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MATH 576: Section H01
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MATH 576 Homework 9

Problem 1 Determine the outcome class of the game

$$-2 + 2 \cdot \uparrow + *7 \pm 6 \pm 5 \pm 4 \pm 3.$$

We proved in class that in positions like these (sums and differences of numbers, switches, $\uparrow$ components, and nimbers), the best move for the $\mathcal{N}$ext player is to move in the switch of hottest temperature if there are any switches at all. So if L moves first, the game after each player adopts this strategy will eventually reach

$$-2 + 2 \cdot \uparrow + *7 + 6 - 5 + 4 - 3 = 2 \cdot \uparrow + *7.$$

We proved in class that $2 \cdot \uparrow$ is greater than any nimber, so L is winning. So the game is in $\mathcal{N}^L$. If R moves first, the game will eventually reach

$$-2 + 2 \cdot \uparrow + *7 - 6 + 5 - 4 + 3 = -4 + 2 \cdot \uparrow + *7,$$

which is clearly winning for R as it is infinitesimally close to $-4 \in \mathcal{R}$. So the game is in $\mathcal{N}^R$. Therefore, the outcome class of the game is $\mathcal{N}$. $\square$

Problem 2 Let G be a finite partisan game and let ε be an infinitesimal. Prove that $L(G + \varepsilon) = L(G)$ and $R(G + \varepsilon) = R(G)$.

From Theorem 19.2.1a, we have $L(G) + L(\varepsilon) \geq L(G + \varepsilon) \geq L(G) + R(\varepsilon)$. From Theorem 16.2.1, we have $L(\varepsilon) = R(\varepsilon) = 0$, so we have

$$L(G + \varepsilon) \leq L(G) + 0 = L(G)$$

and

$$L(G + \varepsilon) \geq L(G) + 0 = L(G).$$

Thus, we have $L(G + \varepsilon) = L(G)$.

Similarly, from Therom 19.2.1b we have $R(\varepsilon) + R(G) \leq R(\varepsilon + G) \leq L(\varepsilon) + R(G)$. So we have

$$R(G + \varepsilon) = R(\varepsilon + G) \leq 0 + R(G) = R(G)$$

and

$$R(G + \varepsilon) = R(\varepsilon + G) \geq 0 + R(G) = R(G).$$

Thus, we have $R(G + \varepsilon) = R(G)$. $\square$

Problem 3 Find the mean value of the game $\{5 \mid 3\} + -_3$.

Let $G := \{5 \mid 3\} + -_3 = 4 \pm 1 + -_3$. We will compute

$$m(G) := \lim_{n \to \infty} \frac{L(n \cdot G)}{n}.$$

Consider $n \cdot G$ when n is even. Since ± 1 is its own negative, we have

$$n \cdot G = 4n + \left(\pm 1 \cdot \frac{n}{2}\right) - \left(\pm 1 \cdot \frac{n}{2}\right) + n \cdot -_3 = 4n + n \cdot -_3.$$

Since $-_3$ is an infinitesimal and the finite sum of infinitesimals is infinitesimal, from Problem 2 we have

$$L(n \cdot G) = L(4n + n \cdot -_3) = L(4n) = 4n,$$

and thus $\frac{L(n \cdot G)}{n} = \frac{4n}{n} = 4$. So we certainly have that $\langle \frac{L(n \cdot G)}{n} \rangle_{n \in 2\mathbb{Z}}$ converges to 4. Since $m(G)$ exists by Fekete's lemma, and a subsequence converges to 4, we have $m(G) = 4$. $\square$

Problem 4 Let x_1 and x_2 be numbers with $x_1 > x_2 > 0$. Find the mean value of

$$\pm x_1 \pm x_2.$$

Let $G := \pm x_1 \pm x_2$. We use a similar technique to Problem 3. Consider $n \cdot G$ when n is even. Then we have

$$n \cdot G = \pm x_1 \cdot n \pm x_2 \cdot n = \left(\pm x_1 \cdot \frac{n}{2}\right) - \left(\pm x_1 \cdot \frac{n}{2}\right) + \left(\pm x_2 \cdot \frac{n}{2}\right) - \left(\pm x_2 \cdot \frac{n}{2}\right) = 0 + 0 = 0,$$

which we can use to write

$$\frac{L(n \cdot G)}{n} = \frac{L(0)}{n} = \frac{0}{n} = 0.$$

Clearly, then, $\langle \frac{L(n \cdot G)}{n} \rangle_{n \in 2\mathbb{Z}}$ converges to 0, so $m(G) = 0$ as $m(G)$ exists and a subsequence converges to 0. $\square$

Problem 5 Let $G = \{1 \mid -1\}$, $H = \{3 \mid -3\}$ and $J = \{3, \{4 \mid 2\} \mid -3\}$. Show that $H \neq J$, but $H_2 = J_2 = G$.

We show $J - H \notin \mathcal{P}$. We have

$$J - H = \{3, \{4 \mid 2\} \mid -3\} - (\pm 3) = \{3, 3 \pm 1 \mid -3\} \pm 3.$$

If L goes first, they can move to $3 \pm 1 \pm 3$. Then, R's best move is to $3 - 3 \pm 1 = \pm 1$, but this is an $\mathcal{N}$-position, so L wins. So $J - H \in \mathcal{N}^L$, so $J - H \neq 0$ and thus $H \neq J$.

We first compute

$$H_2 = \{3_2 - 2 \mid (-3)_2 + 2\} = \{3 - 2 \mid -3 + 2\} = \{1 \mid -1\} = G.$$

We now compute $\{4 \mid 2\}_2$ (doing so will be useful for computing J_2). We note that

$$\{4 \mid 2\}_1 = \{4_1 - 1 \mid 2_1 + 1\} = \{4 - 1 \mid 2 + 1\} = \{3 \mid 3\} = 3 + *,$$

which is infinitesimally close to 3. Since $1 \leq 2$, we have $\{4 \mid 2\}_2 = 3$. We can use this to write

$$J_2 = \{3_2 - 2, \{4 \mid 2\}_2 - 2 \mid (-3)_2 + 2\} = \{3 - 2, 3 - 2 \mid -3 + 2\} = \{1 \mid -1\} = G.$$

Therefore, we have $H \neq J$, but $H_2 = J_2 = G$. $\square$