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MATH 544 Homework 6

Problem 1 Let $V = \left\{ \vec{x} = \begin{pmatrix} x_1 \\ x_2 \end{pmatrix} : x_2 > 0 \right\} \subset \mathbb{R}^2$. Suppose that we define two operations on V:

- For all $\vec{u} = \begin{pmatrix} u_1 \\ u_2 \end{pmatrix}$, $\vec{v} = \begin{pmatrix} v_1 \\ v_2 \end{pmatrix} \in V$, we have $\vec{u} \oplus \vec{v} = \begin{pmatrix} u_1 + v_1 \\ u_2 v_2 \end{pmatrix}$.
- For all $\vec{v} = \begin{pmatrix} v_1 \\ v_2 \end{pmatrix} \in V$ and for all $c \in \mathbb{R}$, we have $c \star \vec{v} = \begin{pmatrix} cv_1 \\ {v_2}^c \end{pmatrix}$.

Show that $(V, \oplus, \star)$ is a real vector space by verifying that the ten vector space axioms hold.

Solution.

We have defined $(V, \oplus, \star)$ as a vector space over a field F if it satisfies the following for all $\vec{u}, \vec{v}, \vec{w} \in V$ and for all $a, b \in F$:

1. $\vec{u} \oplus \vec{v} \in V$.
2. $(\vec{u} \oplus \vec{v}) \oplus \vec{w} = \vec{u} \oplus (\vec{v} \oplus \vec{w})$.
3. There exists $\vec{0_V} \in V$ such that for all $\vec{v} \in V$, we have $\vec{0_V} \oplus \vec{v} = \vec{v} = \vec{v} \oplus \vec{0_V}$.
4. For all $\vec{v} \in V$, there exists $\vec{-v} \in V$ such that $\vec{v} \oplus \vec{-v} = \vec{0_V} = \vec{-v} \oplus \vec{v}$.
5. $\vec{u} \oplus \vec{v} = \vec{v} \oplus \vec{u}$.
6. $a \star \vec{v} \in V$.
7. $a \star (b \star \vec{v}) = (ab) \star \vec{v} = b \star (a \star \vec{v})$.
8. $a \star (\vec{u} \oplus \vec{v}) = (a \star \vec{u}) \oplus (a \star \vec{v})$.
9. $(a + b) \star \vec{u} = (a \star \vec{u}) \oplus (b \star \vec{u})$.
10. For all $\vec{v} \in V$, there exists $1_F \in F$ such that $1_F \star \vec{v} = \vec{v}$.

Let

$$\vec{u} = \begin{pmatrix} u_1 \\ u_2 \end{pmatrix}, \vec{v} = \begin{pmatrix} v_1 \\ v_2 \end{pmatrix}, \vec{w} = \begin{pmatrix} w_1 \\ w_2 \end{pmatrix},$$

where, $u_1, v_1, w_1 \in \mathbb{R}$ and $u_2, v_2, w_2 \in \mathbb{R}^+$. By definition, these vectors are in V. Let $a, b \in F$, where $F = \mathbb{R}$. We will prove that the axioms are satisfied.

1. We have $\vec{u} \oplus \vec{v} = \begin{pmatrix} u_1 + v_1 \\ u_2 v_2 \end{pmatrix}$. Since $u_1, v_1 \in \mathbb{R} \implies u_1 + v_1 \in \mathbb{R}$ and $u_2 > 0, v_2 > 0 \implies u_2 v_2 > 0$, we have $\vec{u} \oplus \vec{v} \in V$.

2. We have

$$\begin{aligned}(\vec{u}\oplus\vec{v})\oplus\vec{w} &= \begin{pmatrix}u_1+v_1\\ u_2v_2\end{pmatrix}\oplus\begin{pmatrix}w_1\\ w_2\end{pmatrix}\\ &= \begin{pmatrix}(u_1+v_1)+w_1\\ (u_2v_2)w_2\end{pmatrix}\\ &= \begin{pmatrix}u_1+(v_1+w_1)\\ u_2(v_2w_2)\end{pmatrix} && \text{(associativity)}\\ &= \begin{pmatrix}u_1\\ u_2\end{pmatrix}\oplus\begin{pmatrix}v_1+w_1\\ v_2w_2\end{pmatrix}\\ &= \vec{u}\oplus(\vec{v}\oplus\vec{w}).\end{aligned}$$

