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MATH 546: Section
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# MATH 546 Homework 4

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**Problem 1** Let $G$ be a group, and let $H$ and $K$ be subgroups of $G$. If $H \not\subseteq K$ and $K \not\subseteq H$, prove that $H \cup K$ cannot be a subgroup of $G$.

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Solution.

Suppose toward contradiction that $H \cup K$ is a subgroup of $G$. Since $H \not\subseteq K$, there must exist some $h \in H, h \notin K$. Similarly since $K \not\subseteq H$, there must exist some $k \in K$, $k \notin H$. By definition, we have $h, k \in H \cup K$, and since we have closure, we have $h * k \in H \cup K$. So we either have $h * k \in H$ or $h * k \in K$.

Case 1: $h * k \in H$. Since $H$ is a subgroup, we have $h^{-1} \in H$, and from closure we have

$$h^{-1} * (h * k) = (h^{-1} * h) * k = e * k = k \in H,$$

a contradiction since we chose $k \notin H$.

Case 2: $h * k \in K$. Similarly to case 1, we have $k^{-1} \in K$, so we have

$$(h * k) * (k^{-1}) = h * (k * k^{-1}) = h * e = h \in K,$$

a contradiction.

So $H \cup K$ cannot be a subgroup of $G$ if $H \not\subseteq K$ and $K \not\subseteq H$. □

**Problem 2** Let $G$ be an abelian group, and let $H$ and $K$ be subgroups of $G$. Consider the set $HK$ defined below:

$$HK = \{hk \mid h \in H, k \in K\}$$

Prove that $HK$ is a subgroup of $G$.

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Solution.

Let $x, y \in HK$. We will show $xy^{-1} \in HK$. Let $h_x, h_y \in H$ and $k_x, k_y \in K$ such that $x = h_x k_x$ and $y = h_y k_y$. Then the inverse of $y$ in $G$ is ${k_y}^{-1} {h_y}^{-1}$. Since $G$ is abelian, we can write

$$xy^{-1} = h_x k_x {k_y}^{-1} {h_y}^{-1} = h_x {h_y}^{-1} k_x {k_y}^{-1}.$$

Since $H$ and $K$ are subgroups, we have ${h_y}^{-1} \in H$ and ${k_y}^{-1} \in K$. From closure, we have $h_x {h_y}^{-1} \in H$ and $k_x {k_y}^{-1} \in K$. So $xy^{-1} = hk$ for $h = h_x {h_y}^{-1}$ and $k = k_x {k_y}^{-1}$, and therefore $xy^{-1} \in HK$ by definition. So $HK$ is a subgroup from the result we showed in class. □

**Problem 3** Let $G$ be an abelian group. Consider the sets defined below:

$$H = \{a \in G \mid a^3 = e\}, \quad K = \{a^3 \mid a \in G\}.$$

(a) Prove that $H$ and $K$ are subgroups of $G$.

(b) If $G = \mathbb{Z}_{24}$, list all the elements of $H$ and list all the elements of $K$.

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Solution.

**(a)** First, let $x, y \in H$. Then $x^3 = y^3 = e$. Since $G$ is abelian, we can write

$$(xy^{-1})^3 = x^3(y^{-1})^3 = x^3(y^3)^{-1} = ee^{-1} = e,$$

so $xy^{-1} \in H$. Thus, $H$ is a subgroup.

Next, let $x, y \in K$. Then there exist $a, b \in G$ such that $x = a^3$ and $y = b^3$. Since $G$ is abelian, we can write

$$xy^{-1} = a^3(b^3)^{-1} = (ab^{-1})^3,$$

and since $ab^{-1} \in G$, we have $xy^{-1} \in K$. Thus, $K$ is a subgroup.

**(b)** We have

$$H = \left\{[0]_{24}, [8]_{24}, [16]_{24}\right\} \text{ and } K = \left\{[0]_{24}, [3]_{24}, [6]_{24}, [9]_{24}, [12]_{24}, [15]_{24}, [18]_{24}, [21]_{24}\right\}.$$

**Problem 4** Let $G$ be a group, and let $a \in G$ be a fixed element. Consider the set defined below:

$$C(a) = \{x \in G \mid a * x = x * a\}$$

(a) Prove that $C(a)$ is a subgroup of $G$.

