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MATH 300: Section H01
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MATH 300 Homework 12

Problem 1 For each $i \in \mathbb{N}$ let $A_i = \{z \in \mathbb{Z} : |z| < i\}$ and $B_i = \{z \in \mathbb{Z} : |z| \geq i\}$.

(a)

$A_0 = \emptyset$

$A_1 = \{0\}$

$A_2 = \{-1, 0, 1\}$

$A_5 = \{-4, -3, -2, -1, 0, 1, 2, 3, 4\}$

(b)

$B_0 = \mathbb{Z}$

$B_1 = \mathbb{Z} - \{0\}$

$B_2 = \mathbb{Z} - \{-1, 0, 1\}$

$B_5 = \mathbb{Z} - \{-4, -3, -2, -1, 0, 1, 2, 3, 4\}$

(c) $\displaystyle\bigcup_{i \in \{0,1,2,5\}} A_i = A_5$

(d) $\displaystyle\bigcap_{i \in \{0,1,2,5\}} A_i = A_0$

(e) $\displaystyle\bigcup_{i \in \{0,1,2,5\}} B_i = B_0$

(f) $\displaystyle\bigcap_{i \in \{0,1,2,5\}} B_i = B_5$

(g) We claim that $\displaystyle\bigcup_{i=0}^{n} A_i = A_n$.

First, assume $x \in A_n$. So $x \in A_i$ for some $i \in \{z \in \mathbb{N} : z \leq n\}$ because n is in the set, and by definition $x \in \displaystyle\bigcup_{i=0}^{n} A_i$. Thus $A_n \subseteq \displaystyle\bigcup_{i=0}^{n} A_i$.

Next, assume $x \in \displaystyle\bigcup_{i=0}^{n} A_i$. So by definition, $x \in A_i$ for some $i \in \{z \in \mathbb{N} : z \leq n\}$. By definition of A_i, $|x| < i$. Since n is the upper bound of the sum, $i \leq n$, so $|x| < n$. Since A_n contains all the elements whose absolute value is less than n, $x \in A_n$. Thus $\displaystyle\bigcup_{i=0}^{n} A_i \subseteq A_n$.

Therefore, since the two sets are subsets of each other, they are equal. □

(h) We claim that $\displaystyle\bigcap_{i=0}^{n} A_i = \emptyset$.

Assume $\bigcap_{i=0}^{n} A_i \neq \emptyset$. Then, there is an $x \in \bigcap_{i=0}^{n} A_i$. By the definition of intersection, $x \in A_i$ for all $i \in \{z \in \mathbb{N} : z \leq n\}$, and thus $x \in A_0$. But A_0 is the empty set and thus has no elements, so a contradiction ensues. Therefore, the intersection is empty. $\square$

(i) We claim that $\bigcup_{i=0}^{n} B_i = \mathbb{Z}$.

First, assume $x \in \bigcup_{i=0}^{n} B_i$. Since B_i is defined as $\{z \in \mathbb{Z} : |z| \geq i\}$, x must be an integer because every set in the union contains only integers. Thus $\bigcup_{i=0}^{n} B_i \subseteq \mathbb{Z}$.

Next, assume $x \in \mathbb{Z}$. Since $B_0 = \mathbb{Z}$, $\mathbb{Z} \subseteq B_0$, and thus by the definition of subset $x \in B_0$. So x is in B_i for some $i \in \{z \in \mathbb{N} : z \leq n\}$, so by definition of union $x \in \bigcup_{i=0}^{n} B_i$. Thus $\mathbb{Z} \subseteq \bigcup_{i=0}^{n} B_i$.

Therefore, since the two sets are subsets of each other, they are equal. $\square$

(j) We claim that $\bigcap_{i=0}^{n} B_i = B_n$.

First, assume $x \in \bigcap_{i=0}^{n} B_i$. Then, $x \in B_i$ for every $i \in \{z \in \mathbb{N} : z \leq n\}$, so $x \in B_n$. Thus $\bigcap_{i=0}^{n} B_i \subseteq B_n$.

