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MATH 575: Section H01
Professor: Dr. Luo
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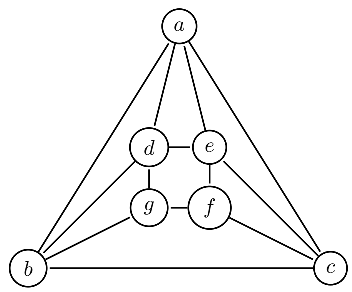
MATH 575 Homework 8

Collaboration:

Problem 1 Let G be the graph below.

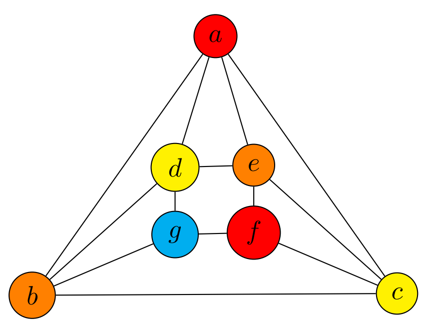


(a) Determine $\chi(G)$.

(b) Is G color-critical? If so, prove that $\chi(G - e) < \chi(G)$ for every edge $e \in E(G)$. If not, find a color-critical subgraph of G.

Solution.

(a) We claim $\chi(G) = 4$. To see that $\chi(G) \leq 4$, we create a proper coloring:

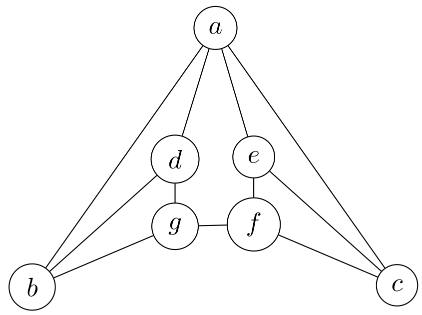


To see that $\chi(G) \geq 4$, we observe that a, b, c is an odd cycle, so we must use 3 colors to color this. Suppose we color this cycle as in the picture above, and can properly color the rest of the vertices with these colors.

- d must be yellow since it is adjacent to red (a) and orange (b).
- e must be orange since it is adjacent to red (a) and yellow (d).
- g must be red since it is adjacent to orange (b) and yellow (d).

But then f is adjacent to red (g), orange (e), and yellow (c), so we must use a fourth color, a contradiction. □

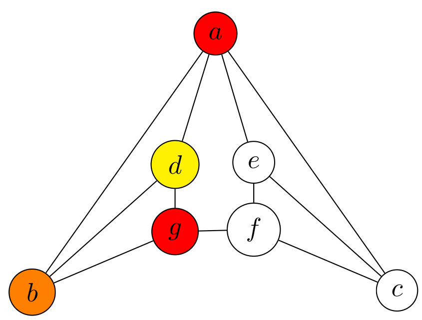
(b) We claim that G is not color-critical, and that it has a color-critical subgraph G' with $\chi(G') = 4$. Consider this subgraph G' of G:



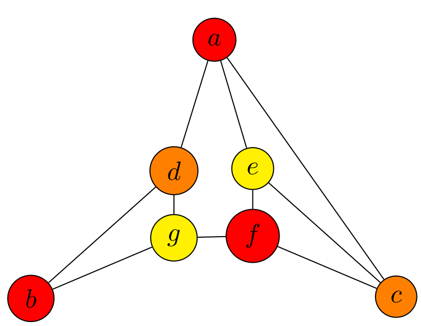
Clearly, G' is 4-colorable since G is 4-colorable. Now, suppose we can color this subgraph with red, orange, and yellow. We arbitrarily color a red and b orange, since they are adjacent and both need colors. Then,

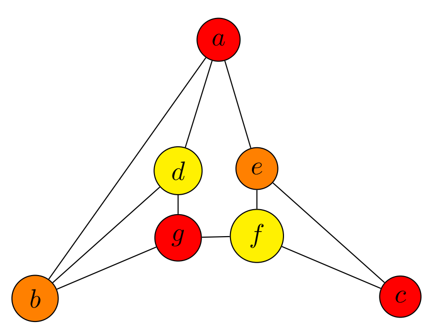
- d must be yellow since it is adjacent to red (a) and orange (b).
- g must be red since it is adjacent to orange (b) and yellow (d).

