MATH 552: Section 001
Professor: Dr. Miller
January 28, 2022

MATH 552 Homework Set 3ˆ

Problem 3.30.10
(a) Show that if e^z is real, then $\operatorname{Im} z = n\pi \quad (n = 0, \pm 1, \pm 2, \dots)$.
(b) If e^z is pure imaginary, what restriction is placed on z?

Solution.

(a) For all z, $z = \operatorname{Re} z + (\operatorname{Im} z)i$. Thus, $e^z = e^{\operatorname{Re} z} e^{(\operatorname{Im} z)i}$. Since $e^{\operatorname{Re} z}$ is always real, e^z is real iff $e^{(\operatorname{Im} z)i}$ is real.

$$\begin{aligned}
e^{(\operatorname{Im} z)i} &= \cos(\operatorname{Im} z) + i\sin(\operatorname{Im} z) && \text{(using de Moivre's formula)}\\
i\sin(\operatorname{Im} z) &= 0 && \text{(setting imaginary part to 0)}\\
\sin(\operatorname{Im} z) &= 0 && \text{(dividing by i)}\\
\operatorname{Im} z &= n\pi, n \in \mathbb{Z} && \text{(solving for } \operatorname{Im} z\text{)}
\end{aligned}$$

(b) Using the same reasoning:

$$\begin{aligned}
e^{(\operatorname{Im} z)i} &= \cos(\operatorname{Im} z) + i\sin(\operatorname{Im} z) && \text{(using de Moivre's formula)}\\
\cos(\operatorname{Im} z) &= 0 && \text{(setting real part to 0)}\\
\operatorname{Im} z &= \frac{\pi}{2} + \pi n, n \in \mathbb{Z} && \text{(solving for } \operatorname{Im} z\text{)}
\end{aligned}$$

As shown, and somewhat intuitively, the restriction is the restriction from (a) + $\frac{\pi}{2}$: $\operatorname{Im} z = \frac{\pi}{2} + \pi n, n \in \mathbb{Z}$.

Problem Supplemental B After you do problem #9 from 2.18, examine what can go wrong if we do not have the hypothesis that $|g(z)| \leq M$ in a neighborhood of z_0. Show, by explicit example from z_0, $f(z)$ and $g(z)$ that it is possible that $\lim_{z\to z_0} f(z)g(z)$ is not equal to 0 even though $\lim_{z\to z_0} f(z) = 0$.

Solution. Take $z_0 = 0$, $f(z) = z$, and $g(z) = \frac{1}{z^2}$. Because $g(z)$ is not bounded, there is no M that satisfies $|g(z)| \leq M$ in a neighborhood of z_0.

$$\begin{aligned}
\lim_{z\to z_0} f(z) &= \lim_{z\to 0} z\\
\lim_{z\to z_0} f(z) &= 0 && \text{(direct substitution)}\\
\lim_{z\to z_0} f(z)g(z) &= \lim_{z\to 0} \frac{z}{z^2}\\
\lim_{z\to z_0} f(z)g(z) &= \lim_{z\to 0} \frac{1}{z} && \text{(we can cancel out z because we're not interested in } z = 0\text{)}\\
\lim_{z\to z_0} f(z)g(z) &= \infty && \text{(using a Riemann sphere)}
\end{aligned}$$

This shows that

$$\lim_{z\to z_0} f(z) = 0 \implies \lim_{z\to z_0} f(z)g(z) = 0$$

is not necessarily true if $g(z)$ is not bounded.