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MATH 300: Section H01
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February 25, 2022

MATH 300 Homework 6

Problem 1

(a) **False:**

Take $a = 1$ and $b = -1$. There exist $m, n \in \mathbb{Z}$ such that $am = b$ and $bn = a$ by choosing $m = n = -1$, so $1|-1$ and $-1|1$. However, $1 \neq -1$, so the claim is false. $\square$

(b) **True:**

Let $a, b \in \mathbb{N}$.

Assume $a|b$ and $b|a$. Then, there exist $m, n \in \mathbb{Z}$ such that $am = b$ and $bn = a$. As such, both a and b are either both non-zero or both 0, because there is no $m \in \mathbb{Z}$ that satisfies $0m = b$ if b is not zero, and there is no $n \in \mathbb{Z}$ that satisfies $0n = a$ if a is not zero.

Case 1: Suppose $a = b = 0$.

$0|0$ is true and $0 = 0$, so the claim is true.

Case 2: Suppose $a \neq 0$ and $b \neq 0$.

Since $am = b$ and $a = bn$, $bnm = b$. Thus, dividing by b (since it is nonzero), $nm = 1$. Since $m, n \in \mathbb{N}$, the only way this is possible is if $m = n = 1$. Therefore, $a(1) = b$ and $b(1) = a$, so $a = b$. $\square$

Problem 2

(a) $\{x \in \mathbb{N} : (\exists k \in \mathbb{Z})(x = 3k \wedge x \leq 12)\}$

(b) $\{x \in \mathbb{Z} : -3 \leq x \leq 3\}$

(c) $\{x : x = m \vee x = n \vee x = o \vee x = p\}$

Problem 3

- B is a subset of A, because 2 and 6 are the only members of B and are both in A.
- C is a subset of A, because 4 and 6 are the only members of C and are both in A.
- C is a subset of D, because 4 and 6 are the only members of D and are both in D.

Problem 4

(a) **True:** $\emptyset$ is the only member of the set.

(b) **True:** $\emptyset$ is the first member of the set.

(c) **False:** The set containing $\emptyset$ is not a member of the set.

(d) **True:** The set containing $\emptyset$ is the only member of the set.

(e) **True:** Every member of the first set ($\emptyset$) is in the second set, and there is at least one member of the second set ($\{\emptyset\}$) not in the first set.

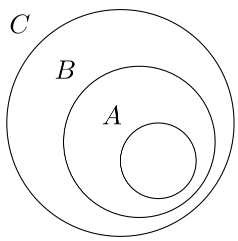
(f) **True:** Every member of the first set ($\{\emptyset\}$) is in the second set, and there is at least one member of the second set ($\emptyset$) not in the first set.

(g) **False:** There is no member of the second set that is not in the first set.

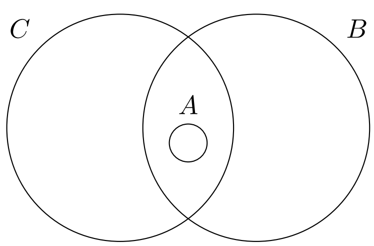
(h) **True:** $\emptyset = \{\}$, so the second set just has two instances of the same element. Thus, the sets are subsets of each other and are therefore equal.

Problem 5

(a)



(b)



Problem 6

$A = \{1, 2\}$

$B = \{1, 2, \{1, 2\}\}$

Problem 7

(a) 0

(b) 1

(c) 2

(d) 3

(e) 2

Problem 8

(a) This cannot be a power set, because for any set S, $\emptyset \in S$. Thus, any power set must at least contain $\emptyset$.

(b) This is the power set of $\{a\}$, because it has only one element and thus the only subsets are $\{a\}$ and $\emptyset$.

(c) This cannot be a power set. If $\{\emptyset, a\}$ is a member of a power set, then $\{\emptyset\}$ must also be a member of the power set, but it is not.

(d) This is the power set of $\{a, b\}$. It contains every subset of $\{a, b\}$.

Problem 9

(a) $A \times B = \{(a, x), (a, y), (b, x), (b, y), (c, x), (c, y)\}$

$(A \times B) \times C = \{(a, x, 0), (a, x, 1), (a, y, 0), (a, y, 1), (b, x, 0), (b, x, 1), (b, y, 0), (b, y, 1), (c, x, 0), (c, x, 1), (c, y, 0), (c, y, 1)\}$

(b) $B \times A = \{(x, a), (x, b), (x, c), (y, a), (y, b), (y, c)\}$

$C \times (B \times A) = \{(0, x, a), (0, x, b), (0, x, c), (0, y, a), (0, y, b), (0, y, c), (1, x, a), (1, x, b), (1, x, c), (1, y, a), (1, y, b), (1, y, c)\}$

Problem 10

(a) The square of some real number is -1. This is false for the real numbers (but true for the complex numbers by choosing $x = i$).

(b) The square of some integer is 2. This is false because 2 is not a perfect square and $\sqrt{2}$ is not an integer (or even rational, as we proved in class).

