Nathan Bickel
MATH 575: Section H01
Professor: Dr. Luo
January 24, 2023

MATH 575 Homework 1

Problem 1 Recall that $\Delta(G)$ and $\delta(G)$ denote the maximum degree and minimum degree of a graph G respectively. Suppose G has n vertices.
(a) Prove that if $\delta(G) \geq \lceil (n-1)/2 \rceil$, then G is connected.
(b) For all $n \geq 3$, give an example of an n-vertex disconnected graph G with $\delta(G) = \lfloor (n-2)/2 \rfloor$.
(c) Prove or disprove: if $\delta(G) = \lfloor (n-2)/2 \rfloor$ and $\Delta(G) \geq \lceil n/2 \rceil$, then G is connected.

Solution.

(a) Let G be a graph with $\delta(G) \geq \lceil (n-1)/2 \rceil$, and assume G is not connected. Then, G has at least two connected components, and thus, there must be a connected component $G' \subset G$ with $|V(G')| \leq \lfloor \frac{n}{2} \rfloor$ (there cannot be two components each with more than half the vertices).

Case 1: If n is even, then $|V(G')| \leq \lfloor \frac{n}{2} \rfloor = \frac{n}{2}$ and $\delta(G) \geq \lceil (n-1)/2 \rceil = \frac{n}{2}$. Then, each vertex in G' must connect to at least $\frac{n}{2}$ vertices, but there are only $\frac{n}{2} - 1$ other vertices in G' to connect to, a contradiction.

Case 2: If n is odd, then $|V(G')| \leq \lfloor \frac{n}{2} \rfloor = \frac{n-1}{2}$ and $\delta(G) \geq \lceil (n-1)/2 \rceil = \frac{n-1}{2}$. Similarly, each vertex in G' must connect to at least $\frac{n-1}{2}$ vertices, but there are only $\frac{n-1}{2} - 1$ other vertices in G' to connect to, a contradiction. $\square$

(b) We note that

$$\lfloor (n-2)/2 \rfloor = \begin{cases} \frac{n-2}{2} & \text{if } 2 \mid n \\ \frac{n-3}{2} & \text{else} \end{cases}.$$

Case 1: Suppose n is even. Then, let G be two copies of $K_{n/2}$, which will have $\frac{n}{2} + \frac{n}{2} = n$ vertices. The two components will not be connected to each other, and since $\delta(K_{n/2}) = \frac{n}{2} - 1 = \frac{n-2}{2}$, we have $\delta(G) = \frac{n-2}{2} = \lfloor (n-2)/2 \rfloor$.

Case 2: Suppose n is odd. Then, let G be a $K_{(n-1)/2}$ and a $K_{(n+1)/2}$, which will have $\frac{n-1}{2} + \frac{n+1}{2} = n$ vertices. The two components will not be connected to each other, and since $\delta(K_{(n-1)/2}) = \frac{n-1}{2} - 1 = \frac{n-3}{2}$, we have $\delta(G) = \frac{n-3}{2} = \lfloor (n-2)/2 \rfloor$. $\square$

(c) Let G be a graph with $\delta(G) = \lfloor (n-2)/2 \rfloor$ and $\Delta(G) \geq \lceil n/2 \rceil$, and assume G is not connected.

Case 1: Suppose n is even. Then, $\delta(G) = \frac{n-2}{2}$ and $\Delta(G) = \frac{n}{2}$. The only way to satisfy $\delta(G) = \frac{n}{2} - 1$ is to have two connected components, each of which is a $K_{n/2}$. But then no vertex can have degree of $\frac{n}{2}$, because it would have to connect to a vertex in the other component, a contradiction.

Case 2: Suppose n is odd. Then, $\delta(G) = \frac{n-3}{2}$ and $\Delta(G) = \frac{n+1}{2}$. To satisfy $\delta(G) = \frac{n-1}{2} - 1$, each connected component must have at least $\frac{n-1}{2}$ vertices. Let the smallest component be G', and since $|V(G')| \geq \frac{n-1}{2}$, the rest of the graph has $n - \frac{n-1}{2} = \frac{n+1}{2}$ vertices. But then no vertex can have degree $\frac{n+1}{2}$, because it must connect to at least one vertex in G', a contradiction. $\square$

Problem 2 Prove that if G is an n-vertex bipartite graph, then $|E(G)| \leq \frac{n^2}{4}$.

