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Linear Algebra: Section 1
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Linear Algebra Homework 1

Problem 1

The dimensions of

- A_1 are 2×3.
- A_2 are 2×1.
- A_3 are 3×2.
- A_4 are 1×2.

There are 16 permutations for multiplication:

1. $A_1 A_1$ is undefined because there are 3 columns in A_1 but 2 rows in A_1, which is incompatible.
2. $A_1 A_2$ is undefined because there are 3 columns in A_1 but 2 rows in A_2, which is incompatible.
3. We have

$$\begin{aligned} A_1 A_3 &= \begin{bmatrix} 1 & -1 & 2 \\ 0 & 1 & 3 \end{bmatrix} \begin{bmatrix} 2 & -1 \\ 0 & 2 \\ -2 & 1 \end{bmatrix} \\ &= \begin{bmatrix} (1)(2) + (-1)(0) + (2)(-2) & (1)(-1) + (-1)(2) + (2)(1) \\ (0)(2) + (1)(0) + (3)(-2) & (0)(-1) + (1)(2) + (3)(1) \end{bmatrix} \\ &= \begin{bmatrix} -2 & -1 \\ -6 & 5 \end{bmatrix}. \end{aligned}$$

4. $A_1 A_4$ is undefined because there are 3 columns in A_1 but 1 row in A_4, which is incompatible.
5. $A_2 A_1$ is undefined because there is 1 column in A_2 but 2 rows in A_1, which is incompatible.
6. $A_2 A_2$ is undefined because there is 1 column in A_2 but 2 rows in A_2, which is incompatible.
7. $A_2 A_3$ is undefined because there is 1 column in A_2 but 3 rows in A_3, which is incompatible.
8. We have

$$\begin{aligned} A_2 A_4 &= \begin{bmatrix} 1 \\ -2 \end{bmatrix} \begin{bmatrix} 1 & 2 \end{bmatrix} \\ &= \begin{bmatrix} (1)(1) & (1)(2) \\ (-2)(1) & (-2)(2) \end{bmatrix} \\ &= \begin{bmatrix} 1 & 2 \\ -2 & -4 \end{bmatrix}. \end{aligned}$$

9. We have

$$\begin{aligned} A_3A_1 &= \begin{bmatrix} 2 & -1 \\ 0 & 2 \\ -2 & 1 \end{bmatrix} \begin{bmatrix} 1 & -1 & 2 \\ 0 & 1 & 3 \end{bmatrix} \\ &= \begin{bmatrix} (2)(1)+(-1)(0) & (2)(-1)+(-1)(1) & (2)(2)+(-1)(3) \\ (0)(1)+(2)(0) & (0)(-1)+(2)(1) & (0)(2)+(2)(3) \\ (-2)(1)+(1)(0) & (-2)(-1)+(1)(1) & (-2)(2)+(1)(3) \end{bmatrix} \\ &= \begin{bmatrix} 2 & -3 & 1 \\ 0 & 2 & 6 \\ -2 & 3 & -1 \end{bmatrix}. \end{aligned}$$

10. We have

$$\begin{aligned} A_3A_2 &= \begin{bmatrix} 2 & -1 \\ 0 & 2 \\ -2 & 1 \end{bmatrix} \begin{bmatrix} 1 \\ -2 \end{bmatrix} \\ &= \begin{bmatrix} (2)(1)+(-1)(-2) \\ (0)(1)+(2)(-2) \\ (-2)(1)+(1)(-2) \end{bmatrix} \\ &= \begin{bmatrix} 4 \\ -4 \\ -4 \end{bmatrix}. \end{aligned}$$

11. A_3A_3 is undefined because there are 2 columns in A_3 but 3 rows in A_3, which is incompatible.
12. A_3A_4 is undefined because there are 2 columns in A_3 but 1 row in A_4, which is incompatible.
13. We have

$$\begin{aligned} A_4A_1 &= \begin{bmatrix} 1 & 2 \end{bmatrix} \begin{bmatrix} 1 & -1 & 2 \\ 0 & 1 & 3 \end{bmatrix} \\ &= \begin{bmatrix} (1)(1)+(2)(0) & (1)(-1)+(2)(1) & (1)(2)+(2)(3) \end{bmatrix} \\ &= \begin{bmatrix} 1 & 1 & 8 \end{bmatrix}. \end{aligned}$$

14. We have

$$\begin{aligned} A_4A_2 &= \begin{bmatrix} 1 & 2 \end{bmatrix} \begin{bmatrix} 1 \\ -2 \end{bmatrix} \\ &= \begin{bmatrix} (1)(1)+(2)(-2) \end{bmatrix} \\ &= \begin{bmatrix} -3 \end{bmatrix}. \end{aligned}$$

15. A_4A_3 is undefined because there are 2 columns in A_4 but 3 rows in A_3, which is incompatible.
16. A_4A_4 is undefined because there are 2 columns in A_4 but 1 row in A_4, which is incompatible.

