Nathan Bickel
SCHC 501: Section H01
Professor: Dr. Dubinsky
September 3, 2024

SCHC 501 Homework 4

Problem 4 Suppose that the following assumptions are true of English:

(i) There is a finite alphabet for writing sentences, consisting of 26 letters, a set of punctuation marks and a space.

(ii) Every sentence is a finite string in the alphabet given in (i).

(iii) There is no upper bound on the length on sentences of English. E.g. given any sentence, a longer once can be made by conjoining it with another one.

What then is the cardinality of the set of all sentences of English? Motivate your answer.

We claim that the set of all sentences in English has cardinality $\aleph_0$. To see this, let Σ be the English alphabet and Σ^* be the set of all sentences in English. Since Σ is finite by assumption (i), we have that for all lengths $k \in \mathbb{N}$, there are finitely many sentences containing k characters (in particular, there are $|\Sigma|^k$ English sentences of length k). Thus, we can define a bijection from $\mathbb{N}$ to Σ^* by first listing the empty sentence, then all the sentences with one symbol, then all the sentences with two, and so forth. Every sentence will be reached in this list eventually by assumption (ii), but the list never ends by assumption (iii). Therefore, Σ^* has the same cardinality as $\mathbb{N}$, $\aleph_0$.

Problem 6 Assume that the earth rests on the back of a giant turtle, and that the turtle sits on the back of two giant turtles, and those two on three, etc. 'all the way down' (i.e. there is no bottom layer of turtles).

(a) Suppose each turtle is the sole deity of some monotheistic sect (exactly one sect per turtle). What is the cardinality of the set of all such sects?

(b) Suppose each *subset* of all these earth-supporting turtles forms the deity-group of some one sect (a-, mono-, or polytheistic, with the latter including both finite and infinite numbers of deities). What is the cardinality of the set of all such sects?

(a) Let T be the set of turtles. Clearly, there is a bijection from T to the set of monotheistic sects, so it suffices to determine the cardinality of T. We can define a bijection from $\mathbb{N}$ to T by listing the turtles starting from the top, working from left to right in each row, and going to the next row once reaching the rightmost turtle of a row. So T has cardinality $\aleph_0$, and therefore the set of monotheistic sets also has cardinality $\aleph_0$.

(b) There is a clear bijection from $\mathcal{P}(T)$ to the sects described, so it suffices to determine the cardinality of $\mathcal{P}(T)$. Since we have $|T| = \aleph_0$ from **(a)**, it follows that $\mathcal{P}(T) = 2^{\aleph_0}$. Therefore, the set of described sects has cardinality $2^{\aleph_0}$.