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CSCE 350: Section H01
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February 10, 2023

CSCE 350 Homework 2

Problem 1 For each of the following functions, find a function $g(n)$ such that $f(n) \in \Theta(g(n))$. You must use the simplest $g(n)$ possible in your answers such as n, $\log n$, $n \log n$, n^2, n^3, a^n, and product of them. Prove your assertion.
(a) $(n^3+1)^2$
(b) $\sqrt{9n} + 9 \log n$
(c) $2n \log n^2 + (n+1)^2 \log n$
(d) $3^{n+2} + 4^{n-2}$

Solution.

(a) We have $(n^3+1)^2 = n^6 + 2n^3 + 1 \in \Theta(n^6)$. Observe that

$$\begin{aligned}
\lim_{n\to\infty} \frac{(n^3+1)^2}{n^6} &= \lim_{n\to\infty} \frac{6n^2(n^3+1)}{6n^5} && \text{(L'Hôpital's rule)} \\
&= \lim_{n\to\infty} \frac{n^3+1}{n^3} && \text{(simplifying)} \\
&= \lim_{n\to\infty} \frac{3n^2}{3n^2} = 1, && \text{(L'Hôpital's rule)}
\end{aligned}$$

so n^6 has the same growth order because the limit is constant.

(b) We have $\sqrt{9n} + 9 \log n \in \Theta(\sqrt{n})$. Observe that

$$\begin{aligned}
\lim_{n\to\infty} \frac{3\sqrt{n} + 9\log n}{\sqrt{n}} &= \lim_{n\to\infty} \frac{3\sqrt{n}}{\sqrt{n}} + \lim_{n\to\infty} \frac{9 \log n}{\sqrt{n}} && \text{(splitting fraction/limits)} \\
&= 3 + \lim_{n\to\infty} \frac{9/n}{1/(2\sqrt{n})} && \text{(L'Hôpital's rule)} \\
&= 3 + \lim_{n\to\infty} \frac{18}{\sqrt{n}} && \text{(simplifying)} \\
&= 3 + 0 = 3, && \text{(denominator tends to } \infty\text{)}
\end{aligned}$$

so $\sqrt{n}$ has the same growth order because the limit is constant.

(c) We have $2n \log n^2 + (n+1)^2 \log n \in \Theta(n^2 \log n)$. Observe that

$$\begin{aligned}
\lim_{n\to\infty} \frac{2n \log n^2 + (n+1)^2 \log n}{n^2 \log n} &= \lim_{n\to\infty} \frac{n^2 \log n}{n^2 \log n} + \lim_{n\to\infty} \frac{6n \log n}{n^2 \log n} + \lim_{n\to\infty} \frac{\log n}{n^2 \log n} && \text{(simplifying)} \\
&= 1 + \lim_{n\to\infty} \frac{6}{n} + \lim_{n\to\infty} \frac{1}{n^2} && \text{(simplifying)} \\
&= 1 + 0 + 0 = 1, && \text{(second/third denominators tend to } \infty\text{)}
\end{aligned}$$

so $n^2 \log n$ has the same growth order because the limit is constant.

(d) We have $3^{n+2} + 4^{n-2} \in \Theta(4^n)$. Observe that

$$\lim_{n\to\infty} \frac{3^{n+2} + 4^{n-2}}{4^n} = 9 \lim_{n\to\infty} \left(\frac{3}{4}\right)^n + \frac{1}{4} \lim_{n\to\infty} 1 \qquad \text{(splitting limits)}$$

$$= 0 + \frac{1}{4} = \frac{1}{4}, \qquad (a^n \to 0 \text{ when } a < 1)$$

so 4^n has the same growth order because the limit is constant.

Problem 2 Prove that
(a) every polynomial of degree k, $p(n) = a_k n^k + a_{k-1}n^{k-1} + \ldots + a_0$, with $a_k > 0$ belongs to $\Theta(n^k)$.
(b) exponential functions a^n have different orders of growth for different values of base $a > 0$.

Solution.

