Nathan Bickel
Linear Algebra: Section 1
Professor: Munoz
August 4, 2023

Linear Algebra Homework 5

Problem 1 Let $V = M_n(\mathbb{C})$ be the vector space of $n \times n$ matrices with complex entries. Let B be a fixed matrix, and define $T : V \to V$ by

$$T(A) = AB - BA.$$

(a) Prove that T is a linear map.

(b) Prove that T is not invertible. Hint: For a given A, what is the trace of $T(A)$?

Solution.

(a) Let $A_1, A_2 \in M_n(\mathbb{C})$ and $c \in R$. We have

$$\begin{aligned} T(cA_1 + A_2) &= (cA_1 + A_2)B - B(cA_1 + A_2) \\ &= cA_1B + A_2B - cBA_1 - BA_2 && \text{(distributivity)} \\ &= cA_1B - cBA_1 + A_2B - BA_2 && \text{(commutativity of addition)} \\ &= c(A_1B - BA_1) + (A_2B - BA_2) && \text{(distributivity)} \\ &= cT(A_1) + T(A_2), \end{aligned}$$

so T is a linear map.

(b) A linear map is invertible only if it is injective. But then T cannot be invertible, because we have

$$T(B) = B^2 - B^2 = O_{n\times n} = T(O_{n\times n})$$

but B need not equal $O_{n\times n}$, so it is not injective.

□

Problem 2 Let $T : V \to V$ be a linear operator on a vector space V, and let λ be a scalar. The *eigenspace* $V^{(\lambda)}$ is the set of eigenvectors of T with eigenvalue λ, together with 0. Prove that $V^{(\lambda)}$ is a T-invariant subspace.

Solution.

We will first prove that $V^{(\lambda)}$ is a subspace. Let $c \in \mathbb{R}$ and $u, v \in V^{(\lambda)}$. Then, we have

$$T(cu + v) = cT(u) + T(v) = c\lambda u + \lambda v = \lambda(cu + v),$$

so $cu+v$ is an eigenvector of T with eigenvalue λ. Thus, $cu+v \in V^{(\lambda)}$, which shows that $V^{(\lambda)}$ is a subspace.

Next, let $v \in V^{(\lambda)}$. Then, $T(v) = \lambda(v)$. Since $V^{(\lambda)}$ is a subspace, is it closed under multiplication and thus contains $\lambda(v)$. So $T(v) \in V^{(\lambda)}$, and thus $V^{(\lambda)}$ is T-invariant. □

Problem 3 Compute the characteristic polynomials and the complex eigenvalues and eigenvectors of

(a) $\begin{pmatrix} -2 & 2 \\ -2 & 3 \end{pmatrix}$

(b) $\begin{pmatrix} 1 & i \\ -i & 1 \end{pmatrix}$

(c) $\begin{pmatrix} \cos(\theta) & -\sin(\theta) \\ \sin(\theta) & \cos(\theta) \end{pmatrix}$.

Solution.

We will call each matrix A. We will first compute the characteristic polynomial $p_A(\lambda) = \det(A - \lambda I)$, whose roots will be the eigenvalues. Then, for each eigenvalues λ, we will find the associated eigenspace E_λ, which will equal $\mathrm{Null}(A - \lambda I)$ (which we will compute using row-reduction). The eigenvectors will then be the union of the eigenspaces.

(a) From the characteristic polynomial

$$p_A(\lambda) = \begin{vmatrix} -2-\lambda & 2 \\ -2 & 3-\lambda \end{vmatrix} = (-2-\lambda)(3-\lambda) + 4 = \lambda^2 - \lambda - 2 = (\lambda+1)(\lambda-2),$$

we see that A has eigenvalues -1 and 2.

We have $E_{-1} = \mathrm{Null}\begin{pmatrix} -2+1 & 2 \\ -2 & 3+1 \end{pmatrix}$, and because

$$\begin{aligned} \left(\begin{array}{cc|c} -1 & 2 & 0 \\ -2 & 4 & 0 \end{array}\right) &\sim \left(\begin{array}{cc|c} -1 & 2 & 0 \\ 0 & 0 & 0 \end{array}\right) && (\rho_2 \mapsto \rho_2 - 2\rho_1) \\ &\sim \left(\begin{array}{cc|c} 1 & -2 & 0 \\ 0 & 0 & 0 \end{array}\right), && (\rho_1 \mapsto -\rho_1) \end{aligned}$$

we conclude that $E_{-1} = \mathrm{Span}\left\{\begin{pmatrix} 2 \\ 1 \end{pmatrix}\right\}$.