Problem 2

The dimensions of

- A_1 are 2×3.
- A_2 are 2×3.
- A_3 are 3×2.
- A_4 are 3×2.

For A_1, we have

$$a_{11} = 1, a_{12} = -1, a_{13} = 2, a_{21} = 0, a_{22} = 1, a_{23} = 3,$$

and similarly for A_3, we have

$$a_{11} = 2, a_{12} = -1, a_{21} = 0, a_{22} = 2, a_{31} = -2, a_{23} = 1.$$

We compute

$$\begin{aligned} 2A_1 - A_2 &= 2\begin{bmatrix} 1 & -1 & 2 \\ 0 & 1 & 3 \end{bmatrix} - \begin{bmatrix} -2 & 5 & 3 \\ 0 & 1 & 0 \end{bmatrix} \\ &= \begin{bmatrix} 2(1) - (-2) & 2(-1) - 5 & 2(2) - 3 \\ 2(0) - 0 & 2(1) - 1 & 2(3) - 0 \end{bmatrix} \\ &= \begin{bmatrix} 4 & -7 & 1 \\ 0 & 1 & 6 \end{bmatrix}. \end{aligned}$$

Similarly,

$$\begin{aligned} 5A_3 + 4A_4 &= 5\begin{bmatrix} 2 & -1 \\ 0 & 2 \\ -2 & 1 \end{bmatrix} + 4\begin{bmatrix} 0 & 1 \\ 3 & 4 \\ 1 & 2 \end{bmatrix} \\ &= \begin{bmatrix} 5(2) + 4(0) & 5(-1) + 4(1) \\ 5(0) + 4(3) & 5(2) + 4(4) \\ 5(-2) + 4(1) & 5(1) + 4(2) \end{bmatrix} \\ &= \begin{bmatrix} 10 & -1 \\ 12 & 26 \\ -6 & 13 \end{bmatrix} \end{aligned}$$

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Problem 3

- Let $m, n \in \mathbb{Z}^+$ and let A be an $m \times n$ matrix. We will show that $AI_n = A$ and $I_m A = A$. Let $A = \left[a_{ij}\right]$, $I_n = \left[\iota_{ij}\right]$, and $I_m = \left[\iota_{ij}\right]$. Then, we have

$$\begin{aligned} AI_n &= \left[\textstyle\sum_{k=1}^{n} a_{ik}\iota_{kj}\right] \\ &= \left[a_{ij}\iota_{jj}\right] && (\iota_{kj} \text{ is 0 for all } k \neq j) \end{aligned}$$

$$\begin{aligned}
&= \left[(a_{ij})(1)\right] && (\iota_{kk} \text{ is 1 for all } 1 \le k \le n)\\
&= \left[a_{ij}\right] = A.
\end{aligned}$$

We note that this holds for the case where A is square and $m = n$.

Using similar reasoning, we have

$$\begin{aligned}
I_m A &= \left[\textstyle\sum_{k=1}^{m} \iota_{ik} a_{kj}\right]\\
&= \left[\iota_{ii} a_{ij}\right]\\
&= \left[(1)(a_{ij})\right]\\
&= \left[a_{ij}\right] = A.
\end{aligned}$$

$\square$

- For AB to be well-defined, B must have m rows since A has m columns. For $(AB)C$ to be well-defined, AB must have p columns since C has p rows. Since AB will only have p columns if B has p columns, we must have that B is an $m \times p$ matrix. Let $A = \left[a_{ij}\right]$, $B = \left[b_{ij}\right]$, $C = \left[c_{ij}\right]$. We now show associativity by noting that

$$\begin{aligned}
(AB)C &= \left[\textstyle\sum_{r=1}^{p} \left[AB\right]_{ir} c_{rj}\right]\\
&= \left[\textstyle\sum_{r=1}^{p} \left(\sum_{k=1}^{m} a_{ik} b_{kr}\right) c_{rj}\right] && \text{(definition of } AB\text{)}\\
&= \left[\textstyle\sum_{r=1}^{p} \sum_{k=1}^{m} a_{ik} b_{kr} c_{rj}\right] && \text{(sum property)}\\
&= \left[\textstyle\sum_{k=1}^{m} \sum_{r=1}^{p} a_{ik} b_{kr} c_{rj}\right] && \text{(switching sums)}\\
&= \left[\textstyle\sum_{k=1}^{m} a_{ik} \sum_{r=1}^{p} b_{kr} c_{rj}\right] && \text{(sum property)}\\
&= \left[\textstyle\sum_{k=1}^{m} a_{ik} \left[BC\right]\right] && \text{(definition of } BC\text{)}\\
&= A(BC). && \text{(definition of matrix multiplication)}
\end{aligned}$$

