MATH 552: Section 001
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MATH 552 Homework 4ˆ

Problem 20.4+ Suppose that $f(z_0) = g(z_0) = 0$ and that $f'(z_0)$ and $g'(z_0)$ exist, where $g'(z_0) \neq 0$. Use definition (1), Sec. 19, of derivative to show that

$$\lim_{z \to z_0} \frac{f(z)}{g(z)} = \frac{f'(z_0)}{g'(z_0)}.$$

Solution.

$$\begin{aligned}
\frac{f'(z_0)}{g'(z_0)} &= \frac{\lim_{z\to z_0} \frac{f(z)-f(z_0)}{z-z_0}}{\lim_{z\to z_0} \frac{g(z)-g(z_0)}{z-z_0}} && \text{(using definition of derivative)} \\
\frac{f'(z_0)}{g'(z_0)} &= \lim_{z\to z_0} \frac{f(z)-f(z_0)}{g(z)-g(z_0)} && \text{(using Theorem 16.2 and multiplying by } \tfrac{z-z_0}{z-z_0}\text{)} \\
\frac{f'(z_0)}{g'(z_0)} &= \lim_{z\to z_0} \frac{f(z)}{g(z)} && \text{(using } f(z_0) = g(z_0) = 0\text{)}
\end{aligned}$$

The equality is shown.

Problem 20.6+ Derive expression (2), Sec. 20, for the derivative of z^n when n is a positive integer by using
(a) mathematical induction and expression (4), Sec. 20, for the derivative for the product of two functions;
(b) definition (3), Sec. 19, of derivative and the binomial formula (Sec. 3).

Solution.

(a) We claim $\frac{d}{dz}[z^n] = nz^{n-1}$ for all $n \in \mathbb{Z}^+$.

Let $n = 1$.

$$\begin{aligned}
\frac{d}{dz}[z] &= \lim_{\Delta z\to 0} \frac{(z+\Delta z) - z}{\Delta z} && \text{(using limit definition)} \\
&= \lim_{\Delta z\to 0} \frac{\Delta z}{\Delta z} \\
&= \lim_{\Delta z\to 0} 1 \\
&= 1
\end{aligned}$$

Using the claim:

$$\begin{aligned}
\frac{d}{dz}[z] &= (1)z^{1-0} \\
&= 1
\end{aligned}$$

Thus, the claim is true for $n = 1$.

Let $k \in \mathbb{Z}^+$ be given and suppose the claim is true for $n = k$. Then,

$$
\begin{aligned}
\frac{d}{dz}[z^{k+1}] &= \frac{d}{dz}[z(z^k)] \\
&= z\frac{d}{dz}[z^k] + \frac{d}{dz}[z]z^k && \text{(using product rule)} \\
&= z(kz^{k-1}) + z^k && \text{(by } z^k = kz^{k-1} \text{ for } k\text{)} \\
&= kz^k + z^k \\
&= (k+1)z^{(k+1)-1} && \text{(rearranging)}
\end{aligned}
$$

Thus, the claim holds for $n = k + 1$. By induction, the claim is true for all $n \in \mathbb{Z}^+$.

(b)

$$
\begin{aligned}
\frac{d}{dz}[z^n] &= \lim_{\Delta z \to 0} \frac{(z+\Delta z)^n - z^n}{\Delta z} && \text{(using limit definition)} \\
&= \lim_{\Delta z \to 0} \frac{1}{\Delta z}\left(\sum_{k=0}^{n}\left[\binom{n}{k} z^{n-k}(\Delta z)^k\right] - z^n\right) && \text{(using binomial theorem)} \\
&= \lim_{\Delta z \to 0} \frac{1}{\Delta z}\left(z^n + \sum_{k=1}^{n}\left[\binom{n}{k} z^{n-k}(\Delta z)^k\right] - z^n\right) && \text{(since } \tbinom{n}{0} z^{n-0}(\Delta z)^0 = z^n\text{)} \\
&= \lim_{\Delta z \to 0} \frac{1}{\Delta z}\left(\sum_{k=1}^{n}\binom{n}{k} z^{n-k}(\Delta z)^k\right) && \text{(subtracting)} \\
&= \lim_{\Delta z \to 0} \sum_{k=1}^{n}\binom{n}{k} z^{n-k}(\Delta z)^{k-1} && \text{(dividing each term by } \Delta z\text{)} \\
&= \sum_{k=1}^{n}\binom{n}{k} z^{n-k}(0)^{k-1} && \text{(direct substitution)} \\
&= \binom{n}{1} z^{n-1} && \text{(every term is 0 except for when } k = 1\text{)} \\
&= nz^{n-1}
\end{aligned}
$$

