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MATH 544: Section H01
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MATH 544 Homework 3

Problem 1 Determine whether the following systems are consistent. If they are consistent, give the solution in parametric form. If the system has infinitely many solutions, give the solution in vector form $\vec{x} = \vec{c} + a_1\vec{v_1} + \cdots + a_j\vec{v_j}$, where $a_1, \ldots, a_j \in \mathbb{R}$ are arbitrary.

(a)
$$\begin{array}{rcrcrcr} x_1 & + & 2x_2 & + & 3x_3 & = & 1 \\ 2x_1 & + & 4x_2 & + & 5x_3 & = & 1 \\ x_1 & + & 2x_2 & + & 2x_3 & = & 0 \end{array}$$

(b)
$$\begin{array}{rcrcrcrcr} 3x_1 & - & 6x_2 & - & x_3 & + & x_4 & = & 6 \\ -x_1 & + & 2x_2 & + & 2x_3 & + & 3x_4 & = & 3 \\ 4x_1 & - & 8x_2 & - & 3x_3 & - & 2x_4 & = & 3 \end{array}$$

Solution.

(a) We write the augmented matrix $(A \mid \vec{b})$ and obtain the reduced row-echelon matrix $(C \mid \vec{d})$:

$$(A \mid \vec{b}) = \left(\begin{array}{ccc|c} 1 & 2 & 3 & 1 \\ 2 & 4 & 5 & 1 \\ 1 & 2 & 2 & 0 \end{array}\right)$$

$$\sim \left(\begin{array}{ccc|c} 1 & 2 & 3 & 1 \\ 0 & 0 & -1 & -1 \\ 1 & 2 & 2 & 0 \end{array}\right) \qquad (\rho_2 \mapsto \rho_2 - 2\rho_1)$$

$$\sim \left(\begin{array}{ccc|c} 1 & 2 & 3 & 1 \\ 0 & 0 & -1 & -1 \\ 0 & 0 & -1 & -1 \end{array}\right) \qquad (\rho_3 \mapsto \rho_3 - \rho_1)$$

$$\sim \left(\begin{array}{ccc|c} 1 & 2 & 3 & 1 \\ 0 & 0 & -1 & -1 \\ 0 & 0 & 0 & 0 \end{array}\right) \qquad (\rho_3 \mapsto \rho_3 - \rho_2)$$

$$\sim \left(\begin{array}{ccc|c} 1 & 2 & 3 & 1 \\ 0 & 0 & 1 & 1 \\ 0 & 0 & 0 & 0 \end{array}\right) \qquad (\rho_2 \mapsto -\rho_2)$$

$$\sim \left(\begin{array}{ccc|c} 1 & 2 & 0 & -2 \\ 0 & 0 & 1 & 1 \\ 0 & 0 & 0 & 0 \end{array}\right) = (C \mid \vec{d}). \qquad (\rho_1 \mapsto \rho_1 - 3\rho_2)$$

There are no leading ones in $\vec{d}$, so the system is consistent, and we have

$$\begin{cases} x_1 = -2 - 2x_2 \\ x_2 \in \mathbb{R} \\ x_3 = 1 \end{cases}$$

and the solution set

$$\left\{ \vec{x} = \begin{pmatrix} -2 \\ 0 \\ 1 \end{pmatrix} + a_1 \begin{pmatrix} -2 \\ 1 \\ 0 \end{pmatrix} : a_1 \in \mathbb{R} \right\}.$$

(b) We write the augmented matrix $(A \mid \vec{b})$ and obtain the reduced row-echelon matrix $(C \mid \vec{d})$:

$$(A \mid \vec{b}) = \left(\begin{array}{cccc|c} 3 & -6 & -1 & 1 & 6 \\ -1 & 2 & 2 & 3 & 3 \\ 4 & -8 & -3 & -2 & 3 \end{array}\right)$$

$$\sim \left(\begin{array}{cccc|c} 1 & -2 & -\frac{1}{3} & \frac{1}{3} & 2 \\ -1 & 2 & 2 & 3 & 3 \\ 4 & -8 & -3 & -2 & 3 \end{array}\right) \quad (\rho_1 \mapsto \tfrac{1}{3}\rho_1)$$

$$\sim \left(\begin{array}{cccc|c} 1 & -2 & -\frac{1}{3} & \frac{1}{3} & 2 \\ 0 & 0 & \frac{5}{3} & \frac{10}{3} & 5 \\ 4 & -8 & -3 & -2 & 3 \end{array}\right) \quad (\rho_2 \mapsto \rho_2 + \rho_1)$$

