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MATH 576: Section H01
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MATH 576 Homework 3

Problem 1 The *Subtraction*-(S_L, S_R) game is the partisan version of Subtraction where on their turn Left can remove m tokens from the heap if and only if $m \in S_L$, and Right can remove m tokens if and only if $m \in S_R$. For example, if $S_L = \{1,2,4\}$ and $S_R = \{1,3,4\}$, then Left can remove $1,2$ or 4 tokens from the heap on their turn, while Right can remove $1,3$ or 4 tokens from the heap on their turn.

Let $\underset{S_L,S_R}{n}$ denote the position in the Subtraction-(S_L, S_R) game with n tokens in a single heap. Here are the outcome classes of the first few positions in Subtraction-$(\{1,2,4\},\{1,3,4\})$:

Size	0	1	2	3	4	5	6	7	8	9	10
Outcome Class	$\mathcal{P}$	$\mathcal{N}$	$\mathcal{L}$	$\mathcal{N}$	$\mathcal{N}$						

Fill out the remainder of the table. What pattern do you observe? Prove this pattern by strong induction.

Let $\underline{n} := \underset{\{1,2,4\},\{1,3,4\}}{n}$, and let $\equiv$ denote equivalence $\pmod 5$. We claim that for all $n \in \mathbb{N}$, we have

$$\underline{n} \in \begin{cases} \mathcal{P} & \text{if } n \equiv 0; \\ \mathcal{N} & \text{if } n \equiv 1,3,4; \\ \mathcal{L} & \text{if } n \equiv 2. \end{cases} \tag{$\star$}$$

We prove this by induction on n. The base case is given in the table. Now, suppose $n \geq 5$ and that for all $n' < n$, $(\star)$ holds for n'. We show that $(\star)$ holds for n. Observe that we have:

$$\underline{n} = \{\underline{n-1}, \underline{n-2}, \underline{n-4} \mid \underline{n-1}, \underline{n-3}, \underline{n-4}\}.$$

Case 0: $n \equiv 0$. Then we have $n-1 \equiv 4$, $n-2 \equiv 3$, and $n-4 \equiv 1$, so by the IH we have $\underline{n-1}, \underline{n-2}, \underline{n-4} \in \mathcal{N} \subseteq \mathcal{N}^R$ and thus $\underline{n} \in \mathcal{P}^R$. We can also write $n-3 \equiv 2$, so by the IH we have $\underline{n-3} \in \mathcal{L} \subseteq \mathcal{N}^L$ and $\underline{n-1}, \underline{n-4} \in \mathcal{N} \subseteq \mathcal{N}^L$ and thus $\underline{n} \in \mathcal{P}^L$. So $\underline{n} \in \mathcal{P}^L \cap \mathcal{P}^R = \mathcal{P}$.

Case 1: $n \equiv 1$. Then $n-1 \equiv 0$, so by the IH we have $\underline{n-1} \in \mathcal{P}$. Since both players have $\underline{n-1}$ as an option, we have $\underline{n} \in \mathcal{N}$.

Case 2: $\mathcal{N} \equiv 2$. Then we have $n-1 \equiv 1$, $n-3 \equiv 4$, and $n-4 \equiv 3$, so by the IH we have $\underline{n-1}, \underline{n-3}, \underline{n-4} \in \mathcal{N} \subseteq \mathcal{N}^L$ and thus $\underline{n} \in \mathcal{P}^L$. Also, $n-2 \equiv 0$, so by the IH we have $\underline{n-2} \in \mathcal{P} \subseteq \mathcal{P}^L$ and thus $\underline{n} \in \mathcal{N}^L$. So $\underline{n} \in \mathcal{P}^L \cap \mathcal{N}^L = \mathcal{L}$.

Case 3: $n \equiv 3$. Then we have $n-1 \equiv 2$, so by the IH we have $\underline{n-1} \in \mathcal{L} \subseteq \mathcal{P}^L$ and thus $\underline{n} \in \mathcal{N}^L$. Also, we have $n-3 \equiv 0$, so by the IH we have $\underline{n-3} \in \mathcal{P} \subseteq \mathcal{P}^R$ and thus $\underline{n} \in \mathcal{N}^R$. So $\underline{n} \in \mathcal{N}^L \cap \mathcal{N}^R = \mathcal{N}$.