Similarly, we have $E_2 = \mathrm{Null}\begin{pmatrix} -2-2 & 2 \\ -2 & 3-2 \end{pmatrix}$, and because

$$\begin{aligned} \left(\begin{array}{cc|c} -4 & 2 & 0 \\ -2 & 1 & 0 \end{array}\right) &\sim \left(\begin{array}{cc|c} -4 & 2 & 0 \\ 0 & 0 & 0 \end{array}\right) && (\rho_2 \mapsto \rho_2 - \tfrac{1}{2}\rho_1) \\ &\sim \left(\begin{array}{cc|c} 1 & -\frac{1}{2} & 0 \\ 0 & 0 & 0 \end{array}\right), && (\rho_1 \mapsto -\tfrac{1}{4}\rho_1) \end{aligned}$$

we conclude that $E_2 = \mathrm{Span}\left\{\begin{pmatrix} \frac{1}{2} \\ 1 \end{pmatrix}\right\} = \mathrm{Span}\left\{\begin{pmatrix} 1 \\ 2 \end{pmatrix}\right\}$.

(b) From the characteristic polynomial

$$p_A(\lambda) = \begin{vmatrix} 1-\lambda & i \\ -i & 1-\lambda \end{vmatrix} = (1-\lambda)^2 + i^2 = 1 - 2\lambda + \lambda^2 = \lambda^2 - 2\lambda = \lambda(\lambda-2),$$

we see that A has eigenvalues 0 and 2.

We have $E_0 = \text{Null}\begin{pmatrix} 1-0 & i \\ -i & 1-0 \end{pmatrix}$, and because

$$\left(\begin{array}{cc|c} 1 & i & 0 \\ -i & 1 & 0 \end{array}\right) \sim \left(\begin{array}{cc|c} 1 & i & 0 \\ 1 & i & 0 \end{array}\right) \qquad (\rho_2 \mapsto i\rho_2)$$
$$\sim \left(\begin{array}{cc|c} 1 & i & 0 \\ 0 & 0 & 0 \end{array}\right), \qquad (\rho_2 \mapsto \rho_2 - \rho_1)$$

we conclude that $E_0 = \text{Span}\left\{\begin{pmatrix} -i \\ 1 \end{pmatrix}\right\}$.

Similarly, we have $E_2 = \text{Null}\begin{pmatrix} 1-2 & i \\ -i & 1-2 \end{pmatrix}$, and because

$$\left(\begin{array}{cc|c} -1 & i & 0 \\ -i & -1 & 0 \end{array}\right) \sim \left(\begin{array}{cc|c} -1 & i & 0 \\ 1 & -i & 0 \end{array}\right) \qquad (\rho_2 \mapsto i\rho_2)$$
$$\sim \left(\begin{array}{cc|c} -1 & i & 0 \\ 0 & 0 & 0 \end{array}\right) \qquad (\rho_2 \mapsto \rho_2 + \rho_1)$$
$$\sim \left(\begin{array}{cc|c} 1 & -i & 0 \\ 0 & 0 & 0 \end{array}\right), \qquad (\rho_1 \mapsto -\rho_1)$$

we conclude that $E_2 = \text{Span}\left\{\begin{pmatrix} i \\ 1 \end{pmatrix}\right\}$.

