MATH 552: Section 001
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MATH 552 Homework 12ˆ

Problem 72.8 Prove that if f is analytic at z_0 and $f(z_0) = f'(z_0) = ... = f^{(m)}(z_0) = 0$, then the function g defined by means of the equations

$$g(z) = \begin{cases} \frac{f(z)}{(z-z_0)^{m+1}} & \text{when } z \neq z_0 \\ \frac{f^{(m+1)}(z_0)}{(m+1)!} & \text{when } z = z_0 \end{cases}$$

is analytic at z_0.

Solution.

Since f is analytic at z_0, we can use Taylor's theorem to write

$$f(z) = f(z_0)+\frac{f'(z_0)(z-z_0)}{1!}+...+\frac{f^{(m)}(z_0)(z-z_0)^m}{m!}+\frac{f^{(m+1)}(z_0)(z-z_0)^{m+1}}{(m+1)!}+\frac{f^{(m+2)}(z_0)(z-z_0)^{m+2}}{(m+2)!}+...$$

Since $f(z_0) = f'(z_0) = ... = f^{(m)}(z_0) = 0$,, only the terms with derivatives of $m+1$ or higher contribute to the series.

When $z \neq 0$, we can divide $f(z)$ by $(z-z_0)^{m+1}$ to obtain $g(z)$:

$$\frac{f^{(m+1)}(z_0)(z-z_0)^{m+1}}{(z-z_0)^{m+1}(m+1)!}+\frac{f^{(m+2)}(z_0)(z-z_0)^{m+2}}{(z-z_0)^{m+1}(m+2)!}+... = \frac{f^{(m+1)}(z_0)}{(m+1)!}+\frac{f^{(m+2)}(z_0)(z-z_0)}{(m+2)!}+...$$

Thus, $g(z)$ has a removable singular point because the singular part has all zero coefficients.

When $z = z_0$, the series is equal to how $g(z)$ is defined when $z = z_0$, so the series converges to $g(z)$ for some neighborhood of $z = z_0$. Thus, the function is analytic at z_0.

Problem 79.3 Suppose that a function f is analytic at z_0, and write $g(z) = f(z)/(z-z_0)$. Show that
(a) if $f(z_0) \neq 0$, then z_0 is a simple pole of g, with residue $f(z_0)$;
(b) if $f(z_0) = 0$, then z_0 is a removable singular point of g.
Suggestion: As pointed out in Sec. 62, there is a Taylor series for $f(z)$ about z_0 since f is analytic there. Start each part of this exercise by writing out a few terms of that series.

Solution.

Since f is analytic at z_0, we can write

$$f(z) = f(z_0)+\frac{f'(z_0)(z-z_0)}{1!}+\frac{f''(z_0)(z-z_0)^2}{2!}+...$$

and by how $g(z)$ is defined,

$$g(z) = \frac{f(z_0)}{z-z_0}+f'(z_0)+\frac{f''(z_0)(z-z_0)}{2}+...$$

So the principal part of $g(z)$ is $\frac{f(z_0)}{z-z_0}$.

(a) If $f(z_0) \neq 0$, then one of the coefficients of the principal part is nonzero and $g(z)$ thus has a pole of order 1, or a simple pole. Since the coefficient is $f(z_0)$, the residue is $f(z_0)$.

(b) If $f(z_0) = 0$, then every coefficient of the principal part is zero and $g(z)$ has a removable singular point at z_0.