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MATH 554 Homework 4

Problem 2.36 Prove that in the real numbers, every nonempty set that is bounded below has a greatest lower bound.

Let $S \subset \mathbb{R}$ be nonempty, let $s \in S$, let b be a lower bound of S, and define $-S := \{-s : s \in S\}$. We have that $b \leq s$, which implies that $-s \leq -b$. Thus, $-b$ is an upper bound for $-S$. By the least upper bound property, then, there exists $c \in \mathbb{R}$ such that $c = \sup(-S)$. By definition, $c \geq -s$ and $c \leq -b$. Therefore, $-c \leq s$ and $-c \geq b$, and since s and b are arbitrary, this implies that $-c$ is a greatest upper bound for S. $\square$

Problem 2.37 Prove that for any real number x, there is a natural n with $x < n$.

Suppose (toward contradiction) that there is some $x \in \mathbb{R}$ such that for all $n \in \mathbb{N}$, $x \geq n$. Let $n \in \mathbb{N}$. Thus, x is an upper bound for $\mathbb{N}$, and so by the least upper bound property there exists $b := \sup(\mathbb{N}) \in \mathbb{R}$. So $b \geq n$, and since $n \in \mathbb{N} \implies n+1 \in \mathbb{N}$, $b \geq n+1$. Thus, $b-1 \geq n$, so $b-1$ is an upper bound. Since $b-1 < b$, this is a contradiction because we chose b to be the least upper bound. $\square$

Problem 2.38 Let $a > 1$ be a real number. Prove that for any real number x, there is a natural number n such that $a^n > x$.

Suppose (toward contradiction) that there is some $x \in \mathbb{R}$ such that for all $n \in \mathbb{N}$, $a^n \leq x$. Then

$$A = \{a^n : n \in \mathbb{N}\}$$

is bounded above by x, so by the least upper bound property there exists $b := \sup(A) \in \mathbb{R}$. Let $n \in \mathbb{N}$. Since b is an upper bound and $n+1 \in \mathbb{N}$, we have both $b \geq a^n$ and $b \geq a^{n+1} = a(a^n)$. The latter implies that $\frac{b}{a} \geq a^n$, so $\frac{b}{a}$ is an upper bound. Since $a > 1$, we have $\frac{b}{a} < b$, a contradiction because we chose b to be the least upper bound. $\square$

Problem 2.39 Let $a > 0$ be a positive real number. Without using the first version of Archimedes's axiom, prove that there is a natural number, n, such that $1/n < a$.

Suppose (toward contradiction) that there exists $a > 0$ such that for all $n \in \mathbb{N}$, $a \leq \frac{1}{n}$. Then a is a lower bound for $S = \{\frac{1}{n} : n \in \mathbb{N}\}$, and thus by the greatest lower bound property there exists $c := \inf(S) \in \mathbb{R}$, which implies $c \geq a > 0$ (and thus $c > 0$). Let $n \in \mathbb{N}$. Since $2n \in \mathbb{N}$ and c is the greatest lower bound, we have both $c \leq \frac{1}{n}$ and $c \leq \frac{1}{2n}$. The latter implies that $2c < \frac{1}{n}$, so $2c$ is a lower bound. But since $c > 0$, we have $2c > c$, a contradiction because we chose c to be the greatest upper bound. $\square$

Problem 2.40 Let a be a real number with $0 < a < 1$. Using problem 2.38, prove that for any positive real number x, there is a natural number n such that $a^n < x$.

Let $x \in \mathbb{R}^+$. Since $0 < a < 1$, we have $\frac{1}{a} > 1$. By problem 2.38, there exists some $n \in \mathbb{N}$ such that $\left(\frac{1}{a}\right)^n > \frac{1}{x}$. We can rewrite this as $\frac{1}{a^n} > \frac{1}{x}$, and therefore it follows that $a^n < x$ (since x and a^n are both positive). $\square$

Problem 2.41 Prove that for any real number x, there is a unique integer n such that $n \leq x < n+1$.