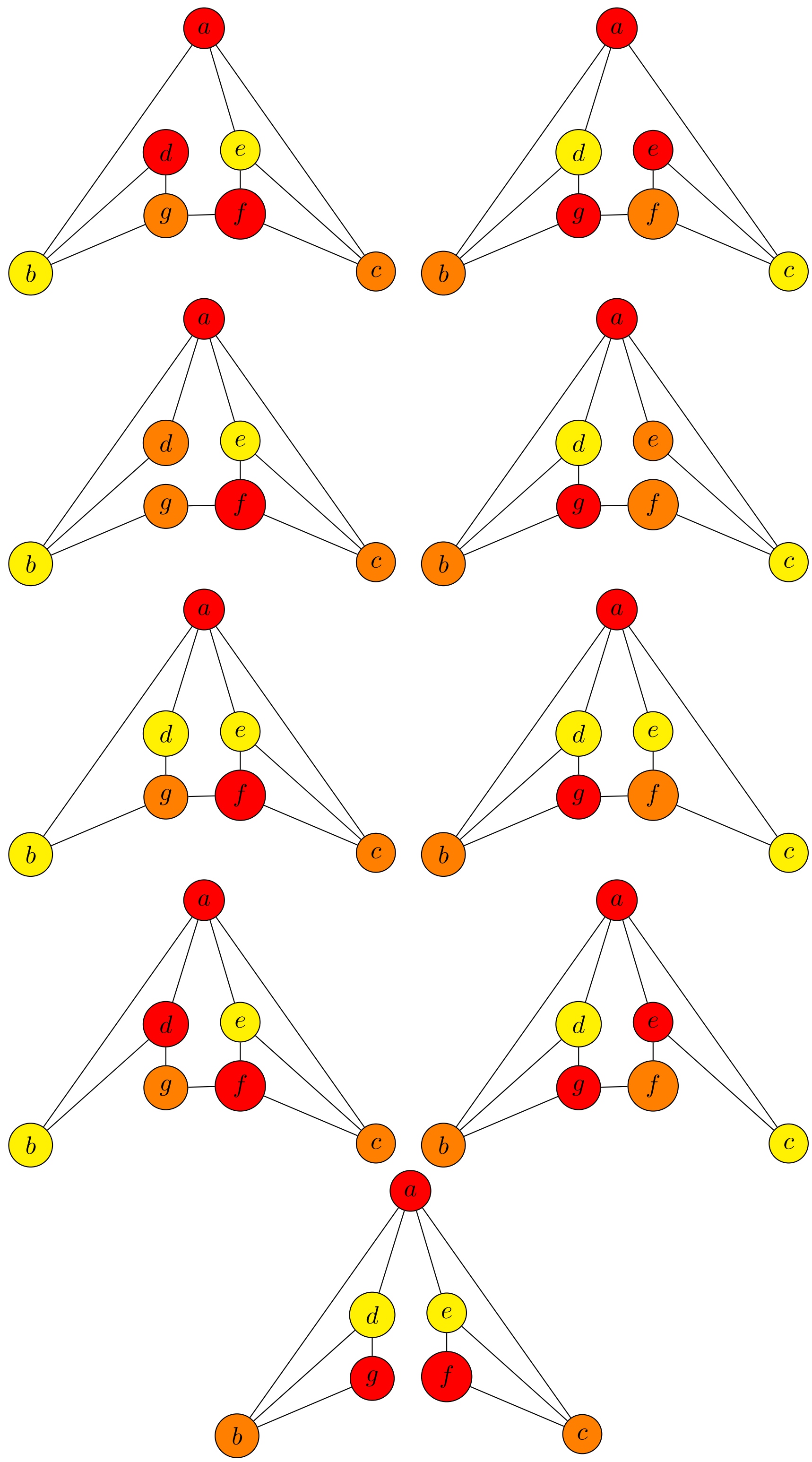
So we have this graph:



If we color f orange, then e must be yellow since it is adjacent to red (a) and orange (f). Then, we must use a fourth color for c since it is adjacent to red (a), orange (f), and yellow (e), a contradiction. Alternatively, if we color f yellow, then we must use a fourth color for c by a symmetrical argument. So we must use four colors to color G', and thus $\chi(G') = 4$. We claim this is also color critical. To see this, we observe the following proper colorings:







Since each possible subgraph resulting from removing an edge from G' is 3-colorable, we have exhaustively (and exhaustingly) proved that G' is color-critical. $\square$

Problem 2 Prove that for any graph G, there exists an ordering of $V(G)$ for which the greedy algorithm uses exactly $\chi(G)$ colors.

Solution.

We will induct on $\chi(G)$. First, let G be a graph with $\chi(G) = 1$. Then, any ordering of the vertices will result in all vertices being colored the same, since there are no edges. Next, let $k \in \mathbb{N}$, $k > 1$, and assume that for all G' with $\chi(G') < k$, there exists an ordering of $V(G')$ for which the greedy algorithm uses exactly $\chi(G')$ colors. Let G be a graph with $\chi(G) = k$ with color classes $V_1, V_2, \ldots, V_k$ colored with colors $c_1, c_2, \ldots, c_k$ respectively. Then, $\chi(G - V_k) < \chi(G)$ since we don't have to color these vertices anymore, so by the IH there exists an ordering of $V(G - V_k)$ for which the greedy algorithm uses $\chi(G - V_k)$ colors.

