MATH 552: Section 001
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MATH 552 Homework 5*

Problem 26.7 Let a function f be analytic everywhere in domain D. Prove that if $f(z)$ is real-valued for all z in D, then $f(z)$ must be constant throughout D.

Solution.

We write $z = x + iy$ and $f(z) = u(x, y) + iv(x, y)$. If f is real-valued for all z in D, then it does not have an imaginary component, so $v(x, y) = 0$. Since f is analytic, the Cauchy-Riemann equations tell us

$$u_x = v_y, u_y = -v_x.$$

Since $v(x, y) = 0 \implies v_x = v_y = 0$, we also know by the Cauchy-Riemann equations that $u_x = u_y = 0$.

Because $f'(z) = u_x + iu_y$, $f'(z) = 0$. We know that a function is constant throughout its domain if and only if its derivative is 0 everywhere, so f must be constant throughout D.

Problem 29.1 Use the theorem in Sec. 28 to show that if $f(z)$ is analytic and not constant throughout a domain D, then it cannot be constant throughout any neighborhood lying in D.
Suggestion: Suppose that $f(z)$ does have a constant value w_0 throughout some neighborhood in D.

Solution.

Suppose that $f(z)$ does has a constant value throughout some neighborhood in D. According to the theorem, a function that is analytic in a domain D is uniquely determined over D by its values in a domain contained in D. Since a neighborhood in D would be a domain contained in D, and $f(z)$ is analytic, $f(z)$ must be constant throughout D.

Since the contrapositive is true, the claim must also be true.

Problem 30.11 Describe the behavior of $e^z = e^x e^{iy}$ as (a) x tends to $-\infty$; (b) y tends to ∞.

Solution.

Since $e^z = e^x e^{iy}$, the function will map to a complex number w such that $|w| = e^x$ and $\arg(w) = y$ by Euler's formula. Thus, the magnitude of w depends on x while changing y causes w to move around a circle with radius e^x.

(a) Since

$$\lim_{x \to -\infty} e^x = 0,$$

the magnitude of w will tend toward 0 and thus w approaches 0.

(b) Since a change in y causes a periodic change in w, w does not approach any particular value as y tends toward ∞.