Nathan Bickel
MATH 555: Section H01
Professor: Dr. Howard
January 15, 2024

MATH 555 Homework 1

Problem 1 Let $\langle a_n \rangle_{n=1}^{\infty}$ and $\langle b_n \rangle_{n=1}^{\infty}$ be sequences of real numbers with

$$\lim_{n\to\infty} a_n = L, \lim_{n\to\infty} b_n = \infty.$$

(a) Prove that

$$\lim_{n\to\infty} (a_n + b_n) = \infty.$$

(b) Now suppose $L > 0$. Prove that

$$\lim_{n\to\infty} a_n b_n = \infty.$$

(a) Since $\langle a_n \rangle$ converges to L, we have that there exists some $N_1 > 0$ such that for all $n > N_1$, $|a_n - L| < 1$. So $-(a_n - L) < 1$, and therefore $a_n > L - 1$ for all $n > N_1$.

Let $M \in \mathbb{R}$. Since $\langle b_n \rangle$ diverges to ∞, we have that there exists some $N_2 > 0$ such that for all $n > N_2$, $b_n > M - L + 1$. Set $N := \max\{N_1, N_2\}$ and let $n > N$.

Then, since $n > N_1$ we have $a_n > L - 1$, and since $n > N_2$ we have $b_n > M - L + 1$, so

$$a_n + b_n > L - 1 + M - L + 1 = M.$$

Therefore, we have

$$\lim_{n\to\infty} (a_n + b_n) = \infty.$$

(b) Since $\langle a_n \rangle$ converges to L, we have that there exists some $N_1 > 0$ such that for all $n > N_1$, $|a_n - L| < \frac{L}{2}$. So $-(a_n - L) < \frac{L}{2}$, and therefore $a_n > \frac{L}{2}$ for all $n > N_1$.

Let $M \in \mathbb{R}$. Since $\langle b_n \rangle$ diverges to ∞, we have that there exists some $N_2 > 0$ such that for all $n > N_2$, $b_n > \frac{2M}{L}$. Set $N := \max\{N_1, N_2\}$ and let $n > N$.

Then, since $n > N_1$ we have $a_n > \frac{L}{2}$, and since $n > N_2$ we have $b_n > \frac{2M}{L}$, so

$$a_n b_n > \left(\frac{L}{2}\right)\left(\frac{2M}{L}\right) = M.$$

Therefore, we have

$$\lim_{n\to\infty} (a_n b_n) = \infty.$$

Problem 2 Prove that the function $f(x) = \sqrt{x}$ is differentiable on $(0, \infty)$ and

$$f'(x) = \frac{1}{2\sqrt{x}}.$$

We will first prove a proposition: For all $n \in \mathbb{N}$, $n \geq 1$, we have that $f(x) = x^{1/n}$ is differentiable for all $a \in (0, \infty)$ with

$$f'(a) = \frac{1}{n} a^{\frac{1-n}{n}}.$$

To see this, let $n \in \mathbb{N}$ and $a \in (0, \infty)$. We have seen before (and it is not hard to check) that

$$\begin{aligned} x - a &= \left(x^{1/n} - a^{1/n}\right)\left(x^{\frac{n-1}{n}} + x^{\frac{n-2}{n}} a^{\frac{1}{n}} + \cdots + x^{\frac{1}{n}} a^{\frac{n-2}{n}} + a^{\frac{n-1}{n}}\right) \\ &= \left(x^{1/n} - a^{1/n}\right) \sum_{k=0}^{n-1} \left(x^{\frac{n-k-1}{n}} a^{\frac{k}{n}}\right). \end{aligned}$$