(a) We observe that

$$\begin{aligned}
\lim_{n\to\infty} \frac{a_k n^k + a_{k-1}n^{k-1} + \ldots + a_0}{n^k} &= \lim_{n\to\infty} \frac{a_k n^k}{n^k} + \lim_{n\to\infty} \frac{a_{k-1}n^{k-1}}{n^k} + \ldots + \lim_{n\to\infty} \frac{a_0}{n^k} \\
&= a_k \lim_{n\to\infty} 1 + a_{k-1} \lim_{n\to\infty} \frac{1}{n} + \ldots + a_0 \lim_{n\to\infty} \frac{1}{n^k} \\
&= a_k, \qquad \text{(denominators of all but the first limits tend to } \infty\text{)}
\end{aligned}$$

so a^k has the same growth order because the limit is constant.

(b) Let $a_1, a_2 \in \mathbb{R}^+$ such that $a_1 \neq a_2$. Then, we observe that

$$\lim_{n\to\infty} \frac{a_1^n}{a_2^n} = \lim_{n\to\infty} \left(\frac{a_1}{a_2}\right)^n.$$

We have that for any a^n tends to ∞ for $a > 1$ and tends to 0 for $a < 1$, so the only way this limit is constant is if $\frac{a_1}{a_2} = 1$, which implies they are equal. But we assumed they are not equal, so the limit is 0 or ∞ and the growth orders must be different.

Problem 3 Find the order of growth of the following sums. You need to indicate the class $\Theta(g(n))$ the function belongs to. You must use the simplest $g(n)$ possible in your answers.
(a) $\sum_{i=0}^{n} (i^2 + 1)^2$
(b) $\sum_{i=1}^{n} n \log(i^2)$
(c) $\sum_{i=0}^{n} (i + 2)2^i$
(d) $\sum_{i=0}^{n} \sum_{j=0}^{i-1} (i + j)$

Solution.

We will use the results from the appendix that as $n \to \infty$,

$$\sum_{i=0}^{n} i^k = \frac{i^{k+1}}{k+1} \tag{1}$$

$$\sum_{i=1}^{n} \log n = n \log n \tag{2}$$

$$\sum_{i=1}^{n} i2^i = (n-1)2^{n+1} + 2 \tag{3}$$

(a) We have

$$\begin{aligned}
\sum_{i=0}^{n} (i^2 + 1)^2 &= \sum_{i=0}^{n} i^4 + 2\sum_{i=0}^{n} i^2 + \sum_{i=0}^{n} 1 && \text{(splitting sum)} \\
&\approx \frac{n^5}{5} + \frac{2n^3}{3} + (n-1), && \text{(from (1))}
\end{aligned}$$

so this is in $\Theta(n^5)$.

(b) We have

$$\begin{aligned}
\sum_{i=1}^{n} n\log(i^2) &= 2n\sum_{i=1}^{n}\log n && \text{(splitting sum)}\\
&= 2n^2\log n, && \text{(from 2)}
\end{aligned}$$

so this is in $\Theta(n^2\log n)$.

(c) We have

$$\begin{aligned}
\sum_{i=1}^{n}(i+2)2^i &= \sum_{i=1}^{n} i2^i + \sum_{i=1}^{n} 2^{i+1} && \text{(splitting sum)}\\
&= \left[(n-1)2^{n+1}+2\right] + 2\sum_{i=0}^{n-1} 2^i && \text{(from 3)}\\
&= \left[(n-1)2^{n+1}+2\right] + 2\left(\frac{2^n-1}{2-1}\right) && \text{(geometric series)}\\
&= n2^{n+1} - 2^{n+1} + 2 + 2^{n+1} - 2 && \text{(simplifying)}\\
&= 2n2^n, && \text{(simplifying)}
\end{aligned}$$

so this is in $\Theta(n2^n)$.

