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MATH 555 Homework 2

Problem 1 Prove the following generalization of Rollê's rule: Let $f : (a, b) \to \mathbb{R}$ so that f' and f'' exist on (a, b) and assume there are $x_0, x_1, x_2 \in (a, b)$ with $x_0 < x_1 < x_2$ and

$$f(x_0) = f(x_1) = f(x_2) = 0.$$

Then there is a point ξ between x_0 and x_2 with $f''(\xi) = 0$.

Since f' exists on (a, b), f is continuous and differentiable on (a, b). So by Rollê's theorem, since $f(x_0) = f(x_1)$ and $f(x_1) = f(x_2)$, there exist $\xi_1 \in (x_0, x_1)$, $\xi_2 \in (x_1, x_2)$ with $f'(\xi_1) = f'(\xi_2) = 0$.

Further, since f'' exists on (a, b), f' is continuous and differentiable on (a, b). So we can apply Rollê's theorem again, and conclude that since $f'(\xi_1) = f'(\xi_2)$, there exists $\xi \in (\xi_1, \xi_2) \subset (x_0, x_2)$ with $f''(\xi) = 0.\square$

Problem 2 Let $f : (a, b) \to \mathbb{R}$ so that f' and f'' exist on (a, b). Let $p(x) = ax^2 + bx + c$ be a quadratic polynomial. Assume there are distinct points $x_0, x_1, x_2 \in (a, b)$ with $x_0 < x_1 < x_2$ and

$$f(x_0) = p(x_0), \quad f(x_1) = p(x_1), \quad f(x_2) = p(x_2).$$

Prove there is a point ξ bewteen x_0 and x_2 with $f''(\xi) = 2a$.

Let $h(x) = f(x) - p(x)$. Since p is a polynomial, it is twice differentiable everywhere, so h' and h'' exist on (a, b). Also, by the assumption, we have $h(x_0) = h(x_1) = h(x_2) = 0$. So by Problem 1, there exists a ξ between x_0 and x_2 with $h''(\xi) = 0$. We compute $p'(x) = 2ax + b$, so differentiating again we have $p''(x) = 2a$. Therefore, we have

$$\begin{aligned}
0 &= h''(\xi) && \text{(by choice of } \xi) \\
&= f''(\xi) - p''(\xi) && \text{(sum rule of derivatives)} \\
&= f''(\xi) - 2a && \text{(as computed above)} \\
\implies f''(\xi) &= 2a. && \text{(algebra)}
\end{aligned}$$

$\square$

Problem 3 Let $f : (a, b) \to \mathbb{R}$ be so that for some $x_0 \in (a, b)$ that $f'(x) < 0$ for $x < x_0$ and $f'(x) > 0$ for $x > x_0$. Show $f(x) \geq f(x_0)$ for all $x \in (a, b)$ (that is x_0 is a global minimizer of f on (a, b)).

Let $x \in (a, b)$. By the MVT, there exists a ξ between x and x_0 such that

$$f(x) - f(x_0) = f'(\xi)(x - x_0).$$

Case 1: $x < x_0$. Then $x - x_0 < 0$, and $\xi < x_0$ so $f'(\xi) < 0$ by our assumption. Since a negative times a negative is positive, we have $f(x) - f(x_0) > 0$ and thus $f(x) > f(x_0)$.

Case 2: $x > x_0$. Then $x - x_0 > 0$, and $\xi > x_0$ so $f'(\xi) > 0$ by our assumption. Since a positive times a positive is positive, we have $f(x) - f(x_0) > 0$ and thus $f(x) > f(x_0)$.

Case 3: $x = x_0$. Then $f(x) = f(x_0)$.

So in all cases, $f(x) \geq f(x_0)$. $\square$

Problem 2.21 Prove the following:

(b) If $a, b > 1$ then
$$\left|\sqrt{b^2-1} - \sqrt{a^2-1}\right| \geq |b-a|.$$

(c) If $x > 0$ then
$$e^x - 1 > x.$$

(b) Let $a, b > 1$, and let $f : (1, \infty) \to \mathbb{R}$ defined by $f(x) = \sqrt{x^2-1}$. By the MVT, there exists a ξ between a and b with
$$\left|f(b) - f(a)\right| = \left|f'(\xi)(b-a)\right|.$$
By the chain and power rule, we have
$$f'(x) = \frac{1}{2}(x^2-1)^{-1/2}(2x) = \frac{x}{\sqrt{x^2-1}},$$
so we can write
$$f'(\xi) = \frac{\xi}{\sqrt{\xi^2-1}} > \frac{\xi}{\sqrt{\xi^2}} = \frac{\xi}{\xi} = 1.$$
Using this, we have
$$\left|\sqrt{b^2-1} - \sqrt{a^2-1}\right| = \left|f(b) - f(a)\right| \geq (1)\,|b-a| = |b-a|.$$
$\square$

(c) Let $x > 0$, and let $f : (0, \infty) \to \mathbb{R}$ defined by $f(x) = e^x$. By the MVT, there exists a ξ between 0 and x with
$$f(x) - f(0) = f'(\xi)(x-0).$$
We know that $f'(x) = e^x$, so we can write
$$f'(\xi) = e^\xi > e^0 = 1$$
since $\xi > 0$. Using this, we have
$$e^x - 1 = f(x) - f(0) > (1)(x-0) = x.$$
$\square$

Problem 2.29a Use L'Hôpital's rule to compute
$$\lim_{x\to 0} \frac{\sin(x) - x}{x^3}.$$