Solution.

Let $X \cup Y$ be a bipartition of G. Then, $n = |X| + |Y|$, and by the product rule of combinatorics, $|E(G)| \le |X||Y|$ (there are $|Y|$ options for $|X|$ vertices, or vice versa). We observe that

$$\begin{aligned} & 0 \le \big(|X| - |Y|\big)^2 && \text{(squares are non-negative)} \\ \implies & 0 \le |X|^2 - 2|X||Y| + |Y|^2 \\ \implies & 4|X||Y| \le |X|^2 + 2|X||Y| + |Y|^2 \\ \implies & |X||Y| \le \frac{\big(|X| + |Y|\big)^2}{4} \\ \implies & |E(G)| \le \frac{n^2}{4}. && \text{(since } |E(G)| \le |X||Y|\text{)} \end{aligned}$$

□

Problem 3 A graph is called k-partite if its vertex set can be partitioned into sets $V_1, V_2, \ldots, V_k$ such that for each $1 \le i \le k$, there are no edges between vertices in the set V_i. Prove that for all integers $k \ge 2$ every graph G has a k-partite subgraph with at least $\frac{(k-1)|E(G)|}{k}$ edges.

Solution.

We will induct on n. First, let $n = 1$. Then, any graph on n vertices is clearly k-partite for any $k \in \mathbb{N}$, since there are no edges. So the claim holds for $n = 1$.

Next, let $n \in \mathbb{N}$, $n > 1$, and assume that the claim holds for all $n' \in \mathbb{N}$ such that $n' < n$. Let G be a graph on n vertices. Then, choose a vertex v and let H be the graph with $V(H) = V(G) - \{v\}$ and $E(H) = E(G) - N(v)$ (so we have removed one vertex and the edges it is incident to).

Since $|V(H)| = n - 1 < n$, the claim holds by the IH. Let $k \in \mathbb{N}$ such that $k \ge 2$. Then, there exists a k-partition $V_1, V_2, \ldots, V_k$ of some subgraph H' of H with at least $\frac{(k-1)|E(H)|}{k}$ edges. Since $|E(H)| = |E(G)| - d(v)$, we note that we have

$$|E(H')| \ge \frac{(k-1)|E(H)|}{k} = \frac{(k-1)(|E(G)| - d(v))}{k} = \frac{(k-1)|E(G)|}{k} - \frac{d(v)(k-1)}{k}.$$

We now add v to H' to obtain a spanning subgraph G' of G, and we will show that we can do so such that $V_1, V_2, \ldots, V_k$ is a k-partition of G' and G' has at least $\frac{(k-1)|E(G)|}{k}$ edges. We first need to choose a partition to place v in. Since we want to remove as few edges as possible, we place v in a partition V_i such that no other partition has fewer vertices that connect to v in G (for example, if $k = 3$ and v connects to 1 vertex in V_1, 2 vertices in V_2, and 1 vertex in V_3, we would put v in either V_1 or V_3).

We now remove the edges between v and the vertices in V_i. Since every other partition has at least as many vertices connected to v, we will have to remove at most $\frac{d(v)}{k}$ edges. Thus, since we initially added $d(v)$ edges to H', we have

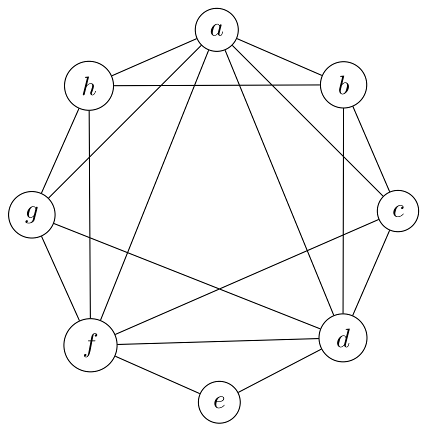
$$|E(G')| \ge |E(H')| + d(v) - \frac{d(v)}{k} = |E(H')| + \frac{d(v)(k-1)}{k}.$$

