Nathan Bickel
MATH 546: Section
Professor: Dr. Vraciu
November 10, 2023

MATH 546 Homework 8

Problem 1

(a) List all the elements of A_4 that have order equal to 2.

(b) Does A_4 have any cyclic subgroup of order 4?

(c) Does A_4 have any non-cyclic subgroup of order 4?

Solution.

We can write

$$A_4 = \{(1), (1\ 2\ 3), (1\ 3\ 2), (1\ 2\ 4), (1\ 4\ 2), (1\ 3\ 4), (1\ 4\ 3), (2\ 3\ 4), (2\ 4\ 3), (1\ 2)(3\ 4), (1\ 3)(2\ 4), (1\ 4)(2\ 3)\}.$$

(a) All of the elements of A_4 that are two disjoint cycles–$(1\ 2)(3\ 4), (1\ 3)(2\ 4), (1\ 4)(2\ 3)$–have order 2: they are not the identity, and the disjoint cycles commute and are their own inverses. The only other elements in A_4 are the identity (which has order 1) and cycles of length 3, which have order 3. Thus, these elements are all the elements with order 2.

(b) No. As discussed in (a), the only orders of elements are 1, 2, and 3, so no element of order 4 exists. This implies that no cyclic subgroup of order 4 exists.

(c) We claim that

$$H = \{(1), (1\ 2)(3\ 4), (1\ 3)(2\ 4), (1\ 4)(2\ 3)\}$$

is a subgroup. We will make a multiplication table:

*	(1)	(1 2)(3 4)	(1 3)(2 4)	(1 4)(2 3)
(1)	(1)	(1 2)(3 4)	(1 3)(2 4)	(1 4)(2 3)
(1 2)(3 4)	(1 2)(3 4)	(1)	(1 4)(2 3)	(1 3)(2 4)
(1 3)(2 4)	(1 3)(2 4)	(1 4)(2 3)	(1)	(1 2)(3 4)
(1 4)(2 3)	(1 4)(2 3)	(1 3)(2 4)	(1 2)(3 4)	(1)

As we can see from the table, closure holds, and each element is its own inverse. Since the identity is in H, we have that H is a subgroup (and since each element has order 2, it is not cyclic).

Problem 2

(a) List all the possible decomposition types of elements in A_8.

(b) List all the possible orders of elements of A_8. For each possible order, give an example of an element that has that order.

Solution.

For any $\sigma \in S_8$, we can find the disjoint cycle decomposition $\sigma = \tau_1\tau_2\ldots\tau_k$ such that $1 \leq k \leq 8$, where τ_i has length ℓ_i, $1 \leq \ell_i \leq 8$ (other than the identity, we will not include cycles of length one). Then, we have $o(\sigma) = \operatorname{lcm}(s_1, s_2, \ldots, s_k)$. Since we cannot have more than 8 elements, we have $s_1 + s_2 + \cdots + s_k \leq 8$.

We can find the partitions of 8 using combinatorics, write the sum of the partition s, number of cycles k, find the set of unique elements used in the partition, and then take the lcm of these elements. For each partition, we will give an example element.