$$\sim \left(\begin{array}{cccc|c} 1 & -2 & -\frac{1}{3} & \frac{1}{3} & 2 \\ 0 & 0 & \frac{5}{3} & \frac{10}{3} & 5 \\ 0 & 0 & -\frac{5}{3} & -\frac{10}{3} & -5 \end{array}\right) \quad (\rho_3 \mapsto \rho_3 - 4\rho_1)$$

$$\sim \left(\begin{array}{cccc|c} 1 & -2 & -\frac{1}{3} & \frac{1}{3} & 2 \\ 0 & 0 & 1 & 2 & 3 \\ 0 & 0 & -\frac{5}{3} & -\frac{10}{3} & 5 \end{array}\right) \quad (\rho_2 \mapsto \tfrac{3}{5}\rho_2)$$

$$\sim \left(\begin{array}{cccc|c} 1 & -2 & -\frac{1}{3} & \frac{1}{3} & 2 \\ 0 & 0 & 1 & 2 & 3 \\ 0 & 0 & 0 & 0 & 0 \end{array}\right) \quad (\rho_3 \mapsto \rho_3 + \tfrac{5}{3}\rho_2)$$

$$\sim \left(\begin{array}{cccc|c} 1 & -2 & 0 & 1 & 3 \\ 0 & 0 & 1 & 2 & 3 \\ 0 & 0 & 0 & 0 & 0 \end{array}\right) = (C \mid \vec{d}). \quad (\rho_1 \mapsto \rho_1 + \tfrac{1}{3}\rho_2)$$

There are no leading ones in $\vec{d}$, so the system is consistent, and we have

$$\begin{cases} x_1 = 3 + 2x_2 - x_4 \\ x_2 \in \mathbb{R} \\ x_3 = 3 - 2x_4 \\ x_4 \in \mathbb{R} \end{cases}$$

and the solution set

$$\left\{ \vec{x} = \begin{pmatrix} 3 \\ 0 \\ 3 \\ 0 \end{pmatrix} + a_1 \begin{pmatrix} 2 \\ 1 \\ 0 \\ 0 \end{pmatrix} + a_2 \begin{pmatrix} -1 \\ 0 \\ -2 \\ 1 \end{pmatrix} : a_1, a_2 \in \mathbb{R} \right\}.$$

Problem 2 For each matrix A, compute $\text{Null}(A)$, the null space of A. (Recall that $\text{Null}(A) = \{\vec{x} \mid A\vec{x} = \vec{0}\}$.) If $\text{Null}(A) \neq \{\vec{0}\}$, express $\text{Null}(A)$ as the span of one or more vectors.

(a) $A = \begin{pmatrix} 2 & -2 & 4 \\ -1 & 1 & -2 \\ 3 & -3 & 6 \end{pmatrix}$

(b) $A = \begin{pmatrix} 1 & -1 & 1 & 1 & 0 \\ 1 & 0 & 2 & 1 & 1 \\ 0 & 2 & 2 & 2 & 0 \\ -1 & 1 & -1 & 0 & -1 \end{pmatrix}$

Solution.

(a) We write the augmented matrix $(A \mid \vec{0})$ and obtain the reduced row-echelon matrix $(C \mid \vec{0})$.

$$(A \mid \vec{0}) = \left(\begin{array}{ccc|c} 2 & -2 & 4 & 0 \\ -1 & 1 & -2 & 0 \\ 3 & -3 & 6 & 0 \end{array}\right)$$

$$\sim \left(\begin{array}{ccc|c} 1 & -1 & 2 & 0 \\ -1 & 1 & -2 & 0 \\ 3 & -3 & 6 & 0 \end{array}\right) \qquad (\rho_1 \mapsto \tfrac{1}{2}\rho_1)$$

$$\sim \left(\begin{array}{ccc|c} 1 & -1 & 2 & 0 \\ 0 & 0 & 0 & 0 \\ 3 & -3 & 6 & 0 \end{array}\right) \qquad (\rho_2 \mapsto \rho_2 + \rho_1)$$

$$\sim \left(\begin{array}{ccc|c} 1 & -1 & 2 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \end{array}\right) = (C \mid \vec{0}). \qquad (\rho_3 \mapsto \rho_3 - 3\rho_1)$$