(d) We have

$$\begin{aligned}
\sum_{i=0}^{n}\sum_{j=0}^{i-1}(i+j) &= \sum_{i=0}^{n}\left(\sum_{j=0}^{i-1} i + \sum_{j=0}^{i-1} j\right)\\
&= \sum_{i=0}^{n}\left(i\sum_{j=0}^{i-1} 1 + \sum_{j=0}^{i-1} j\right)\\
&= \sum_{i=0}^{n}\left(i^2 + \frac{i(i-1)}{2}\right)\\
&= \frac{3}{2}\sum_{i=0}^{n} i^2 - \frac{1}{2}\sum_{i=0}^{n} i\\
&\approx \frac{3}{2}\left(\frac{n^3}{3}\right) - \frac{1}{2}\left(\frac{n(n+1)}{2}\right)\\
&= \frac{n^3}{3} - \frac{n^2}{4} - \frac{n}{4},
\end{aligned}$$

so this is in $\Theta(n^3)$.

Problem 4 Consider the following algorithm.
(a) What does this algorithm compute?
(b) What is its basic operation and what is the efficiency class $\Theta(g(n))$ the function belongs to, of this algorithm?
(c) Make as many improvements as you can in this algorithm. You must write down the pseudo code for your new algorithm. What is the efficiency class $\Theta(g(n))$ the function belongs to, of your new algorithm? If you cannot improve this algorithm, explain why you cannot do it.

Algorithm 1 Enigma($A[1\dots n, 1, \dots, n]$)

```
// Input: A matrix (A[1...n, 1,...,n] of real numbers elements
for i ← 1 to n do
    for j ← 1 to n do
        if A[i, j] ≠ A[j, i] then
            return false
        end if
    end for
end for
return true
```

Solution.

(a) This algorithm computes whether or not a matrix is symmetric, and returns true if it is or false if it is not.

(b) The basic operation is comparison. This is done for all $i, j \in \{1, 2, \dots, n\}$, so the function belongs to $\Theta(n^2)$.

(c) We only need to check above the diagonal, so we change this algorithm to:

Algorithm 2 EnigmaImproved($A[1\dots n, 1, \dots, n]$)

```
// Input: A matrix (A[1...n, 1,...,n] of real numbers elements
for i ← 1 to n − 1 do
    for j ← i + 1 to n do
        if A[i, j] ≠ A[j, i] then
            return false
        end if
    end for
end for
return true
```

However, this is still $\Theta(n^2)$ because the n^2 is simply being divided by a constant. We cannot do better than $\Theta(n^2)$ because we need to check half of an $n \times n$ matrix, and since we have no information about the matrix, we need to check every element.

Problem 5 Solve the following recurrence relations. Give the particular solution to the problem.
(a) $x(n) = x(n-1) + 3$ for $n > 0$, $x(0) = 2$
(b) $x(n) = x(n-1) + 3n$ for $n > 1$, $x(1) = 1$
(c) $x(n) = x(n/4) + n$ for $n > 1$, $x(1) = 1$ (solve for $n = 4^k$)
(d) $x(n) = 2x(n-1) - x(n-2)3$ for $n > 1$, $x(0) = 0$ and $x(1) = 1$

Solution.

(a) We claim that $x(n) = 3n + 2$, and we will prove this with induction. First, let $n = 0$. Then $x(n) = 3(0) + 2 = 2$, so the claim holds for $n = 0$.

Next, let $n \in \mathbb{N}$, $n \geq 0$, and assume $x(n) = 3n+2$. Then,

$$\begin{aligned}
x(n+1) &= x(n) + 3 && \text{(recursive definition)}\\
&= (3n+2) + 3 && \text{(induction hypothesis)}\\
&= 3(n+1) + 2. && \text{(rearranging)}
\end{aligned}$$

So if the claim holds for n, it holds for $n+1$, and thus it holds for all $n \in \mathbb{N}$ by induction. $\square$

(b) We claim that $x(n) = \frac{3}{2}n^2 + \frac{3}{2}n - 2$, and we will prove this with induction. First, let $n = 1$. Then $x(n) = \frac{3}{2} + \frac{3}{2} - 2 = 3 - 2 = 1$, so the claim holds for $n = 1$.

