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MATH 554: Section H01
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MATH 554 Homework 7

Problem 1 Let (E, d) be a metric space and $A \subseteq E$. Let $\overline{A}$ be the set of all points $p \in E$ so that for all $r > 0$ we have $B(p, r) \cap A \neq \emptyset$. Show that $\overline{A}$ is closed.

Consider $p \in \mathcal{C}(\overline{A})$. Since $p \notin \overline{A}$, there exists some $r > 0$ such that $B(p, r) \cap A = \emptyset$. We claim that $B(p, r) \subseteq \mathcal{C}(\overline{A})$. Let $q \in B(p, r)$. Since $B(p, r)$ is open, there exists an $r' > 0$ such that $B(q, r') \subseteq B(p, r)$. Thus, we have $B(q, r') \cap A = \emptyset$ since $B(p, r) \cap A = \emptyset$. So $q \notin \overline{A}$, and thus $q \in \mathcal{C}(\overline{A})$. Thus, $B(p, r) \subseteq \mathcal{C}(\overline{A})$, and so $\mathcal{C}(\overline{A})$ is open. Therefore, by definition $\overline{A}$ is closed. $\square$

Problem 2 Let (E, d) be a metric space. Let $S \subseteq E$ with the property that if $s_1, s_2 \in S$ with $s_1 \neq s_2$, then $d(s_1, s_2) \geq 1$. Show S is closed.

Consider $p \in \mathcal{C}(S)$. We claim that there is either zero or one point from S in $B(p, 1/2)$. If there are $s_1, s_2 \in S \cap B(p, 1/2)$ with $s_1 \neq s_2$, then

$$d(s_1, s_2) \leq d(s_1, p) + d(p, s_2) < \frac{1}{2} + \frac{1}{2} = 1,$$

a contradiction. So these are the two possible cases.

Case 1: There are no points from S in $B(p, 1/2)$. Then $B(p, 1/2) \subseteq \mathcal{C}(S)$.

Case 2: There is some point $s \in S \cap B(p, 1/2)$. Then choose $r := d(p, s)$, and consider $B(p, r)$. We have $B(p, r) \subseteq B(p, 1/2)$ so the only possible point from S that could be in $B(p, r)$ is s, and since $d(p, s) < r$ does not hold by definition, $s \notin B(p, r)$. So $B(p, r) \subseteq \mathcal{C}(S)$.

Since we can choose an appropriate radius in both cases, $\mathcal{C}(S)$ is open. Therefore, S is closed. $\square$

Problem 3 In the plane $\mathbb{R}^2$, show the half plane $H = \{(x, y) \in \mathbb{R}^2 : y > 0\}$ is open.

We will assume the standard Euclidean metric on $\mathbb{R}^2$.

Consider $p = (x, y) \in H$. Then we claim $B(p, y) \subseteq H$. For any $q \in \mathcal{C}(H)$, we will have $q = (x', y')$ where $y' \leq 0$. Then,

$$\begin{aligned}
[d(p, q)]^2 &= (x - x')^2 + (y - y')^2 \\
&\geq (y - y')^2 \\
&\geq y^2 && \text{(since } y > 0 \text{ and } y' \leq 0) \\
\implies d(p, q) &\geq y.
\end{aligned}$$

So $q \notin B(p, y)$, and thus $B(p, y) \subseteq H$. Therefore, H is open. $\square$

Problem 4 Let (E, d) be a metric space and $p, q \in E$ with $p \neq q$. Show that $U := \{x \in E : d(p, x) < d(q, x)\}$ is open.

Let $x \in U$, and consider $r := \frac{d(q,x)-d(p,x)}{2}$. Let $y \in B(x,r)$. Then, we have

$$\begin{aligned}
d(p,y) &\leq d(p,x) + d(x,y) && \text{(triangle inequality)} \\
&< d(p,x) + \frac{d(q,x)-d(p,x)}{2} && (d(x,y) < r) \\
&= \frac{d(p,x)+d(q,x)}{2} && \text{(combining fractions)} \\
&= d(q,x) - \frac{d(q,x)-d(p,x)}{2} && \text{(rewriting fraction)} \\
&< d(q,x) - d(x,y) && (-d(x,y) > -r) \\
&\leq d(q,y). && \text{(reverse triangle inequality)}
\end{aligned}$$

Thus, $q \in U$, so $B(x,r) \subseteq U$ and therefore U is open. □

Problem 5 In $\mathbb{R}$ for the following sets say if they are open, closed, or neither. Prove your answer is correct.

