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MATH 544: Section H01
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MATH 544 Homework 4

Problem 1 For each matrix A, compute (i) Row(A), (ii) Col(A), and (iii) Null(A) as the span of a set of vectors. Then compute rk(A).

$$A = \begin{pmatrix} 1 & 2 & 3 \\ 1 & 3 & 1 \\ 2 & 2 & 10 \end{pmatrix}, \quad B = \begin{pmatrix} 1 & 2 & 3 & -1 \\ 3 & 5 & 8 & -2 \\ 1 & 1 & 2 & 0 \end{pmatrix}, \quad C = \begin{pmatrix} 1 & 2 & 1 & 0 \\ 2 & 5 & 3 & -1 \\ 2 & 2 & 0 & 2 \\ 0 & 1 & 1 & -1 \end{pmatrix}$$

Solution.

(a) We find

$$\begin{aligned} A &= \begin{pmatrix} 1 & 2 & 3 \\ 1 & 3 & 1 \\ 2 & 2 & 10 \end{pmatrix} \\ &\sim \begin{pmatrix} 1 & 2 & 3 \\ 0 & 1 & -2 \\ 2 & 2 & 10 \end{pmatrix} && (\rho_2 \mapsto \rho_2 - \rho_1) \\ &\sim \begin{pmatrix} 1 & 2 & 3 \\ 0 & 1 & -2 \\ 0 & -2 & 4 \end{pmatrix} && (\rho_3 \mapsto \rho_3 - 2\rho_1) \\ &\sim \begin{pmatrix} 1 & 2 & 3 \\ 0 & 1 & -2 \\ 0 & 0 & 0 \end{pmatrix} && (\rho_3 \mapsto \rho_3 + 2\rho_2) \\ &\sim \begin{pmatrix} 1 & 0 & 7 \\ 0 & 1 & -2 \\ 0 & 0 & 0 \end{pmatrix} = \operatorname{rref}(A). && (\rho_1 \mapsto \rho_1 - 2\rho_2) \end{aligned}$$

So using the methods from class, we have

$$\operatorname{Row}(A) = \operatorname{Span}\left\{ \begin{pmatrix} 1 \\ 0 \\ 7 \end{pmatrix}, \begin{pmatrix} 0 \\ 1 \\ -2 \end{pmatrix} \right\} \tag{1}$$

$$\operatorname{Col}(A) = \operatorname{Span}\left\{ \begin{pmatrix} 1 \\ 1 \\ 2 \end{pmatrix}, \begin{pmatrix} 2 \\ 3 \\ 2 \end{pmatrix} \right\} \tag{2}$$

$$\text{Null}(A) = \text{Span}\left\{\begin{pmatrix}-7\\2\\1\end{pmatrix}\right\}. \tag{3}$$

$$\text{rk}(A) = 2 \tag{4}$$

(b) We find

$$B = \begin{pmatrix}1 & 2 & 3 & -1\\3 & 5 & 8 & -2\\1 & 1 & 2 & 0\end{pmatrix}$$

$$\sim \begin{pmatrix}1 & 2 & 3 & -1\\0 & -1 & -1 & 1\\1 & 1 & 2 & 0\end{pmatrix} \qquad (\rho_2 \mapsto \rho_2 - 3\rho_1)$$

$$\sim \begin{pmatrix}1 & 2 & 3 & -1\\0 & 1 & 1 & -1\\1 & 1 & 2 & 0\end{pmatrix} \qquad (\rho_2 \mapsto -\rho_2)$$

$$\sim \begin{pmatrix}1 & 2 & 3 & -1\\0 & 1 & 1 & -1\\0 & -1 & -1 & 1\end{pmatrix} \qquad (\rho_3 \mapsto \rho_3 - \rho_1)$$

$$\sim \begin{pmatrix}1 & 2 & 3 & -1\\0 & 1 & 1 & -1\\0 & 0 & 0 & 0\end{pmatrix} \qquad (\rho_3 \mapsto \rho_3 + \rho_2)$$

$$\sim \begin{pmatrix}1 & 0 & 1 & 1\\0 & 1 & 1 & -1\\0 & 0 & 0 & 0\end{pmatrix} = \text{rref}(B). \qquad (\rho_1 \mapsto \rho_1 - 2\rho_2)$$