Partition	s	k	**Set**	lcm	**Example Element with Order** lcm
$-$	0	0	$-$	$-$	e
2	2	1	$\{2\}$	2	(1 2)
$2+2$	4	2	$\{2\}$	2	(1 2)(3 4)
$2+2+2$	6	3	$\{2\}$	2	(1 2)(3 4)(5 6)
$2+2+2+2$	8	4	$\{2\}$	2	(1 2)(3 4)(5 6)(7 8)
3	3	1	$\{3\}$	3	(1 2 3)
$3+3$	6	2	$\{3\}$	3	(1 2 3)(4 5 6)
4	4	1	$\{4\}$	4	(1 2 3 4)
$4+2$	6	2	$\{2,4\}$	4	(1 2 3 4)(5 6)
$4+2+2$	8	3	$\{2,4\}$	4	(1 2 3 4)(5 6)(7 8)
$4+4$	8	2	$\{4\}$	4	(1 2 3 4)(5 6 7 8)
5	5	1	$\{5\}$	5	(1 2 3 4 5)
$3+2$	5	2	$\{2,3\}$	6	(1 2 3)(4 5)
$3+2+2$	7	3	$\{2,3\}$	6	(1 2 3)(4 5)(6 7)
$3+3+2$	8	3	$\{2,3\}$	6	(1 2 3)(4 5 6)(7 8)
6	6	1	$\{6\}$	6	(1 2 3 4 5 6)
$6+2$	8	2	$\{2,6\}$	6	(1 2 3 4 5 6)(7 8)
7	7	1	$\{7\}$	7	(1 2 3 4 5 6 7)
8	8	1	$\{8\}$	8	(1 2 3 4 5 6 7 8)
$5+2$	7	2	$\{2,5\}$	10	(1 2 3 4 5)(6 7)
$4+3$	7	2	$\{3,4\}$	12	(1 2 3 4)(5 6 7)
$5+3$	8	2	$\{3,5\}$	15	(1 2 3 4 5)(6 7 8)

For each element in A_8, we can express it in terms of an even number of transpositions. We can express a cycle of length ℓ with $\ell - 1$ transpositions, so for a composition of k disjoint cycles with sum of lengths s, we can express it with $s - k$ transpositions. Thus, using the table above, we can find the decomposition types that belong in A_8 by including the elements where $s - k$ is even:

Partition	s	k	**Set**	lcm	**Example Element with Order** lcm
$-$	0	0	$-$	$-$	e
$2+2$	4	2	$\{2\}$	2	(1 2)(3 4)
$2+2+2+2$	8	4	$\{2\}$	2	(1 2)(3 4)(5 6)(7 8)
3	3	1	$\{3\}$	3	(1 2 3)
$3+3$	6	2	$\{3\}$	3	(1 2 3)(4 5 6)
$4+2$	6	2	$\{2,4\}$	4	(1 2 3 4)(5 6)
$4+4$	8	2	$\{4\}$	4	(1 2 3 4)(5 6 7 8)
5	5	1	$\{5\}$	5	(1 2 3 4 5)
$3+2+2$	7	3	$\{2,3\}$	6	(1 2 3)(4 5)(6 7)
$6+2$	8	2	$\{2,6\}$	6	(1 2 3 4 5 6)(7 8)
7	7	1	$\{7\}$	7	(1 2 3 4 5 6 7)
$5+3$	8	2	$\{3,5\}$	15	(1 2 3 4 5)(6 7 8)

(a) In the table, the decomposition types are listed in the **Partition** column.

(b) In the table, the possible orders are listed in the lcm column, and example elements are given.

Problem 3

(a) Let H be a subgroup of S_n. If $H \not\subseteq A_n$, prove that

$$|H \cap A_n| = \frac{|H|}{2}.$$

(b) Using the result in part (a), prove that if H is a subgroup of S_n and $|H|$ is an odd number, then $H \subseteq A_n$.

Solution.

(a) Suppose $H \not\subseteq A_n$. Then there exists an element $\sigma \in H$ that is not in A_n. Consider the function $f : H \cap A_n \to H \backslash A_n$ defined by $f(x) = x\sigma$. This maps elements to $H \backslash A_n$ because any element x in A_n is expressed as an even number of transpositions, and since σ is expressed as an odd number of transpositions, and an even number plus an odd number is odd, $f(x)$ is expressed as an odd number of transpositions.

Since $\sigma \in H$, we also have $\sigma^{-1} \in H$, which will also be expressed as an odd number of tranpositions (if it were even, then $\sigma\sigma^{-1} = e$ would have an odd plus an even number of transpositions, which would be an odd number of transpositions, a contradiction since $e \in A_n$). We can use this to define $f^{-1}(y) = y\sigma^{-1}$, which we can see is well-defined because $f^{-1}(f(x)) = x\sigma\sigma^{-1} = x$ for all $x \in H \cap A_n$. Thus, f is a one-to-one correspondence, so we have $|H \cap A_n| = |H \backslash A_n|$. Therefore, since $|H| = |H \cap A_n| + |H \backslash A_n|$, we can conclude that

$$|H \cap A_n| = \frac{|H|}{2}.$$

□

(b) Suppose (toward contradiction) that $H \not\subseteq A_n$. Then, from part (a) we have $|H \cap A_n| = \frac{|H|}{2}$, but since $|H|$ is odd, we have $|H \cap A_n|$ is not an integer, a contradiction. □

Problem 4 Prove that

$$A_4 = \{\sigma \in S_4 \mid \sigma = \tau^2 \text{ for some } \tau \in S_4\}.$$

Solution.

