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SCHC 501 Homework 7

Problem 3 In each of the following expressions, identify all bound and free occurrences of variables, and underline the scope of the quantifiers.

(a) $(\forall x)P(x) \lor Q(x, y)$

(b) $(\forall y)\big(Q(x) \to (\forall z)P(y, z)\big)$

(c) $(\forall x) \sim \big(P(x) \to (\exists y)(\forall z)Q(x, y, z)\big)$

(d) $(\exists x)Q(x, y) \,\&\, P(y, x)$

(e) $(\forall x)\Big(P(x) \to (\exists y)\big(Q(y) \to (\forall z)R(y, z)\big)\Big)$

For each occurrence of a variable, we write it in blue if it is bound and red if it is free.

(a) $(\forall x)\underline{P(\textcolor{blue}{x})} \lor Q(\textcolor{red}{x}, \textcolor{red}{y})$

(b) $(\forall y)\underline{\Big(Q(\textcolor{red}{x}) \to (\forall z)\underline{P(\textcolor{blue}{y}, \textcolor{blue}{z})}\Big)}$

(c) $(\forall x)\underline{\sim \Big(P(\textcolor{blue}{x}) \to (\exists y)\underline{(\forall z)\underline{Q(\textcolor{blue}{x}, \textcolor{blue}{y}, \textcolor{blue}{z})}}\Big)}$

(d) $(\exists x)\underline{Q(\textcolor{blue}{x}, \textcolor{red}{y})} \,\&\, P(\textcolor{red}{y}, \textcolor{red}{x})$

(e) $(\forall x)\underline{\Big(P(\textcolor{blue}{x}) \to (\exists y)\underline{\Big(Q(\textcolor{blue}{y}) \to (\forall z)\underline{R(\textcolor{blue}{y}, \textcolor{blue}{z})}\Big)}\Big)}$

Problem 4 Each part of this exercise consists of an English sentence followed by a translation of it in predicate logic and a number of additional formulas. Indicate which of the formulas are equivalent to the translation and give the laws of rules necessary to show this equivalence. If a formula is not equivalent to the translation, give a rendition of it in English.

(a) Everything has a father and all odd numbers are integers.

$$(\forall x)(\exists y)F(y, x) \,\&\, \big(\forall z)(O(z) \to I(z)\big)$$

(1) $(\forall z)(\forall x)(\exists y)\Big(F(y, x) \,\&\, \big(O(z) \to I(z)\big)\Big)$

(2) $(\forall z)(\exists y)(\forall x)\Big(F(y, x) \,\&\, \big(O(z) \to I(z)\big)\Big)$

(3) $(\forall x)(\forall z)(\exists y)\Big(F(y, x) \,\&\, \big(O(z) \to I(z)\big)\Big)$

(b) If Adam is a bachelor, then not all men are husbands.

$$B(a) \to \sim (\forall x)\big(M(x) \to H(x)\big)$$

(1) $(\forall x)\Big(B(a) \to \sim \big(M(x) \to H(x)\big)\Big)$

(2) $(\exists x)\left(B(a) \to \sim \big(M(x) \to H(x)\big)\right)$

(3) $\sim \left(B(a) \to (\forall x)\big(M(x) \to H(x)\big)\right)$

(4) $B(a) \to (\exists x)\big(M(x) \,\&\, \sim H(x)\big)$

(c) If there is anything that is evil, then God is not benevolent.

$$(\exists x)E(x) \to \sim B(g)$$

(1) $\sim \big((\exists x)E(x) \,\&\, B(g)\big)$

(2) $(\forall x)\big(E(x) \to \sim B(g)\big)$

(a) (1) Equivalent. To see this, we can write

$$\begin{aligned}
&(\forall z)(\forall x)(\exists y)\left(F(y,x) \,\&\, \big(O(z) \to I(z)\big)\right) \\
\equiv&(\forall x)(\forall z)(\exists y)\left(F(y,x) \,\&\, \big(O(z) \to I(z)\big)\right) && \text{(Law 6)} \\
\equiv&(\forall x)(\forall z)\left((\exists y)F(y,x) \,\&\, (\exists y)\big(O(z) \to I(z)\big)\right) && \text{(Law 3)} \\
\equiv&(\forall x)\left((\forall z)(\exists y)F(y,x) \,\&\, (\forall z)(\exists y)\big(O(z) \to I(z)\big)\right) && \text{(Law 2)} \\
\equiv&(\forall x)(\forall z)(\exists y)F(y,x) \,\&\, (\forall x)(\forall z)(\exists y)\big(O(z) \to I(z)\big) && \text{(Law 2)} \\
\equiv&(\forall x)(\exists y)F(y,x) \,\&\, (\forall z)\big(O(z) \to I(z)\big). && \text{(dropping vacuous quantifiers)}
\end{aligned}$$

(2) Not equivalent. This is a strengthening of (1). A rendition is "something is everything's father and all odd numbers are integers."