Let $f(x) = \sin(x) - x$ and $g(x) = x^3$. Then, we have

$$f'(x) = \cos(x) - 1, f''(x) = -\sin(x), f'''(x) = -\cos(x)$$

and

$$g'(x) = 3x^2, g''(x) = 6x, g'''(x) = 6.$$

We have $f(0) = f'(0) = f''(0) = g(0) = g'(0) = g''(0) = 0$, and since

$$\lim_{x\to 0} \frac{f'''(x)}{g'''(x)} = \lim_{x\to 0} \frac{-\cos(x)}{6} = -\frac{1}{6},$$

through repeated application of L'Hôpital's rule we have

$$\lim_{x\to 0} \frac{\sin(x) - x}{x^3} = \lim_{x\to 0} \frac{f(x)}{g(x)} = -\frac{1}{6}.$$

$\square$

Problem 2.32 (Generalized Rollê's Theorem). Let f be $n+1$ times differentiable on the open interval I. Let $x_0, x_1 \in I$ with $x_0 \neq x_1$. Assume that

- $f(x_0) = f'(x_0) = \cdots = f^{(n)}(x_0) = 0$,
- $f(x_1) = 0$.

Prove that there is a point ξ between x_0 and x_1 with

$$f^{(n+1)}(\xi) = 0.$$

We will induct on n.

Base Case: Let $n = 0$. Then the claim holds directly from Rollê's Theorem.

Induction Step: Let $n \in \mathbb{N}, n > 0$. Suppose that

- $f(x_0) = f'(x_0) = \cdots = f^{(n)}(x_0) = 0$,
- $f(x_1) = 0$,

and that the claim holds for $n-1$: there is a point ξ' between x_0 and x_1 with

$$f^{(n-1+1)}(\xi') = f^{(n)}(\xi') = 0.$$

Then, since f is $n+1$ times differentiable on I, $f^{(n)}$ is differentiable on (x_0, ξ') with $f^{(n)}(x_0) = f^{(n)}(\xi')$. So since the derivative of $f^{(n)}$ is $f^{(n+1)}$, by Rollê's Theorem we have that there exists a $\xi \in (x_0, \xi') \subset (x_0, x_1)$ with $f^{(n+1)}(\xi) = 0$.

Therefore, the claim holds for all $n \in \mathbb{N}$. $\square$

Problem 5 Let f be n times differentiable on an open interval (a, b). Assume there are points $x_0, x_1, \ldots, x_n \in (a, b)$ with $x_0 < x_1 < \cdots < x_n$ such that $f(x_j) = 0$ for all $j \in \{0, 1, \ldots, n\}$. Show there is a ξ between x_0 and x_n with $f^{(n)}(\xi) = 0$.

We will prove a stronger statement: Let f be n times differentiable on an open interval (a, b). Assume there are points $x_0, x_1, \ldots, x_n \in (a, b)$ with $x_0 < x_1 < \cdots < x_n$ such that $f(x_j) = 0$ for all $j \in [n] := \{0, 1, \ldots, n\}$. Then, we claim that

- There are n points in (a, b) at which f' is 0.
- There are $n-1$ points in (a, b) at which f'' is 0.

  $\vdots$
- There are 2 points in (a, b) at which $f^{(n-1)}$ is 0.
- There is 1 point $\xi \in (a, b)$ with $f^{(n)}(\xi) = 0$.

More precisely, for each $i \in [n]$, there exist $\xi_{i,0}, \xi_{i,1}, \ldots, \xi_{i,n-i} \in (a, b)$ with $\xi_{i,0} < \xi_{i,1} < \cdots < \xi_{i,n-i}$ such that $f^{(i)}(\xi_{i,j}) = 0$ for all $j \in [n-i]$. Specifically, we can choose these points such that for all $0 < i \leq n, j \leq n-i$, we have $\xi_{i,j} \in (\xi_{i-1,j}, \xi_{i-1,j+1})$. We will prove this with induction on n.

Base Case: Let $n = 0$. This says that there exist $\xi_{0,0}, \xi_{0,1}, \ldots, \xi_{0,n-0} \in (a, b)$ with $\xi_{0,0} < \xi_{0,1} < \cdots < \xi_{0,n-0}$ such that $f^{(0)}(\xi_{0,j}) = 0$ for all $j \in [n-0]$. Since $f^{(0)} = f$, this is satisfied by $\xi_{0,j} = x_j$ for all $j \in [n]$.

Induction Step: Let $n \in \mathbb{N}$, $n > 0$. Suppose that the claim holds for $n-1$. To show that it holds for n, it suffices to find an appropriate $\xi_{i,n-i}$ for each $i \in [n]$: the rest of the $\xi_{i,j}$s are defined for $0 \leq j < n-i$ from the induction hypothesis.

We already have our $\xi_{0,n-0}$ from the assumption with $\xi_{0,n} = x_n$. Now, let $0 < i \leq n$. Consider $f^{(i-1)}$. Since $i \leq n$, and $f^{(n)}$ exists, $f^{(i-1)}$ is differentiable. Also, we have that

$$f^{(i-1)}(\xi_{i-1,n-i}) = f^{(i-1)}(\xi_{i-1,n-i+1}) = 0,$$

so by Rollê's theorem, there exists $\xi_{i,n-i} \in (\xi_{i-1,n-i}, \xi_{i-1,n-i+1})$ with $f^{(i)}(\xi_{i,n-i}) = 0$. Also, as desired we have $\xi_{i,n-i} > \xi_{i,n-i-1}$ because by the IH, we chose $\xi_{i,n-i-1} \in (\xi_{i-1,n-i-1}, \xi_{i-1,n-i})$, so

$$\xi_{i,n-i-1} < \xi_{i-1,n-i} < \xi_{i,n-i}.$$

Therefore, we have found an appropriate $\xi_{i,n-i}$ as desired, so by induction, our statement holds. So we have that $\xi := \xi_{n,0}$ satisfies $f^{(n)}(\xi) = 0$. $\square$