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MATH 555: Section H01
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MATH 555 Homework 4

Problem 1 Let $f : \mathbb{R} \to \mathbb{R}$ be uniformly continuous. Show that there are constants M and B so that

$$\left|f(x)\right| \leq M|x| + B$$

for all $x \in \mathbb{R}$.[1]

Choose $\varepsilon = 1$. Since f is uniformly continuous, there exists $\delta > 0$ such that for all $x_1, x_2 \in \mathbb{R}$, we have

$$|x_1 - x_2| < \delta \implies \left|f(x_1) - f(x_2)\right| < \varepsilon.$$

Consider $M := \frac{1}{\delta}$ and $B := 1 + \left|f(0)\right|$. Let $x \in \mathbb{R}$, and consider $\frac{|x|}{\delta}$. By Archimedes, there exists some positive $n \in \mathbb{N}$ such that

$$\frac{|x|}{\delta} < n \leq \frac{|x|}{\delta} + 1, \tag{1}$$

which implies that $\frac{|x|}{n} < \delta$. Adding and subtracting, we can write

$$\left|f(x)\right| = \left|f\left(\frac{nx}{n}\right)\right| = \left|f\left(\frac{nx}{n}\right) - f\left(\frac{(n-1)x}{n}\right) + f\left(\frac{(n-1)x}{n}\right) - \cdots - f\left(\frac{x}{n}\right) + f\left(\frac{x}{n}\right) - f(0) + f(0)\right|,$$

or in sum notation,

$$\left|f(x)\right| = \left|\sum_{k=1}^{n}\left[f\left(\frac{(n-k+1)x}{n}\right) - f\left(\frac{(n-k)x}{n}\right)\right] + f(0)\right|. \tag{2}$$

For all $1 \leq k \leq n$, we can use that n is positive and the result from equation (1) to write

$$\left|\frac{(n-k+1)x}{n} - \frac{(n-k)x}{n}\right| = \left|\frac{nx - kx + x - nx + kx}{n}\right| = \left|\frac{x}{n}\right| = \frac{|x|}{n} < \delta.$$

So by our choice of δ, we have for all k that

$$\left|f\left(\frac{(n-k+1)x}{n}\right) - f\left(\frac{(n-k)x}{n}\right)\right| < 1. \tag{3}$$

We can use this to write

$$\begin{aligned}
\left|f(x)\right| &= \left|\sum_{k=1}^{n}\left[f\left(\frac{(n-k+1)x}{n}\right) - f\left(\frac{(n-k)x}{n}\right)\right] + f(0)\right| && \text{(equation 2)} \\
&\leq \sum_{k=1}^{n}\left|f\left(\frac{(n-k+1)x}{n}\right) - f\left(\frac{(n-k)x}{n}\right)\right| + \left|f(0)\right| && \text{(triangle inequality)} \\
&< \sum_{k=1}^{n}(1) + \left|f(0)\right| && \text{(equation 3)}
\end{aligned}$$

[1] Credit to https://math.stackexchange.com/questions/1014598/if-f-is-a-uniformly-continous-function-then-fx-leq-axb

$$
\begin{aligned}
&= n + \left|f(0)\right| \\
&\le \frac{|x|}{\delta} + 1 + \left|f(0)\right| && \text{(equation 1)} \\
&= \frac{1}{\delta}|x| + \left(1 + \left|f(0)\right|\right). \\
&= M|x| + B.
\end{aligned}
$$

□

Problem 2 Let $f : X \to Y$ be a continuous function between metric spaces with X compact. Prove that f is uniformly continuous.

Let $\varepsilon > 0$. Since f is continuous, for all $p \in X$, there exists $\delta_p > 0$ such that for all $q \in X$,

$$d_X(p, q) < \delta_p \implies d_Y(f(p), f(q)) < \frac{\varepsilon}{2}. \tag{1}$$

Let $U := \{B(p, \delta_p/2) : p \in X\}$. Clearly, this is an open cover of X, and since X is compact, there exists a subcover $U_0 := \{B(p_1, \delta_{p_1}/2), \ldots, B(p_n, \delta_{p_n}/2)\}$ of X.