Next, let $n \in \mathbb{N}$, $n \geq 1$, and assume $x(n) = \frac{3}{2}n^2 + \frac{3}{2} - 2$. Then,

$$\begin{aligned}
x(n+1) &= x(n) + 3(n+1) && \text{(recursive definition)}\\
&= \frac{3}{2}n^2 + \frac{3}{2}n - 2 + 3(n+1) && \text{(induction hypothesis)}\\
&= \frac{3}{2}n^2 + \frac{3}{2}n - 2 + 3n + 3 && \text{(distributing)}\\
&= \frac{3}{2}n^2 + 3n + \frac{3}{2} + \frac{3}{2}n + \frac{3}{2} - 2 && \text{(rearranging)}\\
&= \frac{3}{2}(n^2 + 2n + 1) + \frac{3}{2}(n+1) - 2 && \text{(factoring)}\\
&= \frac{3}{2}(n+1)^2 + \frac{3}{2}(n+1) - 2. && \text{(factoring)}
\end{aligned}$$

So if the claim holds for n, it holds for $n+1$, and therefore it holds for all $n \in \mathbb{N}$ by induction. $\square$

(c) We claim that $x(n) = \frac{4n-1}{3}$ for values of $n = 4^k$, $k \in \mathbb{N}$, and we will prove this with induction on k. First, let $k = 0$. Then $n = 4^k = 1$, and $x(n) = \frac{4(1)-1}{3} = \frac{3}{3} = 1$, so the claim holds for $k = 0$.

Next, let $k \in \mathbb{N}$, and assume that for $n = 4^k$, $x(n) = \frac{4n-1}{3}$. Then, consider $k+1$, and define $n' = 4^{k+1} = 4(4^k) = 4n$. We have

$$\begin{aligned}
x(n') &= x(n'/4) + n' && \text{(recursive definition)}\\
&= x(n) + 4n && (n = \tfrac{n'}{4})\\
&= \frac{4n-1}{3} + 4n && \text{(induction hypothesis)}\\
&= \frac{16n-1}{3} && \text{(combining)}\\
&= \frac{4n'-1}{3}. && (n' = 4n)
\end{aligned}$$

So the claim holds for n'. Since $n' = 4^{k+1}$, if the claim holds for k, it also holds for $k+1$, and therefore it holds for all $k \in \mathbb{N}$ and $n = 4^k$ by induction. $\square$

(d) We can write the characteristic polynomial $p(\lambda) = \lambda^2 - 2\lambda + 3$, so we have

$$\lambda = \frac{2 \pm \sqrt{4 - 4(3)(1)}}{2(1)} = 1 \pm i\sqrt{2} = \sqrt{3}e^{\pm i \arctan \sqrt{2}}.$$

Using the method from class then, there exist $\alpha_1, \alpha_2 \in \mathbb{C}$ such that

$$x(n) = \alpha_1 \left(\sqrt{3}e^{i \arctan \sqrt{2}}\right)^n + \alpha_2 \left(\sqrt{3}e^{-i \arctan \sqrt{2}}\right)^n.$$

We can solve the system of equations

$$\begin{array}{ccccccc}
x(0) & = & 0 & = & \alpha_1 & + & \alpha_2\\
x(1) & = & 1 & = & \alpha_1(1+i\sqrt{2}) & + & \alpha_2(1-i\sqrt{2})
\end{array}$$

by noting

$$\begin{aligned}
&\implies \alpha_2 = -\alpha_1 \implies \\
&\quad 1 = \alpha_1(1+i\sqrt{2}) - \alpha_1(1-i\sqrt{2}) \\
&\quad\ = 2\alpha_1 i\sqrt{2} \\
&\implies \alpha_1 = -\frac{i\sqrt{2}}{4} = \frac{\sqrt{2}}{4}e^{-i\frac{\pi}{2}} && \text{(Euler's identity)} \\
&\implies \alpha_2 = \frac{i\sqrt{2}}{4} = \frac{\sqrt{2}}{4}e^{i\frac{\pi}{2}}. && \text{(Euler's identity)}
\end{aligned}$$

