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MATH 575: Section H01
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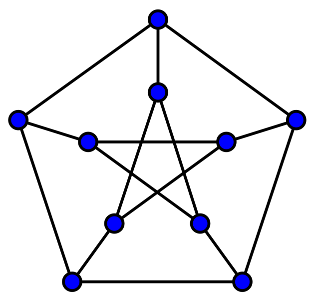
MATH 575 Homework 1

Problem 1 For positive integers n and k, consider the graph $G(n,k)$ which is defined as follows: the vertex set of $G(n,k)$ is the set of subsets of $[n]$ of size k, and two vertices are connected by an edge in $G(n,k)$ if and only if the corresponding subsets are disjoint.
(a) Give a drawing of the graph $G(5,2)$.
(b) Let G be the graph drawn below. Show that $G(5,2)$ is isomorphic to G by relabelling the vertices of G in the drawing below.



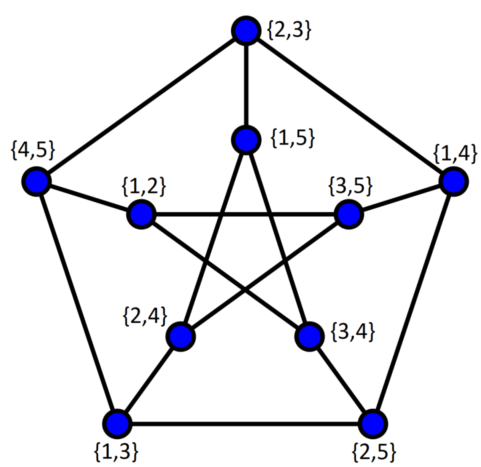
Solution.

We have

$$V(G(5,2)) = \big\{\{1,2\},\{1,3\},\{1,4\},\{1,5\},\{2,3\},\{2,4\},\{2,5\},\{3,4\},\{3,5\},\{4,5\}\big\}$$

and

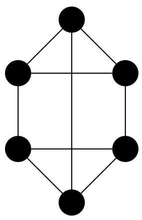
$$\begin{aligned}E(G(5,2)) = \{&\{1,2\}\{3,4\},\{1,2\}\{3,5\},\{1,2\}\{4,5\},\{1,3\}\{2,4\},\{1,3\}\{2,5\},\{1,3\}\{4,5\},\{1,4\}\{2,3\},\\ &\{1,4\}\{2,5\},\{1,4\}\{3,5\},\{1,5\}\{2,3\},\{1,5\}\{2,4\},\{1,5\}\{3,4\},\{2,3\}\{4,5\},\{2,4\}\{3,5\},\{2,5\}\{3,4\}\}.\end{aligned}$$



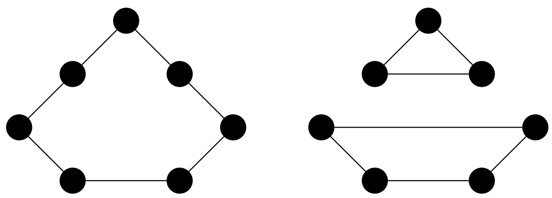
Problem 2 A graph is called **k-regular** if every vertex has degree k.
(a) Draw an example of a 3-regular graph on 6 vertices.
(b) Draw two non-isomorphic 2-regular graphs on 7 vertices.
(c) Prove that if k is odd, then there does not exist a k-regular graph with an odd number of vertices.

Solution.

(a)



(b)



(c) Let k be an odd number and assume there exists a k-regular graph G with an odd number of vertices n. We have from the handshaking lemma that for any graph G,

$$\sum_{v \in V(G)} d(v) = 2|E(G)|.$$

We have n vertices each with degree k, so the sum of the degrees is kn and thus $|E(G)| = \frac{kn}{2}$. However, since k and n are both odd, this is not an integer, a contradiction. $\square$

Problem 3 Prove that every graph G must contain a pair of vertices with the same degree.

Solution.

We note this is clearly only true for graphs with 2 or more vertices.

Let G be a graph with $n \geq 2$ vertices. Each vertex can have degree ranging from 0 to $n-1$. However, G cannot have both a vertex with degree 0 and a vertex with $n-1$, because the latter must connect to everything and the former must connect to nothing. Thus, the n vertices in G have $n-1$ possible degrees, so by the PHP there are 2 vertices with the same degree. $\square$

Problem 4 Let G be a graph with $V(G) = \{v_1, \ldots, v_n\}$. Recall that the *adjacency matrix* of G is the $n \times n$ matrix A such that $A_{ij} = 1$ if $v_i v_j \in E(G)$ and $A_{ij} = 0$ otherwise. Use induction to prove that for all integers $k \geq 1$, the (i, j)-entry of A^k is equal to the number of v_i, v_j-walks of length k in G.

Solution.

First, let $k = 1$, and let $i, j \in [n]$. If v_i and v_j are neighbors, there is 1 $v_i v_j$-walk of length 1, and 0 $v_i v_j$-walks of length 1 otherwise. ${A^1}_{ij} = 1$ if and only if v_i and v_j are neighbors, so the claim holds for $k = 1$.