3. Let $\vec{0_V}=\begin{pmatrix}0\\1\end{pmatrix}$, which is in V. Then, we have

$$\begin{pmatrix}0\\1\end{pmatrix}\oplus\begin{pmatrix}v_1\\v_2\end{pmatrix}=\begin{pmatrix}0+v_1\\1(v_2)\end{pmatrix}=\begin{pmatrix}v_1\\v_2\end{pmatrix}=\begin{pmatrix}v_1+0\\(v_2)1\end{pmatrix}=\begin{pmatrix}v_1\\v_2\end{pmatrix}\oplus\begin{pmatrix}0\\1\end{pmatrix},$$

so $\vec{0_V}\oplus\vec{v}=\vec{v}=\vec{v}+\vec{0_V}$.

4. Let $\vec{-v}=\begin{pmatrix}-v_1\\ \frac{1}{v_2}\end{pmatrix}$ (which is well defined since $0\notin\mathbb{R}^+$), and recall that we have established $\vec{0_V}=\begin{pmatrix}0\\1\end{pmatrix}$. Then, we have

$$\begin{pmatrix}v_1\\v_2\end{pmatrix}\oplus\begin{pmatrix}-v_1\\ \frac{1}{v_2}\end{pmatrix}=\begin{pmatrix}v_1-v_1\\ v_2\left(\frac{1}{v_2}\right)\end{pmatrix}=\begin{pmatrix}0\\1\end{pmatrix}=\begin{pmatrix}-v_1+v_1\\ \left(\frac{1}{v_2}\right)v_2\end{pmatrix}=\begin{pmatrix}-v_1\\ \frac{1}{v_2}\end{pmatrix}\oplus\begin{pmatrix}v_1\\v_2\end{pmatrix},$$

so $\vec{v}\oplus\vec{-v}=\vec{0_V}=\vec{-v}\oplus\vec{v}$.

5. We have

$$\vec{u}\oplus\vec{v}=\begin{pmatrix}u_1\\u_2\end{pmatrix}\oplus\begin{pmatrix}v_1\\v_2\end{pmatrix}=\begin{pmatrix}u_1+v_1\\u_2v_2\end{pmatrix}=\begin{pmatrix}v_1+u_1\\v_2u_2\end{pmatrix}=\begin{pmatrix}v_1\\v_2\end{pmatrix}\oplus\begin{pmatrix}u_1\\u_2\end{pmatrix}=\vec{v}\oplus\vec{u}.$$

6. We have $a\star\vec{v}=\begin{pmatrix}av_1\\ {v_2}^a\end{pmatrix}$. Since $a,v_1\in\mathbb{R}\implies av_1\in\mathbb{R}$ and $a\in\mathbb{R},v_2\in\mathbb{R}^+\implies {v_2}^a\in\mathbb{R}^+$, we have $a\star\vec{v}\in V$.

7. First, we have

$$\begin{aligned}a\star\left(b\star\begin{pmatrix}v_1\\v_2\end{pmatrix}\right) &= a\star\begin{pmatrix}bv_1\\ {v_2}^b\end{pmatrix}\\ &= \begin{pmatrix}a(bv_1)\\ \left({v_2}^b\right)^a\end{pmatrix}\\ &= \begin{pmatrix}(ab)v_1\\ \left({v_2}^b\right)^a\end{pmatrix} && \text{(associativity)}\\ &= \begin{pmatrix}(ab)v_1\\ {v_2}^{ab}\end{pmatrix} && \text{(exponent rule/commutativity)}\end{aligned}$$

$$= (ab) \star \begin{pmatrix} v_1 \\ v_2 \end{pmatrix}.$$

We also have

$$\begin{aligned}
(ab) \star \begin{pmatrix} v_1 \\ v_2 \end{pmatrix} &= \begin{pmatrix} (ab)v_1 \\ {v_2}^{ab} \end{pmatrix} \\
&= \begin{pmatrix} b(av_1) \\ {v_2}^{ab} \end{pmatrix} && \text{(commutativity/associativity)} \\
&= \begin{pmatrix} b(av_1) \\ ({v_2}^{a})^{b} \end{pmatrix} && \text{(exponent rule)} \\
&= b \star \begin{pmatrix} av_1 \\ {v_2}^{a} \end{pmatrix} \\
&= b \star \left(a \star \begin{pmatrix} v_1 \\ v_2 \end{pmatrix} \right).
\end{aligned}$$

So $a \star (b \star \vec{v}) = (ab) \star \vec{v} = b \star (a \star \vec{v})$.