The equality is shown.

Problem 24.2d Use the theorem in Sec. 23 to show that $f'(z)$ and its derivative $f''(z)$ exist everywhere, and find $f''(z)$ when $f(z) = \cos x \cosh y - i \sin x \sinh y$.

Solution.

$$f(z) = u(x, y) + v(x, y) \text{ where } u(x, y) = \cos x \cosh y, v(x, y) = -\sin x \sinh y$$

We can now check the Cauchy-Riemann equations:

$$u_x = -\sin x \cosh y, u_y = \cos x \sinh y$$
$$v_x = -\cos x \sinh y, v_y = -\sin x \cosh y$$

Since the Cauchy-Riemann equations hold, and $u(x,y)$, $v(x,y)$, and their partials are continuous everywhere, $f'(z)$ exists and

$$f'(z) = u_x(x,y) + iv_x(x,y) = -\sin x \cosh y - i \cos x \sinh y.$$

To take the second derivative, let

$$f'(z) = s(x,y) + t(x,y) \text{ where } s(x,y) = -\sin x \cosh y, t(x,y) = -\cos x \sinh y.$$

We can now check the Cauchy-Riemann equations again:

$$\begin{aligned} s_x &= -\cos x \cosh y, s_y = -\cos x \sinh y \\ t_x &= \sin x \sinh y, t_y = -\cos x \cosh y \end{aligned}$$

Since they hold again, and $s(x,y)$, $t(x,y)$, and their partials are continuous everywhere, $f''(z)$ exists and

$$f''(z) = s_x(x,y) + it_x(x,y) = -\cos x \cosh y + i \sin x \sinh y.$$

Problem 24.6 Let a function $f(z) = u + iv$ be differentiable at a nonzero point $z_0 = r_0 \exp(i\theta_0)$. Use the expressions for u_x and v_x found in Exercise 5, together with the polar form (6), Sec. 24, of the Cauchy-Riemann equations, to rewrite the expression

$$f'(z_0) = u_x + iv_x$$

in Sec. 24 as

$$f'(z_0) = e^{-i\theta}(u_r + iv_r),$$

where u_r and v_r are to be evaluated at (r_0, θ_0).

$$\begin{aligned} f'(z) &= u_x + iv_x \\ &= u_r \cos\theta - u_\theta \frac{\sin\theta}{r} + iv_x && \text{(using expressions)} \\ &= u_r \cos\theta - u_\theta \frac{\sin\theta}{r} + iv_x - i[u_r \sin\theta + u_\theta \frac{\cos\theta}{r}] && \text{(using } v_x = -u_y\text{)} \\ &= u_r \cos\theta - u_\theta \frac{\sin\theta}{r} - iu_r \sin\theta - iu_\theta \frac{\cos\theta}{r} && \text{(distributing)} \\ &= u_r \cos\theta + rv_r \frac{\sin\theta}{r} - iu_r \sin\theta + irv_r \frac{\cos\theta}{r} && \text{(using } u_\theta = -rv_r\text{)} \\ &= u_r(cos\theta - i\sin\theta) + iv_r(\cos\theta - i\sin\theta) && \text{(rearranging)} \\ &= (\cos(-\theta) + i\sin(-\theta))(u_r + iv_r) && \text{(rearranging)} \\ &= e^{-i\theta}(u_r + iv_r) && \text{(using Euler's formula)} \end{aligned}$$

The equality is shown.