Next, let $k \in \mathbb{N}$ and assume the claim holds for k. We have $A^{k+1} = \left(A^k\right)(A)$. Let $i, j \in [n]$. By definition, we have

$${A^{k+1}}_{ij} = \sum_{m=1}^{n} {A^k}_{im} A_{mj}.$$

We claim this counts the number of v_iv_j-walks of length $k+1$. To see this, let $m \in [n]$. ${A^k}_{im}$ is the number of v_iv_m-walks of length k. Thus, if $v_mv_j \in E(G)$, there are ${A^k}_{im}$ possible v_iv_j-walks of length $k+1$ whose second-to-last vertices are v_m (simply walk to v_m and then to v_j). Otherwise, there are no possible v_iv_j-walks of length $k+1$ whose second-to-last vertices are v_m.

To get the total number of v_iv_j-walks of length $k+1$, we should sum ${A^k}_{im}$ for all m such that $v_mv_j \in E(G)$. Since $A_{mj} = 1$ if and only if $v_mv_j \in E(G)$, this is equal to the sum above. So if the claim holds for k, it also holds for $k+1$, and therefore it is true for all $k \in \mathbb{N}$. $\square$

Problem 5 Let G be an n-vertex graph with degree sequence $(d_1, d_2, \ldots, d_n)$.
(a) What is the degree sequence of $\overline{G}$?
(b) A graph G is called *self-complementary* if it is isomorphic to its complement. Prove that if G is self-complementary, then either n or $n-1$ is divisible by 4.
(c) Show that for all n divisible by 4, there exists a self-complementary graph on n vertices. (Hint: generalize the structure of the path P_4.)
(d) Show that for all n such that $n-1$ is divisible by 4, there exists a self-complementary graph on n vertices. (Hint: add a vertex to a construction in part (c).)

Solution.

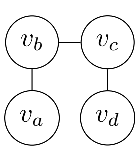
(a) For each vertex, there are $n-1$ possible adjacent edges. In $\overline{G}$, each vertex must be adjacent to all $n-1$ edges except for the edges the vertex is adjacent to in G. Thus, the degree sequence will be $(n-1-d_1, n-1-d_2, \ldots, n-1-d_n)$.

(b) We have that $|E(\overline{G})| = \binom{n}{2} - |E(G)|$, since $E(\overline{G})$ has all $\binom{n}{2}$ possible edges except the edges in $E(G)$. If G is self-complementary, then $|E(\overline{G})| = |E(G)|$, so we have

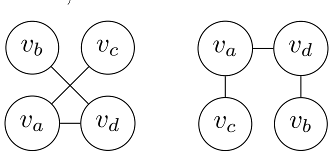
$$\begin{aligned} |E(G)| &= \binom{n}{2} - |E(G)| \\ 2|E(G)| &= \frac{n(n-1)}{2} \\ 4|E(G)| &= n(n-1) \\ 4 &\mid n(n-1). && (|E(G)| \in \mathbb{Z}) \end{aligned}$$

Thus, n or $n-1$ is divisible by 4.

(c) First, consider the path P on vertices $\{v_a, v_b, v_c, v_d\}$:



Then, consider $\overline{P}$:



Thus, P is self-complementary because we have an isomorphism $f_4 : V(P) \to V(\overline{P})$ between P and $\overline{P}$. In particular, $f_4 = \{(v_a, v_c), (v_b, v_a), (v_c, v_d), (v_d, v_b)\}$.

Next, let $n \in \mathbb{N}$ such that $4 \mid n$ and assume there exists a self-complementary graph G with $V(G) =$

$\{v_1, v_2, \ldots, v_n\}$. Then, we have an isomorphism $f_n : V(G) \to V(\overline{G})$ between G and $\overline{G}$. We will show that we can construct a self-complementary graph G' by taking the following steps in $G \cup P$:

- Connect v_a to all $v \in A$, where $A = \{v_1, v_2, \ldots, v_{n/2}\}$.
- Connect $f_4(v_a) = v_c$ to all $v \in C$, where $C = V(\overline{G}) - f_n(A)$. Then, in $\overline{G'}$, we have that for any $v \in V(\overline{G})$, $f_4(v_a)$ will have an edge with $f_n(v)$ if and only if v_a has an edge with v.
- Connect $f_4(v_c) = v_d$ to all $v \in D$, where $D = V(\overline{G}) - f_n(C)$. Similarly, for any $v \in V(\overline{G})$, $f_4(v_c)$ will have an edge with $f_n(v)$ if and only if v_c has an edge with v.
- Connect $f_4(v_d) = v_b$ to all $v \in B$, where $B = V(\overline{G}) - f_n(D)$. Similarly, for any $v \in V(\overline{G})$, $f_4(v_d)$ will have an edge with $f_n(v)$ if and only if v_d has an edge with v.

Then, in G', each vertex in P connects to half of the of the vertices in G. By the way we have connected them, $f_{n+4} = f_n \cup f_4$ is an isomorphism from G' to $\overline{G'}$.

(d) Let $n \in \mathbb{N}$ such that $4 \mid n - 1$. Then, from C there is a graph G on $n - 1$ vertices such that G is self-complementary and thus there exists an isomorphism $f : V(G) \to V(\overline{G})$. Add one vertex v, and connect it to half the vertices in G. Then, v will still be connected to half the vertices in $\overline{G}$, and $f \cup (v, v)$ will be an isomorphism between G and G'.