We have

$$\begin{cases} x_1 = x_2 - 2x_3 \\ x_2 \in \mathbb{R} \\ x_3 \in \mathbb{R} \end{cases}$$

and the solution set

$$\left\{ \vec{x} = a_1 \begin{pmatrix} 1 \\ 1 \\ 0 \end{pmatrix} + a_2 \begin{pmatrix} -2 \\ 0 \\ 1 \end{pmatrix} : a_1, a_2 \in \mathbb{R} \right\}.$$

Thus, we can write

$$\text{Null}(A) = \text{Span}\left\{ \begin{pmatrix} 1 \\ 1 \\ 0 \end{pmatrix}, \begin{pmatrix} -2 \\ 0 \\ 1 \end{pmatrix} \right\}.$$

(b) We write the augmented matrix $(A \mid \vec{0})$ and obtain the reduced row-echelon matrix $(C \mid \vec{0})$.

$$(A \mid \vec{0}) = \left(\begin{array}{ccccc|c} 1 & -1 & 1 & 1 & 0 & 0 \\ 1 & 0 & 2 & 1 & 1 & 0 \\ 0 & 2 & 2 & 2 & 0 & 0 \\ -1 & 1 & -1 & 0 & -1 & 0 \end{array}\right)$$

$$\sim \left(\begin{array}{ccccc|c} 1 & -1 & 1 & 1 & 0 & 0 \\ 0 & 1 & 1 & 0 & 1 & 0 \\ 0 & 2 & 2 & 2 & 0 & 0 \\ -1 & 1 & -1 & 0 & -1 & 0 \end{array}\right) \qquad (\rho_2 \mapsto \rho_2 - \rho_1)$$

$$\sim \left(\begin{array}{ccccc|c} 1 & -1 & 1 & 1 & 0 & 0 \\ 0 & 1 & 1 & 0 & 1 & 0 \\ 0 & 2 & 2 & 2 & 0 & 0 \\ 0 & 0 & 0 & 1 & -1 & 0 \end{array}\right) \qquad (\rho_4 \mapsto \rho_4 + \rho_1)$$

$$\sim \left(\begin{array}{ccccc|c} 1 & -1 & 1 & 1 & 0 & 0 \\ 0 & 1 & 1 & 0 & 1 & 0 \\ 0 & 0 & 0 & 2 & -2 & 0 \\ 0 & 0 & 0 & 1 & -1 & 0 \end{array}\right) \qquad (\rho_3 \mapsto \rho_3 - 2\rho_2)$$

$$\sim \left(\begin{array}{ccccc|c} 1 & -1 & 1 & 1 & 0 & 0 \\ 0 & 1 & 1 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 & -1 & 0 \\ 0 & 0 & 0 & 1 & -1 & 0 \end{array}\right) \qquad (\rho_3 \mapsto \tfrac{1}{2}\rho_3)$$

$$\sim \left(\begin{array}{ccccc|c} 1 & -1 & 1 & 1 & 0 & 0 \\ 0 & 1 & 1 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 & -1 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 \end{array}\right) \qquad (\rho_4 \mapsto \rho_4 - \rho_3)$$

$$\sim \left(\begin{array}{ccccc|c} 1 & 0 & 2 & 1 & 1 & 0 \\ 0 & 1 & 1 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 & -1 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 \end{array}\right) \qquad (\rho_1 \mapsto \rho_1 + \rho_2)$$

$$\sim \left(\begin{array}{ccccc|c} 1 & 0 & 2 & 0 & 2 & 0 \\ 0 & 1 & 1 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 & -1 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 \end{array}\right) = (C \mid \vec{0}). \qquad (\rho_1 \mapsto \rho_1 - \rho_3)$$

We have

$$\begin{cases} x_1 = -2x_3 - 2x_5 \\ x_2 = -x_3 - x_5 \\ x_3 \in \mathbb{R} \\ x_4 = x_5 \\ x_5 \in \mathbb{R} \end{cases}$$

and the solution set

$$\left\{ \vec{x} = a_1 \begin{pmatrix} -2 \\ -1 \\ 1 \\ 0 \\ 0 \end{pmatrix} + a_2 \begin{pmatrix} -2 \\ -1 \\ 0 \\ 1 \\ 1 \end{pmatrix} : a_1, a_2 \in \mathbb{R} \right\}.$$

Thus, we can write

$$\operatorname{Null}(A) = \operatorname{Span}\left\{ \begin{pmatrix} -2 \\ -1 \\ 1 \\ 0 \\ 0 \end{pmatrix}, \begin{pmatrix} -2 \\ -1 \\ 0 \\ 1 \\ 1 \end{pmatrix} \right\}.$$

