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MATH 554 Homework 9

Problem 1 Prove that if (E, d) is a complete metric space and F is a closed subset of E, then (F, d) is also a complete metric space.

Let $\langle p_n \rangle$ be a Cauchy sequence in F. Since $F \subseteq E$, $\langle p_n \rangle$ is Cauchy in E. Since E is complete, $\langle p_n \rangle$ converges to some point p in E. Since F is closed, it contains the limits of all its sequences, so $p \in F$. Thus, $\langle p_n \rangle$ converges in F, so (F, d) is a complete metric space.

Problem 2 Let $\langle p_n \rangle_{n=1}^{\infty}$ be a Cauchy sequence in $\mathbb{R}^3$ with its usual metric. Let $p_n = (x_n, y_n, z_n)$.

(a) Show that each of the sequences $\langle x_n \rangle_{n=1}^{\infty}$, $\langle y_n \rangle_{n=1}^{\infty}$, and $\langle z_n \rangle_{n=1}^{\infty}$ are also Cauchy sequences and explain why this implies the limits

$$x := \lim_{n\to\infty} x_n, y := \lim_{n\to\infty} y_n, z := \lim_{n\to\infty} z_n$$

exist.

(b) Let $p = (x, y, z)$ and show

$$\lim_{n\to\infty} p_n = p.$$

(c) Conclude that $\mathbb{R}^3$ is a complete metric space.

(a) Let $\varepsilon > 0$. Since $\langle p_n \rangle$ is Cauchy, there exists an N such that for all $m, n > N$, we have $\|p_m - p_n\| < \varepsilon$. Let $m, n > N$. Then, we have

$$|x_m - x_n|^2 \le |x_m - x_n|^2 + |y_m - y_n|^2 + |z_m - z_n|^2 = \|p_m - p_n\|^2 \implies |x_m - x_n| \le \|p_m - p_n\| < \varepsilon,$$

$$|y_m - y_n|^2 \le |x_m - x_n|^2 + |y_m - y_n|^2 + |z_m - z_n|^2 = \|p_m - p_n\|^2 \implies |y_m - y_n| \le \|p_m - p_n\| < \varepsilon,$$

$$|z_m - z_n|^2 \le |x_m - x_n|^2 + |y_m - y_n|^2 + |z_m - z_n|^2 = \|p_m - p_n\|^2 \implies |z_m - z_n| \le \|p_m - p_n\| < \varepsilon.$$

So by definition, $\langle x_n \rangle$, $\langle y_n \rangle$, and $\langle z_n \rangle$ are Cauchy sequences in $\mathbb{R}$. We have shown that in $\mathbb{R}$, all Cauchy sequences converge, so these three sequences all converge.

(b) Let $\varepsilon > 0$. Since these three sequences all converge, by definition there exist N_1, N_2, N_3 such that for all $n > N := \max\{N_1, N_2, N_3\}$, we have

$$|x_n - x|, |y_n - y|, |z_n - z| < \frac{\varepsilon}{\sqrt{3}}.$$

Let $n > N$. Then, from the choice of N we can write

$$\begin{aligned}
\|p_n - p\| &= \sqrt{|x_n - x|^2 + |y_n - y|^2 + |z_n - z|^2} \\
&< \sqrt{\frac{\varepsilon^2}{3} + \frac{\varepsilon^2}{3} + \frac{\varepsilon^2}{3}} \\
&= \sqrt{\varepsilon^2} = \varepsilon,
\end{aligned}$$

so $\langle p_n \rangle$ converges to p.

(c) Since $\langle p_n \rangle$ was an arbitrary Cauchy sequence in $\mathbb{R}^3$ and we have shown it must converge, by definition $\mathbb{R}^3$ is complete. $\square$

Problem 3 Let $\langle p_n \rangle_{n=1}^{\infty}$ be a Cauchy sequence in the metric space E. Prove that the sequence is bounded.

Consider $\varepsilon = 1$. Since $\langle p_n \rangle$ is Cauchy, there exists an N such that for all $m, n > N$, $d(p_m, p_n) < 1$. Let $K := \lceil N + 1 \rceil$. Consider

$$M := \max\{1, d(p_1, p_K), d(p_2, p_K), \dots, d(p_{\lfloor N \rfloor}, p_K)\},$$

which is well defined because the set is finite. Then, we claim $\overline{B}(p_K, M)$ contains $\langle p_n \rangle$. To see this, let $n \in \mathbb{N}$. If $n > N$, then $d(p_n, p_K) < 1 \leq M$ by our choice of N, K, and M, so $p_n \in \overline{B}(p_K, M)$. If $n \leq N$, then $d(p_n, p_K) \leq M$ by our choice of M, so $p_n \in \overline{B}(p_K, M)$ in this case as well. $\square$

Problem 4 Let $f : E \to E$ be a contraction and let $\lim_{n\to\infty} p_n = p$ in E. Show $\lim_{n\to\infty} f(p_n) = f(p)$.