8. We have

$$\begin{aligned}
a \star (\vec{u} \oplus \vec{v}) &= a \star \left(\begin{pmatrix} u_1 \\ u_2 \end{pmatrix} \oplus \begin{pmatrix} v_1 \\ v_2 \end{pmatrix} \right) \\
&= a \star \begin{pmatrix} u_1 + v_1 \\ u_2 v_2 \end{pmatrix} \\
&= \begin{pmatrix} a(u_1 + v_1) \\ (u_2 v_2)^a \end{pmatrix} \\
&= \begin{pmatrix} au_1 + av_1 \\ {u_2}^a {v_2}^a \end{pmatrix} && \text{(distributing coefficient/exponent)} \\
&= \begin{pmatrix} au_1 \\ {u_2}^a \end{pmatrix} \oplus \begin{pmatrix} av_1 \\ {v_2}^a \end{pmatrix} \\
&= \left(a \star \begin{pmatrix} u_1 \\ u_2 \end{pmatrix} \right) \oplus \left(a \star \begin{pmatrix} v_1 \\ v_2 \end{pmatrix} \right) \\
&= (a \star \vec{u}) \oplus (a \star \vec{v}).
\end{aligned}$$

9. We have

$$\begin{aligned}
(a + b) \star \vec{u} &= (a + b) \star \begin{pmatrix} u_1 \\ u_2 \end{pmatrix} \\
&= \begin{pmatrix} (a + b)u_1 \\ {u_2}^{a+b} \end{pmatrix} \\
&= \begin{pmatrix} au_1 + bu_1 \\ {u_2}^a {u_2}^b \end{pmatrix} && \text{(distributivity/exponent rules)} \\
&= \begin{pmatrix} au_1 \\ {u_2}^a \end{pmatrix} \oplus \begin{pmatrix} bu_1 \\ {u_2}^b \end{pmatrix}
\end{aligned}$$

$$= \left(a \star \begin{pmatrix} u_1 \\ u_2 \end{pmatrix}\right) \oplus \left(b \star \begin{pmatrix} u_1 \\ u_2 \end{pmatrix}\right)$$
$$= (a \star \vec{u}) \oplus (b \star \vec{u}).$$

10. Let $1_F = 1$. Then, we have

$$1_F \star \vec{v} = 1 \star \begin{pmatrix} v_1 \\ v_2 \end{pmatrix} = \begin{pmatrix} 1(v_1) \\ {v_2}^1 \end{pmatrix} = \begin{pmatrix} v_1 \\ v_2 \end{pmatrix} = \vec{v}.$$

If the reader has not passed away from boredom pages ago, we can see that $(V, \oplus, \star)$ satisfies the axioms for $F = \mathbb{R}$, and therefore it is a real vector space. $\square$

Problem 2 Briefly explain why each of the following subsets W are not subspaces of the vector spaces V.

(a) $W = \left\{ \begin{pmatrix} x_1 \\ x_2 \end{pmatrix} : x_1 \text{ is a rational number} \right\}$, $V = \mathbb{R}^2$.

(b) $W = \left\{ \begin{pmatrix} x_1 \\ x_2 \\ x_3 \end{pmatrix} : x_1 - x_2 + x_3 = 0 \ \textbf{ or } \ x_2 - x_3 = 0 \right\}$, $V = \mathbb{R}^3$.

(c) $W = \{p(x) \in \mathbb{R}_2(x) : p(1)p(3) = 0\}$, $V = \mathbb{R}_2[x]$.

Solution.

(a) Let $k = \pi, \vec{u} = \begin{pmatrix} 1 \\ 0 \end{pmatrix}, \vec{v} = \begin{pmatrix} 0 \\ 0 \end{pmatrix}$. Then, we have $\vec{u}, \vec{v} \in W$ because $0, 1 \in \mathbb{Q}$, but $k\vec{u} + \vec{v} = \begin{pmatrix} \pi \\ 0 \end{pmatrix} \notin W$ because $\pi \notin \mathbb{Q}$. So W cannot be a subspace of V.