We will first prove existence. Let $x \in \mathbb{R}$. We claim there exists $m_0 \in \mathbb{Z}$ such that $m_0 < x$. If $x > 0$, $m_0 = 0$ suffices. If $x \le 0$, then $-x \ge 0$, and by the Archimedean property there exists $m_0' \in \mathbb{Z}$ such that $-x < m_0'$. Thus, $-m_0' < x$, so we can choose $m_0 = -m_0'$.

With this, we can define $S = \{k \in \mathbb{Z} : m_0 \le k \le x\}$, which we claim is non-empty and finite. Since m_0 is in S, S is nonempty. By the Archimedian property, there exists $m_1 \in \mathbb{Z}$ such that $x < m_1$. So $S \subset \{k \in \mathbb{Z} : m_0 \le k \le m_1\}$, which has size $m_1 - m_0 + 1$ and thus is finite. We can then say S is also finite, which implies there exists $n = \max(S) \in \mathbb{Z}$. Then, since $n + 1 > n$ we have $n + 1 \notin S$, which by our definition of S implies $n \le x < n + 1$.

We now prove uniqueness. Suppose that there exist $m, n \in \mathbb{Z}$ such that $n \le x < n + 1$ and $m \le x < m + 1$. We can rewrite this as $n - 1 \le x - 1 < n$ and $m - 1 \le x - 1 < m$, and combining these we have $x - 1 < n \le x$ and $x - 1 < m \le x$. So $n, m \in (x - 1, x]$, which implies that $|n - m| < x - (x - 1) = 1$. Since $n, m \in \mathbb{Z}$, this implies $|n - m| = 0$, so $n = m$. □

Problem 2.42 Prove that between any two real numbers there is a rational number.

Let $a, b \in \mathbb{R}$ with $a < b$. By the Archimedean property, there exists $N \in \mathbb{N}$ such that $\frac{1}{N} < b - a$ since $b - a > 0$. Let $n := \lfloor Na \rfloor$. By definition, $n \le Na < n + 1$, which implies $\frac{n}{N} \le a < \frac{n+1}{N}$. So we have

$$\begin{aligned}
a &< \frac{n+1}{N} \\
&= \frac{n}{N} + \frac{1}{N} \\
&\le a + \frac{1}{N} && (\tfrac{n}{N} \le a) \\
&< b, && (\tfrac{1}{N} < b - a \implies a + \tfrac{1}{N} < b)
\end{aligned}$$

which shows that $r := \frac{n+1}{N} \in \mathbb{Q}$ satisfies $a < r < b$. □

Problem 2.43 Prove that between any two rational numbers there is an irrational number.

We will first prove a lemma: any non-zero rational number multiplied by an irrational number is irrational. Suppose (toward contradiction) that there exist $a \in \mathbb{Q}\backslash\{0\}$, $b \in \mathbb{R}\backslash\mathbb{Q}$ such that $ab \in \mathbb{Q}$. Then there exist $c, d, e, f \in \mathbb{Z}\backslash\{0\}$ such that $a = \frac{c}{d}$ and $ab = \frac{e}{f}$. But then

$$b = \frac{ab}{a} = \frac{c/d}{e/f} = \frac{cf}{de},$$

so $b \in \mathbb{Q}$, a contradiction. A similar argument can be used to show that any rational number plus an irrational number is irrational.

Let $a, b \in \mathbb{Q}$ such that $a < b$. It is known that $\sqrt{2}$ is irrational, so by the lemma we have $n = \frac{(b-a)\sqrt{2}}{2}$ is irrational and so is $a + n$. Since $\frac{\sqrt{2}}{2} < 1$, we have $0 < n < b - a$, and thus $a < a + n < b$ holds. □

Problem 2.44 Let $y_0, y_1 \in \mathbb{R}$ and assume that there is a number $M > 0$ such that for all $\varepsilon > 0$

$$|y_1 - y_0| \le M\varepsilon.$$

Prove that $y_0 = y_1$.

