MATH 552: Section 001
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# MATH 552 Homework 7*

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**Problem 38.7** In Sec. 37 use expressions (13) and (14) to derive expressions (15) and (16) for $|\sin z|^2$ and $|\cos z|^2$.
*Suggestion:* Recall the identities $\sin^2 x + \cos^x = 1$ and $\cosh^2 y - \sinh^2 y = 1$.

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Solution.

(a) We have $\sin z = \sin x \cosh y + i \cos x \sinh y$.

$$\begin{aligned}
|\sin z|^2 &= (\sin z)(\overline{\sin z}) && \text{(identity)}\\
&= (\sin x \cosh y + i \cos x \sinh y)(\sin x \cosh y - i \cos x \sinh y) && \text{(substitution)}\\
&= \sin^2 x \cosh^2 y + \cos^2 x \sinh^2 y\\
&= \sin^2 x(1+\sinh^2 y) + (1-\sin^2 x)\sinh^2 y && \text{(using identities in suggestion)}\\
&= \sin^2 x + \sin^2 x \sinh^2 y + \sinh^2 y - \sin^2 x \sinh^2 y\\
&= \sin^2 x + \sinh^2 y && \text{(expression (15))}
\end{aligned}$$

(b) We have $|\cos z|^2 = \cos^2 x + \sinh^2 y$.

$$\begin{aligned}
|\cos z|^2 &= (\cos z)(\overline{\cos z}) && \text{(identity)}\\
&= (\cos x \cosh y - i \sin x \sinh y)(\cos x \cosh y + i \sin x \sinh y) && \text{(substitution)}\\
&= \cos^2 x \cosh^2 y + \sin^2 x \sinh^2 y && \text{(using identities in suggestion)}\\
&= \cos^2 x(1+\sinh^2 y) + (1-\cos^2 x)\sinh^2 y\\
&= \cos^2 x + \cos^2 x \sinh^2 y + \sinh^2 y - \cos^2 x \sinh^2 y\\
&= \cos^2 x + \sinh^2 y && \text{(expression (16))}
\end{aligned}$$

**Problem 40.2** Prove that $\sinh 2z = 2 \sinh z \cosh z$ by starting with
(a) definitions (1), Sec. 39, of $\sinh z$ and $\cosh z$;
(b) the identity $\sin 2z = 2 \sin z \cos z$ (Sec. 37) and using relations (3) in Sec. 39

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Solution.

(a) We have $\sinh x = \frac{1}{2}(e^z - e^{-z})$ and $\cosh x = \frac{1}{2}(e^z + e^{-z})$.

$$\begin{aligned}
\sinh 2z &= \frac{e^{2z} - e^{-2z}}{2}\\
&= \frac{(e^z)^2 - (e^{-z})^2}{2} && \text{(rewriting exponents)}\\
&= \frac{(e^z + e^{-z})(e^z - e^{-z})}{2} && \text{(factoring difference of squares)}\\
&= 2\frac{(e^z - e^{-z})}{2}\frac{(e^z + e^{-z})}{2} && \text{(splitting fraction)}\\
&= 2 \sinh z \cosh z && \text{(using definitions of } \sinh z \text{ and } \cosh z\text{)}
\end{aligned}$$

(b) We have $\sin z = -i\sinh iz$, $\cos z = \cosh iz$, and $\sin 2z = 2\sin z\cos z$.

$$\begin{aligned}
\sin(-2iz) &= 2\sin(-iz)\cos(-iz) && \text{(replacing } z \text{ by } -iz)\\
\sin(-2iz) &= 2(-i\sinh(-i^2z))(\cosh(-i^2z)) && \text{(using identities)}\\
\sin(-2iz) &= 2(-i\sinh(z))(\cosh(z)) && \text{(using } -i^2 = 1)\\
-i\sinh(i(-2iz)) &= 2(-i\sinh(z))(\cosh(z)) && \text{(using } \sin z = -i\sinh iz)\\
-i\sinh z &= -2i\sinh z\cosh z && \text{(simplifying)}\\
\sinh z &= 2\sinh z\cosh z && \text{(dividing by } -i)
\end{aligned}$$

**Problem 40.3** Show how identities (6) and (8) in Sec. 39 follow from identities (9) and (6), respectively, in Sec. 37.

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Solution.

We write $\sin z = -i\sinh iz$ and $\cos z = \cosh iz$.

(a) We have $\sin^2 z + \cos^2 z = 1$.

$$\begin{aligned}
\sin^2 z &= -\sinh^2 iz\\
\cos^2 z &= \cosh^2 iz\\
-\sinh^2 iz + \cosh^2 iz &= 1 && \text{(using } \sin^2 z + \cos^2 z = 1)\\
\cosh^2 z - \sinh^2 z &= 1 && \text{(letting } z = -iz)
\end{aligned}$$

(b) We have $\cos z_1 + z_2 = \cos z_1\cos z_2 - \sin z_1\sin z_2$.

$$\begin{aligned}
\cos(z_1+z_2) &= \cosh(iz_1+iz_2)\\
&= \cos z_1\cos z_2 - \sin z_1\sin z_2 && \text{(using identity)}\\
&= \cosh(iz_1)\cosh(iz_2) - (-i\sinh(iz_1))(-i\sinh(iz_2))\\
&= \cosh(iz_1)\cosh(iz_2) + \sinh(iz_1)\sinh(iz_2)\\
\cosh(z_1+z_2) &= \cosh(z_1)\cosh(z_2) + \sinh(z_1)\sinh(z_2) && \text{(letting } z_1 = -iz_1 \text{ and } z_2 = -iz_2)
\end{aligned}$$