(b) Let $k = 1, \vec{u} = \begin{pmatrix} 1 \\ 1 \\ 0 \end{pmatrix}, \vec{v} = \begin{pmatrix} 1 \\ 1 \\ 1 \end{pmatrix}$. Then, we have $\vec{u} \in W$ because $1 - 1 + 0 = 0$ and $\vec{v} \in W$ because $1 - 1 = 0$, but $k\vec{u} + \vec{v} = \begin{pmatrix} 2 \\ 2 \\ 1 \end{pmatrix} \notin W$ because $2 - 2 + 1 \neq 0$ and $2 - 1 \neq 0$. So W cannot be a subspace of W.

(c) Let $k = 1, p(x) = x - 1, q(x) = x - 3$. Then, we have $p(x) \in W$ because $p(1)p(3) = (0)(2) = 0$ and $q(x) \in W$ because $q(1)q(3) = (-2)(0) = 0$, but $g(x) = p(x) + q(x) = 2x - 4 \notin W$ because $g(1)g(3) = (-2)(2) = -4 \neq 0$. So W cannot be a subspace of W.

Problem 3 Let

$$W = \{A \in \mathrm{Mat}_{3\times 3} : (A)_{11} + (A)_{22} + (A)_{33} = 0,\ (A)_{12} + (A)_{23} = 0,\ (A)_{21} + (A)_{32} = 0\}.$$

(a) Show that W is a subspace of $\mathrm{Mat}_{3\times 3}$.

(b) Find a set $S \subset W$ with $|S|$ as small as possible such that $W = \mathrm{Span}(S)$.

Solution.

(a) First, we note that $O_{3\times 3} \in W$ because $0+0+0=0$ and $0+0=0$. Next, let

$$k \in \mathbb{R}, A = \begin{pmatrix} a_{11} & a_{12} & a_{13} \\ a_{21} & a_{22} & a_{23} \\ a_{31} & a_{32} & a_{33} \end{pmatrix} \in W, B = \begin{pmatrix} b_{11} & b_{12} & b_{13} \\ b_{21} & b_{22} & b_{23} \\ b_{31} & b_{32} & b_{33} \end{pmatrix} \in W,$$

and consider

$$kA + B = \begin{pmatrix} ka_{11}+b_{11} & ka_{12}+b_{12} & ka_{13}+b_{13} \\ ka_{21}+b_{21} & ka_{22}+b_{22} & ka_{23}+b_{23} \\ ka_{31}+b_{31} & ka_{32}+b_{32} & ka_{33}+b_{33} \end{pmatrix}.$$

We have

$$\begin{aligned} & 0 = a_{11}+a_{22}+a_{33} && (A \in W) \\ \implies & 0 = ka_{11}+ka_{22}+ka_{33} && \text{(algebra)} \\ \implies & 0 = (ka_{11}+ka_{22}+ka_{33}) + (b_{11}+b_{22}+b_{33}) && (B \in W) \\ \implies & 0 = ka_{11}+b_{11}+ka_{22}+b_{22}+ka_{33}+b_{33} && \text{(commutatitivity)} \\ \implies & 0 = (kA+B)_{11} + (kA+B)_{22} + (kA+B)_{33}. \end{aligned}$$

By similar arguments,

$$(kA+B)_{12} + (kA+B)_{23} = 0 \text{ and } (kA+B)_{21} + (kA+B)_{32} = 0.$$

So W is a subspace of V.

(b) Let

$$S = \left\{ \begin{pmatrix} 0 & 0 & 1 \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{pmatrix}, \begin{pmatrix} 0 & 0 & 0 \\ 0 & 0 & 0 \\ 1 & 0 & 0 \end{pmatrix}, \begin{pmatrix} 0 & 1 & 0 \\ 0 & 0 & -1 \\ 0 & 0 & 0 \end{pmatrix}, \begin{pmatrix} 0 & 0 & 0 \\ 1 & 0 & 0 \\ 0 & -1 & 0 \end{pmatrix}, \begin{pmatrix} 1 & 0 & 0 \\ 0 & -1 & 0 \\ 0 & 0 & 0 \end{pmatrix}, \begin{pmatrix} 1 & 0 & 0 \\ 0 & 0 & 0 \\ 0 & 0 & -1 \end{pmatrix} \right\}.$$

The first two matrices allow a_{13} and a_{31} to be anything. The next two matrices ensure the condition that $a_{12}+a_{23} = a_{21}+a_{32} = 0$. The last two matrices ensure that $a_{11}+a_{22}+a_{33} = 0$. Thus, $\text{Span}(S) = W$, and S cannot be smaller because then it would not ensure these conditions.