(b) Find $C(A)$ if $G = \mathrm{GL}_2(\mathbb{R})$, the group of $2 \times 2$ invertible matrices with multiplication of matrices as operation, and

$$A = \begin{bmatrix} 1 & -1 \\ 1 & 0 \end{bmatrix}.$$

Prove your answer.

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Solution.

**(a)** Let $x, y \in C(a)$. We will show $xy^{-1} \in C(a)$. We have $ax = xa$ and $ay = ya$ by definition. As a result,

$$a = axx^{-1} = xax^{-1} \text{ and } a = y^{-1}ya = y^{-1}ay.$$

So we can write

$$\begin{aligned} axy^{-1} &= xax^{-1}xy^{-1} && (a = xax^{-1}) \\ &= xay^{-1} && (x^{-1}x = e) \\ &= xy^{-1}ayy^{-1} && (a = y^{-1}ay) \\ &= xy^{-1}a, && (yy^{-1} = e) \end{aligned}$$

and thus $xy^{-1} \in C(a)$. Therefore, $C(a)$ is a subgroup.

**(b)** Any matrices in $C(A)$ will satisfy

$$\begin{bmatrix} a & b \\ c & d \end{bmatrix} \begin{bmatrix} 1 & -1 \\ 1 & 0 \end{bmatrix} = \begin{bmatrix} 1 & -1 \\ 1 & 0 \end{bmatrix} \begin{bmatrix} a & b \\ c & d \end{bmatrix},$$

which implies

$$\begin{bmatrix} a+b & -a \\ c+d & -c \end{bmatrix} = \begin{bmatrix} a-c & b-d \\ a & b \end{bmatrix}$$

and thus we can solve the system

$$\begin{cases} a + b = a - c \\ -a = b - d \\ c + d = a \\ -c = b \end{cases} \implies \begin{cases} b + c = 0 \\ a + b + \quad - d = 0 \\ a \quad - c - d = 0 \\ b + c = 0 \end{cases}$$

We can use Gauss-Jordan elimination to reduce

$$\left[\begin{array}{cccc|c} 0 & 1 & 1 & 0 & 0 \\ 1 & 1 & 0 & -1 & 0 \\ 1 & 0 & -1 & -1 & 0 \\ 0 & 1 & 1 & 0 & 0 \end{array}\right] \longrightarrow \left[\begin{array}{cccc|c} 1 & 0 & -1 & -1 & 0 \\ 0 & 1 & 1 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 \end{array}\right].$$

Thus, we have $a = c + d$ and $b = -c$ for any arbitrary $c, d \in \mathbb{R}$, so

$$C(A) = \left\{ \begin{bmatrix} c+d & -c \\ c & d \end{bmatrix} : c, d \in \mathbb{R}, cd + c^2 + d^2 \neq 0 \right\},$$

with the second condition in place so that the determinant is non-zero and the matrix is invertible.

**Problem 5** Let $G$ be a group. Consider the set defined below:

$$Z(G) = \{x \in G \mid \forall a \in G, x * a = a * x\}$$

(a) Explain the relationship between the set $Z(G)$ and the sets $C(a)$ defined in problem 4.

(b) Prove that $Z(G)$ is a subgroup of $G$.

(c) What is a necessary and sufficient condition for $Z(G)$ to be equal to $G$? Explain.

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Solution.

**(a)** We can express $Z(G)$ in terms of $C(a)$ with

$$Z(G) = \{a \in G : C(a) = G\}:$$

$Z(G)$ is the set of elements that commute with every element in $G$, and each such element will have $C(a) = G$.

**(b)** This can be done in a similar way to 4. Let $x, y \in Z(G)$, and let $a \in G$. We will show $xy^{-1} \in Z(G)$. We have $ax = xa$ and $ay = ya$ by definition. As a result,

$$a = axx^{-1} = xax^{-1} \text{ and } a = y^{-1}ya = y^{-1}ay.$$

So for all $a \in G$, we can write

$$\begin{aligned} axy^{-1} &= xax^{-1}xy^{-1} && (a = xax^{-1}) \\ &= xay^{-1} && (x^{-1}x = e) \end{aligned}$$

$$
\begin{aligned}
&= xy^{-1}ayy^{-1} && (a = y^{-1}ay) \\
&= xy^{-1}a, && (yy^{-1} = e)
\end{aligned}
$$

and thus $xy^{-1} \in Z(G)$. Therefore, $Z(G)$ is a subgroup.

