MATH 552: Section 001
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MATH 552 Homework 6

Problem 6 Given that the branch $\log z = \ln r + i\theta$ $(r > 0, \alpha < \theta < \alpha + 2\pi)$ of the logarithmic function is analytic at each point z in the stated domain, obtain its derivative by differentiating each side of the identity

$$e^{\log z} = z \quad (|z| > 0, \alpha < \arg z < \alpha + 2\pi)$$

and using the chain rule.

Solution.

Differentiating the right hand side of $e^{\log z}$, we get 1 by the power rule $(1z^{1-1} = 1)$.

We differentiate the left hand side by using $\frac{d}{dz}[e^z] = e^z$ and the chain rule:

$$\frac{d}{dz}[e^{\log z}] = \left(\frac{d}{dz}[\log z]\right)\left(e^{\log z}\right).$$

Since the functions are equal, their derivatives must also be equal, so

$$1 = \left(\frac{d}{dz}[\log z]\right)\left(e^{\log z}\right).$$

And since $e^{\log z} = z$, we can rewrite this as

$$\left(\frac{d}{dz}[\log z]\right) z = 1.$$

Therefore,

$$\frac{d}{dz}[\log z] = \frac{1}{z}.$$

Problem 7 Show that a branch

$$\log z = \ln r + i\theta \quad (r > 0, \alpha < \theta < \alpha + 2\pi)$$

of the logarithmic function can be written

$$\log z = \frac{1}{2}\ln(x^2 + y^2) + i\tan^{-1}\left(\frac{y}{x}\right)$$

in rectangular coordinates. Then, using the theorem in Sec. 23, show that the given branch is analytic in its domain of definition and that

$$\frac{d}{dz}\log z = \frac{1}{z}$$

there.

Solution.

Since r is defined as the distance from the origin, $r = \sqrt{x^2 + y^2}$ where x, y are the real and imaginary components, respectively, by the Pythagorean theorem. To get θ in rectangular coordinates, we use trigonometry to get $\theta = \arctan \frac{y}{x}$.

Rewriting $\log z = \ln r + i\theta$ with these substitutions, we get:

$$\begin{aligned}
\log z &= \ln \sqrt{x^2+y^2} + i \arctan \frac{y}{x} \\
&= \ln (x^2+y^2)^{1/2} + i \tan^{-1}\left(\frac{y}{x}\right) && \text{(rephrasing)} \\
&= \frac{1}{2}\ln (x^2+y^2) + i \tan^{-1}\left(\frac{y}{x}\right) && \text{(can pull the exponent out front since we're on a single branch)}
\end{aligned}$$

We can then use the Cauchy-Riemann equations to show that the given branch is analytic:

$$\begin{aligned}
u(x,y) &= \frac{1}{2}\ln (x^2+y^2) \\
v(x,y) &= \tan^{-1}\left(\frac{y}{x}\right) \\
u_x &= \left(\frac{1}{2}(2x)\right)\left(\frac{1}{x^2+y^2}\right) = \frac{x}{x^2+y^2} \\
u_y &= \left(\frac{1}{2}(2y)\right)\left(\frac{1}{x^2+y^2}\right) = \frac{y}{x^2+y^2} \\
v_x &= \left(\frac{y}{x^2}\right)\left(\frac{1}{\left(\frac{y}{x}\right)^2+1}\right) = -\frac{y}{x^2+y^2} \\
v_y &= \left(\frac{1}{x}\right)\left(\frac{1}{\left(\frac{y}{x}\right)^2+1}\right) = \frac{x}{x^2+y^2}
\end{aligned}$$

So

$$u_x = \frac{x}{x^2+y^2} = v_y \text{ and } u_y = \frac{y}{x^2+y^2} = -v_x,$$

and thus the Cauchy-Riemann equations hold. Other than $(x, y) \neq 0$, which is not in the domain since $r > 0$, the partials are well-defined and differentiable everywhere, so the function is analytic on the domain.

Using $f'(z) = u_x + iv_x$ for $f(z) = \log z$,

$$\frac{d}{dz}[\log z] = \frac{x}{x^2+y^2} - i\frac{y}{x^2+y^2} = \frac{x-iy}{x^2+y^2}.$$

Since

$$\frac{1}{z} = \frac{1}{x+iy} = \frac{x-iy}{(x+iy)(x-iy)} = \frac{x-iy}{x^2+y^2},$$

it follows that for the domain of definition,

$$\frac{d}{dz}[\log z] = \frac{x-iy}{x^2+y^2} = \frac{1}{z}.$$