Problem 3 One might need to find solutions to $A\vec{x} = \vec{b_i}$ for $\vec{b_1}, \dots, \vec{b_k}$. To do this efficiently, reduce the "multi-augmented" matrix $(A \mid \vec{b_1}\vec{b_2} \dots \vec{b_k})$ to reduced row-echelon form. Use this method to solve $A\vec{x} = \vec{b_i}$ for $A = \begin{pmatrix} 1 & 0 & -1 \\ 2 & 1 & -1 \\ -1 & 2 & 2 \end{pmatrix}$ and $\vec{b_1} = \begin{pmatrix} -1 \\ 1 \\ 5 \end{pmatrix}$, $\vec{b_2} = \begin{pmatrix} 1 \\ 3 \\ 2 \end{pmatrix}$. If the system is consistent give the solution in parametric form. If the system has infinitely many solutions, give the solution in vector form following the instructions in (1) above.

Solution.

We write the augmented matrix $(A \mid \vec{b_1}\vec{b_2})$ and obtain the reduced row-echelon matrix $(C \mid \vec{d_1}\vec{d_2})$:

$$\begin{aligned}
(A \mid \vec{b_1}\vec{b_2}) &= \left(\begin{array}{ccc|cc} 1 & 0 & -1 & -1 & 1 \\ 2 & 1 & -1 & 1 & 3 \\ -1 & 2 & 2 & 5 & 2 \end{array}\right) \\
&\sim \left(\begin{array}{ccc|cc} 1 & 0 & -1 & -1 & 1 \\ 0 & 1 & 1 & 3 & 1 \\ -1 & 2 & 2 & 5 & 2 \end{array}\right) && (\rho_2 \mapsto \rho_2 - 2\rho_1) \\
&\sim \left(\begin{array}{ccc|cc} 1 & 0 & -1 & -1 & 1 \\ 0 & 1 & 1 & 3 & 1 \\ 0 & 2 & 1 & 4 & 3 \end{array}\right) && (\rho_3 \mapsto \rho_3 + \rho_1) \\
&\sim \left(\begin{array}{ccc|cc} 1 & 0 & -1 & -1 & 1 \\ 0 & 1 & 1 & 3 & 1 \\ 0 & 0 & -1 & -2 & 1 \end{array}\right) && (\rho_3 \mapsto \rho_3 - 2\rho_2) \\
&\sim \left(\begin{array}{ccc|cc} 1 & 0 & -1 & -1 & 1 \\ 0 & 1 & 1 & 3 & 1 \\ 0 & 0 & 1 & 2 & -1 \end{array}\right) && (\rho_3 \mapsto -\rho_3) \\
&\sim \left(\begin{array}{ccc|cc} 1 & 0 & -1 & -1 & 1 \\ 0 & 1 & 0 & 1 & 2 \\ 0 & 0 & 1 & 2 & -1 \end{array}\right) && (\rho_2 \mapsto \rho_2 - \rho_3) \\
&\sim \left(\begin{array}{ccc|cc} 1 & 0 & 0 & 1 & 0 \\ 0 & 1 & 0 & 1 & 2 \\ 0 & 0 & 1 & 2 & -1 \end{array}\right) = (C \mid \vec{d_1}\vec{d_2}). && (\rho_1 \mapsto \rho_1 + \rho_2)
\end{aligned}$$

There are no leading ones in $\vec{d_1}$ or $\vec{d_2}$, so both systems are consistent. For $A\vec{x} = \vec{b_1}$, we have

$$\begin{cases} x_1 = 1 \\ x_2 = 1 \\ x_3 = 2 \end{cases}$$

and for $A\vec{x} = \vec{b_2}$, we have

$$\begin{cases} x_1 = 0 \\ x_2 = 2 \\ x_3 = -1 \end{cases}.$$

Problem 4 Let $A = \begin{pmatrix} 1 & \alpha \\ \alpha & 3\alpha \end{pmatrix}$. Find all values of α such that $A\vec{x} = \vec{0}$ has infinitely many solutions.

Solution.