Next, assume $x \in B_n$. By definition of B_i, $|x| \geq n$. So $|x| \geq i$ for all $i \in \{z \in \mathbb{Z} : z < n\}$. So by the definition of B_i, $x \in B_i$ for all $i \in \{z \in \mathbb{Z} : z < n\}$, and $x \in \bigcap_{i=0}^{n} B_i$. Thus, $B_n \subseteq \bigcap_{i=0}^{n} B_i$.

Therefore, since the two sets are subsets of each other, they are equal. $\square$

(k) We claim that $\bigcup_{i=0}^{\infty} A_i = \mathbb{Z}$.

First, assume $x \in \bigcup_{i=0}^{\infty} A_i$. Since A_i is defined as $\{z \in \mathbb{Z} : |z| < i\}$, A_i contains only integers for all $i \in \mathbb{N}$, so $x \in \mathbb{Z}$. Thus $\bigcup_{i=0}^{\infty} A_i \subseteq \mathbb{Z}$.

Next, assume $x \in \mathbb{Z}$. Since $|x| < |x+1|$, by the definition of A_i, $x \in A_{|x+1|}$. So $x \in A_i$ for some $i \in \{z \in \mathbb{N}\}$, and by definition of union $x \in \bigcup_{i=0}^{\infty} A_i$. Thus $\mathbb{Z} \subseteq \bigcup_{i=0}^{\infty} A_i$.

Therefore, since the two sets are subsets of each other, they are equal. $\square$

(l) We claim that $\bigcap_{i=0}^{\infty} A_i = \emptyset$.

Assume $\bigcap_{i=0}^{\infty} A_i \neq \emptyset$. Then, there is an $x \in \bigcap_{i=0}^{\infty} A_i$. By the definition of intersection, $x \in A_i$ for all $i \in \{z \in \mathbb{N}\}$, and thus $x \in A_0$. But A_0 is the empty set and thus has no elements, so a contradiction ensues. Therefore, the intersection is empty. $\square$

(m) We claim that $\bigcup_{i=0}^{\infty} B_i = \mathbb{Z}$.

First, assume $x \in \bigcup_{i=0}^{\infty} B_i$. Since B_i is defined as $\{z \in \mathbb{Z} : |z| \geq i\}$, x must be an integer because every set in the union contains only integers. Thus $\bigcup_{i=0}^{\infty} B_i \subseteq \mathbb{Z}$.

Next, assume $x \in \mathbb{Z}$. Since $B_0 = \mathbb{Z}$, $\mathbb{Z} \subseteq B_0$, and thus by the definition of subset $x \in B_0$. So x is in B_i for some $i \in \{z \in \mathbb{N}\}$, so by definition of union $x \in \bigcup_{i=0}^{\infty} B_i$. Thus $\mathbb{Z} \subseteq \bigcup_{i=0}^{\infty} B_i$.

Therefore, since the two sets are subsets of each other, they are equal. $\square$

(n) We claim that $\bigcap_{i=0}^{\infty} B_i = \emptyset$.

Assume $\bigcap_{i=0}^{\infty} B_i \neq \emptyset$. So there is an $x \in \bigcap_{i=0}^{\infty} B_i$. Thus, $x \in B_i$ for every $i \in \{z \in \mathbb{N}\}$, and thus $x \in B_{|x+1|}$. By the definition of $B_{|x+1|}$, then, $|x| > |x+1|$. But this is a contradiction, so there cannot be an $x \in \bigcap_{i=0}^{\infty} B_i$ and the set is therefore empty. $\square$

(o) We claim that $\bigcup_{i=n}^{\infty} A_i = \mathbb{Z}$.

First, assume $x \in \bigcup_{i=n}^{\infty} A_i$. Since A_i is defined as $\{z \in \mathbb{Z} : |z| < i\}$, A_i contains only integers for all $i \in \mathbb{N}$, so $x \in \mathbb{Z}$. Thus $\bigcup_{i=n}^{\infty} A_i \subseteq \mathbb{Z}$.