Suppose (toward contradiction) that there exist $y_0, y_1 \in \mathbb{R}$ such that $y_0 \neq y_1$, and that there exists $M > 0$ such that $|y_1 - y_0| \leq M\varepsilon$ for all $\varepsilon > 0$. Since $y_0 \neq y_1$, $|y_1 - y_0| > 0$, so $\varepsilon = \frac{|y_1 - y_0|}{2M} > 0$. But then

$$|y_1 - y_0| < M\varepsilon = M\left(\frac{|y_1 - y_0|}{2M}\right) = \frac{|y_1 - y_0|}{2},$$

so $1 < \frac{1}{2}$, a contradiction. □

Problem 2.45 Prove that if $f : [a, b] \to \mathbb{R}$ is Lipschitz with Lipschitz constant M then for any $x, x_0 \in [a, b]$ the inequalities

$$-M|x - x_0| \leq f(x) - f(x_0) \leq M|x - x_0|$$

and

$$f(x_0) - M|x - x_0| \leq f(x) \leq f(x_0) + M|x - x_0|$$

hold.

Suppose $f : [a, b] \to \mathbb{R}$ is Lipschitz with Lipschitz constant M, and let $x, x_0 \in [a, b]$. By definition, we have $|f(x) - f(x_0)| \leq M|x - x_0|$. By proposition 2.24, it directly follows that

$$-M|x - x_0| \leq f(x) - f(x_0) \leq M|x - x_0|,$$

and by adding $f(x_0)$ we obtain

$$f(x_0) - M|x - x_0| \leq f(x) \leq f(x_0) + M|x - x_0|.$$

□

Problem 1 Let A and B be subsets of $\mathbb{R}$ that are each bounded above. Let

$$S = A + B = \{a + b : a \in A \text{ and } b \in B\}.$$

(a) Show that S is bounded above.

(b) Prove $\sup(S) = \sup(A) + \sup(B)$.

(a) Let $x \in S$. Then $s = a + b$ for $a \in A$, $b \in B$. Since $a \leq \sup(A)$ and $b \leq \sup(B)$, we have $x = a + b \leq \sup(A) + \sup(B) = s$ from an inequality property we proved. So $\sup(A) + \sup(B)$ is an upper bound for S.

(b) Suppose (toward contradiction) there exists $s' < \sup(A)+\sup(B)$ such that s' is an upper bound. Then there exists an $\varepsilon > 0$ such that $s' = \sup(A)+\sup(B)-\varepsilon$. By definition, there must exist $0 < \varepsilon_1, \varepsilon_2 < \frac{\varepsilon}{2}$ such that $a = \sup(A) - \varepsilon_1 \in A$ and $b = \sup(B) - \varepsilon_2 \in B$. We then have $x = a + b \in S$. But then

$$x = a + b = \sup(A) - \varepsilon_1 + \sup(B) - \varepsilon_2 > \sup(A) - \frac{\varepsilon}{2} + \sup(B) - \frac{\varepsilon}{2} = \sup(A) + \sup(B) - \varepsilon = s',$$

a contradiction since we chose s' to be an upper bound. □

Problem 2 Let $S \subseteq \mathbb{R}$ be a subset that satisfies the two conditions

(a) S is bounded above.

(b) If $s_1, s_2 \in S$ with $s_1 \neq s_2$, then
$$|s_1 - s_2| \geq 1.$$
Show $\sup(S) \in S$ and therefore S has a maximum.

Since S is bounded above, it has a supremum $s \in \mathbb{R}$ by the least upper bound property. Suppose (toward contradiction) that $s \notin S$. Since s is the least upper bound, we must have $S \cap (s - \frac{1}{2}, s) \neq \emptyset$ (or $s - \frac{1}{2}$ would be an upper bound). So we have $s_1 \in S$, $s - \frac{1}{2} < s_1 < s$. By the same reason, we must have $S \cap (s_1, s) \neq \emptyset$ (or s_1 would be an upper bound). So we have $s_2 \in S$, $s_1 < s_2 < s$. Thus, we have $s_1, s_2 \in (s - \frac{1}{2}, s)$, which we have shown implies $|s_1 - s_2| < \frac{1}{2}$. But this is a contradiction, because $|s_1 - s_2| \geq 1$ for all $s_1 \neq s_2 \in S$.$\square$