MATH 552: Section 001
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# MATH 552 Homework 8ˆ

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**Problem 46.9** Let $C$ denote the positively oriented unit circle $|z| = 1$ about the origin.
(a) Show that if $f(z)$ is the principal branch

$$z^{-3/4} = \exp\left[-\frac{3}{4}\operatorname{Log} z\right] \quad (|z| > 0, -\pi < \operatorname{Arg} z < \pi)$$

of $z^{-3/4}$, then

$$\int_C f(z)dz = 4\sqrt{2}i.$$

(b) Show that if $g(z)$ is the branch

$$z^{-3/4} = \exp\left[-\frac{3}{4}\operatorname{Log} z\right] \quad (|z| > 0, 0 < \arg z < 2\pi)$$

of the same power function as in part (a), then

$$\int_C g(z)dz = -4 + 4i.$$

This exercise demonstrates how the value of an integral of a power function depends in general on the branch that is used.

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Solution.

(a) Let $z = e^{i\theta}$, and consequentially $\frac{dz}{d\theta} = ie^{i\theta}$:

$$\begin{aligned}
\int_C e^{-\frac{3}{4}\log z} dz &= \int_{-\pi}^{\pi} e^{-\frac{3}{4}[i\theta]} ie^{i\theta} d\theta && \text{(parameterizing } z\text{)} \\
&= \int_{-\pi}^{\pi} ie^{i\frac{\theta}{4}} d\theta && \text{(combining exponents)} \\
&= 4\int_{-\pi}^{\pi} \frac{1}{4} ie^{i\frac{\theta}{4}} d\theta && \text{(multiplying by } \tfrac{4}{4}\text{)} \\
&= 4\left[e^{i\frac{\theta}{4}}\right]_{-\pi}^{\pi} && \text{(integrating)} \\
&= 4\left[e^{i\frac{\pi}{4}} - e^{-i\frac{\pi}{4}}\right] \\
&= 4\left[\frac{\sqrt{2}}{2} + i\frac{\sqrt{2}}{2} - \left(\frac{\sqrt{2}}{2} - i\frac{\sqrt{2}}{2}\right)\right] && \text{(using Euler's formula)} \\
&= 4\sqrt{2}i.
\end{aligned}$$

(b) Let $z = e^{i\theta}$, and consequentially $\frac{dz}{d\theta} = ie^{i\theta}$:

$$\begin{aligned}
\int_C e^{-\frac{3}{4}\log z} dz &= \int_0^{2\pi} e^{-\frac{3}{4}[i\theta]} ie^{i\theta} d\theta && \text{(parameterizing } z\text{)} \\
&= \int_0^{2\pi} ie^{i\frac{\theta}{4}} d\theta && \text{(combining exponents)} \\
&= 4\int_0^{2\pi} \frac{1}{4} ie^{i\frac{\theta}{4}} d\theta && \text{(multiplying by } \tfrac{4}{4}\text{)} \\
&= 4\left[e^{i\frac{\theta}{4}}\right]_0^{2\pi} && \text{(integrating)} \\
&= 4\left[e^{i\frac{\pi}{2}} - e^{-i0}\right] \\
&= 4[i-1] && \text{(using Euler's formula)} \\
&= -4+4i.
\end{aligned}$$