Since f is a contraction, there exists $\rho \in [0, 1)$ such that $d(f(x), f(y)) \leq \rho d(x, y)$ for all $x, y \in E$.

Case 1: $\rho = 0$. Then the distance between all points is 0, so the limit holds because E has only one point.

Case 2: $0 < \rho < 1$. Let $\varepsilon > 0$. Since $\langle p_n \rangle$ converges to p, there exists an N such that for all $n > N$, $d(p_n, p) < \frac{\varepsilon}{\rho}$. Then, for all $n > N$, we have

$$d(f(p_n), f(p)) \leq \rho d(p_n, p) < \rho \left(\frac{\varepsilon}{\rho}\right) = \varepsilon.$$

So the limit holds by definition.

Problem 5 (Banach Fixed Point Theorem) Let E be a complete metric space and $f : E \to E$ a contraction. Prove that there is a unique point $p_* \in E$ with $f(p_*) = p_*$ along the following lines. To start choose any point $p_0 \in E$ and define a sequence $\langle p_n \rangle_{n=1}^{\infty}$ by

$$p_1 = f(p_0), p_2 = f(p_1), p_3 = f(p_2), \dots, p_{n+1} = f(p_n), \dots.$$

(a) Show for $k \geq 1$ that
$$d(p_k, p_{k+1}) \leq \rho^k d(p_0, p_1).$$

(b) If $m < n$ show
$$d(p_m, p_n) \leq \frac{\rho^m - \rho^n}{1 - \rho} d(p_0, p_1) \leq \frac{\rho^m}{1 - \rho} d(p_0, p_1).$$

(c) Show if N is a natural number and $m, n \geq N$, then
$$d(p_m, p_n) \leq \frac{\rho^N}{1 - \rho} d(p_0, p_1).$$

(d) We have shown that if $0 \leq \rho < 1$, then $\lim_{N\to\infty} = 0$. It follows that
$$\lim_{N\to\infty} \frac{\rho^N}{1 - \rho} d(p_0, p_1) = 0.$$
Use this to show $\langle p_n \rangle_{n=1}^{\infty}$ is a Cauchy sequence.

(e) As E is complete, this implies $\langle p_n \rangle_{n=1}^{\infty}$ is convergent. Let $p_* := \lim_{n\to\infty} p_n$. Use $p_{n+1} = f(p_n)$ and problem 4 to show $f(p_*) = p*$. This shows the existence of a fixed point for f.

(f) Show the fixed point is unique.

(a) Since f is a contraction, there exists $\rho \in [0,1)$ such that for all $x, y \in E$, $d(f(x), f(y)) \leq \rho d(x,y)$. We will induct on k to show the desired inequality.

Base Case: Let $k = 1$. Then, we can use our choice of ρ to write

$$d(p_k, p_{k+1}) = d(p_1, p_2) = d(f(p_0), f(p_1)) \leq \rho d(p_0, p_1) = \rho^k d(p_0, p_1).$$

Induction Step: Let $k \in \mathbb{N}$, $k > 1$. Assume that we have $d(p_{k-1}, p_k) \leq \rho^{k-1} d(p_0, p_1)$. Then, we have

$$\begin{aligned} d(p_k, p_{k+1}) &= d(f(p_{k-1}), f(p_k)) \\ &\leq \rho d(p_{k-1}, p_k) && \text{(by choice of } \rho) \\ &\leq \rho \rho^{k-1} d(p_0, p_1) && \text{(induction hypothesis)} \\ &= \rho^k d(p_0, p_1). \end{aligned}$$

So the inequality holds for all $k \geq 1$.