(c) From the characteristic polynomial

$$p_A(\lambda) = \begin{vmatrix} \cos\theta - \lambda & -\sin\theta \\ \sin\theta & \cos\theta - \lambda \end{vmatrix} = (\cos\theta - \lambda)^2 + \sin^2\theta = \cos^2\theta - 2\lambda\cos\theta + \lambda^2 + \sin^2\theta = \lambda^2 - 2\lambda\cos\theta + 1,$$

we can compute

$$\begin{aligned} \lambda &= \frac{2\cos\theta \pm \sqrt{(2\cos\theta)^2 - 4(1)(1)}}{2(1)} && \text{(quadratic formula)} \\ &= \frac{2\cos\theta \pm \sqrt{4\cos^2\theta - 4}}{2} \\ &= \cos\theta \pm \sqrt{\cos^2\theta - 1} \\ &= \cos\theta \pm \sqrt{-\sin^2\theta} && \text{(trig identity)} \\ &= \cos\theta \pm i\sin\theta. \end{aligned}$$

We have $E_{\cos\theta + i\sin\theta} = \text{Null}\begin{pmatrix} \cos\theta - (\cos\theta + i\sin\theta) & -\sin\theta \\ \sin\theta & \cos\theta - (\cos\theta + i\sin\theta) \end{pmatrix}$, and because

$$\left(\begin{array}{cc|c} -i\sin\theta & -\sin\theta & 0 \\ \sin\theta & -i\sin\theta & 0 \end{array}\right) \sim \left(\begin{array}{cc|c} -i\sin\theta & -\sin\theta & 0 \\ i\sin\theta & \sin\theta & 0 \end{array}\right) \qquad (\rho_2 \mapsto i\rho_2)$$
$$\sim \left(\begin{array}{cc|c} -i\sin\theta & -\sin\theta & 0 \\ 0 & 0 & 0 \end{array}\right) \qquad (\rho_2 \mapsto \rho_2 + \rho_1)$$

$$\sim \left(\begin{array}{cc|c} 1 & -i & 0 \\ 0 & 0 & 0 \end{array}\right), \qquad (\rho_1 \mapsto \tfrac{i}{\sin\theta}\rho_1\text{: only care about } \sin\theta \neq 0)$$

we conclude that $E_{\cos\theta + i\sin\theta} = \operatorname{Span}\left\{\begin{pmatrix} i \\ 1 \end{pmatrix}\right\}$.

Similarly, we have $E_{\cos\theta - i\sin\theta} = \operatorname{Null}\begin{pmatrix} \cos\theta - (\cos\theta - i\sin\theta) & -\sin\theta \\ \sin\theta & \cos\theta - (\cos\theta - i\sin\theta) \end{pmatrix}$, and because

$$\begin{aligned} \left(\begin{array}{cc|c} i\sin\theta & -\sin\theta & 0 \\ \sin\theta & i\sin\theta & 0 \end{array}\right) &\sim \left(\begin{array}{cc|c} i\sin\theta & -\sin\theta & 0 \\ i\sin\theta & -\sin\theta & 0 \end{array}\right) && (\rho_2 \mapsto i\rho_2) \\ &\sim \left(\begin{array}{cc|c} i\sin\theta & -\sin\theta & 0 \\ 0 & 0 & 0 \end{array}\right) && (\rho_2 \mapsto \rho_2 - \rho_1) \\ &\sim \left(\begin{array}{cc|c} 1 & i & 0 \\ 0 & 0 & 0 \end{array}\right), && (\rho_1 \mapsto -\tfrac{i}{\sin\theta}\rho_1\text{: only care about } \sin\theta \neq 0) \end{aligned}$$

we conclude that $E_{\cos\theta + i\sin\theta} = \operatorname{Span}\left\{\begin{pmatrix} -i \\ 1 \end{pmatrix}\right\}$.

Problem 4 Determine all the real numbers that may be eigenvalues of a matrix satisfying $A^2 - 5A + 6I = 0$.

Solution.

We would like to find all $\lambda \in \mathbb{R}$ such that $\left|A - \lambda I\right| = 0$. Since the determinant of the 0-matrix is 0, we have

$$0 = \left|O\right| = \left|A^2 - 5A + 6I\right| = \left|(A - 2I)(A - 3I)\right| = \left|A - 2I\right|\left|A - 3I\right|,$$

which happens only when $\left|A - 2I\right| = 0$ or $\left|A - 3I\right| = 0$. Thus, the only real numbers that can satisfy $\left|A - \lambda I\right| = 0$ are $\lambda = 2$ and $\lambda = 3$. $\square$

Problem 5 Let V be a vector space (of dimension $n+1$) with basis $(v_0, v_1, \dots, v_n)$, and let $a_0, \dots, a_n$ be scalars. Define a linear operator T on V by the rules $T(v_i) = v_{i+1}$ if $i < n$ and $T(v_n) = a_0v_0 + a_1v_1 + \cdots + a_nv_n$. Determine the matrix of T with respect to the given basis, and compute its characteristic polynomial.