Next, assume $x \in \mathbb{Z}$. Since $|x| < |x+1|$, by the definition of A_i, $x \in A_{|x+1|}$. If $|x+1| < n$, then $|x| < |x+1| < n$, so $x \in A_n$ in this case. Thus $x \in A_i$ for some $i \in \{z \in \mathbb{N} : z \geq n\}$ by choosing the maximum of $|x+1|$ and n, and by definition of union $x \in \bigcup_{i=0}^{\infty} A_i$. Thus $\mathbb{Z} \subseteq \bigcup_{i=n}^{\infty} A_i$.

Therefore, since the two sets are subsets of each other, they are equal. $\square$

(p) We claim that $\bigcap_{i=n}^{\infty} A_i = A_n$.

First, assume $x \in \bigcap_{i=n}^{\infty} A_i$. So $x \in A_i$ for every $i \in \{z \in \mathbb{N} : z \geq n\}$. So $x \in A_n$. Thus, $\bigcap_{i=n}^{\infty} A_i \subseteq A_n$.

Next, assume $x \in A_n$. By the definition of A_n, $|x| < n$. So $|x| < i$ for every $i \in \{z \in \mathbb{N} : z \geq n\}$. So by definition of intersection, $x \in \bigcap_{i=n}^{\infty} A_i$. Thus, $A_n \subseteq \bigcap_{i=n}^{\infty} A_i$.

Therefore, since the two sets are subsets of each other, they are equal. $\square$

(q) We claim that $\bigcup_{i=n}^{\infty} B_i = B_n$.

First, assume $x \in B_n$. So $x \in B_i$ for some $i \in \{z \in \mathbb{N} : z \geq n\}$. So by definition of union, $x \in \bigcup_{i=n}^{\infty} B_i$. Thus $B_n \subseteq \bigcup_{i=n}^{\infty} B_i$.

Next, assume $x \in \bigcup_{i=n}^{\infty} B_i$. So $x \in B_i$ for some $i \in \{z \in \mathbb{N} : z \geq n\}$. By the definition of B_i, $|x| > i$, and since n is the least element in the set, $|x| > i \geq n$. So $x \in B_n$, and thus $\bigcup_{i=n}^{\infty} B_i \subseteq B_n$.

Therefore, since the two sets are subsets of each other, they are equal. $\square$

(r) We claim that $\bigcap_{i=n}^{\infty} B_i = \emptyset$.

Assume $\bigcap_{i=n}^{\infty} B_i \neq \emptyset$. So there is an $x \in \bigcap_{i=n}^{\infty} B_i$. Thus, $x \in B_i$ for every $i \in \{z \in \mathbb{N} : z \geq n\}$. Since this guarantees that $x \in B_n$, $|x| \geq n$. Then, $|x+1| > |x| \geq n$, and so $|x+1| \in \{z \in \mathbb{N} : z \geq n\}$.

So $x \in B_{|x+1|}$. By the definition of $B_{|x+1|}$, then, $|x| > |x+1|$. But this is a contradiction, so there cannot be an $x \in \bigcap_{i=n}^{\infty} B_i$ and the set is therefore empty. $\square$

Problem 2 Let $A_i = \{0\} \cup \left\{ \frac{1}{z} : z \in \mathbb{Z}^+, z \leq i \right\}$ for $i \in \mathbb{N}$, and let $n \in \mathbb{N}$.

(a) We claim that $\bigcap_{i=0}^{n} A_i = A_0$.

First, assume $x \in \bigcap_{i=0}^{n} A_i$. So $x \in A_i$ for every $i \in \{z \in \mathbb{N} : z \leq n\}$, and thus $x \in A_0$. Thus $\bigcap_{i=0}^{n} A_i \subseteq A_0$.