Then, we can simply place the vertices in V_k in any order at the end of the ordering. Let $v \in V_k$. When v is reached by the greedy algorithm, it will be colored the first color c_i such that no vertices in $N(v)$ have color c_i. We will never have $i > k$, because S is an independent set and thus no vertices in $N(v)$ have color k. So we will use at most k colors to color G, and since we can't color G with fewer than k colors, the greedy algorithm uses $\chi(G) = k$ colors given this ordering. □

Problem 3 Let G be a graph. Prove or disprove the following statements.

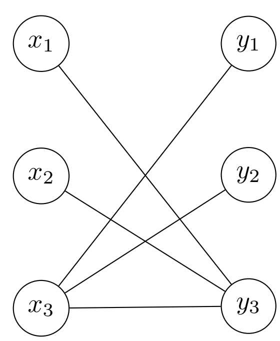
(a) There exists a $\chi(G)$-coloring of G in which one color class contains $\alpha(G)$ vertices.

(b) $\chi(G) \leq 1 + \overline{d}(G)$ where $\overline{d}(G)$ is the *average degree* in G.

(c) $\chi(G) \leq n - \alpha(G) + 1$.

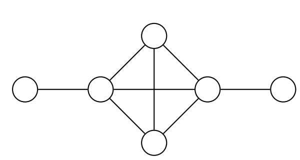
Solution.

(a) This is false. Consider this bipartite graph $G = X \cup Y$:



We have $\{x_1, x_2, y_1, y_2\}$ is an independent set, so $\alpha(G) = 4$ (there is clearly no independent set of size 5). However, the only way to get a coloring with four vertices colored the same is to color this set all one color. Then, since x_3 and y_3 both have edges in this set, we must use a different color for them, and since they are adjacent themselves, we must use distinct colors for x_3 and y_3. So the only way to do this is to use 3 colors, but since G is bipartite $\chi(G) \leq 2 \leq 3$. So there is no $\chi(G)$-coloring with a color class containing $\alpha(G)$ vertices. □

(b) This is false. Consider this graph G:



Since G has a K_4 as a subgraph, $\chi(G) \geq 4$. We have

$$1+\overline{d}(G) = 1+\frac{1+4+3+3+4+1}{6} = \frac{11}{3} < 4 \leq \chi(G),$$

so $\chi(G) \leq 1+\overline{d}(G)$ does not hold. □

(c) This is true. Let $S \cup V(G)$ be a maximum independent set in G. Then, we can color all vertices in S one color, and the remaining $n-\alpha(G)$ vertices all different colors. So we will at most use $n-\alpha(G)$ colors for $V(G)-S$ and 1 color for S, and thus $\chi(G) \leq n-\alpha(G)+1$. □

Problem 4 Let G be an n-vertex graph. Prove that $\chi(G)\cdot\chi(\overline{G}) \geq n$, and use this to prove $\chi(G)+\chi(\overline{G}) \geq 2\sqrt{n}$. For each n where $\sqrt{n}$ is an integer, construct a graph that achieves both equalities.

Hint: using a $\chi(G)$-coloring of G and a $\chi(\overline{G})$-coloring of $\overline{G}$, construct a proper coloring of K_n. Also you may find it useful to use the AM-GM inequality which states that if x and y are non-negative real numbers, then $\sqrt{xy} \leq \frac{x+y}{2}$.

Solution.

Let $\chi(G) = k$. We can partition $V(G)$ into non-empty independent sets $V_1, V_2, \ldots, V_k$ in G. Since there are n vertices and k sets, by the extended PHP, there will be a V_i in the partition such that $|V_i| \geq \lceil \frac{n}{k} \rceil$. By definition of complement, $V_1, V_2, \ldots, V_k$ will be cliques in $\overline{G}$ since they are independent sets in G. So $\chi(\overline{G}) \geq \max\{|V_1|, |V_2|, \ldots, |V_k|\} \geq \lceil \frac{n}{k} \rceil$, since we need to use a different color for every vertex in each clique. So we have

$$\chi(G)\chi(\overline{G}) = k\chi(\overline{G}) \geq k\max\{|V_1|, |V_2|, \ldots, |V_k|\} \geq k\left\lceil \frac{n}{k} \right\rceil \geq k\left(\frac{n}{k}\right) \geq n.$$

It follows that

$$\begin{aligned}
\chi(G)\chi(\overline{G}) &\geq n \\
\implies \sqrt{\chi(G)\chi(\overline{G})} &\geq \sqrt{n} \\
\implies \frac{\chi(G)+\chi(\overline{G})}{2} \geq \sqrt{\chi(G)\chi(\overline{G})} &\geq \sqrt{n} && \text{(AM-GM inequality)} \\
\implies \chi(G)+\chi(\overline{G}) &\geq 2\sqrt{n}. && \text{(algebra)}
\end{aligned}$$