We have

$$(A \mid \vec{0}) = \left(\begin{array}{cc|c} 1 & \alpha & 0 \\ \alpha & 3\alpha & 0 \end{array}\right) \sim \left(\begin{array}{cc|c} 1 & \alpha & 0 \\ 0 & 3\alpha - \alpha^2 & 0 \end{array}\right)$$

by performing the row operation $\rho_2 \mapsto \rho_2 - \alpha\rho_1$. For any $3\alpha - \alpha^2 \neq 0$, we can perform the row operation $\rho_2 \mapsto \frac{1}{3\alpha-\alpha^2}\rho_2$ and we will obtain $\mathrm{rref}(A) = I_2$. However, if $3\alpha - \alpha^2 = 0$, x_2 will be an independent variable and we will have infinitely many solutions. Solving the quadratic, this will happen when $\alpha \in \{0, 3\}$.

Problem 5 For each of the following vectors $\vec{b} \in \mathbb{R}^4$, decide whether $\vec{b} \in \mathrm{Span}\left\{\begin{pmatrix} 1 \\ 0 \\ 1 \\ -2 \end{pmatrix}, \begin{pmatrix} 0 \\ -1 \\ 0 \\ 1 \end{pmatrix}, \begin{pmatrix} 1 \\ -2 \\ 1 \\ 0 \end{pmatrix}\right\}$.

If the answer is "yes", then express $\vec{b}$ as a linear combination of the vectors $\vec{v_1}$,$\vec{v_2}$,$\vec{v_3}$. There could be many ways to do this.

(a) $\vec{b_a} = \begin{pmatrix} 1 \\ 1 \\ 1 \\ 1 \end{pmatrix}$.

(b) $\vec{b_b} = \begin{pmatrix} 1 \\ -1 \\ 1 \\ -1 \end{pmatrix}$.

(c) $\vec{b_c} = \begin{pmatrix} 1 \\ -1 \\ 0 \\ -2 \end{pmatrix}$.

Note: You may consider using the method of problem 3 to handle (a), (b), (c) all at once.

Solution.

For $i \in \{a, b, c\}$, we would like to find some $a_1, a_2, a_3 \in \mathbb{R}$ such that $\vec{b_i} = a_1\vec{v_1} + a_2\vec{v_2} + a_3\vec{v_3}$. Let

$$A = (\vec{v_1}\vec{v_2}\vec{v_3}) = \begin{pmatrix} 1 & 0 & 1 \\ 0 & -1 & -2 \\ 1 & 0 & 1 \\ -2 & 1 & 0 \end{pmatrix}.$$

We can reframe this problem as a system of linear equations, where we are trying to solve $A\vec{x} = \vec{b_i}$. We write the augmented matrix $(A \mid \vec{b_a}\vec{b_b}\vec{b_c})$ and obtain the reduced row-echelon matrix $(C \mid \vec{d_a}\vec{d_b}\vec{d_c})$:

$$\begin{aligned}(A \mid \vec{b_a}\vec{b_b}\vec{b_c}) &= \left(\begin{array}{ccc|ccc} 1 & 0 & 1 & 1 & 1 & 1 \\ 0 & -1 & -2 & 1 & -1 & -1 \\ 1 & 0 & 1 & 1 & 1 & 0 \\ -2 & 1 & 0 & 1 & -1 & -2 \end{array}\right) \\ &\sim \left(\begin{array}{ccc|ccc} 1 & 0 & 1 & 1 & 1 & 1 \\ 0 & -1 & -2 & 1 & -1 & -1 \\ 0 & 0 & 0 & 0 & 0 & -1 \\ -2 & 1 & 0 & 1 & -1 & -2 \end{array}\right) && (\rho_3 \mapsto \rho_3 - \rho_1)\end{aligned}$$

$$\sim \left(\begin{array}{ccc|ccc} 1 & 0 & 1 & 1 & 1 & 1 \\ 0 & -1 & -2 & 1 & -1 & -1 \\ 0 & 0 & 0 & 0 & 0 & -1 \\ 0 & 1 & 2 & 3 & 1 & 0 \end{array}\right) \qquad (\rho_4 \mapsto \rho_4 + \rho_1)$$

$$\sim \left(\begin{array}{ccc|ccc} 1 & 0 & 1 & 1 & 1 & 1 \\ 0 & 1 & 2 & -1 & 1 & 1 \\ 0 & 0 & 0 & 0 & 0 & -1 \\ 0 & 1 & 2 & 3 & 1 & 0 \end{array}\right) \qquad (\rho_2 \mapsto -\rho_2)$$