Next, assume $x \in A_0$. Then, x must be 0, because $\left\{ \frac{1}{z} : z \in \mathbb{Z}^+, z \leq i \right\}$ is the empty set when $i = 0$, as it does in this case, and thus the only element of A_0 is 0. For every $i \in \mathbb{N}$, $\{0\} \subseteq A_i$ by definition, so 0 must be in the intersection. So $x = 0 \in \bigcap_{i=0}^{n} A_i$. Thus, $A_0 \subseteq \bigcap_{i=0}^{n} A_i$. $\square$

Therefore, since the two sets are subsets of each other, they are equal. $\square$

(b) We claim that $\bigcup_{i=0}^{n} A_i = A_n$.

First, assume $x \in A_n$. Then, $x \in A_i$ for some $i \in \{z \in \mathbb{N} : z \leq n\}$, and $x \in \bigcup_{i=0}^{n} A_i$. Thus, $A_n \subseteq \bigcup_{i=0}^{n} A_i$.

Next, assume $x \in \bigcup_{i=0}^{n} A_i$. So for some $i \in \{z \in \mathbb{N} : z \leq n\}$, $x = 0$ or $x = \frac{1}{z}$ where $z \leq i$. Since $i \leq n$, $z \leq i \leq n$ for every i. Since $z \leq n$, by definition $x \in A_n$. Thus, $\bigcup_{i=0}^{n} A_i \subseteq A_n$.

Therefore, since the two sets are subsets of each other, they are equal. $\square$

(c) We claim that $\bigcap_{i=0}^{\infty} A_i = A_0$.

First, assume $x \in \bigcap_{i=0}^{\infty} A_i$. So $x \in A_i$ for every $i \in \{z \in \mathbb{N}\}$, and thus $x \in A_0$. Thus $\bigcap_{i=0}^{\infty} A_i \subseteq A_0$.

Next, assume $x \in A_0$. Then, x must be 0, because $\left\{ \frac{1}{z} : z \in \mathbb{Z}^+, z \leq i \right\}$ is the empty set when $i = 0$, as it does in this case, and thus the only element of A_0 is 0. For every $i \in \mathbb{N}$, $\{0\} \subseteq A_i$ by definition, so 0 must be in the intersection. So $x = 0 \in \bigcap_{i=0}^{\infty} A_i$. Thus, $A_0 \subseteq \bigcap_{i=0}^{\infty} A_i$. $\square$

Therefore, since the two sets are subsets of each other, they are equal. $\square$

(d) We claim that $\bigcup_{i=0}^{\infty} A_i = \{0\} \cup \left\{\frac{1}{z} : z \in \mathbb{Z}^+\right\}$.

First, assume $x \in \bigcup_{i=0}^{\infty} A_i$. So $x \in A_i$ for some $i \in \mathbb{N}$. If $x = 0$, then it is certainly in the union of $\{0\}$ with any other set. If $x = \frac{1}{z}$ where $z \leq i$, then it is certainly in the union of $\left\{\frac{1}{z} : z \in \mathbb{Z}^+\right\}$ with any other set, so $x \in \{0\} \cup \left\{\frac{1}{z} : z \in \mathbb{Z}^+\right\}$, Thus, $\bigcup_{i=0}^{\infty} A_i \subseteq \{0\} \cup \left\{\frac{1}{z} : z \in \mathbb{Z}^+\right\}$.

Next, assume $x \in \{0\} \cup \left\{\frac{1}{z} : z \in \mathbb{Z}^+\right\}$. If $x = 0$, then $x \in A_0$. If $x = \frac{1}{z}$ for some $z \in \mathbb{Z}^+$, then $x \in A_z$ by definition because $z \leq z$. So $x \in A_i$ for some $i \in \mathbb{N}$, and by definition of the union $x \in \bigcup_{i=0}^{\infty} A_i$. Thus $\{0\} \cup \left\{\frac{1}{z} : z \in \mathbb{Z}^+\right\} \subseteq \bigcup_{i=0}^{\infty} A_i$.

Therefore, since the two sets are subsets of each other, they are equal. $\square$