So we have

$$x(n) = \frac{\sqrt{2}}{4}e^{-i\frac{\pi}{2}}\left(\sqrt{3}e^{i\arctan\sqrt{2}}\right)^n + \frac{\sqrt{2}}{4}e^{i\frac{\pi}{2}}\left(\sqrt{3}e^{-i\arctan\sqrt{2}}\right)^n.$$

We want a real valued function, so we will use some trigonometry:

$$\begin{aligned}
x(n) &= \frac{\sqrt{2}\sqrt{3}^n}{4}e^{-i\frac{\pi}{2}}e^{in\arctan\sqrt{2}} + \frac{\sqrt{2}\sqrt{3}^n}{4}e^{i\frac{\pi}{2}}e^{-in\arctan\sqrt{2}} && \text{(distributing exponent)} \\
&= \frac{\sqrt{2}\sqrt{3}^n}{4}\left(e^{i(n\arctan\sqrt{2}-\frac{\pi}{2})} + e^{i(\frac{\pi}{2}-n\arctan\sqrt{2})}\right) && \text{(factoring/combining exponents)} \\
&= \frac{\sqrt{2}\sqrt{3}^n}{2}\left(\frac{e^{i(n\arctan\sqrt{2}-\frac{\pi}{2})} + e^{-i(n\arctan\sqrt{2}-\frac{\pi}{2})}}{2}\right) && \text{(rearranging)} \\
&= \frac{\sqrt{2}\sqrt{3}^n}{2}\cosh\left(i\left(n\arctan\sqrt{2}-\frac{\pi}{2}\right)\right) && (\cosh(z) = \tfrac{e^z+e^{-z}}{2}) \\
&= \frac{\sqrt{2}\sqrt{3}^n}{2}\cos\left(-\left(n\arctan\sqrt{2}-\frac{\pi}{2}\right)\right) && (\cosh(x) = \cos(ix) \implies \cosh(ix) = \cos(-x)) \\
&= \frac{\sqrt{3}^n}{\sqrt{2}}\cos\left(\frac{\pi}{2}-n\arctan\sqrt{2}\right) && \text{(rearranging)} \\
&= \frac{\sqrt{3}^n}{\sqrt{2}}\left(\cos\left(\frac{\pi}{2}\right)\cos\left(n\arctan\sqrt{2}\right) + \sin\left(\frac{\pi}{2}\right)\sin\left(n\arctan\sqrt{2}\right)\right) && \text{(difference rule)} \\
&= \frac{\sqrt{3}^n}{\sqrt{2}}\sin\left(n\arctan\sqrt{2}\right). && (\cos(\pi/2) = 0,\ \sin(\pi/2) = 1)
\end{aligned}$$

Problem 6 Let $f(n)$ and $g(n)$ be asymptotically nonnegative functions. Using the basic definition of Θ-notation, prove that $\max(f(n), g(n)) = \Theta(f(n) + g(n))$.

Solution.

We have that a function $t(n) \in \Theta(h(n))$ if there exist $c_1, c_2 \in \mathbb{R}^+$ and $n_0 \in \mathbb{N}$ such that for all $n \geq n_0$, $c_1 h(n) \leq t(n) \leq c_1 h(n)$.

Since $f(n)$ and $g(n)$ are asymptotically nonnegative, there exist $n_f, n_g \in \mathbb{N}$ such that for all $n \in \mathbb{N}$, $f(n) \geq 0$ if $n \geq n_f$ and $g(n) \geq 0$ if $n \geq n_g$. Let $n_0 = \max(n_f, n_g)$. So $f(n)$ and $g(n)$ are nonnegative for all $n \geq n_0$.

For all $n \geq n_0$, we have $\max(f(n), g(n)) \leq f(n) + g(n)$, because the LHS equals one of the functions and the other function is nonnegative. We also have that $\frac{1}{2}(f(n), g(n)) \leq \max(f(n), g(n))$, because the LHS is the average of the functions and the maximum value cannot be less than the average.

