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MATH 554: Section H01
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MATH 554 Homework 10

Problem 1 Let E be a metric space, $p \in E$ and $r_1, r_2, \dots, r_n$ positive real numbers. Let $\mathcal{U}$ be the collection of open balls

$$\mathcal{U} = \left\{ B(p, r_1), B(p, r_2), \dots, B(p, r_n) \right\}.$$

Let $r_{\max} = \max\{r_1, r_2, \dots, r_n\}$ and $r_{\min} = \min\{r_1, r_2, \dots, r_n\}$. Show

$$\text{(a)} \ \bigcup \mathcal{U} = B(p, r_{\max}) \quad \text{(b)} \ \bigcap \mathcal{U} = B(p, r_{\min}).$$

We will prove a double-inclusion for each part.

(a) ($\supseteq$) Since $B(p, r_{\max}) \in \mathcal{U}$ by our choice of $r_{\max}$, by definition of union $\bigcup \mathcal{U} \supseteq B(p, r_{\max})$ is clear.

($\subseteq$) Let $q \in \bigcup \mathcal{U}$. By definition of union, there exists $i \in \{1, 2, \dots, n\}$ such that $q \in B(p, r_i)$. By our choice of $r_{\max}$, we have $r_i \leq r_{\max}$, so we have

$$d(p, q) < r_i \leq r_{\max}.$$

So $q \in B(p, r_{\max})$, and therefore $\bigcup \mathcal{U} \subseteq B(p, r_{\max})$. $\square$

(b) ($\subseteq$) Since $B(p, r_{\min}) \in \mathcal{U}$ by our choice of $r_{\min}$, by definition of intersection $\bigcap \mathcal{U} \subseteq B(p, r_{\min})$ is clear.

($\supseteq$) Let $q \in B(p, r_{\min})$. By our choice of $r_{\min}$, we have $r_{\min} \leq r_i$ for all $i \in \{1, 2, \dots, n\}$, so we have

$$d(p, q) < r_{\min} \leq r_i$$

for all i. Thus, $q \in B(p, r_i)$ for all i, so $q \in \bigcap \mathcal{U}$. Therefore, $\bigcap \mathcal{U} \supseteq B(p, r_{\min})$. $\square$

Problem 2 Let E be a metric space and $p \in E$. Let R be a set of positive real numbers which is bounded and let

$$\mathcal{U} = \left\{ B(p, r) : r \in R \right\}.$$

Let $r_0 = \inf(R)$ and $r_1 = \sup(R)$.

(a) Show

$$\bigcup \mathcal{U} = B(p, r_1).$$

(b) Give an example where

$$\bigcap \mathcal{U} \neq B(p, r_0).$$

(a) We will show a double inclusion.

($\subseteq$) Let $q \in \bigcup \mathcal{U}$. By definition of union, there exists $r \in R$ such that $q \in B(p, r)$. By our choice of r_1, we have $r \leq r_1$, so we have

$$d(p, q) < r \leq r_1.$$

So $q \in B(p, r_1)$, and therefore $\bigcup \mathcal{U} \subseteq B(p, r_1)$.

($\supseteq$) Suppose (toward contradiction) that there exists some $q \in E$ such that

$$q \in B(p, r_1), q \notin \bigcup \mathcal{U}.$$

Since $q \in B(p, r_1)$, we have $d(p, q) < r_1$. So there exists some $\varepsilon > 0$ such that $d(p, q) = r_1 - \varepsilon$. Also, since q is not in the intersection, for all $r \in R$ we have $d(p, q) \geq r$. Thus, for all $r \in R$ we have

$$r \leq d(p, q) = r_1 - \varepsilon,$$

so $r_1 - \varepsilon$ is an upper bound for R. But then $r_1 - \varepsilon$ is a less upper bound for R than the supremum r_1, a contradiction.

So both inclusions hold, and therefore $\bigcup \mathcal{U} = B(p, r_1)$. (We note this is a more general version of problem 1a, since finite sets are closed and thus contain their supremums.) □

(b) Consider $E = \mathbb{R}$, $p = 0$, and $R = (1, 2)$. Then $r_0 := \inf(R) = 1$. So we have

$$\bigcap \mathcal{U} = \bigcap_{r \in (1,2)} (-r, r), \quad B(0, r_0) = (-1, 1).$$

We claim that 1 is in the intersection, which would disprove the inequality since $1 \notin B(0, 1) = (-1, 1)$. To see this, let $r \in (1, 2)$. Then $1 < r < 2$, so in particular $1 < r$ and thus $1 \in B(0, r)$. Since this is true for all $r \in (1, 2)$, 1 is in the intersection. Therefore, in this case

$$\bigcap \mathcal{U} \neq B(p, r_0).$$

□

Problem 3 Let $\mathcal{U} = \{(-n, n) : n \in \mathbb{N}\}$. Show that $\mathcal{U}$ is an open cover of $\mathbb{R}$ if and only if Archimedes' axiom holds.

