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MATH 554 Homework 5

Problem 1 Let $f : [a, b] \to \mathbb{R}$ be a function such that $f(a) \leq 0$, $f(b) \geq 0$, and there exists an $M > 0$ such that for all $x_1, x_2 \in [a, b]$, the inequality $|f(x_2) - f(x_1)| \leq M|x_2 - x_1|$ holds. Prove that there is a number $\xi \in [a, b]$ with $f(\xi) = 0$.

Let $S = \{x \in [a, b] : f(x) < 0\}$. Since $f(a) < 0$, a is in S, and since $f(b) > 0$, b is an upper bound for S. Thus, S has a supremum $\xi \in \mathbb{R}$. Let $\varepsilon > 0$. Since $\xi = \sup(S)$, there exists some $x_1 \in S$ such that $\xi - \varepsilon < x_1 < \xi$ (or $\xi - \varepsilon$ would be an upper bound). Similarly, there exists some $x_2 \notin S$ such that $\xi < x_2 < \xi + \varepsilon$ (or ξ would not be an upper bound). Since the function is Lipschitz, there exists some M where we can write

$$\begin{aligned}
f(\xi) &= f(x_1) + f(\xi) - f(x_1) \\
&< f(\xi) - f(x_1) && (x_1 \in S \implies f(x_1) < 0) \\
&\leq |f(\xi) - f(x_1)| \\
&\leq M|\xi - x_1| && (f \text{ is Lipschitz}) \\
&< M\varepsilon, && (\text{chose } \xi - \varepsilon < x_1 < \xi)
\end{aligned}$$

and similarly

$$\begin{aligned}
f(\xi) &= f(x_2) + f(\xi) - f(x_2) \\
&\geq f(\xi) - f(x_2) && (x_1 \notin S \implies f(x_2) \geq 0) \\
&\geq -|f(\xi) - f(x_2)| \\
&\geq -M|\xi - x_2| \\
&> -M\varepsilon.
\end{aligned}$$

This implies that for all $\varepsilon > 0$, $|f(\xi)| < M\varepsilon$, which we have previously shown implies that $f(\xi) = 0$. $\square$

Problem 2 Prove that on a bounded interval $[a, b]$, the function $f(x) = x^n$ is Lipschitz for any positive integer n.

Let $n \in \mathbb{N}$, and let $f : [a, b] \to \mathbb{R}$ be defined by $f(x) = x^n$ for all $x \in [a, b]$. Let $C = \max\{a, b\}$. Then, $|x| \leq C$ for all x. Let $x_1, x_2 \in [a, b]$. We can then write

$$\begin{aligned}
|f(x_1) - f(x_2)| &= |{x_1}^n - {x_2}^n| \\
&= |x_1 - x_2| \left| \sum_{k=0}^{n-1} {x_1}^{n-k-1} {x_2}^k \right| && (\text{factoring}) \\
&\leq |x_1 - x_2| \sum_{k=0}^{n-1} |x_1|^{n-k-1} |x_2|^k && (\text{triangle inequality}) \\
&\leq |x_1 - x_2| \sum_{k=0}^{n-1} C^{n-k-1} C^k && (|x_1|, |x_2| < C)
\end{aligned}$$

$$= |x_1 - x_2|(nC^{n-1}). \qquad \text{(evaluating sum)}$$

So with the choice of $M = nC^{n-1}$, we have that $|f(x_1) - f(x_2)| \leq M|x_1 - x_2|$ for all $x_1, x_2 \in [a, b]$. Therefore f is Lipschitz on $[a, b]$. □

Problem 3 Show that $f(x) = x^2$ is not Lipschitz on the interval $[0, \infty)$.