So by choosing $c_1 = \frac{1}{2}, c_2 = 1, n_0 = \max(n_f, n_g)$, we have that for all $n \geq n_0$,

$$\frac{1}{2}(f(n) + g(n)) \leq \max(f(n), g(n)) \leq f(n) + g(n),$$

and therefore by the definition $\max(f(n), g(n)) \in \Theta(f(n) + g(n))$. $\square$

Problem 7 Show that for any real constants a and b, where $b > 0$, $(n+a)^b = \Theta(n^b)$.

Solution.

Let $a, b \in \mathbb{R}$. Then, we have

$$\begin{aligned}
\lim_{n\to\infty} \frac{(n+a)^b}{n^b} &= \lim_{n\to\infty} \left(\frac{n+a}{n}\right)^b && \text{(fraction property)} \\
&= \left(\lim_{n\to\infty} \frac{n+a}{n}\right)^b && \text{(limit property)} \\
&= 1^b = 1. && \text{(L'Hôpital's rule)}
\end{aligned}$$

So b^n has the same growth order and $(n+a)^b \in \Theta(n^b)$.

Problem 8 Is $2^{n+1} \in O(2^n)$? Is $2^{2n} \in O(2^n)$? Justify your answers.

Solution.

- We have
$$\lim_{n\to\infty} \frac{2^{n+1}}{2^n} = \lim_{n\to\infty} 2 = 2,$$
so 2^{n+1} is the same growth rate as 2^n and thus $2^{n+1} \in \Theta(2^n) \implies 2^{n+1} \in O(2^n)$.
- We have
$$\lim_{n\to\infty} \frac{2^{2n}}{2^n} = \lim_{n\to\infty} 2^n = \infty,$$
which we have from the textbook implies that $2^{2n} \notin O(2^n)$.

Problem Bonus 1 Prove that the running time of an algorithm is $\Theta(g(n))$ if and only if its worst-case running time is $O(g(n))$ and its best-case running time is $\Omega(g(n))$.

We have the following definitions:

- $t(n) \in O(g(n))$ iff there exist $c_O \in \mathbb{R}^+$, $n_O \in \mathbb{N}$ such that for all $n \geq n_O$, $t(n) \leq c_O g(n)$.
- $t(n) \in \Omega(g(n))$ iff there exist $c_\Omega \in \mathbb{R}^+$, $n_\Omega \in \mathbb{N}$ such that for all $n \geq n_\Omega$, $t(n) \geq c_\Omega g(n)$.
- $t(n) \in \Theta(g(n))$ iff there exist $c_1, c_2 \in \mathbb{R}^+$, $n_\Theta \in \mathbb{N}$ such that for all $n \geq n_\Theta$, $c_1 g(n) \leq t(n) \leq c_2 g(n)$.

First, assume that the running time of an algorithm $t(n)$ is $\Theta(g(n))$. By choosing $c_\Omega = c_1$ and $n_\Omega = n_\Theta$ from the definition of $\Theta(g(n))$, we have that for all $n \geq n_\Omega$, $t(n) \geq c_\Omega g(n)$, and thus the algorithm is $\Omega(g(n))$. By choosing $c_O = c_2$ and $n_O = n_\Theta$ from the definition of $\Theta(g(n))$, we have that for all $n \geq n_O$, $t(n) \leq c_O g(n)$, and thus the algorithm is also $O(g(n))$.

Next, assume that the algorithm's worst-case running time $t(n)$ is $O(g(n))$ and its best-case running time is $\Omega(g(n))$. By choosing $c_1 = c_\Omega$, $c_2 = c_O$, and $n_\Theta = \max(n_\Omega, n_O)$ from the definitions of $\Omega(n)$ and $O(n)$, we have that for all $n \geq n_\Theta$, $c_1 g(n) \leq t(n) \leq c_2 g(n)$, and thus the algorithm is $\Theta(g(n))$.

Therefore, the two statements are equivalent. $\square$