MATH 552: Section 001
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MATH 552 Homework 12*

Problem 72.4 Show that the function defined by means of the equations

$$f(z) = \begin{cases} (1-\cos z)/z^2 & \text{when } z \neq 0, \\ 1/2 & \text{when } z = 0 \end{cases}$$

is entire.

Solution.

$$\begin{aligned} \cos z &= 1 - \frac{z^2}{2!} + \frac{z^4}{4!} + ... && \text{(Maclaurin series)} \\ 1 - \cos z &= \frac{z^2}{2!} - \frac{z^4}{4!} + \frac{z^6}{6!} + ... \\ \frac{1-\cos z}{z^2} &= \frac{1}{2!} - \frac{z^2}{4!} + \frac{z^4}{6!} + ... \end{aligned}$$

Since the Maclaurin series representation of $\cos z$ is valid for every value of z, the series obtained by subtracting it from 1 and dividing by z^2 converges to $f(z)$ when $z \neq 0$. Also, since the series equals $\frac{1}{2!} = \frac{1}{2}$ when $z = 0$, the series converges to $f(z)$ when $z = 0$.

Therefore, since $f(z)$ is represented by the convergent series for all z, f is an entire function.

Problem 77.3 In the example in Sec. 76, two residues were used to evaluate the integral

$$\int_C \frac{4z-5}{z(z-1)} dz$$

where C is the positively oriented circle $|z| = 2$. Evaluate this integral once again by using the theorem in Sec. 77 and finding only one residue.

Solution. With the integrand being $f(z)$:

$$\begin{aligned} \int_C \frac{4z-5}{z(z-1)} dz &= 2\pi i \operatorname*{Res}_{z=0} \left[\frac{1}{z^2} f\left(\frac{1}{z}\right) \right] \\ &= 2\pi i \operatorname*{Res}_{z=0} \left[\frac{1}{z^2} \left(\frac{4\left(\frac{1}{z} - 5\right)}{\frac{1}{z}\left(\frac{1}{z} - 1\right)} \right) \right] \\ &= 2\pi i \operatorname*{Res}_{z=0} \left[\frac{\frac{4}{z} - 5}{1 - z} \right] \\ &= 2\pi i \operatorname*{Res}_{z=0} \left[\left(\frac{4-5z}{z} \right) \left(\frac{1}{1-z} \right) \right] \\ &= 2\pi i \operatorname*{Res}_{z=0} \left[(4-5z) \left(\frac{1}{z} + 1 + z + z^2 + z^3 + ... \right) \right] \end{aligned}$$

$$
\begin{aligned}
&= 2\pi i \operatorname*{Res}_{z=0}\left[\left(\frac{4}{z}+4+4z+...\right)-(5+5z+5z^2+...)\right]\\
&= 2\pi i(4) && \text{(coefficient of } \tfrac{1}{z} \text{ term is 4)}\\
&= 8\pi i
\end{aligned}
$$

Problem 77.4a Use the theorem in Sec. 77, involving a single residue, to evaluate the integral of this function around the circle $|z| = 2$ in the positive sense:

$$f(z) = \frac{z^5}{1-z^3}.$$

Solution. With the integrand being $f(z)$:

$$
\begin{aligned}
\int_C \frac{z^5}{1-z^3}dz &= 2\pi i \operatorname*{Res}_{z=0}\left[\frac{1}{z^2}f\left(\frac{1}{z}\right)\right]\\
&= 2\pi i \operatorname*{Res}_{z=0}\left[\frac{1}{z^2}\left(\frac{(1/z)^5}{1-(1/z)^3}\right)\right]\\
&= 2\pi i \operatorname*{Res}_{z=0}\left[\frac{1}{z^7}\left(\frac{1}{1-(1/z^3)}\right)\right]\\
&= 2\pi i \operatorname*{Res}_{z=0}\left[-\frac{1}{z^4}\left(\frac{1}{1-z^3}\right)\right]\\
&= 2\pi i \operatorname*{Res}_{z=0}\left[-\frac{1}{z^4}(1+z^3+z^6+...)\right] && \text{(using Taylor series expansion)}\\
&= 2\pi i \operatorname*{Res}_{z=0}\left[-\frac{1}{z^4}-\frac{1}{z}-z^2-z^5+...\right]\\
&= 2\pi i(-1) && \text{(coefficient of } \tfrac{1}{z} \text{ term is -1)}\\
&= -2\pi i
\end{aligned}
$$