We will prove a double inclusion.

($\supseteq$) Let σ be in the set above. Then there exists $\tau \in S_4$ such that $\sigma = \tau^2$. If τ is expressed as an odd number of transpositions, then τ^2 is expressed as 2 times an odd number of transpositions, which is even. Since 2 times an even number is also even, this is true if τ is expressed as an even number of transpositions as well. Thus, $\sigma \in A_4$ by definition.

($\subseteq$) We will check each element in A_4, as listed in Problem 1:

- $(1) = (1)^2$
- $(1\ 2\ 3) = (1\ 3\ 2)^2$
- $(1\ 3\ 2) = (1\ 2\ 3)^2$
- $(1\ 2\ 4) = (1\ 4\ 2)^2$
- $(1\ 4\ 2) = (1\ 2\ 4)^2$
- $(1\ 3\ 4) = (1\ 4\ 3)^2$
- $(1\ 4\ 3) = (1\ 3\ 4)^2$
- $(2\ 3\ 4) = (2\ 4\ 3)^2$
- $(2\ 4\ 3) = (2\ 3\ 4)^2$
- $(1\ 2)(3\ 4) = (1\ 3\ 2\ 4)^2$
- $(1\ 3)(2\ 4) = (1\ 2\ 3\ 4)^2$
- $(1\ 4)(2\ 3) = (1\ 2\ 4\ 3)^2$

Thus, every element in A_4 can be expressed as the square of some element in S_4.

Therefore, we have $A_4 = \{\sigma \in S_4 \mid \sigma = \tau^2 \text{ for some } \tau \in S_4\}$.

Problem 5 Let $a = (1\ 2)(3\ 4)$ and $b = (1\ 2\ 3)$. If H is a subgroup of A_4 with $a, b \in H$, prove that $H = A_4$.

Solution.

Clearly, $H \subseteq A_4$, so it suffices to show that $A_4 \subseteq H$. We have

- $(1) \in H$ because H is a subgroup.
- $(1\ 2\ 3) \in H$ is given.
- $(1\ 3\ 2) = (1\ 2\ 3)^2 \in H$ from closure.
- $(1\ 2\ 4) = (1\ 2\ 3)(1\ 2)(3\ 4)(1\ 2\ 3) \in H$ from closure.
- $(1\ 4\ 2) = (1\ 2)(3\ 4)(1\ 2\ 3)(1\ 2)(3\ 4) \in H$ from closure.
- $(1\ 3\ 4) = (1\ 2)(3\ 4)(1\ 2\ 3) \in H$ from closure.
- $(1\ 4\ 3) = (1\ 2\ 3)^2(1\ 2)(3\ 4) \in H$ from closure.

- $(2\ 3\ 4) = (1\ 2)(3\ 4)(1\ 2\ 3)^2 \in H$ from closure.
- $(2\ 4\ 3) = (1\ 2\ 3)(1\ 2)(3\ 4) \in H$ from closure.
- $(1\ 2)(3\ 4) \in H$ is given.
- $(1\ 3)(2\ 4) = (1\ 2\ 3)^2(1\ 2)(3\ 4)(1\ 2\ 3)(1\ 2)(3\ 4) \in H$ from closure.
- $(1\ 4)(2\ 3) = (1\ 2)(3\ 4)(1\ 2\ 3)^2(1\ 2)(3\ 4)(1\ 2\ 3)(1\ 2)(3\ 4) \in H$ from closure.

Therefore, we have $A_4 = H$. $\square$