(b) Assume $m < n$. We can write

$$\begin{aligned} d(p_m, p_n) &\leq \sum_{k=m}^{n-1} d(p_k, p_{k+1}) && \text{(triangle inequality)} \\ &\leq \sum_{k=m}^{n-1} \rho^k d(p_0, p_1) && \text{(bound from (a))} \\ &= d(p_0, p_1) \sum_{k=m}^{n-1} \rho^k && \text{(pulling out constant)} \\ &= d(p_0, p_1) \frac{\rho^m - \rho^{n-1+1}}{1-\rho} && \text{(evaluating geometric sum)} \\ &= \frac{\rho^m - \rho^n}{1-\rho} d(p_0, p_1) && \text{(rearranging)} \\ &= \frac{\rho^m}{1-\rho} d(p_0, p_1) - \frac{\rho^n}{1-\rho} d(p_0, p_1) && \text{(splitting fraction)} \\ &\leq \frac{\rho^m}{1-\rho} d(p_0, p_1). && \text{(no longer subtracting non-negative term)} \end{aligned}$$

(c) Let $N \in \mathbb{N}$ and $m, n \geq N$. Without loss of generality, assume $m < n$. (If $m = n$, then $d(p_m, p_n) = 0$, so the inequality trivially holds). Because $\rho < 1$, we can write $\rho^m \leq \rho^N$ because $m \geq N$. So we can use (b) to write

$$d(p_m, p_n) \leq \frac{\rho^m}{1-\rho} d(p_0, p_1) \leq \frac{\rho^N}{1-\rho} d(p_0, p_1).$$

(d) Let ε. Since $\lim\limits_{N \to \infty} \frac{\rho^N}{1-\rho} d(p_0, p_1) = 0$, there exists an M such that for all $n > M$, we have

$$\left| \frac{\rho^N}{1-\rho} d(p_0, p_1) \right| < \varepsilon.$$

Let $m, n > M$. Then, from part (c), we can write

$$d(p_m, p_n) \leq \frac{\rho^N}{1-\rho} d(p_0, p_1) \leq \left| \frac{\rho^N}{1-\rho} d(p_0, p_1) \right| < \varepsilon.$$

So $\langle p_n \rangle$ is Cauchy by definition.

(e) Since $\langle p_n \rangle$ is Cauchy and E is complete, $p_* := \lim\limits_{n\to\infty} p_n$ exists. From problem 4, we have

$$\lim_{n\to\infty} f(p_n) = f(p_*).$$

But since $p_{n+1} = f(p_n)$ for all $n \geq 1$, we have $\langle f(p_n) \rangle_{n=1}^{\infty}$ is the same as $\langle p_n \rangle_{n=2}^{\infty}$. Since the sequences have the same end behavior, we can use problem 4 to write

$$\begin{aligned} f(p_*) &= \lim_{n\to\infty} f(p_n) && \text{(problem 4)} \\ &= \lim_{n\to\infty} p_n && \text{(same end behavior)} \\ &= p_*. && \text{(as defined)} \end{aligned}$$

Therefore, we have found a fixed point $p_* \in E$ such that $f(p_*) = p_*$.

(f) Suppose there exist two fixed points $p_*, p_{**} \in E$. Then, $f(p_*) = p_*$ and $f(p_{**}) = p_{**}$. We claim $d(p_*, p_{**}) = 0$. If not, then because f is a contraction we would have

$$d(f(p_*), f(p_{**})) \leq \rho d(p_*, p_{**}) = \rho d(f(p_*), f(p_{**})),$$

which would imply $\rho = 1$, a contradiction since we must have $0 \leq \rho < 1$. Since $d(p_*, p_{**}) = 0$, we have $p_* = p_{**}$ by the metric space axioms. Therefore, our fixed point is unique. $\square$

Problem 6 Let $a \geq 1$ and define $f : [0, \infty) \to [0, \infty)$ by $f(x) = \sqrt{a + x}$.

(a) Show for $x, y \in [0, \infty)$ that

$$\big|f(x) - f(y)\big| = \frac{|x - y|}{\sqrt{a + x} + \sqrt{a + y}} \leq \frac{|x - y|}{2\sqrt{a}} \leq \frac{1}{2}|x - y|$$

and therefore f is a contraction. The space $[0, \infty)$ is a complete metric space as it is a closed subset of the complete space $\mathbb{R}$.