(3) Equivalent. This is equivalent to (1) via Law 6.

(b) (1) Not equivalent. A rendition is "If Adam is a bachelor, then no man is a husband."

(2) Equivalent. To see this, we can write

$$\begin{aligned}
&(\exists x)\left(B(a) \to \sim \big(M(x) \to H(x)\big)\right) \\
\equiv&B(a) \to (\exists x) \sim \big(M(x) \to H(x)\big) && \text{(Law 10)} \\
\equiv&B(a) \to \sim (\forall x)\big(M(x) \to H(x)\big). && \text{(Law 1)}
\end{aligned}$$

(3) Not equivalent. Negating the implication, a rendition is "Adam is a bachelor and there is a man who is not a husband (which changes from the sentence from a hypothetical to a statement of current fact).

(4) Equivalent. To see this, we can write

$$\begin{aligned}
&B(a) \to (\exists x)\big(M(x) \,\&\, \sim H(x)\big) \\
\equiv&B(a) \to (\exists x) \sim \big(M(x) \to H(x)\big) && \text{(definition of implication)} \\
\equiv&B(a) \to \sim (\forall x)\big(M(x) \to H(x)\big) && \text{(Law 1)}
\end{aligned}$$

(c) (1) Equivalent. We can write

$$\begin{aligned}
&\sim \big((\exists x)E(x) \,\&\, B(g)\big) \\
\equiv&\sim (\exists x)E(x) \lor \sim B(g) && \text{(De Morgan)} \\
\equiv&\ (\exists x)E(x) \to \sim B(g) && \text{(definition of implication)}
\end{aligned}$$

(2) Equivalent. This follows directly from Law 12.

Problem 5 Find two equivalent but different formulas translating each of the sentences below, using the predicates given.

(a) For every integer, there is a larger integer.

$$I(x), L(x,y)$$

(b) Either every prime number is odd or some integers are even, or both.

$$P(x), I(x), O(x)$$

(c) If there is a prime number which is even, then all prime numbers greater than 7 are odd.

$$P(x), O(x), G(x,y)$$

(d) If all men are mortal, then Socrates is mortal.

$$H(x), M(x)$$

(a)
1. $(\forall x)\big(I(x) \to (\exists y)L(x,y)\big)$
2. $(\forall x)(\exists y)\big(I(x) \to L(x,y)\big)$

(b)
1. $(\forall x)\big(P(x) \to O(x)\big) \vee (\exists y)\big(I(y) \,\&\, \sim O(y)\big)$
2. $(\forall x)\big(P(x) \to O(x)\big) \vee \sim (\forall y)\big(I(y) \to O(y)\big)$

(c)
1. $(\exists x)\big(P(x) \,\&\, \sim O(x)\big) \to (\forall x)\Big(\big(P(x) \,\&\, G(x,7)\big) \to O(x)\Big)$
2. $\sim (\forall x)\big(P(x) \to O(x)\big) \to (\forall x)\Big(\big(P(x) \,\&\, G(x,7)\big) \to O(x)\Big)$

(d)
1. $(\forall x)\big(M(x) \to H(x)\big) \to M(\text{Socrates})$
2. $\sim M(\text{Socrates}) \to (\exists x)\big(M(x) \to \sim H(x)\big)$

Problem 6 Give the Prenex Normal Forms of these formulas:

(a) $\big((\exists x)A(x) \,\&\, (\exists x)B(x)\big) \to C(x)$

(b) $(\forall x)A(x) \leftrightarrow (\exists x)B(x)$

(a) $(\forall y)(\forall z)\Big(\big(A(y) \,\&\, B(z)\big) \to C(x)\Big)$

(b) $(\exists y)(\exists x)\big(A(x) \to B(y)\big) \,\&\, (\forall x)\big(\forall y)(B(y) \to A(x)\big)$