So we have

$$\text{Row}(B) = \text{Span}\left\{\begin{pmatrix}1\\0\\1\\1\end{pmatrix}, \begin{pmatrix}0\\1\\1\\-1\end{pmatrix}\right\} \tag{5}$$

$$\text{Col}(B) = \text{Span}\left\{\begin{pmatrix}1\\3\\1\end{pmatrix}, \begin{pmatrix}2\\5\\1\end{pmatrix}\right\} \tag{6}$$

$$\text{Null}(B) = \text{Span}\left\{\begin{pmatrix}-1\\-1\\1\\0\end{pmatrix}, \begin{pmatrix}-1\\1\\0\\1\end{pmatrix}\right\} \tag{7}$$

$$\text{rk}(B) = 2 \tag{8}$$

(c) We find

$$\begin{aligned}
C &= \begin{pmatrix} 1 & 2 & 1 & 0 \\ 2 & 5 & 3 & -1 \\ 2 & 2 & 0 & 2 \\ 0 & 1 & 1 & -1 \end{pmatrix} \\
&\sim \begin{pmatrix} 1 & 2 & 1 & 0 \\ 0 & 1 & 1 & -1 \\ 2 & 2 & 0 & 2 \\ 0 & 1 & 1 & -1 \end{pmatrix} && (\rho_2 \mapsto \rho_2 - 2\rho_1) \\
&\sim \begin{pmatrix} 1 & 2 & 1 & 0 \\ 0 & 1 & 1 & -1 \\ 0 & -2 & -2 & 2 \\ 0 & 1 & 1 & -1 \end{pmatrix} && (\rho_3 \mapsto \rho_3 - 2\rho_1) \\
&\sim \begin{pmatrix} 1 & 2 & 1 & 0 \\ 0 & 1 & 1 & -1 \\ 0 & 0 & 0 & 0 \\ 0 & 1 & 1 & -1 \end{pmatrix} && (\rho_3 \mapsto \rho_3 + 2\rho_2) \\
&\sim \begin{pmatrix} 1 & 2 & 1 & 0 \\ 0 & 1 & 1 & -1 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \end{pmatrix} && (\rho_4 \mapsto \rho_4 + \rho_2) \\
&\sim \begin{pmatrix} 1 & 0 & -1 & 2 \\ 0 & 1 & 1 & -1 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \end{pmatrix} = \mathrm{rref}(C). && (\rho_4 \mapsto \rho_4 + \rho_2)
\end{aligned}$$

So we have

$$\mathrm{Row}(C) = \mathrm{Span}\left\{ \begin{pmatrix} 1 \\ 0 \\ -1 \\ 2 \end{pmatrix}, \begin{pmatrix} 0 \\ 1 \\ 1 \\ -1 \end{pmatrix} \right\} \tag{9}$$

$$\mathrm{Col}(C) = \mathrm{Span}\left\{ \begin{pmatrix} 1 \\ 2 \\ 2 \\ 0 \end{pmatrix}, \begin{pmatrix} 2 \\ 5 \\ 2 \\ 1 \end{pmatrix} \right\} \tag{10}$$

$$\mathrm{Null}(C) = \mathrm{Span}\left\{ \begin{pmatrix} 1 \\ -1 \\ 1 \\ 0 \end{pmatrix}, \begin{pmatrix} -2 \\ 1 \\ 0 \\ 1 \end{pmatrix} \right\} \tag{11}$$

$$\mathrm{rk}(C) = 2 \tag{12}$$

Problem 2 Let $S = \left\{ \begin{pmatrix} 1 \\ 2 \\ 2 \end{pmatrix}, \begin{pmatrix} 1 \\ 5 \\ 3 \end{pmatrix}, \begin{pmatrix} 0 \\ 6 \\ 2 \end{pmatrix}, \begin{pmatrix} 1 \\ -1 \\ 1 \end{pmatrix} \right\}$. Find a subset $T \subseteq S$ for which $\text{Span}(S) = \text{Span}(T)$. (Think about how to frame this as a question about a certain matrix.)

Solution.