$\square$

- We will compute the product both ways to verify that we obtain the same result. First, we compute

$$\begin{aligned}
\left(\begin{bmatrix} 1 & 2 \\ 0 & 1 \end{bmatrix} \begin{bmatrix} 0 & 1 & 2 \\ 1 & 1 & 3 \end{bmatrix}\right) \begin{bmatrix} 1 \\ 4 \\ 3 \end{bmatrix} &= \begin{bmatrix} (1)(0)+(2)(1) & (1)(1)+(2)(1) & (1)(2)+(2)(3) \\ (0)(0)+(1)(1) & (0)(1)+(1)(1) & (0)(2)+(1)(3) \end{bmatrix} \begin{bmatrix} 1 \\ 4 \\ 3 \end{bmatrix}\\
&= \begin{bmatrix} 2 & 3 & 8 \\ 1 & 1 & 3 \end{bmatrix} \begin{bmatrix} 1 \\ 4 \\ 3 \end{bmatrix}\\
&= \begin{bmatrix} (2)(1)+(3)(4)+(8)(3) \\ (1)(1)+(1)(4)+(3)(3) \end{bmatrix}\\
&= \begin{bmatrix} 38 \\ 14 \end{bmatrix}.
\end{aligned}$$

Next, we compute

$$\begin{bmatrix} 1 & 2 \\ 0 & 1 \end{bmatrix} \left(\begin{bmatrix} 0 & 1 & 2 \\ 1 & 1 & 3 \end{bmatrix} \begin{bmatrix} 1 \\ 4 \\ 3 \end{bmatrix} \right) = \begin{bmatrix} 1 & 2 \\ 0 & 1 \end{bmatrix} \begin{bmatrix} (0)(1)+(1)(4)+(2)(3) \\ (1)(1)+(1)(4)+(3)(3) \end{bmatrix}$$
$$= \begin{bmatrix} 1 & 2 \\ 0 & 1 \end{bmatrix} \begin{bmatrix} 10 \\ 14 \end{bmatrix}$$
$$= \begin{bmatrix} (1)(10)+(2)(14) \\ (0)(10)+(1)(14) \end{bmatrix}$$
$$= \begin{bmatrix} 38 \\ 14 \end{bmatrix},$$

which matches our first result.

Problem 4

Let $n \in \mathbb{Z}^+$, A be an $n \times n$ matrix, and suppose that A has two inverses B and B'. Then, we have

$$\begin{aligned} & AB = I_n && \text{(true by definition of inverse)} \\ \implies & B'AB = B'I_n && \text{(left-multiplying both sides by } B'\text{)} \\ \implies & I_nB = B'I_n && (B'A = I_n \text{ since } B' \text{ is an inverse}) \\ \implies & B = B'. && \text{(identity property)} \end{aligned}$$

□

Problem 5

We compute

$$\begin{bmatrix} a_1 & \dots & a_n \end{bmatrix} \begin{bmatrix} b_1 \\ \vdots \\ b_n \end{bmatrix} = \begin{bmatrix} a_1b_1 + \cdots + a_nb_n \end{bmatrix}.$$

This is equivalent to the inner product of two vectors.

Problem 6

We know that we have the distributive property so that $A(B+C) = AB + AC$ and $(A+B)C = AC + BC$ for appropriate matrices A, B, C. Thus,

$$\begin{aligned} (A+B)^2 &= (A+B)(A+B) \\ &= A(A+B) + B(A+B) \\ &= A^2 + AB + BA + B^2. \end{aligned}$$

Similarly, we can expand

$$\begin{aligned} (A-B)(A+B) &= A(A+B) - B(A+B) \\ &= A^2 + AB - BA - B^2. \end{aligned}$$

So $(A-B)(A+B) = A^2 - B^2$ if and only if $AB - BA = 0$, or equivalently $AB = BA$ and A and B commute.