(a) The set, $\mathbb{Q}$, of rational numbers.

(b) The set $S := \{1/n : n \in \mathbb{Z}^+\}$.

(c) The set $S := \{0\} \cup \{1/n : n \in \mathbb{Z}^+\}$.

(a) This is neither nor open closed in $\mathbb{R}$. First, consider $0 \in \mathbb{Q}$. Then for all $r > 0$, $B(0,r) = (-r,r)$ is not a subset of $\mathbb{Q}$. This is because (as we have proved before) there is an irrational number between any two real numbers, so in particular there is one between $-r$ and r. So $\mathbb{Q}$ is not open.

Next, consider $\sqrt{2} \in \mathcal{C}(\mathbb{Q})$. Then for all $r > 0$, $B(\sqrt{2},r) = (\sqrt{2}-r, \sqrt{2}+r)$ is not a subset of $(\mathbb{Q})$. This is because (as we have also proved before) there is a rational number between any two real numbers, so in particular there is one between $\sqrt{2}-r$ and $\sqrt{2}+r$. So $\mathcal{C}(\mathbb{Q})$ is not open, and thus $\mathbb{Q}$ is not closed.

(b) This is neither open nor closed in $\mathbb{R}$. First, consider $1 \in S$. Clearly, $B(1,r)$ is not a subset of S for any $r > 0$, because there is an irrational number between $1-r$ and $1+r$, and this will not be in S since $S \subseteq \mathbb{Q}$. So S is not open.

Next, consider $0 \in \mathcal{C}(S)$. For any $r > 0$, we claim $B(0,r)$ is not a subset of $\mathcal{C}(S)$. This is because, by the Archimedean property, there exists $n \in \mathbb{N}$ such that $\frac{1}{n} < r$, which is in S and thus not in $\mathcal{C}(S)$. So $\mathcal{C}(S)$ is not open, and thus S is not closed.

(c) This is closed. Consider $x \in \mathcal{C}(S)$.

Case 1: $x < 0$. Then $B(x,x) \subseteq \mathcal{C}(S)$, because every $s \in S$ satisfies $s \geq 0$.

Case 2: $x > 1$. Then $B(x,x-1) \subseteq \mathcal{C}(S)$, because every $s \in S$ satisfies $s \leq 1$.

Case 3: $0 < x < 1$. Since $x \notin S$, we have $\lfloor \frac{1}{x} \rfloor < \frac{1}{x} < \lceil \frac{1}{x} \rceil$. Let $r := \min\left\{\frac{1}{\lfloor 1/x \rfloor}, \frac{1}{\lceil 1/x \rceil}\right\}$. Then $B(x,r) \subseteq \mathcal{C}(S)$, because $\frac{1}{\lfloor 1/x \rfloor}, \frac{1}{\lceil 1/x \rceil}$ are the points in S closest to x and they are outside the ball.

So for any $x \in \mathcal{C}(S)$, we can choose an appropriate radius. So $\mathcal{C}(S)$ is open, and thus S is closed. □

Problem 3.18 Let $\lim\limits_{n\to\infty} p_n = p$ in the metric space E. Let $a_n = p_{2n}$. Show that $\lim\limits_{n\to\infty} a_n = p$ also holds.

Let $\varepsilon > 0$. Since $\langle p_n \rangle$ converges to p, there exists an N such that $d(p_n, p) < \varepsilon$ for all $n > N$. So we have $d(a_n, p) = d(p_{2n}, p) < \varepsilon$ because $2n > n > N$ for all $n > N$. Thus, $\langle a_n \rangle$ converges to p. □

Problem 3.19 Let $\langle x_n\rangle_{n=1}^{\infty}$ and $\langle y_n\rangle_{n=1}^{\infty}$ be sequences in $\mathbb{R}$ with

$$\lim_{n\to\infty} x_n = x \text{ and } \lim_{n\to\infty} y_n = y.$$

Prove that for any real numbers a and b,

$$\lim_{n\to\infty}(ax_n + by_n) = ax + by.$$

Let $\varepsilon > 0$. Since $\langle x_n\rangle$ and $\langle y_n\rangle$ converge, there exist N_x, N_y such that for all $n > N := \max\{N_x, N_y\}$,

$$|x_n - x| < \frac{\varepsilon}{2a+1} \text{ and } |y_n - y| < \frac{\varepsilon}{2b+1}.$$