$$\sim \left(\begin{array}{ccc|ccc} 1 & 0 & 1 & 1 & 1 & 1 \\ 0 & 1 & 2 & -1 & 1 & 1 \\ 0 & 0 & 0 & 0 & 0 & -1 \\ 0 & 0 & 0 & 4 & 0 & -1 \end{array}\right) \qquad (\rho_4 \mapsto \rho_4 - \rho_2)$$

$$\sim \left(\begin{array}{ccc|ccc} 1 & 0 & 1 & 1 & 1 & 1 \\ 0 & 1 & 2 & -1 & 1 & 1 \\ 0 & 0 & 0 & 4 & 0 & -1 \\ 0 & 0 & 0 & 0 & 0 & -1 \end{array}\right) \qquad (\rho_3 \leftrightarrow \rho_4)$$

$$\sim \left(\begin{array}{ccc|ccc} 1 & 0 & 1 & 1 & 1 & 1 \\ 0 & 1 & 2 & -1 & 1 & 1 \\ 0 & 0 & 0 & 4 & 0 & -1 \\ 0 & 0 & 0 & 0 & 0 & 1 \end{array}\right) \qquad (\rho_4 \mapsto -\rho_4)$$

$$\sim \left(\begin{array}{ccc|ccc} 1 & 0 & 1 & 1 & 1 & 1 \\ 0 & 1 & 2 & -1 & 1 & 1 \\ 0 & 0 & 0 & 4 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 1 \end{array}\right) \qquad (\rho_3 \mapsto \rho_3 + \rho_4)$$

$$\sim \left(\begin{array}{ccc|ccc} 1 & 0 & 1 & 1 & 1 & 1 \\ 0 & 1 & 2 & -1 & 1 & 1 \\ 0 & 0 & 0 & 1 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 1 \end{array}\right) \qquad (\rho_3 \mapsto \tfrac{1}{4}\rho_3)$$

$$\sim \left(\begin{array}{ccc|ccc} 1 & 0 & 1 & 1 & 1 & 1 \\ 0 & 1 & 2 & 0 & 1 & 1 \\ 0 & 0 & 0 & 1 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 1 \end{array}\right) \qquad (\rho_2 \mapsto \rho_2 + \rho_3)$$

$$\sim \left(\begin{array}{ccc|ccc} 1 & 0 & 1 & 1 & 1 & 1 \\ 0 & 1 & 2 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 1 \end{array}\right) \qquad (\rho_2 \mapsto \rho_2 - \rho_4)$$

$$\sim \left(\begin{array}{ccc|ccc} 1 & 0 & 1 & 0 & 1 & 1 \\ 0 & 1 & 2 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 1 \end{array}\right) \qquad (\rho_2 \mapsto \rho_1 - \rho_3)$$

$$\sim \left(\begin{array}{ccc|ccc} 1 & 0 & 1 & 0 & 1 & 0 \\ 0 & 1 & 2 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 1 \end{array}\right) = (C \mid \vec{d_a}\vec{d_b}\vec{d_c}). \qquad (\rho_2 \mapsto \rho_1 - \rho_4)$$

Since $\vec{d_a}$ and $\vec{d_c}$ contain leading ones, these systems are inconsistent and thus there is no way to express $\vec{b_a}$ or $\vec{b_c}$ as a linear combination of the vectors $\vec{v_1}$,$\vec{v_2}$,$\vec{v_3}$. However, we can write the solution set of $A\vec{x} = \vec{b_2}$ as

$$\left\{ \vec{x} = \begin{pmatrix} 1 \\ 1 \\ 0 \end{pmatrix} + a_1 \begin{pmatrix} -1 \\ -2 \\ 1 \end{pmatrix} : a_1 \in \mathbb{R} \right\}.$$

Then in particular for $a_1 = 0$, we can write

$$\begin{pmatrix} 1 \\ -1 \\ 1 \\ -1 \end{pmatrix} = \begin{pmatrix} 1 \\ 0 \\ 1 \\ -2 \end{pmatrix} + \begin{pmatrix} 0 \\ -1 \\ 0 \\ 1 \end{pmatrix}.$$

Problem 6 Let $A \in \mathrm{Mat}_{m\times n}(\mathbb{R})$. Prove or give a counterexample to the following statements:
(a) Suppose that there exists $\vec{x} \in \mathbb{R}^n$ such that $A\vec{x} = \vec{0}$. Then we have either $A = O_{m\times n}$ or $\vec{x} = \vec{0}$.
(b) Suppose, for every $\vec{x} \in \mathbb{R}^n$, we have $A\vec{x} = \vec{0}$. Then we have $A = O_{m\times n}$.

Solution.