(c) For every integer, subtracting 1 yields another integer. This is true because the difference of an integer and 1 (another integer) is an integer.

(d) The square of every integer is another integer. This is true because the product of two integers is an integer.

Problem 11

Yes. $P(A)$ contains the set A (because $A \subseteq A$), and $P(B)$ contains the set B (because $B \subseteq B$). Since $P(A) = P(B)$, $P(B)$ contains A and $P(A)$ contains B. By the definition of the power set, then, $A \subseteq B$ and $B \subseteq A$. Therefore, $A = B$.

Problem 12

Let A, B be sets.

Assume $A \neq \emptyset$ and $B \neq \emptyset$. Then, A contains at least one element a and B contains at least one element b.

Then, by the definition of the Cartesian product, $A \times B$ contains the element (a, b). Because $A \times B$ contains an element, it cannot be the empty set. Thus, if $A \neq \emptyset$ and $B \neq \emptyset$, $A \times B \neq \emptyset$.

Therefore, by the contrapositive, if $A \times B = \emptyset$, then $A = \emptyset$ or $B = \emptyset$. $\square$

Problem 13

Let A, B be sets.

Assume $A \neq \emptyset$, $B \neq \emptyset$, and $A \neq B$. Then, $A \not\subseteq B$ or $B \not\subseteq A$.

Case 1: Suppose $B \subseteq A$. Since $A \neq B$, $B \subsetneq A$. Then there is an a_0 that is in A but not in B. Because $A \times B = \{(a, b) : a \in A \wedge b \in B\}$, it follows that $A \times B$ contains an ordered pair whose first component is a_0. Since a_0 is not in B, $B \times A$ cannot have an ordered pair whose first component is a_0, so $B \times A \not\subseteq A \times B$ and thus $A \times B \neq B \times A$.

Case 2: Suppose $A \subseteq B$. Since $A \neq B$, $A \subsetneq B$. Then there is a b_0 that is in B but not in A, and it follows that $B \times A$ contains an ordered pair whose first component is b_0. Since b_0 is not in A, $A \times B$ cannot have an ordered pair whose first component is b_0, so $A \times B \not\subseteq B \times A$ and thus $B \times A \neq A \times B$.

Case 3: Suppose $A \not\subseteq B$ and $B \not\subseteq A$. Then, there is an a_0 that is in A but not in A and a b_0 that is in B but not in A, and it follows that $A \times B$ contains an ordered pair whose first component is a_0. Since a_0 is not in B, $B \times A$ cannot have an ordered pair whose first component is a_0, so $B \times A \not\subseteq A \times B$ and thus $A \times B \neq B \times A$.

In all three possible cases, $A \times B \neq B \times A$, so therefore if $A \neq \emptyset$ and $B \neq \emptyset$, then $A \times B \neq B \times A$ unless $A = B$. $\square$

Problem 14

(a) $A \cap B$

(b) $A \cap \overline{B}$

(c) $A \cup B$

(d) $\overline{A} \cup \overline{B}$

Problem 15

(a) $\{a, b, c, d, e, f, g, h\}$

(b) $\{a, b, c, d, e\}$

(c) $\{f, g, h\}$

(d) $\emptyset$

Problem 16

(a) Assume $A \cup \emptyset = \{x : x \in A \vee x \in \emptyset\}$. Since there are no elements in $\emptyset$, the second half of the proposition is always false. Therefore by the identity laws of logic, $A \cup \emptyset = A$. $\square$

(b) Assume $A \cup \emptyset = \{x : x \in A \wedge x \in \emptyset\}$. Since there are no elements in $\emptyset$, the second half of the proposition is always false, so by the domination laws of logic, the set of elements that satisfy the proposition is empty. Therefore, $A \cap \emptyset = \emptyset$. $\square$

(c) Assume $A \cup A = \{x : x \in A \vee x\}$. An element is in A if and only if it is in A or if it is in A by the idempotent laws of logic. Therefore, $A \cup A = A$. $\square$

(d) Assume $A \cup A = \{x : x \in A \wedge x \in A\}$. An element is in A if and only if it is in A and if it is in A by the idempotent laws of logic. Therefore, $A \cap A = A$. $\square$

(e) Assume $A - \emptyset = \{x : x \in A \wedge x \notin \emptyset\}$. Since $\emptyset$ has no elements, the second half of the proposition is always true. Therefore by the identity laws of logic, $A - \emptyset = A$. $\square$

(f) Assume $A \cup U = \{x : x \in A \vee x \in U\}$. Since $(\forall x)(x \in A \Rightarrow x \in U)$ by definition, $x \in A \vee x \in U \equiv x \in U$ by the domination laws of logic. Therefore, by the domination laws of logic, $A \cup U = U$. $\square$

(g) Assume $A \cup U = \{x : x \in A \wedge x \in U\}$. If an element is in A, it is in U by definition, so $x \in A \wedge x \in U \equiv x \in A$ by the identity laws of logic. Therefore, $A \cap U = A$. $\square$