Using this, we can write

$$\begin{aligned} f'(a) &= \lim_{x \to a} \frac{f(x) - f(a)}{x - a} \\ &= \lim_{x \to a} \frac{x^{1/n} - a^{1/n}}{x - a} \\ &= \lim_{x \to a} \frac{x^{1/n} - a^{1/n}}{\left(x^{1/n} - a^{1/n}\right) \sum_{k=0}^{n-1} \left(x^{\frac{n-k-1}{n}} a^{\frac{k}{n}}\right)} && \text{(from above)} \\ &= \lim_{x \to a} \frac{1}{\sum_{k=0}^{n-1} \left(x^{\frac{n-k-1}{n}} a^{\frac{k}{n}}\right)} \\ &= \frac{1}{\sum_{k=0}^{n-1} \left(a^{\frac{k}{n}} \lim_{x \to a} x^{\frac{n-k-1}{n}}\right)} && \text{(limits distribute over sums/products)} \\ &= \frac{1}{\sum_{k=0}^{n-1} \left(a^{\frac{k}{n}} a^{\frac{n-k-1}{n}}\right)} && (x^{\frac{n-k-1}{n}} \text{ is continuous for all } k) \\ &= \frac{1}{n a^{\frac{n-1}{n}}} && (x^{\frac{n-k-1}{n}} \text{ is continuous for all } k) \\ &= \frac{1}{n} a^{\frac{1-n}{n}}. \end{aligned}$$

The derivative of $f(x) = \sqrt{x}$ follows with $n = 2$: equivalently $f(x) = x^{1/2}$, so f is differentiable at a with

$$f'(a) = \frac{1}{2} a^{\frac{1-2}{2}} = \frac{a^{-1/2}}{2} = \frac{1}{2\sqrt{a}}.$$

$\square$

Problem 3 Prove that $f(x) = x^{\frac{1}{3}}$ is differentiable at all points $x = a \neq 0$ and that $f'(a) = \frac{1}{3} a^{\frac{-2}{3}}$.

The proposition in problem 2 can be extended when n is odd to be true for all $a \in \mathbb{R} \backslash \{0\}$ with the same reasoning given above: the only issue when n is even is that $a^{1/n}$ is not defined for even n. Once this is done, the proof follows immediately from the $n = 3$ case. $\square$

Problem 4 Prove that $f(x) = x^{\frac{1}{4}}$ is differentiable at all points $a > 0$ and that $f'(a) = \frac{1}{4} a^{\frac{-3}{4}}$.

This follows immediately from the proposition in problem 2 with $n = 4$. $\square$

Problem 5 Let f and g be defined in an interval containing a and assume they are both differentiable at a. Prove that the product $p(x) = f(x)g(x)$ is differentiable at a and

$$p'(a) = f'(a)g(a) + f(a)g'(a).$$

We showed in class that since g is differentiable at a point a, then g is continuous at a. Using this, we can write

$$\begin{aligned}
p'(x) &= \lim_{x\to a} \frac{p(x)-p(a)}{x-a} \\
&= \lim_{x\to a} \frac{f(x)g(x)-f(a)g(a)}{x-a} \\
&= \lim_{x\to a} \frac{f(x)g(x)-f(a)g(x)+f(a)g(x)-f(a)g(a)}{x-a} \\
&= \lim_{x\to a} \left[\left(\frac{f(x)-f(a)}{x-a}\right) g(x) + f(a)\left(\frac{g(x)-g(a)}{x-a}\right)\right] \\
&= \lim_{x\to a} \frac{f(x)-f(a)}{x-a} \lim_{x\to a} g(x) + f(a) \lim_{x\to a} \frac{g(x)-g(a)}{x-a} \\
&= f'(a)g(a) + f(a)g'(a). && \text{(definition of derivative and } g \text{ is continuous)}
\end{aligned}$$

□

Problem 6 Let $f : \mathbb{R} \to \mathbb{R}$ be defined by

$$f(x) = \begin{cases} x^2, & x \in \mathbb{Q}, \\ 0, & x \notin \mathbb{Q} \end{cases}$$

where $\mathbb{Q}$ is the set of rational numbers.

(a) Show that for any $a \neq 0$ that f is not continuous at a.

(b) Show f is differentiable at $x = 0$.