Substituting from above, then,

$$|E(G')| \ge |E(H')| + \frac{d(v)(k-1)}{k} \ge \left(\frac{(k-1)|E(G)|}{k} - \frac{d(v)(k-1)}{k}\right) + \frac{d(v)(k-1)}{k} = \frac{(k-1)|E(G)|}{k}.$$

So G' has at least $\frac{(k-1)|E(G)|}{k}$ edges, and therefore by strong induction, the claim holds for all $n \in \mathbb{N}$. □

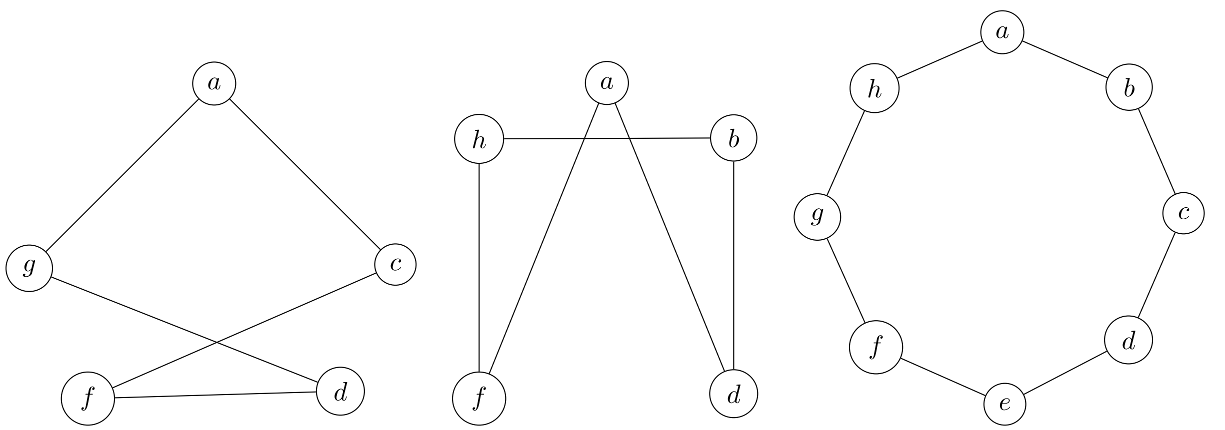
Problem 4 Consider the graph G below.



(a) Partition the edge set of G into a collection of edge-disjoint cycles. List the vertices of each cycle.
(b) Splice together the cycles in part (a) to find an Eulerian circuit of G.

Solution.

(a)



(b)

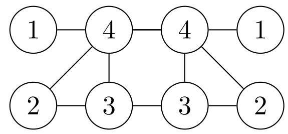
- First Cycle: $(a,\ g,\ d,\ f,\ c,\ a)$
- Adding Second Cycle: $(a,\ g,\ d,\ a,\ f,\ h,\ b,\ d,\ f,\ c,\ a)$
- Adding Third Cycle: $(a,\ g,\ d,\ a,\ f,\ h,\ a,\ b,\ c,\ d,\ e,\ f,\ g,\ h,\ b,\ d,\ f,\ c,\ a)$

Problem 5 Determine if each of the following sequences is graphic. If it is not, give a reason why. If it is, draw a graph that realizes the degree sequence.
(a) $(4, 4, 3, 3, 2, 2, 1, 1, 1)$
(b) $(4, 4, 3, 3, 2, 2, 1, 1)$
(c) $(8, 7, 6, 5, 4, 3, 2, 1)$
(d) $(7, 4, 4, 4, 4, 3, 3, 3)$

Solution.

(a) The sum of the sequence is odd, so by the handshaking lemma, the sequence cannot be graphic.

(b) This graph realizes the sequence, with the degrees labelled:



(c) This sequence cannot be graphic, because there would only be 8 vertices and thus no vertex can be incident to 8 edges since there are only 7 other vertices to connect to.

(d) This graph realizes the sequence, with the degrees labelled: