Nathan Bickel
MATH 554: Section H01
Professor: Dr. Howard
November 25, 2023

MATH 554 Homework 15

Problem 1 Let (E, d) and (E', d') be metric spaces and $f : E \to E'$ be a function. Prove that the following are equivalent:

(a) f is continuous.

(b) For every $p_0 \in E$, the limit $\lim_{p \to p_0} f(p) = f(p_0)$ holds.

(c) If $\langle p_n \rangle_{n=1}^{\infty}$ is a sequence in E with $\lim_{n \to \infty} p_n = p_0$, then $\lim_{n \to \infty} f(p_n) = f(p_0)$.

(d) If V is an open subset of E', then the preimage $f^{-1}[V]$ is an open subset of E.

We have $(a) \iff (b) \iff (c)$ from the thereom we proved in class, so it is enough to prove $(a) \iff (d)$. We will prove both directions.

$(\Rightarrow)$ Let V be an open subset of E'. Let $p \in f^{-1}[V]$. Since V is open, there exists an $\varepsilon > 0$ such that $B(f(p), \varepsilon) \subseteq V$. Since f is continuous, it is continuous at p, so there exists $\delta > 0$ such that for all $x \in E$,

$$d(p, x) < \delta \implies d'(f(p), f(x)) < \varepsilon$$

holds. Equivalently, for all $x \in E$ we have

$$x \in B(p, \delta) \implies f(x) \in B(f(p), \varepsilon).$$

Let $x \in B(p, \delta)$. Since this implies $f(x) \in B(f(p), \varepsilon) \subseteq V$, we have $f(x) \in V$ and thus $x \in f^{-1}[V]$. Therefore, we have $B(p, \delta) \subseteq f^{-1}[V]$ and so by definition $f^{-1}[V]$ is open.

$(\Leftarrow)$ Let $p \in E$. We will show f is continuous at p. Let $\varepsilon > 0$, and consider $V := B(f(p), \varepsilon)$. Since V is open in E', the preimage $f^{-1}[V]$ is open in E. By the assumption, there exists some $\delta > 0$ such that $B(p, \delta) \subseteq f^{-1}[V]$. Let $x \in E$ such that $d(p, x) < \delta$. Then $x \in B(p, \delta) \subseteq f^{-1}[V]$, so $f(x) \in V = B(f(p), \varepsilon)$. Thus, $d'(f(p), f(x)) < \varepsilon$, so by definition f is continuous at p. Therefore, since p was arbitrary, f is continuous everywhere. $\square$

Problem 2 Let $f : E \to E'$ be a continuous function between metric spaces and $\mathcal{U}$ an open cover of E'. Prove $\mathcal{Y} := \{f^{-1}[V] : V \in \mathcal{U}\}$ is an open cover of E.

From Problem 1, we have that every set in $\mathcal{Y}$ is open since f is continuous, so it suffices to show that $E \subseteq \bigcup \mathcal{Y}$. Let $p \in E$. Then $f(p) \in V$ for some $V \in \mathcal{U}$ since $\mathcal{U}$ is an open cover of E'. Thus, $p \in f^{-1}[V]$, and so by definition p is in $\bigcup \mathcal{Y}$. Therefore, $\mathcal{Y}$ is an open cover. $\square$

Problem 3 Find the error in a given "proof" (the proof is a bit long to include here).

The issue with the "proof" is the choice of δ. The definition of a limit states that for all $\varepsilon > 0$, there exists some $\delta > 0$ such that for all $x \in \mathbb{R}$, an implication holds, so it is not well-defined to have δ depend on x.

Problem 4 Show that

$$\lim_{x \to 1} \frac{1}{x-1}$$

does not exist.

Let $f(x) = \frac{1}{x-1}$. We first note that f is monotone decreasing on $(1, \infty)$. To see this, let $x_\ell > x_s > 1$. Then $x_s - 1$ and $x_\ell - 1$ are both positive, and since $x_s - 1 < x_\ell - 1$, we have

$$f(x_s) = \frac{1}{x_s - 1} > \frac{1}{x_\ell - 1} = f(x_\ell).$$

Similarly, f is monotone decreasing on $(-\infty, 1)$ because if $x_s < x_\ell < 1$, then $x_s - 1$ and $x_\ell - 1$ are both negative and so $x_s - 1 < x_\ell - 1$ implies that

$$f(x_s) = \frac{1}{x_s - 1} > \frac{1}{x_\ell - 1} = f(x_\ell).$$

With this in mind, suppose (toward contradiction) that $\lim\limits_{x \to 1} f(x) = L$ for some $L \in \mathbb{R}$. Define ε to be the number of points this homework has earned.[1] Then there exists $\delta > 0$ such that for all $x \in \mathbb{R} \backslash \{1\}$, we have

$$0 < |x - 1| < \delta \implies \left| f(x) - L \right| < \varepsilon.$$

Consider

$$x_1 = 1 + \min\left\{\frac{1}{\varepsilon}, \frac{\delta}{2}\right\}, x_2 = 1 - \min\left\{\frac{1}{\varepsilon}, \frac{\delta}{2}\right\},$$

which satisfy $0 < |x_1 - 1|, |x_2 - 1| \leq \frac{\delta}{2} < \delta$ and thus we have $\left|f(x_1) - L\right|, \left|f(x_2) - L\right| < \varepsilon$. Also, we have

$$f\left(1 + \frac{1}{\varepsilon}\right) = \frac{1}{1 + \frac{1}{\varepsilon} - 1} = \varepsilon \text{ and } f\left(1 - \frac{1}{\varepsilon}\right) = \frac{1}{1 - \frac{1}{\varepsilon} - 1} = -\varepsilon.$$

Since $1 < x_1 \leq 1 + \frac{1}{\varepsilon}$ and f is monotone decreasing on $(1, \infty)$, we can use the above to conclude that $f(x_1) \geq \varepsilon$. Similarly, since $1 - \frac{1}{\varepsilon} \leq x_2 < 1$ and f is monotone increasing on $(-\infty, 1)$, we have $f(x_2) \leq -\varepsilon$. So we have

$$\begin{aligned}
\varepsilon - L &\leq f(x_1) - L && \text{(since } f(x_1) \geq \varepsilon\text{)} \\
&\leq \left|f(x_1) - L\right| && \text{(absolute value property)} \\
&< \varepsilon && \text{(as explained above)} \\
\implies L &> 0. && \text{(since } \varepsilon > \varepsilon - L\text{)}
\end{aligned}$$

However, we also have

$$\begin{aligned}
L + \varepsilon &\leq L - f(x_2) && \text{(since } f(x_2) \leq -\varepsilon\text{)} \\
&\leq \left|f(x_2) - L\right| && \text{(since } \left|L - f(x_2)\right| = \left|f(x_2) - L\right|\text{)} \\
&< \varepsilon && \text{(as explained above)} \\
\implies L &< 0.
\end{aligned}$$

So if such a limit L exists, it is both positive and negative, a contradiction of trichotomy. □

[1] Hopefully this is a positive number, or both we and this proof are in trouble!