Let $n = k^2$ for some $k \in \mathbb{N}$, and construct a graph G with k distinct copies of K_k. In G, we will need k colors to color each copy, but we can reuse the same colors for each copy so $\chi(G) = k$. In $\overline{G}$, all k copies of K_k in G become an independent sets, so in each set, we can color every vertex the same color. However, each vertex in the set will be adjacent to everything outside the set, so we cannot reuse any colors and thus $\chi(\overline{G}) = k$. Therefore, $\chi(G)\chi(\overline{G}) = k^2 = n$, and $\chi(G)+\chi(\overline{G}) = k+k = \sqrt{n}+\sqrt{n} = 2\sqrt{n}$, so we have achieved both equalities. □

Problem 5 Let G be an n-vertex graph. Prove that $\chi(G)+\chi(\overline{G}) \leq n+1$ and conclude that $\chi(G)\cdot\chi(\overline{G}) \leq [(n+1)/2]^2$. For each odd n, give an example of a graph G that achieves both equalities. (From questions 3 and 4, we obtain that for every graph G on n vertices, either G or its complement has chromatic number at least $\sqrt{n}$, and either G or its complement has chromatic number at most $(n+1)/2$.)

Hint: use induction to prove $\chi(G)+\chi(\overline{G}) \leq n+1$.

Solution.

We will induct on n. First, let G be the empty graph on $n = 0$ vertices. Then, $\chi(G) + \chi(\overline{G}) = 0 + 0 \leq 0 + 1$. Next, let $n \in \mathbb{N}$, $n > 0$, and assume that for any graph G' on any $n' < n$ vertices, we have $\chi(G') + \chi(\overline{G'}) \leq n' + 1$. Let G be a graph on n vertices, and let $k = \chi(G)$. Now, we choose a vertex $v \in V(G)$ and consider $G' = G - v$. By the IH, we have $\chi(G') + \chi(\overline{G'}) \leq (n-1) + 1 = n$.

Case 1: $\chi(G') + \chi(\overline{G'}) \leq n - 1$. Then, we can use an extra color in both G and G' to color v when we add v back to G, so we have

$$\chi(G) + \chi(\overline{G}) \leq \chi(G') + 1 + \chi(\overline{G'}) + 1 \leq (n-1) + 1 + 1 = n + 1.$$

Case 2: $\chi(G') + \chi(\overline{G'}) = n$. Suppose (toward contradiction) that when we add v back to G, we must use an extra color to color G and an extra color to color $\overline{G}$ (so we have $\chi(G) = \chi(G') + 1$ and $\chi(\overline{G}) = \chi(\overline{G'}) + 1$). The only way this would happen is if v is adjacent to vertices in all $\chi(G)$ color classes in G and all $\chi(\overline{G})$ color classes in $\overline{G}$. Since an edge cannot be present in both G and G', this means that $|N(v)| \geq \chi(G') + \chi(\overline{G'})$. It follows from our assumption that $|N(v)| \geq n$, a contradiction since no vertex can have more than $n - 1$ neighbors. So we will never need to use an extra color in both G and $\overline{G}$, and thus we have

$$\chi(G) + \chi(\overline{G}) \leq \chi(G') + \chi(\overline{G'}) + 1 = n + 1.$$

So it follows that

$$\begin{aligned}
\chi(G) + \chi(\overline{G}) &\leq n + 1 \\
\implies \frac{\chi(G) + \chi(\overline{G})}{2} &\leq \frac{n+1}{2} \\
\implies \sqrt{\chi(G)\chi(\overline{G})} \leq \frac{\chi(G) + \chi(\overline{G})}{2} &\leq \frac{n+1}{2} && \text{(AM-GM inequality)} \\
\implies \chi(G)\chi(\overline{G}) \leq \left(\frac{\chi(G) + \chi(\overline{G})}{2}\right)^2 &\leq \left(\frac{n+1}{2}\right)^2.
\end{aligned}$$

Let n be an odd integer, and $k = \frac{n+1}{2}$. Construct a graph G such that we have a K_k, an independent set of size k, and half of the edges possible (rounded down) between the clique and the independent set. Then, we need k colors to color the clique, and we can reuse one of the colors in the independent set. So we have $\chi(G) = k$. In $\overline{G}$, the clique will become an independent set and visa versa, and the other half of the possible edges (rounding up) will be present. So by the same reasoning, we have $\chi(\overline{G}) = k$. Therefore, $\chi(G) + \chi(\overline{G}) = 2k = n = 1$, and $\chi(G)\chi(\overline{G}) = k^2 = \left(\frac{n+1}{2}\right)^2$.

$\square$