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MATH 554: Section H01
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# MATH 554 Homework 8

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**Problem 3.35** Prove that a set is closed if and only if it contains all its adherent points.

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Let $(E, d)$ be a metric space and $S \subset E$. We have that a point $p$ is an adherent point of $S$ if for all $r > 0$, $B(p, r) \cap S \neq \emptyset$.

($\Rightarrow$) Suppose $S$ is closed. Let $q \in \mathcal{C}(S)$. Since $\mathcal{C}(S)$ is open by definition, there exists an $r_q > 0$ such that $B(q, r_q) \subseteq \mathcal{C}(S)$. Thus, $B(q, r_q) \cap S = \emptyset$, so $q$ is not an adherent point of $S$. Therefore, every adherent point of $S$ must be contained in $S$.

($\Leftarrow$) Suppose $S$ contains all its adherent points. Let $q \in \mathcal{C}(S)$. Since $q$ is not an adherent point, there exists some $r_q > 0$ such that $B(q, r_q) \cap S = \emptyset$. So $B(q, r_q) \subseteq \mathcal{C}(S)$, and thus $\mathcal{C}(S)$ is open. So $S$ is closed. $\square$

**Problem 3.36** Let $S$ be a set in a metric space and $p$ an adherent point of $S$. Prove that there is a sequence of points $\langle p_n \rangle_{n=1}^{\infty}$ from $S$ that converges to $p$.

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Since $p$ is an adherent point of $S$, we have that $B(p, r)$ contains a point of $S$ for all $r > 0$. Thus, we can construct $\langle p_n \rangle$ such that for all $n \in \mathbb{Z}^+$, $p_n$ is a point from $S$ in $B(p, 1/n)$. We claim $\langle p_n \rangle$ converges to $p$. To see this, let $\varepsilon > 0$. Then, for all $n > N := \frac{1}{\varepsilon}$, we have $\frac{1}{n} < \varepsilon$. Since $p_n \in B(p, 1/n)$, we $d(p_n, p) < \frac{1}{n} < \varepsilon$, so $\langle p_n \rangle$ converges to $p$. $\square$

**Problem 3.37** Let $S$ be a set in a metric space and $p$ a point that is a limit of a sequence of points from $S$. Prove that $p$ is an adherent point of $S$.

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Suppose $\langle p_n \rangle$ is in $S$ and converges to $p$. Let $r > 0$. By definition of convergence, there exists an $N$ such that $d(p_N, p) < r$. So $p_N \in B(p, r)$, so $B(p, r) \cap S \neq \emptyset$. Thus, $p$ is an adherent point of $S$. $\square$

**Problem 3.38** Let $S$ be a subset of the metric space $E$. Prove that $S$ is closed if and only if $S$ contains the limits of its sequences in the sense that if $\langle p_n \rangle_{n=1}^{\infty}$ is a sequence of points from $S$ that converges to $x$, then $x \in S$.

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($\Rightarrow$) Suppose $S$ is closed. Let $\langle p_n \rangle$ be a sequence in $S$ that converges to some point $x$. By problem 3.37, $x$ is an adherent point of $S$, and since $S$ contains all its adherent points by problem 3.35, $x$ is in $S$. Therefore, $S$ contains the limits of its sequences.

($\Leftarrow$) Suppose $S$ contains the limits of its sequences. Let $p$ be an adherent point of $S$. By problem 3.36, we can find a sequence of points $\langle p_n \rangle$ in $S$ that converges to $p$. Since $S$ contains the limits of its sequences, $p$ is in $S$. So $S$ contains all its adherent points, and therefore by problem 3.35, $S$ is closed. $\square$

**Problem 3.39** Let $F$ be a closed subset of $\mathbb{R}$ and $f : \mathbb{R} \to \mathbb{R}$ a polynomial. Show that

$$S := f^{-1}[F] = \{x : f(x) \in F\}$$

is a closed subset of $\mathbb{R}$.

