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MATH 546: Section
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MATH 546 Homework 9

Problem 1 Let

$$G_1 = \left\{ f_{m,b} : \mathbb{R} \to \mathbb{R} \mid f_{m,b}(x) = mx + b, m \neq 0 \right\}$$

be the group of affine functions, with composition of functions $\circ$ as the operation, and let

$$G_2 = \left\{ \begin{bmatrix} m & b \\ 0 & 1 \end{bmatrix} \middle| m, b \in \mathbb{R}, m \neq 0 \right\}$$

with multiplication of matrices $\cdot$ as the operation. Prove that $G_1 \cong G_2$.

Solution.

Consider $\phi : G_1 \to G_2$ defined by

$$\phi(f_{m,b}) = \begin{bmatrix} m & b \\ 0 & 1 \end{bmatrix}.$$

Clearly, the range of ϕ lies in G_2, since $m \neq 0$ is specified for G_1 and the form fits the specification for the set of G_2.

We first claim that ϕ satisfies the homomorphism property. Let $f_{m,b}, f_{m',b'} \in G_1$. Then, we have

$$\begin{aligned}
\phi(f_{m,b} \circ f_{m',b'}) &= \phi(m(m'x + b') + b) \\
&= \phi(mm'x + mb' + b) \\
&= \begin{bmatrix} mm' & mb' + b \\ 0 & 1 \end{bmatrix} \\
&= \begin{bmatrix} (m)(m') + (b)(0) & (m)(b') + (b)(1) \\ (0)(m') + (1)(0) & (0)(b') + (1)(1) \end{bmatrix} \\
&= \begin{bmatrix} m & b \\ 0 & 1 \end{bmatrix} \cdot \begin{bmatrix} m' & b' \\ 1 & 0 \end{bmatrix} \\
&= \phi(f_{m,b}) \cdot \phi(f_{m',b'}),
\end{aligned}$$

so the property holds.

We next claim that ϕ is bijective, and which is true if it has an inverse $\phi^{-1} : G_2 \to G_1$. Consider

$$\phi^{-1}\left(\begin{bmatrix} m & b \\ 0 & 1 \end{bmatrix}\right) = f_{m,b}.$$

Then, we have

$$\phi^{-1}(\phi(f_{m,b})) = \phi^{-1}\left(\begin{bmatrix} m & b \\ 1 & 0 \end{bmatrix}\right) = f_{m,b} \text{ and } \phi\left(\phi^{-1}\left(\begin{bmatrix} m & b \\ 0 & 1 \end{bmatrix}\right)\right) = \phi(f_{m,b}) = \begin{bmatrix} m & b \\ 0 & 1 \end{bmatrix},$$

so ϕ^{-1} is a well-defined inverse and therefore ϕ is bijective. Therefore, ϕ is an isomorphism and we have $G_1 \cong G_2$. $\square$

Problem 2 Let $C = \{-1, 1\}$ with multiplication $\cdot$ as the operation. Let $G_1 = \mathbb{R}^*$ with multiplication $\cdot$ as the operation, and let $G_2 = C \times \mathbb{R}^+$, where $\mathbb{R}^+$ has multiplication $\cdot$ and G_2 has usual operation $*$ by using the operations of the components. Prove that $G_1 \cong G_2$.

Solution.

Consider $\phi : G_1 \to G_2$ defined by

$$\phi(x) = \left(\frac{x}{|x|}, |x|\right).$$

This function is well defined with range lying in G_2 because $|x| \neq 0$ (as $x \in \mathbb{R}^*$), $\frac{x}{|x|}$ takes on only values 1 and -1, and $|x| \in \mathbb{R}^+$ for $x \in \mathbb{R}^*$.

We first claim that ϕ satisfies the homomorphism property. Let $x, x' \in G_1$. Then, we have

$$\begin{aligned}
\phi(x \cdot y) &= \left(\frac{x \cdot y}{|x \cdot y|}, |x \cdot y|\right) \\
&= \left(\frac{x}{|x|} \cdot \frac{y}{|y|}, |x| \cdot |y|\right) && \text{(multiplication property)} \\
&= \left(\frac{x}{|x|}, |x|\right) * \left(\frac{y}{|y|}, |y|\right) \\
&= \phi(x) * \phi(y),
\end{aligned}$$

so the property holds.

We next claim that ϕ is bijective. First, let $x_1, x_2 \in G_1$ such that $\phi(x_1) = \phi(x_2)$. Then, we have

$$\left(\frac{x_1}{|x_1|}, |x_1|\right) = \left(\frac{x_2}{|x_2|}, |x_2|\right),$$

so $|x_1| = |x_2|$. Also, we have $\frac{x_1}{|x_1|} = \frac{x_2}{|x_2|}$, so we can use $|x_1| = |x_2|$ to conclude that $x_1 = x_2$. So ϕ is injective. Next, let $(c, y) \in G_2$. Since this implies $y > 0$ (so $y = |y|$) and $|c| = 1$, we have

$$\phi(c \cdot y) = \left(\frac{c \cdot y}{|c \cdot y|}, |c \cdot y|\right) = \left(\frac{c}{|c|} \cdot \frac{|y|}{|y|}, |c| \cdot |y|\right) = (c, y).$$

So ϕ is surjective, and thus ϕ is bijective.