**(c)** We claim that $G$ is abelian if and only if $Z(G) = G$.

($\Rightarrow$) Suppose $G$ is abelian. Let $x \in G$. Since $G$ is abelian, $xa = ax$ for all $a \in G$, so $x \in Z(G)$. So $G \subseteq Z(G)$, and clearly $Z(G) \subseteq G$, so $Z(G) = G$.

($\Leftarrow$) Suppose $Z(G) = G$. Let $a, b \in G$. Then in particular, $a \in Z(G)$. Since $ax = xa$ for all $x \in G$, $ab = ba$. So $G$ is commutative by definition.

Thus $G$ being abelian is a necessary and sufficient condition for $Z(G)$ to be equal to $G$. $\square$

**Problem 6** Let $G = \mathrm{GL}_2(\mathbb{R})$, the group of $2 \times 2$ invertible matrices with multiplication of matrices as operation. Find $Z(G)$. Prove your answer.

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Solution.

We claim that $Z(G) = \{\lambda I_2 \mid \lambda \in \mathbb{R}^*\}$. We will show a double-inclusion.

($\supseteq$) We have

$$
\begin{bmatrix} \lambda & 0 \\ 0 & \lambda \end{bmatrix} \begin{bmatrix} a & b \\ c & d \end{bmatrix} = \begin{bmatrix} \lambda a & \lambda b \\ \lambda c & \lambda d \end{bmatrix} = \begin{bmatrix} a & b \\ c & d \end{bmatrix} \begin{bmatrix} \lambda & 0 \\ 0 & \lambda \end{bmatrix},
$$

and since $\det(\lambda I) = \lambda^2 > 0$ since $\lambda \neq 0$, we have $\lambda I \in \mathrm{LG}(\mathbb{R})_2$. So $\lambda I_2 \in Z(G)$ since all matrices commute with it. Thus, $Z(G) \supseteq \{\lambda I_2 \mid \lambda \in \mathbb{R}^*\}$

($\subseteq$) We will find all matrices $X = \begin{bmatrix} a & b \\ c & d \end{bmatrix}$ such that $X$ commutes with every matrix in $G$. Since

$$
\begin{vmatrix} 0 & 0 \\ 1 & 0 \end{vmatrix} = \begin{vmatrix} 0 & 1 \\ 0 & 0 \end{vmatrix} = \begin{vmatrix} 0 & 1 \\ 1 & 0 \end{vmatrix} = -1 \neq 0,
$$

all of these matrices are in $G$ and thus $X$ must commute for all of them. So we have

$$
\begin{bmatrix} 0 & 0 \\ 1 & 0 \end{bmatrix} \begin{bmatrix} a & b \\ c & d \end{bmatrix} = \begin{bmatrix} a & b \\ c & d \end{bmatrix} \begin{bmatrix} 0 & 0 \\ 1 & 0 \end{bmatrix} \implies \begin{bmatrix} 0 & 0 \\ a & b \end{bmatrix} = \begin{bmatrix} b & 0 \\ d & 0 \end{bmatrix} \implies b = 0,
$$

$$
\begin{bmatrix} 0 & 1 \\ 0 & 0 \end{bmatrix} \begin{bmatrix} a & b \\ c & d \end{bmatrix} = \begin{bmatrix} a & b \\ c & d \end{bmatrix} \begin{bmatrix} 0 & 1 \\ 0 & 0 \end{bmatrix} \implies \begin{bmatrix} c & d \\ 0 & 0 \end{bmatrix} = \begin{bmatrix} 0 & a \\ 0 & c \end{bmatrix} \implies c = 0,
$$

$$
\begin{bmatrix} 0 & 1 \\ 1 & 0 \end{bmatrix} \begin{bmatrix} a & b \\ c & d \end{bmatrix} = \begin{bmatrix} a & b \\ c & d \end{bmatrix} \begin{bmatrix} 0 & 1 \\ 1 & 0 \end{bmatrix} \implies \begin{bmatrix} c & d \\ a & b \end{bmatrix} = \begin{bmatrix} b & a \\ d & c \end{bmatrix} \implies a = d.
$$

Thus, for any $a \in \mathbb{R}^*$, $X = aI_2$. Thus, $Z(G) \subseteq \{\lambda I_2 \mid \lambda \in \mathbb{R}^*\}$.

Therefore, $Z(G) = \{\lambda I_2 \mid \lambda \in \mathbb{R}^*\}$. $\square$