($\Rightarrow$) Suppose that $\mathcal{U}$ is an open cover of $\mathbb{R}$. So $\mathbb{R} \subseteq \bigcup \mathcal{U}$. Let $x \in \mathbb{R}$. Then $x \in \bigcup \mathcal{U}$, so there exists $n \in \mathbb{N}$ such that $x \in (-n, n)$. So $-n < x < n$, so in particular $x < n$. Thus, Archidemes' axiom holds.

($\Leftarrow$) Suppose Archidemes' axiom holds. Let $x \in \mathbb{R}$. Then from Archimedes, there exists $n \in \mathbb{N}$ such that $|x| < n$. So $x \in (-n, n)$, and thus $x \in \bigcup \mathcal{U}$. So $\mathbb{R} \subseteq \bigcup \mathcal{U}$. Therefore, since every element in $\mathcal{U}$ is an open set, $\mathcal{U}$ is an open cover of $\mathbb{R}$. □

Problem 4 Let S be a subset of a metric space E. For each $p \in S$ let $r_p > 0$ be a positive number. Show $\mathcal{U} = \{B(p, r_p) : p \in S\}$ is an open cover of S.

Let $s \in S$. Then $s \in B(s, r_s)$ since $d(s, s) = 0 < r_s$, so $s \in \bigcup \mathcal{U}$. Therefore $S \subseteq \bigcup \mathcal{U}$, and since $\mathcal{U}$ consists of all open balls, $\mathcal{U}$ is an open cover of S. □

Problem 5 Let E be a metric space and $S \subseteq E$. Let $p \in E$. Show $\mathcal{U} = \{B(p, r) : r > 0\}$ is an open cover of S.

Let $q \in S$. Then $r := d(p, q) \in [0, \infty)$ by definition, so we have $r + 1 > 0$. So $q \in B(p, r+1)$, so $q \in \bigcup \mathcal{U}$. Therefore $S \subseteq \bigcup \mathcal{U}$, and since $\mathcal{U}$ consists of all open balls, $\mathcal{U}$ is an open cover of S. $\square$

Problem 6 Let S be a set that has a finite open cover $\mathcal{U}$. Assume that for each $U \in \mathcal{U}$, $U \cap S$ is finite. Show that S is finite.

Since $\mathcal{U}$ is an open cover for S, we have $S \subseteq \bigcup \mathcal{U}$. Also, since $\mathcal{U}$ is finite, we have $\mathcal{U} = \{U_1, U_2, \ldots, U_n\}$ for some $n \in \mathbb{N}$. So we can write

$$\begin{aligned} S &= S \cap \bigcup \mathcal{U} && \text{(since } S \subseteq \textstyle\bigcup \mathcal{U}\text{)} \\ &= S \cap (U_1 \cup U_2 \cup \ldots \cup U_n) \\ &= (S \cap U_1) \cup (S \cap U_2) \cup \ldots \cup (S \cap U_n). && \text{(distributive property of sets)} \end{aligned}$$

By the principle of inclusion-exclusion, then, we have

$$|S| \leq |S \cap U_1| + |S \cap U_2| + \ldots + |S \cap U_n|.$$

Since $|U_i \cap S|$ is finite for all $i \in \{1, 2, \ldots, n\}$, this implies that $|S|$ is finite. $\square$

Problem 7 Let S be a subset of the metric space E that has the property that if $\mathcal{U}$ is an open cover of S, then $\mathcal{U}$ has a finite subset $\mathcal{U}_0$ which is also a cover of S. Show that S is bounded.

Choose $p \in S$. We have from problem 5 that $\mathcal{U} = \{B(p, r) : r > 0\}$ is an open cover of S. By the property, there must exist a finite subcover $\mathcal{U}_0$ so $S \subseteq \mathcal{U}_0 \subseteq \mathcal{U}$. Since $\mathcal{U}$ is finite, there exist positive $r_1, r_2, \ldots r_n$ such that

$$\mathcal{U}_0 = \{B(p, r_1), B(p, r_2), \ldots, B(p, r_n)\}.$$

Let $r_{\max} := \max\{r_1, r_2, \ldots, r_n\}$. Then, we have from problem 1 that $\bigcup \mathcal{U}_0 = B(p, r_{\max})$. So $S \subseteq B(p, r_{\max})$, and therefore S is bounded. $\square$