Therefore, ϕ is an isomorphism and we have $G_1 \cong G_2$. $\square$

Problem 3 Let G_1 be $\mathbb{R}$ with operation $*$ defined by $a * b = a + b - 1$. Prove that $G_1 \cong \mathbb{R}$, where $\mathbb{R}$ has the usual operation $+$.

Solution.

Consider $\phi : G_1 \to \mathbb{R}$ defined by $\phi(x) = x - 1$. Clearly this is well defined by closure of $\mathbb{R}$.

We first claim that ϕ satisfies the homomorphism property. Let $x, x' \in G_1$. Then, we have

$$\phi(x * x') = \phi(x + x' - 1) = x + x' - 1 - 1 = x - 1 + x' - 1 = \phi(x) + \phi(x'),$$

so the property holds.

We next claim that ϕ is bijective, which is true if it has an inverse $\phi^{-1} : \mathbb{R} \to G_1$. Consider $\phi^{-1}(y) = y + 1$. Then, we have

$$\phi^{-1}(\phi(x)) = \phi^{-1}(x - 1) = x - 1 + 1 = x \text{ and } \phi(\phi^{-1}(y)) = \phi(y + 1) = y + 1 - 1 = y,$$

so ϕ^{-1} is a well-defined inverse and therefore ϕ is bijective. Therefore, ϕ is an isomorphism and we have $G_1 \cong \mathbb{R}$. $\square$

Problem 4 Let $G = \mathbb{R}\backslash\{-1\}$, with operation $*$ defined by $a * b = a + b + ab$. Prove that $G \cong \mathbb{R}^*$, where $\mathbb{R}^*$ has the usual operation $\cdot$ of multiplication.

Solution.

Consider $\phi : G \to \mathbb{R}^*$ defined by $\phi(x) = x + 1$. Since $-1 \notin G$, the range of ϕ will never include 0, so ϕ is well-defined.

We first claim that ϕ satisfies the homomorphism property. Let $x, x' \in G_1$. Then, we have

$$\phi(x * x') = \phi(x + x' + xx') = x + x' + xx' + 1 = (x + 1) \cdot (x' + 1) = \phi(x) \cdot \phi(x'),$$

so the property holds.

We next claim that ϕ is bijective, which is true if it has an inverse $\phi^{-1} : \mathbb{R} \to G$. Consider $\phi^{-1}(y) = y - 1$, which will be in G because $0 \notin \mathbb{R}^*$. Then, we have

$$\phi^{-1}(\phi(x)) = \phi^{-1}(x + 1) = x + 1 - 1 = x \text{ and } \phi(\phi^{-1}(y)) = \phi(y - 1) = y - 1 + 1 = y,$$

so ϕ^{-1} is a well-defined inverse and therefore ϕ is bijective. Therefore, ϕ is an isomorphism and we have $G \cong \mathbb{R}$. $\square$

Problem 5 Let $G = \{(a, b) \in \mathbb{Z} \times \mathbb{Z} \mid a \equiv b \pmod 7\}$, with component-wise addition as the operation $+$. Prove that $G \cong \mathbb{Z} \times \mathbb{Z}$, where $\mathbb{Z} \times \mathbb{Z}$ has usual component-wise addition as the operation $+$.

Solution.

We will show $\mathbb{Z} \times \mathbb{Z} \cong G$, which is equivalent by symmetry. Consider $\phi : \mathbb{Z} \times \mathbb{Z} \to G$ defined by

$$\phi\big((a, b)\big) = (a, a - 7b).$$

Then, we have $a - (a - 7b) = 7b$, so since $b \in \mathbb{Z}$, we have $7 \mid [a - (a - 7b)]$. Thus $a \equiv a - 7b \pmod 7$, so $\phi\big((a, b)\big) \in G$ and thus ϕ is a well-defined function.

We first claim that ϕ satisfies the homomorphism property. Let $(a, b), (a', b') \in \mathbb{Z} \times \mathbb{Z}$. Then, we have

$$\begin{aligned}
\phi\big((a, b) + (a', b')\big) &= \phi\big((a + a', b + b')\big) \\
&= \big(a + a', (a + a') - 7(b + b')\big) \\
&= (a + a', a + a' - 7b - 7b') && \text{(distributive property)} \\
&= (a, a - 7b) + (a', a' - 7b') \\
&= \phi(a, b) + \phi(a', b'),
\end{aligned}$$

so the property holds.

We next claim that ϕ is bijective, which is true if it has an inverse $\phi^{-1} : G \to \mathbb{Z} \times \mathbb{Z}$. Consider

$$\phi^{-1}\left((m, n)\right) = \left(m, \frac{m - n}{7}\right),$$

which is well defined because $m \equiv n \pmod 7$ by definition and thus $m - n$ is divisible by 7. We have

$$\phi^{-1}\left(\phi\left((a,b)\right)\right) = \phi^{-1}\left((a, a-7b)\right) = \left(a, \frac{a-(a-7b)}{7}\right) = \left(a, \frac{7b}{7}\right) = (a,b)$$

and

$$\phi\left(\phi^{-1}\left((m,n)\right)\right) = \phi\left(m, \frac{m-n}{7}\right) = \left(m, m - 7\left(\frac{m-n}{7}\right)\right) = (m, m-(m-n)) = (m,n),$$

so ϕ^{-1} is a well-defined inverse and therefore ϕ is bijective. Therefore, ϕ is an isomorphism and we have $Z \times Z \cong G$. $\square$