Problem 4 Let $W = \{p(x) \in \mathbb{R}_3[x] : p(1) = p(-1),\ p(2) = p(-2)\}$.

(a) Show that W is a subspace of $\mathbb{R}_3[x]$.

(b) Find a set $S \subset W$ with $|S|$ as small as possible such that $W = \text{Span}(S)$.

Solution.

(a) First, we note that the zero function is in W, because the degree is $-\infty \leq 3$, and because the value is equal everywhere (in particular, for $x \in \{1, -1, 2, -2\}$). Next, let

$$k \in \mathbb{R}, \quad p(x) = p_3x^3 + p_2x^2 + p_1x + p_0 \in W, \quad q(x) = q_3x^3 + q_2x^2 + q_1x + q_0 \in W.$$

Since $p(x) \in W$, we have

$$p(1) = p(-1)$$

$$
\begin{aligned}
&\implies p_3(1)^3 + p_2(1)^2 + p_1(1) + p_0 = p_3(-1)^3 + p_2(-1)^2 + p_1(-1) + p_0 \\
&\implies p_3 + p_2 + p_1 + p_0 = -p_3 + p_2 - p_1 + p_0 \\
&\implies p_3 + p_1 = -p_3 - p_1 \\
&\implies p_3 + p_1 = 0. \\
p(2) &= p(-2) \\
&\implies p_3(2)^3 + p_2(2)^2 + p_1(2) + p_0 = p_3(-2)^3 + p_2(-2)^2 + p_1(-2) + p_0 \\
&\implies 8p_3 + 4p_2 + 2p_1 + p_0 = -8p_3 + 4p_2 - 2p_1 + p_0 \\
&\implies 4p_3 + p_1 = -4p_3 - p_1 \\
&\implies 4p_3 + p_1 = 0.
\end{aligned}
$$

Using elimination, we find that $3p_3 = 0$, so $p_3 = p_1 = 0$ and $p(x) = p_2x^2 + p_0$. By the same argument, we have $q(x) = q_2x^2 + q_0$. So

$$kp(x) + q(x) = k(p_2x^2 + p_0) + (q_2x^2 + q_0) = (kp_2 + q_2)x^2 + (kp_0 + q_0).$$

Thus, we have

$$kp(1) + q(1) = (kp_2 + q_2)(1)^2 + (kp_0 + q_0) = (kp_2 + q_2)(-1)^2 + (kp_0 + q_0) = kp(-1) + q(-1)$$

and

$$kp(2) + q(2) = (kp_2 + q_2)(2)^2 + (kp_0 + q_0) = (kp_2 + q_2)(-2)^2 + (kp_0 + q_0) = kp(-2) + q(-2),$$

so $kp(x) + q(x) \in W$. Therefore, W is a subspace of $\mathbb{R}_3[x]$.

(b) Any polynomial in W has only a degree 2 term and a constant term. Thus, $S = \{x^2, 1\}$ is a subset of W (both polynomials have degree ≤ 3) and any polynomial in W can be expressed as a linear combination of elements in S. So $W = \text{Span}(S)$.

Problem 5 Let $A \in \text{Mat}_{m\times n}$, and let $V \in \mathbb{R}^n$ be a subspace of $\mathbb{R}^n$.

(a) Show that $W = \{\vec{y} \in \mathbb{R}^m : \exists\, \vec{x} \in V \text{ such that } \vec{y} = A\vec{x}\}$ is a subspace of $\mathbb{R}^m$.

(b) Now, suppose that $V = \mathbb{R}^n$, and let W be as in part (a). Show that $W = \text{Col}(A)$. When $V = \mathbb{R}^n$, the subspace W is sometimes called the **range** of A. This part of the problem shows that the range of A and the column space of A coincide.

Note: Suppose that we define a map $T : V \longrightarrow \mathbb{R}^m$ by $T(\vec{x}) = A\vec{x}$. The set W in the problem is the *range* of the map T. (The subspace V is the *domain* of T.) Therefore, this problem shows that the range of the map defined by multiplication with the matrix A is a subspace of $\mathbb{R}^m$. We will discuss maps of this type in Chapters 6 and 7.

Solution.