(b) Define a sequence $x_0 = a$ and $x_{n+1} = f(x_n)$. The Banach Fixed Point Theorem tells us this converges to the unique fixed point of f. Find this fixed point. Note this limit can be interpreted as giving meaning to

$$\sqrt{a + \sqrt{a + \sqrt{a + \sqrt{a + \ldots}}}}.$$

(a) Let $x, y \in [0, \infty)$. We can write

$$\begin{aligned} \big|f(x) - f(y)\big| &= \left|\sqrt{a + x} - \sqrt{a + y}\right| \\ &= \left|\frac{(\sqrt{a + x} - \sqrt{a + y})(\sqrt{a + x} + \sqrt{a + y})}{\sqrt{a + x} + \sqrt{a + y}}\right| && \text{(multiplying by conjugate)} \\ &= \left|\frac{(a + x) - (a + y)}{\sqrt{a + x} + \sqrt{a + y}}\right| && \text{(difference of squares)} \\ &= \left|\frac{1}{\sqrt{a + x} + \sqrt{a + y}}\right| |x - y| \\ &\leq \left|\frac{1}{\sqrt{1 + 0} + \sqrt{1 + 0}}\right| |x - y| && \text{(because } a \geq 1 \text{ and } x, y \geq 0) \end{aligned}$$

$$= \frac{1}{2}|x-y|.$$

Since $0 \le \frac{1}{2} < 1$, f is a contraction. Since $[0,\infty)$ is closed and thus complete in $\mathbb{R}$, there exist a fixed point by problem 5.

(b) Let a_* be the fixed point of f. By definition, we have $a_* = f(a_*) = \sqrt{a+a_*}$. So we can write

$${a_*}^2 = a + a_* \implies {a_*}^2 - a_* - a = 0,$$

and solve for a_* using the quadratic formula. Since the square root is non-negative, the positive solution

$$a_* = \frac{-(-1)+\sqrt{(-1)^2-4(1)(-a)}}{2(1)} = \frac{\sqrt{4a+1}+1}{2}$$

is the limit of the sequence.

Problem 7 We will compute numerically a root of the equation $x^3-5x-1=0$. We can rewrite this as $\frac{x^3-1}{5} = x$, so we are looking for a fixed point of f given by

$$f(x) = \frac{x^3-1}{5}.$$

Let $E = [-1,1]$. This is a closed subset of $\mathbb{R}$ and therefore is a complete metric space.

(a) If $|x| \le 1$ show

$$\big|f(x)\big| \le \frac{2}{5}$$

and therefore f maps E into E.

(b) Show if $x,y \in E$ (that is $|x|,|y| \le 1$) then

$$\big|f(x)-f(y)\big| \le \frac{3}{5}|x-y|$$

and therefore f is a contraction on $E=[-1,1]$.

(a) We observe that $(-1)^3-5(-1)-1 = 1+5-1 = 3$ and $(1)^3-5(1)-1 = 1-5-1 = -5$, so by the intermediate value theorem there is a root of the equation in $[-1,1]$. Thus, it would be convenient to show $f(x)$ is a contraction on $[-1,1]$, because we can then use the fixed point theorem to find a root.

Let $x \in [-1,1]$. Then $|x| \le 1$. So we can write

$$\begin{aligned}
\big|f(x)\big| &= \left|\frac{x^3-1}{5}\right| \\
&= \frac{1}{5}\left|x^3+(-1)\right| \\
&\le \frac{1}{5}(|x|^3+|-1|) && \text{(triangle inequality)} \\
&\le \frac{1}{5}(|1|^3+1) && (|x| \le 1) \\
&= \frac{2}{5}.
\end{aligned}$$

So $|f(x)| \le \frac{2}{5}$ on E, and therefore f maps E into E.

(b) We can write

$$
\begin{aligned}
\left|f(x)-f(y)\right| &= \left|\frac{x^3-1}{5}-\frac{y^3-1}{5}\right| \\
&= \left|\frac{x^3-y^3}{5}\right| \\
&= \left|\frac{(x-y)(x^2+xy+y^2)}{5}\right| \\
&= \frac{1}{5}|x-y||x^2+xy+y^2| \\
&\leq \frac{1}{5}|x-y|(|x|^2+|x||y|+|y|^2) && \text{(triangle inequality)} \\
&\leq \frac{1}{5}|x-y|(|1|^2+|1||1|+|1|^2) && (|x|,|y|\leq 1) \\
&= \frac{3}{5}|x-y|.
\end{aligned}
$$

Therefore, since $0 \leq \frac{3}{5} < 1$, f is a contraction on $[-1,1]$. Therefore, we have from the Banach fixed point theorem that there is a unique fixed point where $x = \frac{x^3-1}{5}$, which implies x is a root of the equation in $[-1,1]$. We can then use the sequence defined in problem 5 to calculate the fixed point to an arbitrary level of accuracy, as shown in the given table. $\square$