(h) Assume $\emptyset - A = \{x : x \in \emptyset \wedge x \notin A\}$. Since the empty set has no elements, the first half of the proposition is always false and thus the set of elements that satisfy the proposition is empty by the domination laws of logic. Therefore, $\emptyset - A = \emptyset$. $\square$

(i) Assume $A \cap B = \{x : x \in A \wedge x \in B\}$. Since this is a conjunction, $x \in A$ is always true when the proposition is true. Thus, every element of $A \cap B$ is a member of A, so therefore $A \cap B \subseteq A$. $\square$

(j) Assume $A \cup B = \{x : x \in A \vee x \in B\}$. Thus, every member of A is in the union, so therefore $A \subseteq (A \cup B)$. $\square$

(k) Assume $A - B = \{x : x \in A \wedge x \notin B\}$. If $(A - B) \nsubseteq A$, there would be some element not in A, and then the first half of the proposition would not be satisfied. Therefore, $(A - B) \subseteq A$. $\square$

(l) Assume $A \cap (B - A) = \{x : x \in A \wedge x \in B \wedge x \notin A\}$. Since $x \in A \wedge x \notin A$ is a contradiction by the negation laws of logic, the set of elements satisfying the proposition is empty by the domination laws of logic. Therefore, $A \cap (B - A) = \emptyset$. $\square$

(m) Assume $A \cup (B - A) = \{x : x \in A \vee (x \in B \wedge x \notin A)\}$. Using the distributive laws, $x \in A \vee (x \in B \wedge x \notin A) \equiv (x \in A \vee x \in B) \wedge (x \in A \vee x \notin A)$. The second half of this proposition is a contradiction, so it is equivalent to the first half by the identity laws of logic. This is the definition of a union, so therefore, $A \cup (B - A) = A \cup B$. $\square$

(n) Assume $(A \cap B) \cup (A \cap \overline{B})$. By the distributive laws of sets, $(A \cap B) \cup (A \cap \overline{B}) \equiv A \cap (B \cup \overline{B})$. Following from the definition of a complement, $B \cup \overline{B} = U$. By the result in (g), $A \cap U = A$. Therefore, $(A \cap B) \cup (A \cap \overline{B}) = A$. $\square$

Problem 17

(a) Assume $A \neq \emptyset$ and $B \neq C$. Then, $B \nsubseteq C$ or $C \nsubseteq B$.

Case 1: Suppose $C \subseteq B$. Since $B \neq C$, $C \subsetneq B$. Then there is a b_0 that is in B but not in C. Because $A \times B = \{(a, b) : a \in A \wedge b \in B\}$, it follows that $A \times B$ contains an ordered pair whose second component is b_0. Since b_0 is not in C, $A \times C$ cannot have an ordered pair whose second component is b_0, so $A \times C \nsubseteq A \times B$ and thus $A \times B \neq A \times C$.

Case 2: Suppose $B \subseteq C$. Since $B \neq C$, $B \subsetneq C$. Then there is a c_0 that is in C but not in B, and it follows that $A \times C$ contains an ordered pair whose second component is c_0. Since c_0 is not in B, $A \times B$ cannot have an ordered pair whose first second component is c_0, so $A \times B \nsubseteq A \times C$ and thus $A \times B \neq A \times C$.

Case 3: Suppose $B \nsubseteq C$ and $C \nsubseteq B$. Then, there is an b_0 that is in B but not in C and a c_0 that is in C but not in B, and it follows that $A \times B$ contains an ordered pair whose second component is b_0. Since b_0 is not in C, $A \times C$ cannot have an ordered pair whose second component is b_0, so $A \times C \nsubseteq A \times B$ and thus $A \times B \neq A \times C$.

In all three possible cases, $A \times B \neq A \times C$, so if $A \neq \emptyset$ and $B \neq C$, then $A \times B \neq B \times A$. Therefore, the contrapositive is also true, and if $A \times B = A \times C$ then $A = \emptyset$ or $B = C$. $\square$

(b) Assume $A \subseteq B$. Then, there is no element in A that is not in B. $A \cap C = \{x : x \in A \wedge x \in C\}$, and since $(\forall x)(x \in A \Rightarrow x \in B)$, $\{x : x \in A \wedge x \in C\} \subseteq \{x : x \in B \wedge x \in C\}$. Therefore, by definition, if $A \subseteq B$ then $A \cap C \subseteq B \cap C$. $\square$

(c) Assume $A \subseteq B$. Then, $C - A = \{x : x \in C \wedge x \notin A\}$, and $C - B = \{x : x \in C \wedge x \notin B\}$. Because $A \subseteq B$, the contrapositive states that $(\forall x)(x \notin B \Rightarrow x \notin A)$. As a result, if an element is in C and not in B, it will also be in C and not in A. Therefore, by definition, if $A \subset B$ then $C - B \subseteq C - A$. $\square$