(a) This is not true. For example, take $A = \begin{pmatrix} 2 & -2 & 4 \\ -1 & 1 & -2 \\ 3 & -3 & 6 \end{pmatrix}$. We showed in problem 2a that there are infinitely many nonzero vectors $\vec{x}$ such that $A\vec{x} = \vec{0}$, such as $\vec{x} = \begin{pmatrix} 1 \\ 1 \\ 0 \end{pmatrix}$.

(b) Let

$$A = \begin{pmatrix} a_{11} & a_{12} & \dots & a_{1n} \\ a_{21} & a_{22} & \dots & a_{2n} \\ \vdots & \vdots & \ddots & \vdots \\ a_{m1} & a_{m2} & \dots & a_{mn} \end{pmatrix}, \vec{x} = \begin{pmatrix} x_1 \\ x_2 \\ \vdots \\ x_n \end{pmatrix} \in \mathbb{R}^n.$$

Since $A\vec{x} = \vec{0}$, from multiplying the matrices we have the system

$$\begin{array}{ccccccccc} x_1a_{11} & + & x_2a_{12} & + & \dots & + & x_na_{1n} & = & 0 \\ x_1a_{21} & + & x_2a_{22} & + & \dots & + & x_na_{2n} & = & 0 \\ \vdots & & \vdots & & \ddots & & \vdots & & \vdots \\ x_1a_{m1} & + & x_2a_{m2} & + & \dots & + & x_na_{mn} & = & 0 \end{array}.$$

Let $\vec{a_j} = \begin{pmatrix} a_{1j} \\ a_{2j} \\ \vdots \\ a_{mj} \end{pmatrix}$ for $j \in \{1, 2, \dots, n\}$. Then, we can write this system as $x_1\vec{a_1} + x_2\vec{a_2} + \cdots + x_n\vec{a_n} = \vec{0}$. Since $x_1, x_2, \dots, x_n$ are arbitrary real numbers, we have that $\mathrm{Span}\{\vec{a_1}, \vec{a_2}, \dots, \vec{a_n}\} = \vec{0}$. Assume that $A \neq O_{m\times n}$.

Then, at least one of $\vec{a_1}, \vec{a_2}, \ldots, \vec{a_n}$ is nonzero. But then we can use that vector to span at least one dimension, a contradiction. So we must have $A = O_{m\times n}$.

Problem 7 In each case, either give an example of $m, n \geq 1$ and $A \in \text{Mat}_{m\times n}(\mathbb{R})$ with the stated property, or explain why none can exist.
(a) $A\vec{x} = \vec{b}$ is inconsistent for every $\vec{b} \in \mathbb{R}^m$.
(b) $A\vec{x} = \vec{b}$ has exactly one solution for every $\vec{b} \in \mathbb{R}^m$.
(c) There exists $\vec{b} \in \mathbb{R}^m$ such that $A\vec{x} = \vec{b}$ is inconsistent and $A\vec{x} = \vec{b}$ has infinitely many solutions whenever it is consistent. (Think about possibilities for $\text{rref}(A \mid \vec{b})$.)

Solution.

(a) No such A exists. We have shown in class that every homogeneous system has the solution $\vec{x} = 0$. So no matter the choice of A, we can choose $\vec{b} = \vec{0}$ and then $A\vec{x} = \vec{b}$ is consistent.

(b) Let $m = n = 3$, and $A = I_m$. Then, as we have shown in class, we can reduce $(A \mid \vec{b})$ to the reduced row-echelon form $(C \mid \vec{d})$, and $A\vec{x} = \vec{b}$ will have the unique solution $\vec{x} = \vec{d}$.

(c) Let $m = n = 2$, and $A = \begin{pmatrix} 1 & 1 \\ 0 & 0 \end{pmatrix}$. Then, $A\vec{x} = \vec{b}$ for $\vec{b} = \begin{pmatrix} 1 \\ 1 \end{pmatrix}$ is inconsistent, because $\vec{d}$ has a leading one. However, if $A\vec{x} = \vec{b}$ is consistent, the only leading one will be in the first row of A. So the rank is less than the number of rows, meaning there is one independent variable and thus there are infinitely many solutions.

Problem 8 Let $A, B \in \text{Mat}_{n\times n}(\mathbb{R})$, and suppose that $A\vec{x} = \vec{0}$ and $B\vec{x} = \vec{0}$ both have the only trivial solution, $\vec{x} = \vec{0}$. Show that $(AB)\vec{x} = \vec{0}$ has the only trivial solution.
(In different terms, this problem asks you to show that a product of non-singular matrices is non-singular.)