Case 4: $n \equiv 4$. Then $n-4 \equiv 0$, so by the IH we have $\underline{n-4} \in \mathcal{P}$. Since both players have $\underline{n-4}$ as an option, we have $\underline{n} \in \mathcal{N}$.

So $(\star)$ holds in all cases. This then determines how to fill in the remainder of the table. $\square$

Problem 2 Show that the game position $G := \{\{* \,|\, 0\} \,|\, \{0 \,|\, *\}\}$ is a $\mathcal{P}$-position.

Since $0 \in \mathcal{P} \subseteq \mathcal{P}^R$, we have $G^L := \{* \,|\, 0\} \in \mathcal{N}^R$. Also, since $0 \in \mathcal{P} \subseteq \mathcal{P}^L$, we have $G^R := \{0 \,|\, *\} \in \mathcal{N}^L$. Since $G^R \in \mathcal{N}^L$, we have $G \in \mathcal{P}^L$, and since $G^L \in \mathcal{N}^R$, we have $G \in \mathcal{P}^R$. Therefore, we have $G \in \mathcal{P}^L \cap \mathcal{P}^R = \mathcal{P}$.$\square$

Problem 3 Let G be the COL position and let H be the COL position . What is the outcome class of $G + H$? Justify your answer.

In G, L can only color the second-leftmost vertex and R can only color the second-rightmost vertex. Thus, we have $G \in \mathcal{P}$ because the $\mathcal{N}$ext player must make their only move, allowing the $\mathcal{P}$revious player to make their only move and leave the $\mathcal{N}$ext player with no options. In H, only R has any moves (coloring the rightmost vertex), so we have $H \in \mathcal{R}$. From the theorem we proved in class, then, $G + H \in \mathcal{R}$. $\square$

Problem 4 Determine the inequality relationship between the Toppling Dominoes position $B\,R$ and the Kayles position K_1 with 1 pin in the row. Justify your answer.

Since K_1 is impartial, we have $K_1 = -K_1$. Thus, we have

$$B\,R - K_1 = B\,R + K_1 = \{B\,R, K_1, R + K_1 \mid B\,R, K_1, B + K_1\}.$$

We have $B\,R \in \mathcal{N}$ since the $\mathcal{N}$ext player can topple their domino in the direction of the other domino, and $K_1 \in \mathcal{N}$ since the $\mathcal{N}$ext player can knock over the only pin. Also, we have $R + K_1 \in \mathcal{N}^R$ since R can knock over the pin in K_1 and leave L with no moves, and we have $B + K_1 \in \mathcal{N}^L$ by the symmetric argument. Thus, all of L's options are in $\mathcal{N}^R$ since $\mathcal{N} \subseteq \mathcal{N}^R$, so $BR - K_1 \in \mathcal{P}^R$, and all of R's options are in $\mathcal{N}^L$ since $\mathcal{N} \subseteq \mathcal{N}^L$, so $BR - K_1 \in \mathcal{P}^L$. Therefore, $BR - K_1 \in \mathcal{P}^L \cap \mathcal{P}^R = \mathcal{P}$, so by definition we have $BR = K_1$. $\square$

Problem 5 Node Kayles is the following impartial version of COL: Start with an uncolored graph G. On their turn, a player may color any previously uncolored vertex v green, provided that the vertex v is not adjacent to any vertex which has already been colored green. The loser is the player who cannot make a legal move on their turn. (An alternate rendering of the rules to make the game look more like Kayles: on their turn, a player selects a vertex v in the graph and removes v along with every vertex adjacent to v from the graph).

Prove that the following position in Node Kayles is an $\mathcal{N}$-position:

The $\mathcal{N}$ext player can win by coloring the middle vertex. If the $\mathcal{P}$revious player colors either the leftmost or second-leftmost vertex, the $\mathcal{N}$ext player can respond by coloring the rightmost vertex and leaving the $\mathcal{P}$revious player with no moves. Similarly, if the $\mathcal{P}$revious player colors either the rightmost or second-rightmost vertex, the $\mathcal{N}$ext player can respond by coloring the leftmost vertex and leaving the $\mathcal{P}$revious player with no moves. Therefore, the position is an $\mathcal{N}$-position. $\square$