Then, for $n > N$, we can write

$$\begin{aligned}
|(ax_n + by_n) - (ax + by)| &= |ax_n - ax + by_n - by| \\
&\leq |ax_n - ax| + |by_n - by| && \text{(triangle inequality)} \\
&= a|x_n - x| + b|y_n - y| \\
&< a\left(\frac{\varepsilon}{2a+1}\right) + b\left(\frac{\varepsilon}{2b+1}\right) \\
&< \frac{\varepsilon}{2} + \frac{\varepsilon}{2} = \varepsilon.
\end{aligned}$$

Therefore, $\lim\limits_{n\to\infty}(ax_n + by_n) = ax + by$ by definition. □

Problem 3.20 Let $\langle x_n\rangle$ be a convergent sequence in $\mathbb{R}$. Prove that $\langle x_n\rangle$ is bounded (there is a constant M such that $|x_n| \leq M$ for all n.

Suppose $\langle x_n\rangle$ converges to $x \in \mathbb{R}$. Fix $\varepsilon > 0$. Then, there exists some N such that for all $n > N$, $|x_n - x| < \varepsilon$. Consider $M' := \max\{|x_n| : n \leq N\}$. Then, for all $n \leq N$, $|x_n| \leq M'$, and for all $n > N$, $|x_n| < |x| + \varepsilon$. Therefore, $\langle x_n\rangle$ is bounded by $M := \max\{M', |x| + \varepsilon\}$ for all n. □

Problem 3.21 Let

$$\lim_{n\to\infty} x_n = x \text{ and } \lim_{n\to\infty} y_n = y$$

in $\mathbb{R}$. Prove that

$$\lim_{n\to\infty} x_n y_n = xy.$$

Let $\varepsilon > 0$. Since $\langle x_n\rangle$, $\langle y_n\rangle$ converge, we have from problem 20 that there exists some M such that $|x_n|, |y_n| \leq M$ for all n. Further, by definition there exist N_x, N_y such that for all $n > N := \max N_x, N_y$, we have

$$|x_n - x| < \frac{\varepsilon}{2|y|+1} \text{ and } |y_n - y| < \frac{\varepsilon}{2M+1}.$$

With this, we can write

$$\begin{aligned}
|x_n y_n - xy| &= |x_n y_n - x_n y + x_n y - xy| \\
&\leq |x_n y_n - x_n y| + |x_n y - xy| && \text{(triangle inequality)} \\
&\leq |x_n||y_n - y| + |y||x_n - x| \\
&\leq M|y_n - y| + |y||x_n - x| && \text{(using bound)}
\end{aligned}$$

$$< M\left(\frac{\varepsilon}{2M+1}\right) + |y|\left(\frac{\varepsilon}{2|y|+1}\right) \qquad \text{(from above)}$$
$$< \frac{\varepsilon}{2} + \frac{\varepsilon}{2} = \varepsilon.$$

Therefore, $\lim\limits_{n\to\infty} x_n y_n = xy$ by definition. □

Problem 3.23 Let $f : \mathbb{R} \to \mathbb{R}$ be the quadratic polynomial $f(x) = ax^2 + bx + c$ where a, b, c are constants. Let $\langle p_n \rangle$ be a convergent sequence, $\lim\limits_{n\to\infty} p_n = p$. Prove that

$$\lim_{n\to\infty} f(p_n) = f(p).$$

We can use the properties we have proved to write

$$\begin{aligned}
\lim_{n\to\infty} f(p_n) &= \lim_{n\to\infty} ap_n^2 + bp_n + c \\
&= a \lim_{n\to\infty} p_n^2 + b \lim_{n\to\infty} p_n + \lim_{n\to\infty} c && \text{(problem 3.19)} \\
&= a\left(\lim_{n\to\infty} p_n\right)^2 + b \lim_{n\to\infty} p_n + \lim_{n\to\infty} c && \text{(problem 3.21)} \\
&= ap^2 + bp + c = f(p).
\end{aligned}$$