(a) Let $a \neq 0$. Consider $\varepsilon = \frac{a^2}{2}$, which is positive since $a \neq 0$. Suppose (toward contradiction) that there exists δ such that for all $x \in \mathbb{R}$,

$$|x-a| < \delta \implies \big|f(x)-f(a)\big| < \varepsilon.$$

Case 1: $a \in \mathbb{Q}$. We have shown before that there is some $b \notin \mathbb{Q}$ with $a < b < a+\delta$, so $|a-b| < \delta$ but

$$|f(a)-f(b)| \geq f(a)-f(b) = a^2 - 0 \geq \frac{a^2}{2} = \varepsilon,$$

a contradiction.

Case 2: $a \notin \mathbb{Q}$. We have shown before that there is some $b \in \mathbb{Q}$ with $a < b < a+\delta$, so $|a-b| < \delta$ but

$$\big|f(a)-f(b)\big| \geq f(b)-f(a) = f(b) = b^2 \geq a^2 \geq \frac{a^2}{2} = \varepsilon,$$

a contradiction. So f is not continuous at any $a \neq 0$. □

(b) We claim that $f'(0)$ and consequently that

$$0 = \lim_{x\to 0} \frac{f(x)-f(0)}{x-0} = \lim_{x\to 0} \frac{f(x)}{x}.$$

To see this, let $\varepsilon > 0$ and consider $\delta = \varepsilon$. Let $x \in \mathbb{R}$ such that $0 < |x - 0| < \delta$. If $x \in \mathbb{Q}$, then

$$\left|\frac{f(x)}{x} - 0\right| = \frac{x^2}{x} = x \leq |x| < \delta = \varepsilon,$$

and if $x \notin \mathbb{Q}$, then

$$\left|\frac{f(x)}{x} - 0\right| = \frac{0}{x} = 0 < \varepsilon.$$

Therefore, the limit holds and $f'(0) = 0$. □

Problem 7 The functions cosh and sinh are defined by

$$\cosh(x) = \frac{e^x + e^{-x}}{2}, \quad \sinh(x) = \frac{e^x - e^{-x}}{2}.$$

It is given that on $(0, \infty)$, both cosh and sinh are differentiable,

$$\cosh' = \sinh, \quad \sinh' = \cosh,$$

and

$$\cosh^2(x) - \sinh^2(x) = 1.$$

Let $\operatorname{arcsinh} : (0, \infty) \to (0, \infty)$ be the inverse of sinh. Explain why arcsinh is differentiable and find a formula for the derivative $\operatorname{arcsinh}'$.

Let $y_0 \in (0, \infty)$. Since sinh has an inverse on $(0, \infty)$, it is bijective, so there exists a unique $x_0 \in (0, \infty)$ with $y_0 = \sinh(x_0)$. Also, because it is bijective, sinh is onto and strictly increasing or decreasing since it is injective, and it is continuous because it is the composition of continuous functions.

We have that $\sinh'(x_0) = \cosh(x_0)$, which is non-zero (this is because $e^x > 0$ for all $x \in \mathbb{R}$, and since the positive reals are closed under addition and division, $\frac{e^{x_0} + e^{-x_0}}{2}$ is positive). Therefore, by the result in the notes, arcsinh is differentiable at y_0 with

$$\begin{aligned}
\operatorname{arcsinh}'(y_0) &= \frac{1}{\sinh'(x_0)} && \text{(result from notes)} \\
&= \frac{1}{\cosh(x_0)} && \text{(given)} \\
&= \frac{1}{\sqrt{1 + \sinh^2(x_0)}} && \text{(from identity and cosh is positive)} \\
&= \frac{1}{\sqrt{1 + \sinh^2(\operatorname{arcsinh}(y_0)}} && \text{(since } y_0 = \sinh(x_0)) \\
&= \frac{1}{\sqrt{1 + y_0^2}}.
\end{aligned}$$

So for all $x \in (0, \infty)$ we have

$$\operatorname{arcsinh}'(x) = \frac{1}{\sqrt{1 + x^2}}.$$

□