Solution.

Suppose that $(AB)\vec{x} = \vec{0}$ has a non-trivial solution $\vec{c} \neq \vec{0}$. Then, $(AB)\vec{c} = \vec{0}$, and by associativity, $A(B\vec{c}) = \vec{0}$. Since $\vec{c} \neq \vec{0}$, and $B\vec{x} = \vec{0}$ only if $\vec{x} = \vec{0}$, we have $B\vec{c} \neq \vec{0}$. But then $B\vec{c}$ is a non-trivial solution to $A\vec{x} = \vec{0}$, a contradiction. So $(AB)\vec{x} = \vec{0}$ must only have the trivial solution.

Problem 9 Let $A = \begin{pmatrix} 1 & \alpha & \alpha \\ \alpha & 2 & 1 \\ \alpha & \alpha & 1 \end{pmatrix}$. Find all values of α such that $A\vec{x} = \vec{0}$ has infinitely many solutions.

Solution.

For most values of α, we have

$$(A \mid \vec{0}) = \left(\begin{array}{ccc|c} 1 & \alpha & \alpha & 0 \\ \alpha & 2 & 1 & 0 \\ \alpha & \alpha & 1 & 0 \end{array}\right)$$

$$\sim \left(\begin{array}{ccc|c} 1 & \alpha & \alpha & 0 \\ 0 & 2-\alpha^2 & 1-\alpha^2 & 0 \\ \alpha & \alpha & 1 & 0 \end{array}\right) \qquad (\rho_2 \mapsto \rho_2 - \alpha\rho_1)$$

$$\sim \left(\begin{array}{ccc|c} 1 & \alpha & \alpha & 0 \\ 0 & 2-\alpha^2 & 1-\alpha^2 & 0 \\ 0 & \alpha-\alpha^2 & 1-\alpha^2 & 0 \end{array}\right) \qquad (\rho_3 \mapsto \rho_3 - \alpha\rho_1)$$

$$\sim \left(\begin{array}{ccc|c} 1 & \alpha & \alpha & 0 \\ 0 & 1 & \frac{1-\alpha^2}{2-\alpha^2} & \\ 0 & \alpha-\alpha^2 & 1-\alpha^2 & 0 \end{array}\right) \qquad (\rho_2 \mapsto \tfrac{1}{2-\alpha^2}\rho_2)$$

$$\sim \left(\begin{array}{ccc|c} 1 & \alpha & \alpha & 0 \\ 0 & 1 & \frac{1-\alpha^2}{2-\alpha^2} & 0 \\ 0 & 0 & 1-\alpha^2-\frac{(\alpha-\alpha^2)(1-\alpha^2)}{2-\alpha^2} & 0 \end{array}\right). \qquad (\rho_3 \mapsto \rho_3 - (\alpha-\alpha^2)\rho_2)$$

If we have $2-\alpha^2 \neq 0$ and $1-\alpha^2-\frac{(\alpha-\alpha^2)(1-\alpha^2)}{2-\alpha^2} \neq 0$, then the operations are valid and we can reduce $\operatorname{rref}(A)$ to I_3 using the algorithm we showed in class. If $2-\alpha^2 = 0$, then we can switch rows 2 and 3 and we will still obtain the identity matrix. However, if $1-\alpha^2-\frac{(\alpha-\alpha^2)(1-\alpha^2)}{2-\alpha^2} = 0$, then we have a rank of 2 and we will have an independent variable, leading to infinite solutions.

$$\begin{aligned} 1-\alpha^2-\frac{(\alpha-\alpha^2)(1-\alpha^2)}{2-\alpha^2} &= 0 \\ \implies 1-\alpha^2 &= \frac{(\alpha-\alpha^2)(1-\alpha^2)}{2-\alpha^2} \\ \implies (1-\alpha^2)(2-\alpha^2) &= (\alpha-\alpha^2)(1-\alpha^2) \\ \implies 2-\alpha^2 &= \alpha-\alpha^2 \textbf{ or } 1-\alpha^2 = 0 \\ \implies 2 &= \alpha \textbf{ or } \alpha^2 = 1 \\ \implies \alpha &= 2 \textbf{ or } \alpha = \pm 1 \end{aligned}$$

So $A\vec{x} = \vec{0}$ will have infinitely many solutions when $\alpha \in \{\pm 1, 2\}$.