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Let $\langle p_n \rangle$ be a sequence of points in $S$ that converge to $p$. By definition of $S$, we have $f(p_n) \in F$ for all $n$. We have also previously shown that $\lim_{n\to\infty} f(p_n) = f(p)$, and since $F$ is closed, we have from problem 3.39 that $f(p) \in F$ since $\langle f(p_n) \rangle$ is in $F$. Thus, $p \in S$ by definition, so $p$ is the limit of a sequence in $S$. Therefore, $S$ contains the limits of its sequences, so by problem 3.39, $S$ is closed. $\square$

**Problem 3.40** Prove that every convergent sequence is a Cauchy sequence.

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Suppose $\langle p_n \rangle$ is a sequence that converges to $p$. Let $\varepsilon > 0$. By definition, there exists some $N$ such that for all $n > N$, $d(p_n, p) < \frac{\varepsilon}{2}$. So for all $m, n > N$, we have from the triangle inequality that

$$d(p_n, p_m) \leq d(p_n, p) + d(d_m, p) < \frac{\varepsilon}{2} + \frac{\varepsilon}{2} = \varepsilon.$$

Therefore, by definition $\langle p_n \rangle$ is Cauchy. $\square$

**Problem 3.41** Let $E = (0, 1)$ be the open unit interval with metric $d(x, y) = |x - y|$. Then show that the sequence $\langle 1/n \rangle_{n=1}^{\infty}$ is a Cauchy sequence that is not convergent to any point of $E$.

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Suppose (toward contradiction) that there does exist some $x \in (0, 1)$ such that $\langle 1/n \rangle$ converges to $x$. So $x > 0$. Consider $\varepsilon := \frac{x}{2}$. Since the sequence converges to $x$, there exists an $N$ such that for all $n > N$, we have $\left|\frac{1}{n} - x\right| < \varepsilon$. So we have

$$x - \frac{1}{n} \leq \left|\frac{1}{n} - x\right| < \frac{x}{2} \implies \frac{x}{2} < \frac{1}{n}$$

for all $n > N$, but this contradicts Archimedes. $\square$

**Problem 3.42** Let $\langle p_n \rangle_{n=1}^{\infty}$ be a Cauchy sequence in the metric space $E$, such that some subsequence of $\langle p_{n_k} \rangle_{k=1}^{\infty}$ converges. Prove that the original sequence $\langle p_n \rangle_{n=1}^{\infty}$ converges.

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Since $\langle p_{n_k} \rangle$ converges to some $p$, there exists a $K$ such that for all $k > K$, we have $d(p_{n_k}, p) < \frac{\varepsilon}{2}$. Since $\langle p_n \rangle$ is Cauchy, there exists an $N$ such that for all $m, n > N$, we have $d(p_m, p_n) < \frac{\varepsilon}{2}$. Let $n > N$, and choose a $k$ such that $k > K$ and $n_k > N$ (which will exist because $n_k$ is strictly increasing and discrete). Since $n, n_k > N$, we have $d(p_n, p_{n_k}) < \frac{\varepsilon}{2}$, and since $k > K$, we have $d(p_{n_k}, p) < \frac{\varepsilon}{2}$. Thus, from the triangle inequality we have

$$d(p_n, p) \leq d(p_n, p_{n_k}) + d(p_{n_k}, p) < \frac{\varepsilon}{2} + \frac{\varepsilon}{2} = \varepsilon.$$

Therefore, $\langle p_n \rangle$ converges to $p$ by definition.

**Problem 3.43** Let $\langle p_n \rangle_{n=1}^{\infty}$ be a convergent sequence in the metric space $E$. Let $\langle p_{n_k} \rangle_{k=1}^{\infty}$ be a subsequence of this sequence. Prove that $\langle p_{n_k} \rangle_{k=1}^{\infty}$ is also convergent and has the same limit as the original sequence.

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Let $\varepsilon > 0$. Since $\langle p_n \rangle$ converges, there exists $p \in E$, $N$ such that for all $n > N$, $d(p_n, p) < \varepsilon$. Let $k > N$. Then, as we have shown in class, we have $n_k \geq k > N$, so we have $d(p_{n_k}, p) < \varepsilon$ by our choice of $N$. Therefore, $\langle p_{n_k} \rangle$ converges to $p$. $\square$