Problem 3.24 Let $a \in \mathbb{R}$ with $a \neq 0$. Let $|x - a| < \frac{|a|}{2}$. Prove that

$$\frac{|a|}{2} < |x| < \frac{3|a|}{2},$$

$$\frac{1}{|x|} < \frac{2}{|a|},$$

and

$$\left|\frac{1}{x} - \frac{1}{a}\right| \leq \frac{2|x-a|}{|a|^2}.$$

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1. We can use the triangle inequality to write

$$\begin{aligned}
|x| &= |a + x - a| \\
&\leq |a| + |x - a| \\
&< |a| + \frac{|a|}{2} \\
&= \frac{3|a|}{2}
\end{aligned}$$

and the reverse triangle inequality to write

$$\begin{aligned}
|x| &= |a + x - a| \\
&\geq \big||a| - |x - a|\big| \\
&> \left||a| - \frac{|a|}{2}\right| \\
&= |a| - \frac{|a|}{2} && \text{(above is guaranteed to be positive)} \\
&= \frac{|a|}{2}.
\end{aligned}$$

Combining these, we have $\frac{|a|}{2} < |x| < \frac{3|a|}{2}$.

2. Since $|a| > 0$ (and thus $|x| > \frac{|a|}{2} > \frac{0}{2} = 0$), we can use the properties of inequalities to write

$$\begin{aligned} |x| &> \frac{|a|}{2} && \text{(from 1)} \\ \implies 2|x| &> |a| \\ \implies \frac{2|x|}{|a|} &> 1 \\ \implies \frac{2}{|a|} &> \frac{1}{|x|}. \end{aligned}$$

3. Finally, we can use 2 to write

$$\begin{aligned} \left|\frac{1}{x} - \frac{1}{a}\right| &= \left|\frac{1}{a} - \frac{1}{x}\right| \\ &= \left|\frac{x-a}{xa}\right| \\ &= \left(\frac{1}{|x|}\right)\left(\frac{1}{|a|}\right)|x-a| \\ &< \left(\frac{2}{|a|}\right)\left(\frac{1}{|a|}\right)|x-a| \\ &= \frac{2|x-a|}{|a|^2}. \end{aligned}$$

□

Problem 3.25 Let $\langle x_n \rangle$ be a sequence $\lim_{n\to\infty} x_n = a$ and $a \neq 0$. Prove that

$$\lim_{n\to\infty} \frac{1}{x_n} = \frac{1}{a}.$$

Let $\varepsilon > 0$. Since $\langle x_n \rangle$ converges to a, there exists N_1 such that for all $n > N_1$,

$$|x_n - a| < \frac{|a|}{2},$$

and there also exists N_2 such that for all $n > N_2$,

$$|x_n - a| < \frac{|a|^2 \varepsilon}{2}.$$

Let $N := \max\{N_1, N_2\}$. Then,

$$\begin{aligned} \left|\frac{1}{x_n} - \frac{1}{a}\right| &< \frac{2|x_n - a|}{|a|^2} && \text{(from lemma since } n > N_1\text{)} \\ &< \frac{2\left(\frac{|a|^2\varepsilon}{2}\right)}{|a|^2} && \text{(since } n > N_2\text{)} \\ &= \frac{|a|^2\varepsilon}{|a|^2} = \varepsilon. && (1) \end{aligned}$$

Therefore, $\lim_{n\to\infty} \frac{1}{x_n} = \frac{1}{a}$ by definition. □

Problem 3.26 Let E be a metric space and $f : E \to \mathbb{R}$ be a Lipschitz map. Let $\langle p_n \rangle$ be a sequence in E with $\lim_{n\to\infty} p_n = p$ where $p \in E$. Then

$$\lim_{n\to\infty} f(p_n) = f(p).$$

Since f is Lipschitz, there exists an $M > 0$ such that $|f(p_n) - f(p)| \leq Md(p_n, p)$. Let $\varepsilon > 0$. Then, since $\langle p_n \rangle$ converges to be p, there exists an N such that for all $n > N$, $d(p_n, p) < \frac{\varepsilon}{M}$. So for all $n > N$, we can write

$$\begin{aligned} |f(p_n) - f(p)| &\leq Md(p_n, p) && \text{(Lipschitz)} \\ &< M\left(\frac{\varepsilon}{M}\right) && \text{(convergence)} \\ &= \varepsilon. \end{aligned}$$

Therefore, $\lim_{n\to\infty} f(p_n) = f(p)$ by definition. □