Let $T = \left\{ \begin{pmatrix} 0 \\ 6 \\ 2 \end{pmatrix}, \begin{pmatrix} 1 \\ -1 \\ 1 \end{pmatrix} \right\}$, and observe that

$$\begin{pmatrix} 1 \\ 2 \\ 2 \end{pmatrix} = \frac{1}{2} \begin{pmatrix} 0 \\ 6 \\ 2 \end{pmatrix} + \begin{pmatrix} 1 \\ -1 \\ 1 \end{pmatrix} \text{ and } \begin{pmatrix} 1 \\ 5 \\ 3 \end{pmatrix} = \begin{pmatrix} 0 \\ 6 \\ 2 \end{pmatrix} + \begin{pmatrix} 1 \\ -1 \\ 1 \end{pmatrix}.$$

However, the two vectors in T are not linear combinations of each other because one has a non-zero x-component and one does not. So we have found a smallest $T \subseteq S$ such that $\text{Span}(S) = \text{Span}(T)$.

Problem 3 You are required to do either part (a) or part (b) (You can do both, if you want.)
(a) Suppose that $\{\vec{v}_1, \vec{v}_2, \vec{v}_3\} \subseteq \mathbb{R}^m$ is linearly **dependent**. Show, for all $\vec{v}_4 \in \mathbb{R}^m$, that $\{\vec{v}_1, \vec{v}_2, \vec{v}_3, \vec{v}_4\} \subseteq \mathbb{R}^m$ must also be linearly **dependent**. (When one adds vectors to a linearly dependent set, the resulting set is linearly dependent.)
(b) Suppose that $\{\vec{v}_1, \vec{v}_2, \vec{v}_3\} \subseteq \mathbb{R}^m$ is linearly **independent**. Show that $\{\vec{v}_1, \vec{v}_2\}$ must also be linearly **independent**. (When one removes some (but not all) vectors from a linearly independent set, the resulting set is linearly independent.)

Solution.

We have shown in class that a set S is linearly dependent if and only there is a vector $v \in S$ such that v can be expressed as a linear combination of vectors in $S - \{v\}$.

(a) Since $\{\vec{v_1}, \vec{v_2}, \vec{v_3}\}$ is linearly dependent, we have some vector $\vec{v}$ in the set that is a linear combination of the other two vectors. After adding any vector $\vec{v_4}$ to the set, v is a still a linear combination of the other vectors (simply add $0\vec{v_4}$ to the linear combination from before). So the set is still linearly dependent.

(b) Since $\{\vec{v_1}, \vec{v_2}, \vec{v_3}\}$ is linearly independent, no vector is a linear combination of the other two, so no two vectors are co-linear. So removing one vector will not change the other vectors to be colinear, and the set will still be linearly independent.

Problem 4
(a) Suppose that $A \in \text{Mat}_{4\times 6}$. Prove that the **columns** of A must be linearly dependent.
(b) Suppose that $A \in \text{Mat}_{6\times 4}$. Prove the the **rows** of A must be linearly dependent.
Your proofs can be very brief.

Solution.

We have shown in class that it takes at most m vectors to span $\mathbb{R}^m$.

(a) The columns of A will be in $\mathbb{R}^4$, and there will be 6 of them. From the result, at least $6 - 4 = 2$ of the vectors will be in the span of the other vectors, so the columns must be linearly dependent.

(b) The rows of A will be in $\mathbb{R}^4$, and there will be 6 of them. By the same reasoning as (a), the rows must be linearly dependent. $\square$

Problem 5 Suppose that $\{\vec{v}_1, \vec{v}_2, \vec{v}_3\} \subseteq \mathbb{R}^m$ is linearly independent. Show that the set

$$S = \{\vec{v}_1, \quad \vec{v}_1 + \vec{v}_2, \quad \vec{v}_1 + \vec{v}_2 + \vec{v}_3\}$$

is also linearly independent. (Write down the dependency vector equation for S; then simplify the equation so that you can use the hypothesis.)

Solution.

Since $\{\vec{v_1}, \vec{v_2}\vec{v_3}\}$ is linearly independent, we have that $x_1\vec{v_1} + x_2\vec{v_2} + x_3\vec{v_3} = \vec{0}$ has only the trivial solution. So $x_1 = x_2 = x_3 = 0$.