Let $M \geq 0$, and consider $x_1 = M + 1, x_2 = M \in [0, \infty)$. Then,

$$\begin{aligned}
|f(x_1) - f(x_2)| &= |f(M+1) - f(M)| \\
&= |(M+1)^2 - M^2| \\
&= |M^2 + 2M + 1 - M^2| \\
&= |2M + 1| \\
&= 2M + 1 && \text{(since } M > 0) \\
&> M \\
&= M|1| \\
&= M|(M+1) - M| \\
&= M|x_1 - x_2|.
\end{aligned}$$

So for all $M > 0$, there exist $x_1, x_2 \in [0, \infty)$ such that $|f(x_1) - f(x_2)| \leq M|x_1 - x_2|$ does not hold. Therefore, f is not Lipschitz on $[0, \infty)$. □

Problem 4 Prove that if n is a positive integer, then every positive real number has a positive n-th root.

Let $c \in \mathbb{R}^+$. We will find a positive n-th root for c. Consider $f(x) = x^n - c$, which is Lipschitz on $[0, c+1]$ because it is a polynomial. Since $c > 0$, we have

$$f(0) = 0^n - c = -c < 0.$$

We also have

$$\begin{aligned}
f(c+1) &= (c+1)^n - c \\
&= \sum_{k=0}^{n} \left[\binom{n}{k} c^{n-k} \right] - c && \text{(binomial theorem)} \\
&= \sum_{k=0}^{n-2} \left[\binom{n}{k} c^{n-k} \right] + \binom{n}{n-1} c + \binom{n}{n} - c && \text{(splitting sum)} \\
&= \sum_{k=0}^{n-2} \left[\binom{n}{k} c^{n-k} \right] + (n-1)c + 1 && (\tbinom{n}{n-1} = n) \\
&> 0. && \text{(since } c^k > 0 \text{ for all } k \in \mathbb{R})
\end{aligned}$$

Thus, by the intermediate value theorem, there exists some $\xi \in (0, c+1)$ such that $f(\xi) = 0$. So $\xi^n - c = 0$, and thus $\xi^n = c$. So $\xi > 0$ is an n-th root for c. □

Problem 5 Let $p(x) = x^3 + ax^2 + bx + c$. Prove that $p(x)$ has at least one real root.

Let $S = |a| + |b| + |c|$, and fix $x \in \mathbb{R}$ such that $x \geq \max\{1, 2S\}$. We can write

$$p(x) = x^3 + ax^2 + bx + c = x^3 \left(1 + \frac{a}{x} + \frac{b}{x^2} + \frac{c}{x^3}\right) = (1 + M)x^3,$$

where $M = \frac{a}{x} + \frac{b}{x^2} + \frac{c}{x^3}$. We have

$$\begin{aligned}
|M| &= \left|\frac{a}{x} + \frac{b}{x^2} + \frac{c}{x^3}\right| \\
&\leq \frac{|a|}{x} + \frac{|b|}{x^2} + \frac{|c|}{x^3} && \text{(triangle inequality)} \\
&\leq \frac{|a|}{x} + \frac{|b|}{x} + \frac{|c|}{x} && (x \leq x^2 \leq x^3 \text{ since } x \geq 1) \\
&= \frac{S}{x} \\
&\leq \frac{S}{2S} && (\text{since } x \geq 2S) \\
&= \frac{1}{2}.
\end{aligned}$$

This implies that $\frac{1}{2} \leq 1 + M \leq \frac{3}{2}$, so $1 + M$ is positive. Since x is positive, so is x^3, and since $1 + M$ is positive, so is $(1 + M)x^3$. Thus, $p(x) > 0$. Similarly, since $-x$ is negative, so is $(-x)^3$, and since $1 + M$ is positive, $(1 + M)(-x)^3$ is negative. Thus, $p(-x) < 0$.

Since every polynomial is Lipschitz over any bounded interval, p is Lipschitz over $[-x, x]$. By the Lipschitz IVT, there exists some $\xi \in [-x, x]$ such that $p(\xi) = 0$ since $p(-x) < 0$ and $p(x) > 0$. Therefore, we have at least one root $\xi \in \mathbb{R}$ of p. $\square$