(a) Let $k \in \mathbb{R}$ and $\vec{y_1}, \vec{y_2} \in W$. Then, there exist $\vec{x_1}, \vec{x_2} \in V$ such that $\vec{y_1} = A\vec{x_1}$ and $\vec{y_2} = A\vec{x_2}$. So we have

$$
\begin{aligned}
k\vec{y_1} + \vec{y_2} &= kA\vec{x_1} + A\vec{x_2} \\
&= Ak\vec{x_1} + A\vec{x_2} && \text{(commutativity)} \\
&= A(k\vec{x_1} + \vec{x_2}). && \text{(distributivity)}
\end{aligned}
$$

Since V is a vector space, we have $k\vec{x_1} + \vec{x_2} \in V$. Then by definition, $k\vec{y_1} + \vec{y_2} = A(k\vec{x_1} + \vec{x_2}) \in W$. So W is a subspace of $\mathbb{R}^m$.

(b) Let

$$A = \begin{pmatrix} a_{11} & a_{12} & a_{13} & \dots & a_{1n} \\ a_{21} & a_{22} & a_{23} & \dots & a_{2n} \\ a_{31} & a_{32} & a_{33} & \dots & a_{3n} \\ \vdots & \vdots & \vdots & \ddots & \vdots \\ a_{m1} & a_{m2} & a_{m3} & \dots & a_{mn} \end{pmatrix}.$$

Let $\vec{y} \in W$. So there exists $\vec{x} \in V$ such that $\vec{y} = A\vec{x}$. Let $\vec{x} = \begin{pmatrix} x_1 \\ x_2 \\ x_3 \\ \vdots \\ x_n \end{pmatrix}$. Then, we have

$$A\vec{x} = \begin{pmatrix} x_1a_{11} + x_2a_{12} + x_3a_{13} + \cdots + x_na_{1n} \\ x_1a_{21} + x_2a_{22} + x_3a_{23} + \cdots + x_na_{2n} \\ x_1a_{31} + x_2a_{32} + x_3a_{33} + \cdots + x_na_{3n} \\ \vdots \\ x_1a_{m1} + x_2a_{m2} + x_3a_{m3} + \cdots + x_na_{mn} \end{pmatrix} = x_1 \begin{pmatrix} a_{11} \\ a_{21} \\ a_{31} \\ \vdots \\ a_{m1} \end{pmatrix} + x_2 \begin{pmatrix} a_{12} \\ a_{22} \\ a_{32} \\ \vdots \\ a_{m2} \end{pmatrix} + x_3 \begin{pmatrix} a_{13} \\ a_{23} \\ a_{33} \\ \vdots \\ a_{m3} \end{pmatrix} + \cdots + x_n \begin{pmatrix} a_{1n} \\ a_{2n} \\ a_{3n} \\ \vdots \\ a_{mn} \end{pmatrix}.$$

Thus, we can write $\vec{y} = A\vec{x}$ as a linear combination of the columns of A, so $y \in \text{Col}(A)$. Therefore, $W \subseteq \text{Col}(A)$.

Next, let $\vec{y} \in \text{Col}(A)$. Then, there exist $x_1, x_2, x_3, \dots, x_n \in \mathbb{R}$ such that

$$\vec{y} = x_1 \begin{pmatrix} a_{11} \\ a_{21} \\ a_{31} \\ \vdots \\ a_{m1} \end{pmatrix} + x_2 \begin{pmatrix} a_{12} \\ a_{22} \\ a_{32} \\ \vdots \\ a_{m2} \end{pmatrix} + x_3 \begin{pmatrix} a_{13} \\ a_{23} \\ a_{33} \\ \vdots \\ a_{m3} \end{pmatrix} + \cdots + x_n \begin{pmatrix} a_{1n} \\ a_{2n} \\ a_{3n} \\ \vdots \\ a_{mn} \end{pmatrix}.$$

Distributing, we can write this as

$$\vec{y} = \begin{pmatrix} x_1a_{11} + x_2a_{12} + x_3a_{13} + \cdots + x_na_{1n} \\ x_1a_{21} + x_2a_{22} + x_3a_{23} + \cdots + x_na_{2n} \\ x_1a_{31} + x_2a_{32} + x_3a_{33} + \cdots + x_na_{3n} \\ \vdots \\ x_1a_{m1} + x_2a_{m2} + x_3a_{m3} + \cdots + x_na_{mn} \end{pmatrix} = \begin{pmatrix} a_{11} & a_{12} & a_{13} & \dots & a_{1n} \\ a_{21} & a_{22} & a_{23} & \dots & a_{2n} \\ a_{31} & a_{32} & a_{33} & \dots & a_{3n} \\ \vdots & \vdots & \vdots & \ddots & \vdots \\ a_{m1} & a_{m2} & a_{m3} & \dots & a_{mn} \end{pmatrix} \begin{pmatrix} x_1 \\ x_2 \\ x_3 \\ \vdots \\ x_n \end{pmatrix} = A\vec{x},$$

where $\vec{x} = \begin{pmatrix} x_1 \\ x_2 \\ x_3 \\ \vdots \\ x_n \end{pmatrix}$. So by definition, $y \in W$, and therefore $\text{Col}(A) \subseteq W$. So we have $W = \text{Col}(A)$. $\square$

Problem 6 Let $A \in \mathrm{Mat}_{n\times n}$. Verify that $W = \{\vec{x} \in \mathbb{R}^n : A\vec{x} = 3\vec{x}\}$ is a subspace of $\mathbb{R}^n$.

Note: Suppose that there exists $\vec{x} \neq \vec{0}$ in W. Then we say that $\vec{x}$ is an **eigenvector** for A with **eigenvalue** 3. The subspace W is the **eigenspace** of A associated to the eigenvector 3. We will discuss these objects in Chapter 9.

Solution.

Let $k \in \mathbb{R}$ and $\vec{u}, \vec{v} \in W$. So we have $A\vec{u} = 3\vec{u}$ and $A\vec{v} = 3\vec{v}$, and thus $\vec{u} = \frac{1}{3}A\vec{u}$ and $\vec{v} = \frac{1}{3}A\vec{v}$. Then,

$$\begin{aligned}
k\vec{u} + \vec{v} &= k\left(\frac{1}{3}A\vec{u}\right) + \frac{1}{3}A\vec{v} \\
&= \frac{Ak\vec{u} + A\vec{v}}{3} && \text{(commutativity of scalars)} \\
&= \frac{A(k\vec{u} + \vec{v})}{3} && \text{(distributivity)} \\
\implies 3(k\vec{u} + \vec{v}) &= A(k\vec{u} + \vec{v})
\end{aligned}$$

So by definition, $k\vec{u} + \vec{v} \in W$ and therefore W is a subspace of $\mathbb{R}^n$.

Problem 9 Let W_1 and W_2 be subspaces of a real vector space, V. We showed that

$$W_1 + W_2 = \{\vec{w}_1 + \vec{w}_2 : \vec{w}_1 \in W_1 \text{ and } \vec{w}_2 \in W_2\}$$

and

$$W_1 \cap W_2 = \{\vec{v} \in V : \vec{v} \in W_1 \text{ and } \vec{v} \in W_2\}$$

are subspaces of V.

Suppose that $W_1 + W_2 = V$ and that $W_1 \cap W_2 = \{\vec{0}\}$. Prove, for every vector $\vec{v} \in V$, that there are *unique* vectors $\vec{w}_1 \in W_1$ and $\vec{w}_2 \in W_2$ such that $\vec{v} = \vec{w}_1 + \vec{w}_2$.

Note: We say that V is the **direct sum** of W_1 and W_2, and we write $V = W_1 \bigoplus W_2$.

Solution.

Existence: Let $\vec{v} \in V$. Since $V = W_1 + W_2$, there must be some vectors $\vec{w_1} \in W_1$ and $\vec{w_2} \in W_2$ such that $\vec{v} = \vec{w_1} + \vec{w_2}$, or v would not be in $W_1 + W_2$, contradicting equality.

Uniqueness: Suppose we have $\vec{w_1}, \vec{w_1'} \in W_1$ and $\vec{w_2}, \vec{w_2'} \in W_2$ such that $\vec{v} = \vec{w_1} + \vec{w_2} = \vec{w_1'} + \vec{w_2'}$. Rearranging, we have $\vec{w_1} - \vec{w_1'} = \vec{w_2'} - \vec{w_2} = \vec{v'}$ for some $\vec{v'} \in V$. Because of closure, we must have $\vec{v'} \in W_1$ because it is the sum of vectors in W_1, and $\vec{v'} \in W_2$ because it is the sum of vectors in W_2. The only vector in both W_1 and W_2 is $\vec{0}$, so we have $\vec{v'} = \vec{0} = \vec{w_1} - \vec{w_1'} = \vec{w_2'} - \vec{w_2}$. This implies that $\vec{w_1} = \vec{w_1'}$ and $\vec{w_2} = \vec{w_2'}$. □