Consider $\delta := \min\{\delta_{p_1}/2, \ldots, \delta_{p_n}/2\}$. Let $x_1, x_2 \in X$ such that $d_X(x_1, x_2) < \delta$. We will show that

$$d_Y(f(x_1), f(x_2)) < \varepsilon.$$

Since U_0 covers X, there is some $j \le n$ such that $x_1 \in B(p_j, \delta_{p_j}/2)$. So

$$d_X(x_1, p_j) < \frac{\delta_{p_j}}{2} < \delta_{p_j}, \tag{2}$$

and we can use the triangle inequality to write

$$d_X(p_j, x_2) \le d_X(p_j, x_1) + d_X(x_1, x_2) < \frac{\delta_{p_j}}{2} + \delta \le \frac{\delta_{p_j}}{2} + \frac{\delta_{p_j}}{2} = \delta_{p_j}. \tag{3}$$

So we can use this to write

$$
\begin{aligned}
d_Y(f(x_1), f(x_2)) &\le d_Y(f(x_1), f(p_j)) + d_Y(f(p_j), f(x_2)) && \text{(triangle inequality)} \\
&< \frac{\varepsilon}{2} + \frac{\varepsilon}{2} && \text{(combining equations 2 and 3 with 1)} \\
&= \varepsilon.
\end{aligned}
$$

Therefore, f is uniformly continuous. □

Problem 1.3 Let

$$p(x) = a_m x^m + a_{m-1}x^{m-1} + \cdots + a_1 x + a_0$$

be a polynomial with $a_m > 0$ and $m \ge 1$. Let $\langle x_n \rangle_{n=1}^{\infty}$ be defined by $x_n = p(n)$. Show

$$\lim_{n \to \infty} x_n = \infty.$$

We can write

$$x_n = n^m \left(a_m + \frac{a_{m-1}}{n} + \cdots + \frac{a_1}{n^{m-1}} + \frac{a_0}{n^m}\right).$$

Since $m \geq 1$, it is easy to show that $\lim_{n\to\infty} n^m = \infty$, and we can use additivity to write

$$\lim_{n\to\infty}\left(\frac{a_{m-1}}{n} + \cdots + \frac{a^0}{n^m}\right) = \lim_{n\to\infty} a_m + \lim_{n\to\infty}\frac{a_{m-1}}{n} + \cdots + \lim_{n\to\infty}\frac{a_0}{n^m} = a_m + 0 + \cdots + 0 = a_m.$$

Since it is given that $a_m > 0$, we have

$$\lim_{n\to\infty} n^m = \infty, \quad \lim_{n\to\infty}\left(\frac{a_{m-1}}{n} + \cdots + \frac{a^0}{n^m}\right) > 0.$$

We showed in Homework 1 that this implies that x_n, the product of these two functions, goes to infinity. So $\lim_{n\to\infty} x_n = \infty$. $\square$

Problem 1.4 If $\lim_{n\to\infty} a_n = \infty$, show $\lim_{n\to\infty} \frac{1}{a_n} = 0$.

Since $\lim_{n\to\infty} a_n = \infty$, for all real M there exists $N \geq 0$ such that

$$n \geq N \implies a_n \geq M.$$

Let $\varepsilon > 0$, and choose N such that $n \geq N \implies a_n > \frac{1}{\varepsilon}$. Then a_n is positive since $\frac{1}{\varepsilon} > 0$, so

$$a_n > \frac{1}{\varepsilon} \implies \frac{1}{a_n} < \varepsilon \implies \left|\frac{1}{a_n} - 0\right| < \varepsilon.$$

Therefore, by definition $\left\langle \frac{1}{a_n} \right\rangle$ converges to 0.