal{F}$, of E with the finite intersection property has nonempty intersection, then E is compact.

(a) Suppose (toward contradiction) that E is compact, but there exists a collection $\mathcal{F}$ of closed subsets of E with the finite intersection property and

$$\bigcap \mathcal{F} = \varnothing.$$

Consider $\mathcal{U} = \{\mathcal{C}(F) : F \in \mathcal{F}\}$. Since every $F \in \mathcal{F}$ is closed, every $U \in \mathcal{U}$ is open. Additionally, from De Morgan we have

$$\bigcup_{U \in \mathcal{U}} U = \bigcup_{F \in \mathcal{F}} \mathcal{C}(F) = \mathcal{C}\left(\bigcap_{F \in \mathcal{F}} F\right) = \mathcal{C}\left(\bigcap \mathcal{F}\right) = \mathcal{C}(\varnothing) = E,$$

so $\mathcal{U}$ is an open cover for E. Since E is compact, there exists a finite subcover

$$\mathcal{U}_0 = \{U_1, U_2, \ldots, U_n\} \subseteq \mathcal{U}$$

of E. By definition of $\mathcal{U}$, there are $F_1, F_2, \ldots, F_n$ such that $U_i = \mathcal{C}(F_i)$ for all $i \in \{1, 2, \ldots, n\}$. Consider $\mathcal{F}_0 = \{F_1, F_2, \ldots, F_n\}$. We can use De Morgan again to write

$$\bigcap \mathcal{F}_0 = \bigcap_{i=1}^{n} F_i = \bigcap_{i=1}^{n} \mathcal{C}(U_i) = \mathcal{C}\left(\bigcup_{i=1}^{n} U_i\right) = \mathcal{C}\left(\bigcup \mathcal{U}_0\right) = \mathcal{C}(E) = \varnothing,$$

a contradiction of the finite intersection property. $\square$

(b) We will prove the contrapositive. Suppose that E is not compact. Then there exists an open cover $\mathcal{U}$ with no finite subcover. Let $\mathcal{F} = \{\mathcal{C}(U) : U \in \mathcal{U}\}$. Since every set in $\mathcal{U}$ is open, $\mathcal{F}$ is a collection of closed sets. Additionally, since $\mathcal{U}$ is a cover, we have

$$\bigcap_{F \in \mathcal{F}} F = \bigcap_{U \in \mathcal{U}} \mathcal{C}(U) = \mathcal{C}\left(\bigcup_{U \in \mathcal{U}} U\right) = \mathcal{C}(E) = \varnothing.$$

Let $\mathcal{F}_0 = \{F_1, F_2, \ldots, F_n\}$ be a finite subset of $\mathcal{F}$. By our definition of $\mathcal{F}$, there exist $U_1, U_2, \ldots, U_n \in \mathcal{U}$ such that $F_i = \mathcal{C}(U_i)$ for all $i \in \{1, 2, \ldots, n\}$. Consider $\mathcal{U}_0 = \{U_1, U_2, \ldots, U_n\}$. Since $\mathcal{U}_0$ is a finite subset of $\mathcal{U}$, it is not a cover by our assumption. So there exists some $p \in E$ with p not covered by $\mathcal{U}_0$, and thus

$$p \in \mathcal{C}\left(\bigcup_{i=1}^{n} Ui\right) = \bigcap_{i=1}^{n} \mathcal{C}(U_i) = \bigcap_{i=1}^{n} F_i = \bigcap \mathcal{F}_0.$$

So we have

$$\bigcap \mathcal{F}_0 \neq \varnothing,$$

and since $\mathcal{F}_0$ was an arbitrary finite set, $\mathcal{F}$ is a collection of closed sets of E with both the finite intersection property and empty intersection. Therefore, not every collection of closed sets with the finite intersection property has nonempty intersection. $\square$