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MATH 576: Section H01
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MATH 576 Homework 2

Problem 1 Prove that $*1 + *2 + *3 + \cdots + *(4k+3) = *0$ for any nonnegative integer k.

Let $\oplus$ denote the nim-sum, let $|$ denote concatenation, and let m_2 represent the binary representation of n for any $n \in \mathbb{N}$. We note that for any $k \in \mathbb{N}$ and $0 \leq a, b \leq 3$, we have

$$*(4k+a) + *(4k+b) = *a + *b. \tag{$\star$}$$

This is because multiplying by 4 corresponds to left-shifting by 2 in binary and a_2 will have length 2 (after left-padding with 0s), so $(4k+a)_2 = k_2|a_2$ and $(4k+b)_2 = k_2|b_2$. We can then write

$$(k_2|a_2) \oplus (k_2|b_2) = (k_2 \oplus k_2)|(a_2 \oplus b_2) = a_2 \oplus b_2,$$

proving $(\star)$.

We now prove the claim by induction on k. For the base case, we have

$$*1 + *2 + *3 = 01_2 \oplus 10_2 \oplus 11_2 = 11_2 \oplus 11_2 = 00_2 = *0. \tag{$\star\star$}$$

Now, let $k \in \mathbb{N}$, $k > 0$, and suppose the claim holds for $k-1$. Then, we have

$$\begin{aligned}
&*1 + *2 + *3 + \cdots + *(4k+3) \\
=&*1 + *2 + *3 + \cdots + *(4k-1) + *(4k) + *(4k+1) + *(4k+2) + *(4k+3) \\
=&\big[*1 + *2 + *3 + \cdots + *(4(k-1)+3)\big] + *(4k) + *(4k+1) + *(4k+2) + *(4k+3) \\
=&*0 + *(4k) + *(4k+1) + *(4k+2) + *(4k+3) && \text{(induction hypothesis)} \\
=&\big[*(4k) + *(4k+1)\big] + \big[*(4k+2) + *(4k+3)\big] && \text{(nim-sum is associative)} \\
=&*0 + *1 + *2 + *3 && \text{(from } (\star) \text{ and associativity)} \\
=&*0 + *0 = *0. && \text{(from } (\star\star))
\end{aligned}$$

□

Problem 2 Find an option from the Nim position $(22, 40, 51)$ that is in $\mathcal{P}$.

We claim that $(22, 37, 51)$ is in $\mathcal{P}$. We compute

$$\begin{aligned}
22 \oplus 37 \oplus 51 &= 010110_2 \oplus 100101_2 \oplus 110011_2 \\
&= 110011_2 \oplus 110011_2 \\
&= 0_2 = 0,
\end{aligned}$$

which verifies the claim. □

Problem 3 *Misère Nim* is the game of Nim played under the misère play convention, so that the player who removes the last token is the **loser**. Describe the winning strategy for Misère Nim.

Let G be a game of Misère Nim. Then the $\mathcal{N}$ext player is winning if and only if one of the following hold:

1. G has at least one heap with more than one token, and the nim-sum is non-zero.
2. G has no heaps with more than one token, and there are an even number of heaps with one token.

To see why this is true, consider any position $(a_1, a_2, \ldots, a_k)$ where exactly one heap has more than one token (without loss of generality, assume it is a_1 and that heaps a_2 through a_k have one token). Then this position must have non-zero nim-sum, and the $\mathcal{N}$ext player has a winning strategy by taking all the tokens in a_1 if $k-1$ is odd and all but one of the tokens in a_1 if $k-1$ is even. The players then alternate taking heaps of size one until the other player loses by taking the last heap.

Since Misère Nim is finite, such a position above must occur eventually. Since it will have non-zero nim-sum, and the nim-sum can be forced to alternate between zero and non-zero by the player on whose turn the sum is non-zero, the $\mathcal{N}$ext player can employ this strategy when, and only when, they are in Case 1 to eventually get to Case 2 and win. $\square$

Problem 4 Determine the nimbers for the Subtraction-$\{1, 2, 4\}$ game. Formally prove your answer by strong induction. (It may be helpful to compute the first 10 or so nimbers to spot the pattern).

Let $\underline{n}$ denote $\underset{\{1,2,4\}}{n}$. We claim that $\underline{n}$ will have nimber $*(n \mod 3)$. We prove this by induction on n.

For the base case, observe that

- Since $\underline{0}$ has no options, it has nimber $*0$,
- Since $\underline{1}$ has $\underline{0}$ as its only option which has nimber $*0$, $\underline{1}$ has nimber $*$.
- Since $\underline{2}$ has $\underline{0}$ and $\underline{1}$ as its options, which have nimbers $*0$ and $*1$ respectively, $\underline{2}$ has nimber $*2$.
- Since $\underline{3}$ has $\underline{1}$ and $\underline{2}$ as its options, which have nimbers $*1$ and $*2$ respectively, $\underline{3}$ has nimber $*0$.

Now, let $n \geq 4$ and suppose that for all $n' < n$, $\underline{n'}$ has nimber $*(n' \mod 3)$.

Case 0: $n \equiv 0 \pmod 3$. Then $\underline{n}$ has options $\underline{n-1}$, $\underline{n-2}$, and $\underline{n-4}$, which have nimbers $*2$, $*1$, and $*2$ respectively by the induction hypothesis. Thus, $\underline{n}$ has nimber $*0$.

Case 1: $n \equiv 1 \pmod 3$. Then $\underline{n}$ has options $\underline{n-1}$, $\underline{n-2}$, and $\underline{n-4}$, which have nimbers $*0$, $*2$, and $*0$ respectively by the induction hypothesis. Thus, $\underline{n}$ has nimber $*$.

Case 2: $n \equiv 2 \pmod 3$. Then $\underline{n}$ has options $\underline{n-1}$, $\underline{n-2}$, and $\underline{n-4}$, which have nimbers $*1$, $*0$, and $*1$ respectively by the induction hypothesis. Thus, $\underline{n}$ has nimber $*2$. $\square$

Problem 5 Let G be the game sum $H + J$, where $H = \{*0, *, *9, *8, *4, *3\}$ and $J = \{*0, *2, *4, *, *12\}$. Find the nimber associated with G.

We compute that the nimber of H is $*2$ and the nimber of J is $*3$ by taking the mex of each set. Then, the nimber of G is $*2 + *3$, which by the nim-sum we calculate to be $*$. $\square$