We now solve the dependency equation for $\{\vec{v_1}, \vec{v_1}+\vec{v_2}, \vec{v_1}+\vec{v_2}+\vec{v_3}\}$. We want to find all $x_1', x_2', x_3' \in \mathbb{R}$ such that $x_1'\vec{v_1}+x_2'(\vec{v_1}+\vec{v_2})+x_3'(\vec{v_1}+\vec{v_2}+\vec{v_3}) = \vec{0}$. Distributing, we have $(x_1'+x_2'+x_3')\vec{v_1}+(x_2'+x_3')\vec{v_2}+x_3'\vec{v_3} = \vec{0}$. Substituting from above, we have the homogenous system

$$\begin{array}{ccccccccc} x_1' & + & x_2' & + & x_3' & = & x_1 & = & 0 \\ & & x_2' & + & x_3' & = & x_2 & = & 0 \\ & & & & x_3' & = & x_3 & = & 0 \end{array}.$$

We can solve it with

$$\left(\begin{array}{ccc|c} 1 & 1 & 1 & 0 \\ 0 & 1 & 1 & 0 \\ 0 & 0 & 1 & 0 \end{array}\right)$$

$$\sim \left(\begin{array}{ccc|c} 1 & 1 & 1 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & 0 \end{array}\right) \qquad (\rho_2 \mapsto \rho_2 - \rho_3)$$

$$\sim \left(\begin{array}{ccc|c} 1 & 1 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & 0 \end{array}\right) \qquad (\rho_1 \mapsto \rho_1 - \rho_3)$$

$$\sim \left(\begin{array}{ccc|c} 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & 0 \end{array}\right), \qquad (\rho_1 \mapsto \rho_1 - \rho_2)$$

implying that the only solution is $x_1' = x_2' = x_3' = 0$. So $x_1'\vec{v_1} + x_2'(\vec{v_1} + \vec{v_2}) + x_3'(\vec{v_1} + \vec{v_2} + \vec{v_3}) = \vec{0}$ has only the trivial solution, and therefore $\{\vec{v_1}, \vec{v_1} + \vec{v_2}, \vec{v_1} + \vec{v_2} + \vec{v_3}\}$ is linearly independent. $\square$

Problem 6 Give an example of matrices $A, B \in \text{Mat}_{2\times 2}$ which are row-equivalent, but which have $\text{Col}(A) \neq \text{Col}(B)$. Your example does not need to be complicated. (This shows that row-equivalence **does not**, in general, preserve column spaces. On the other hand, it **does** preserve row spaces.)

Solution.

Let $A = \begin{pmatrix} 1 & 0 \\ 0 & 0 \end{pmatrix}$ and $B = \begin{pmatrix} 0 & 0 \\ 1 & 0 \end{pmatrix}$. Then, $A \sim B$ because we have simply switched the rows. Then, $\text{Row}(A) = \text{Span}\left\{\begin{pmatrix} 1 \\ 0 \end{pmatrix}\right\} = \text{Row}(B)$. However, $\text{Col}(A) = \text{Span}\left\{\begin{pmatrix} 1 \\ 0 \end{pmatrix}\right\}$ while $\text{Col}(B) = \text{Span}\left\{\begin{pmatrix} 0 \\ 1 \end{pmatrix}\right\}$.

These spans are not equal: in particular, there is no way to express $\begin{pmatrix}1\\0\end{pmatrix}$ and $\begin{pmatrix}0\\1\end{pmatrix}$ as linear combinations of each other because they are not colinear. So $\text{Col}(A) \neq \text{Col}(B)$.

Problem 7 Let $A \in \text{Mat}_{m\times n}$ have rank n, and suppose that $\{\vec{v}_1, \ldots, \vec{v}_k\} \subset \mathbb{R}^n$ is linearly independent. Show that $\{A\vec{v}_1, \ldots, A\vec{v}_k\} \subset R^m$ is linearly independent.

Solution.

We can write the dependency equation of $\{A\vec{v_1}, \ldots, A\vec{v_k}\}$ as $x_1A\vec{v_1} + \ldots + x_kA\vec{v_k} = \vec{0}$. By the distributive property, this is equivalent to $A\vec{v} = \vec{0}$, where $\vec{v} = x_1\vec{v_1} + \ldots + x_k\vec{v_k}$. Since A has rank n and n columns, the equation $A\vec{v} = \vec{0}$ has $n - n = 0$ independent variables, and since it is homogenous, it must have only the solution $\vec{v} = 0$. So we have $x_1\vec{v_1} + \ldots + x_k\vec{v_k} = \vec{0}$, and because $\{\vec{v_1}, \ldots, \vec{v_k}\}$ is linearly independent, this has only the trivial solution $x_1 = \ldots = x_k = 0$. It follows that $x_1A\vec{v_1} + \ldots + x_kA\vec{v_k} = \vec{0}$ also only has the trivial solution, and therefore $\{A\vec{v_1}, \ldots, A\vec{v_k}\}$ is also linearly independent. $\square$