Solution.

The matrix M of T is $(n+1) \times (n+1)$ has the form

$$M = \begin{pmatrix} 0 & 1 & 0 & \dots & 0 & 0 \\ 0 & 0 & 1 & \dots & 0 & 0 \\ 0 & 0 & 0 & \dots & 0 & 0 \\ \vdots & \vdots & \vdots & \ddots & \vdots & \vdots \\ 0 & 0 & 0 & \dots & 0 & 1 \\ a_0 & a_1 & a_2 & \dots & a_{n-1} & a_n \end{pmatrix}.$$

The characteristic polynomial is

$$\begin{vmatrix} \lambda & 1 & 0 & \dots & 0 & 0 \\ 0 & \lambda & 1 & \dots & 0 & 0 \\ 0 & 0 & \lambda & \dots & 0 & 0 \\ \vdots & \vdots & \vdots & \ddots & \vdots & \vdots \\ 0 & 0 & 0 & \dots & \lambda & 1 \\ a_0 & a_1 & a_2 & \dots & a_{n-1} & \lambda - a_n \end{vmatrix} = 0,$$

which can be solved numerically.

Problem 6 Do A and A^T always have the same eigenvectors? The same eigenvalues?

Solution.

It is true that A and A^T always have the same eigenvectors. Since the determinant of any matrix is equal to the determinant of the transpose, we have

$$\det(A - \lambda I) = \det((A - \lambda I)^T) = \det(A^T - \lambda I^T) = \det(A^T - \lambda I),$$

so the characteristic polynomials are the same and thus the eigenvalues are the same. However, A and A^T do not always have the same eigenvectors. For example, it is not hard to show that $\begin{pmatrix} 0 & 1 \\ 0 & 0 \end{pmatrix}$ has eigenvectors in Span $\left\{ \begin{pmatrix} 1 \\ 0 \end{pmatrix} \right\}$, but $\begin{pmatrix} 0 & 0 \\ 1 & 0 \end{pmatrix}$ has eigenvectors in Span $\left\{ \begin{pmatrix} 0 \\ 1 \end{pmatrix} \right\}$, which are different. $\square$

Problem 7 Suppose that a $n \times n$ matrix has distinct eigenvalues $\lambda_1, \dots, \lambda_n$, and let $v_1, \dots, v_n$ be eigenvectors with these eigenvalues.

(a) Show that every eigenvector of A is a multiple of one of these vectors.

(b) If you are given only the eigenvalues and the eigenvectors, explain how you can recover the matrix.

Solution.

(a) Let v be an eigenvector of A. Since there are n distinct eigenvalues, we must have an $i \in \{1, 2, \dots, n\}$ such that λ_i is the eigenvalue of v (if it is different than all of them, than the characteristic polynomial has more than n roots, a contradiction since A is $n \times n$). We showed that $V^{(\lambda_i)}$ is a subspace, and since each eigenvalue is distinct, each eigenspace has dimension 1. So since $v, v_i \in V^{(\lambda_i)}$, which is a subspace, we have that v and v_i are multiples of each other.

(b) Since there are n eigenvectors and they are linearly independent (they are associated with distinct eigenvalues), they form a basis. We know that a transformation can be uniquely determined by the effect on each member of a basis, and for all i, the transformation multiplies v_i by λ_i. So the matrix is

$$\left(\lambda_1 v_1 \;\middle|\; \lambda_2 v_2 \;\middle|\; \dots \;\middle|\; \lambda_n v_n \right).$$

Problem 8 Let $T : V \to V$ be a linear operator that has at least two linearly independent eigenvectors v, w with the same eigenvalue λ. Prove that λ is a repeated root of the characteristic polynomial of T. Hint: