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MATH 546: Section
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MATH 546 Homework 10

Problem 1 Find a subgroup of S_4 that is isomorphic to $\mathbb{Z}_2 \times \mathbb{Z}_2$ by carrying through the procedure we used to prove Cayley's theorem. Show work.

Solution.

We have

$$\mathbb{Z}_2 \times \mathbb{Z}_2 = \left\{ \left([0]_2, [0]_2\right), \left([0]_2, [1]_2\right), \left([1]_2, [0]_2\right), \left([1]_2, [1]_2\right) \right\},$$

which we will label z_1, z_2, z_3, z_4, respectively. For all $z \in \mathbb{Z}_2 \times \mathbb{Z}_2$, define $\sigma_z : \{1,2,3,4\} \to \{1,2,3,4\}$ by $\sigma_z(i) = j$ if and only if $z + z_i = z_j$. Then, from the proof of Cayley's theorem, we have that

$$H = \{\sigma_{z_1}, \sigma_{z_2}, \sigma_{z_3}, \sigma_{z_4}\}$$

is a subgroup of S_n and is isomorphic to $\mathbb{Z}_2 \times \mathbb{Z}_2$ with the isomorphism $\phi : \mathbb{Z}_2 \times \mathbb{Z}_2 \to H$ defined by $\phi(z) = \sigma_z$.

We now compute the four permutations. One can check that the following satisfy the definition of σ_z for each $z \in \mathbb{Z}_2 \times \mathbb{Z}_2$:

- $\sigma_{\left([0]_2,[0]_2\right)} = \begin{pmatrix} 1 & 2 & 3 & 4 \\ 1 & 2 & 3 & 4 \end{pmatrix} = (1)$
- $\sigma_{\left([0]_2,[1]_2\right)} = \begin{pmatrix} 1 & 2 & 3 & 4 \\ 2 & 1 & 4 & 3 \end{pmatrix} = (1\ 2)(3\ 4)$
- $\sigma_{\left([1]_2,[0]_2\right)} = \begin{pmatrix} 1 & 2 & 3 & 4 \\ 3 & 4 & 1 & 2 \end{pmatrix} = (1\ 3)(2\ 4)$
- $\sigma_{\left([1]_2,[1]_2\right)} = \begin{pmatrix} 1 & 2 & 3 & 4 \\ 4 & 3 & 2 & 1 \end{pmatrix} = (1\ 4)(2\ 3).$

So

$$H = \left\{(1), (1\ 2)(3\ 4), (1\ 3)(2\ 4), (1\ 4)(2\ 3)\right\}$$

is isomorphic to $\mathbb{Z}_2 \times \mathbb{Z}_2$. (As a sanity check, we note that we proved on the last midterm that this set is a subgroup of A_4, so it's also one of S_4.) $\square$

Problem 2 Cayley's theorem tells us that there exists a subgroup of S_6 that is isomorphic to $\mathbb{Z}_6$.

(a) Give an example of such a subgroup and justify the isomorphism.

(b) Does there exists any $n < 6$ such that $\mathbb{Z}_6$ is isomorphic to a subgroup of S_n? Find the smallest such value of n.

Solution.

(a) We have

$$\mathbb{Z}_6 = \{[0]_6, [1]_6, [2]_6, [3]_6, [4]_6, [5]_6\},$$

which we will label $z_1, z_2, z_3, z_4, z_5, z_6$, respectively. For all $z \in \mathbb{Z}_6$, define $\sigma_z : \{1, \ldots, 6\} \to \{1, \ldots, 6\}$ by $\sigma_z(i) = j$ if and only if $z + z_i = z_j$. Then, from the proof of Cayley's theorem, we have that $H = \{\sigma_z : z \in \mathbb{Z}_6\}$ is a subgroup of S_n and is isomorphic to $\mathbb{Z}_6$ with the isomorphism $\phi : \mathbb{Z}_6 \to H$ defined by $\phi(z) = \sigma_z$.

We now compute the six permutations. One can check that the following satisfy the definition of σ_z for each $z \in \mathbb{Z}_6$:

- $\sigma_{[0]_6} = \begin{pmatrix} 1 & 2 & 3 & 4 & 5 & 6 \\ 1 & 2 & 3 & 4 & 5 & 6 \end{pmatrix} = (1)$
- $\sigma_{[1]_6} = \begin{pmatrix} 1 & 2 & 3 & 4 & 5 & 6 \\ 2 & 3 & 4 & 5 & 6 & 1 \end{pmatrix} = (1\ 2\ 3\ 4\ 5\ 6)$
- $\sigma_{[2]_6} = \begin{pmatrix} 1 & 2 & 3 & 4 & 5 & 6 \\ 3 & 4 & 5 & 6 & 1 & 2 \end{pmatrix} = (1\ 3\ 5)(2\ 4\ 6)$
- $\sigma_{[3]_6} = \begin{pmatrix} 1 & 2 & 3 & 4 & 5 & 6 \\ 4 & 5 & 6 & 1 & 2 & 3 \end{pmatrix} = (1\ 4)(2\ 5)(3\ 6)$
- $\sigma_{[4]_6} = \begin{pmatrix} 1 & 2 & 3 & 4 & 5 & 6 \\ 5 & 6 & 1 & 2 & 3 & 4 \end{pmatrix} = (1\ 5\ 3)(2\ 6\ 4)$
- $\sigma_{[5]_6} = \begin{pmatrix} 1 & 2 & 3 & 4 & 5 & 6 \\ 6 & 1 & 2 & 3 & 4 & 5 \end{pmatrix} = (1\ 6\ 5\ 4\ 3\ 2).$

So

$$H = \{(1), (1\ 2\ 3\ 4\ 5\ 6), (1\ 3\ 5)(2\ 4\ 6), (1\ 4)(2\ 5)(3\ 6), (1\ 5\ 3)(2\ 6\ 4), (1\ 6\ 5\ 4\ 3\ 2)\}$$

is isomorphic to $\mathbb{Z}_6$. □

(b) There is one for $n = 5$, but not smaller. The element $\sigma = (1\ 2)(3\ 4\ 5)$ has order 6 in S_5 because $\mathrm{lcm}(2, 3) = 6$, so $\langle \sigma \rangle$ has order 6. Thus, it is isomorphic to $\mathbb{Z}_6$ since $\mathbb{Z}_6$ has generator $[1]_6$, as we've proved before that cyclic groups with the same order are isomorphic. (This also would have worked in S_6, but tragically we didn't think about that until after finishing part (a).)

However, no smaller n works. If there were such a subgroup H of S_n, it would be cyclic since $\mathbb{Z}_6$ is cyclic, and thus an element would need to have order 6 in H. However, no element in S_n for any $n < 5$ has order 6, which can be seen by considering the decomposition types, so no such cyclic subgroup can exist. □

Problem 3 For the group G and the subgroup H, list all the cosets with respect to H. For each coset, list the elements of the coset. How many distinct cosets are there?

(a) $G = S_3$, $H = \{e, (1\ 2)\}$

(b) $G = \mathbb{Z}_4 \times \mathbb{Z}_4$, $H = \langle([1]_4, [1]_4)\rangle$

Solution.

(a) We have the following cosets:

1. $eH = \{e, (1\ 2)\}$
2. $(2\ 3)H = \{(2\ 3), (1\ 3\ 2)\}$
3. $(1\ 3)H = \{(1\ 3), (1\ 2\ 3)\}$

This is exhaustive since the 3 cosets contain 6 elements between them, and $|S_3| = 3! = 6$.

(b) We have the following cosets:

1. $([0]_4, [0]_4) + H = \{([1]_4, [1]_4), ([2]_4, [2]_4), ([3]_4, [3]_4), ([0]_4, [0]_4)\}$
2. $([0]_4, [1]_4) + H = \{([1]_4, [2]_4), ([2]_4, [3]_4), ([3]_4, [0]_4), ([0]_4, [1]_4)\}$
3. $([0]_4, [2]_4) + H = \{([1]_4, [3]_4), ([2]_4, [0]_4), ([3]_4, [1]_4), ([0]_4, [2]_4)\}$
4. $([0]_4, [3]_4) + H = \{([1]_4, [0]_4), ([2]_4, [1]_4), ([3]_4, [2]_4), ([0]_4, [3]_4)\}$

This is exhaustive since the 4 cosets contain 16 elements between them, and $|\mathbb{Z}_4 \times \mathbb{Z}_4| = (4)(4) = 16$.

Problem 4 For the group G and the subgroup H, decide whether H is a normal subgroup of G or not. Prove your answers.

(a) $G = S_3$, $H = \{e, (1\ 2)\}$

(b) $G = S_4$, $H = A_4$

Solution.

(a) H is not a normal subgroup. Consider $h = (1\ 2) \in H$ and $g = (1\ 3) \in G$. Then $g^{-1} = (1\ 3)$, and we have $ghg^{-1} = (1\ 3)(1\ 2)(1\ 3) = (2\ 3)$, which is not in H.

(b) H is a normal subgroup. Let $h \in A_4$ and $g \in S_4$, and let $t_1, t_2, \ldots, t_k$ and $\tau_1, \tau_2, \ldots, \tau_m$ be transpositions such that

$$h = \tau_1\tau_2 \ldots \tau_m,\ g = t_1 t_2 \ldots t_k.$$

Since a transposition is its own inverse, we have $g^{-1} = t_k t_{k-1} \ldots t_1$. So ghg^{-1} is composed of $2k + m$ transpositions, and since $h \in A_4$, we have that m is even. So we have that $2k + m$ is even, so ghg^{-1} is even and therefore it is in A_4. $\square$

Problem 5 Let $G = \mathbb{Z}_4 \times \mathbb{Z}_6$, and let $H = \langle([1]_4, [0]_6)\rangle$. Consider the factor group G/H.

(a) What is the order of the element $([1]_4, [2]_6)$ as an element of G?

(b) What is the order of the element $([1]_4, [2]_6) + H$ as an element of G/H?

Solution.

(a) The order of $[1]_4$ in $\mathbb{Z}_4$ is 4, and the order of $[2]_6$ in $\mathbb{Z}_6$ is 3, so the order of $([1]_4, [2]_6)$ in G is $\text{lcm}(4, 3) = 12$.

(b) We have

$$H = \{([1]_4, [0]_6), ([2]_4, [0]_6), ([3]_4, [0]_6), ([0]_4, [0]_6)\},$$

so we can choose $([0]_4, [0]_6)$ as the representative from H and $([1]_4, [2]_6)$ as the representative from $([1]_4, [2]_6) + H$. We observe that

- $1\left[([1]_4, [2]_6) + ([0]_4, [0]_6)\right] = ([1]_4, [2]_6)$
- $2\left[([1]_4, [2]_6) + ([0]_4, [0]_6)\right] = ([2]_4, [4]_6)$
- $3\left[([1]_4, [2]_6) + ([0]_4, [0]_6)\right] = ([3]_4, [0]_6)$,

and that $([1]_4, [2]_6), ([2]_4, [4]_6) \notin H$ but $([3]_4, [0]_6) \in H$. So the coset of H with respect to $([3]_4, [0]_6)$ is H, which is the identity of H/G. Therefore, since the least n where the coset of H with respect to $n\left[([1]_4, [2]_6) + ([0]_4, [0]_6)\right]$ is equal to H is $n = 3$, we have that the order of $([1]_4, [2]_6)$ is 3. $\square$