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MATH 701: Section 001
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MATH 701 Exam 2

Problem 1 Prove that if H has finite index n then there is a normal subgroup K of G with $K \leq H$ and $[G : K] \leq n!$.

Let $H \leq G$ with $[G : H] = n$, let π_H be the permutation representation afforded by multiplication on the set of left cosets of H, and let $K := \ker \pi_H$. By Theorem 4.3 from the textbook, K is a normal subgroup of G contained in H (so $K \leq H$). By the first isomorphism theorem, G/K is isomorphic to $\pi_H(G)$. We have that $\pi_H(G)$ is isomorphic to a subgroup of S_n, since there are n left cosets of H in G. Therefore, we have

$$[G : K] = |G/K| = |\pi_H(G)| \leq |S_n| = n!.$$

$\square$

Problem 2 Prove that if there exists a chain of subgroups $G_1 \leq G_2 \leq \ldots \leq G$ such that $G = \bigcup_{i=1}^{\infty} G_i$ and each G_i is simple then G is simple.

Suppose there is some $N \trianglelefteq G$ with $N \neq \{e\}$ and $N \neq G$. Then there exists some $x \in G \setminus N$. Define $f : G \to \mathbb{N}$ by, for all $a \in G$, $f(a) := \min\{i \in \mathbb{N} \mid a \in G_i\}$, which is well-defined by G being a union of the G_i's and by the well-ordering principle.

Case 1: $f(N)$ has a maximum element m. We claim not every G_i can be simple. By choice of m, we have $N \subseteq G_{m'}$ for all $m' \geq m$, and since $N \trianglelefteq G$, we have $N \trianglelefteq G_{m'}$ for all $m' \geq m$. Consider $n := \max\{m, f(x)\}$. Then $N \trianglelefteq G_n$ since $n \geq m$, and since $x \in G_n$ we have $N \neq G_n$. We assumed $N \neq \{e\}$, so N is a non-trivial normal subgroup of G_n and thus G_n is not simple, a contradiction.

Case 2: $f(N)$ has no maximum element. We claim we must have $N = G$. Let $a \in G$, and let $b \in N$ such that $f(a) < f(b)$. Consider $H := G_{f(b)}$ (so $a, b \in H$). We have $N \cap H \trianglelefteq H$ by the second isomorphism theorem, and since H is simple and $N \cap H \neq 1$ (since $b \in N$ and $b \neq e$ as $f(b) > 1$) we must have $N \cap H = H$. Thus, $H \subseteq N$, so $a \in N$. Therefore, $G \subseteq N$, so $N = G$, a contradiction. $\square$

Problem 3 Let $G = \mathbb{Z}_{60} \times \mathbb{Z}_{45} \times \mathbb{Z}_{12} \times \mathbb{Z}_{36}$. Find the number of elements of order 2 and the number of subgroups of index 2 in G.

Each of $\mathbb{Z}_{60}$, $\mathbb{Z}_{12}$, and $\mathbb{Z}_{36}$ has exactly one element with order 2 ($\overline{30}$, $\overline{6}$, and $\overline{18}$, respectively), and $\mathbb{Z}_{45}$ has no elements of order 2. It is known that the order of an element direct product is the least common multiple of the order of the elements in the tuple: thus, any tuple with order 2 must have at least one entry of order 2 and all entries with order at most 2. So the elements of order 2 in G are in correspondence with the non-zero bit strings of length 3, and therefore, there are $2^3 - 1 = 7$ elements of order 2 in G. In particular, they are

1. $(\overline{30}, \overline{0}, \overline{0}, \overline{0})$
2. $(\overline{0}, \overline{0}, \overline{6}, \overline{0})$
3. $(\overline{0}, \overline{0}, \overline{0}, \overline{18})$

4. $(\overline{30}, \overline{0}, \overline{6}, \overline{0})$

5. $(\overline{30}, \overline{0}, \overline{0}, \overline{18})$

6. $(\overline{0}, \overline{0}, \overline{6}, \overline{18})$

7. $(\overline{30}, \overline{0}, \overline{6}, \overline{18})$.

It is known (cf. Problem 5.2.8) that for any prime p and any finite abelian group, the number of subgroups of order p equals the number of subgroups of index p. Every element of G of order 2 generates a subgroup of order 2, and no other subgroups are possible, so there are 7 subgroups of order 2. Therefore, there are also 7 subgroups of index 2. $\square$

Problem 4 Classify groups G of the following orders: (i) $|G| = 6$, (ii) $|G| = 9$, (iii) $|G| = 15$.

(i) Let G be a group with $|G| = 6$. Since $6 = 3 \cdot 2$, by Sylow's theorem we have $n_3 \equiv 1 \pmod 3$. So $n_3 \in \{1, 4\}$, and since 4 does not divide 6 we have $n_3 = 1$. So there is a unique $H \leq G$ with $|H| = 3$ and $H = \langle x \rangle$ for some $x \in G$, and H is normal in G by Corollary 4.20. Also by Sylow's theorem, there is a Sylow 2-subgroup of G. Let y be the generator of such a subgroup (so $y^2 = e$). Since H is normal, we have $yxy^{-1} \in H = \{e, x, x^2\}$.

Case 1: $yxy^{-1} = e$. Then $x = y^{-1}y = e$, contradicting $|x| = 3$, so this is not possible.

Case 2: $yxy^{-1} = x$. Then $yx = xy$. We compute the order of xy, which will be 1, 2, 3, or 6 by Lagrange. First, $(xy)^1 \neq e$ since this would imply $|x| = |y|$. Also, we have $(xy)^2 = x^2y^2 = x^2 \neq e$ using $|y| = 2$ and $|x| = 3$. Similarly, we have $(xy)^3 = x^3y^3 = y^3 = y \neq e$. So $|xy| = 6 = |G|$ is the only option, and thus G is cyclic, so $G \cong Z_6$ in this case.

Case 3: $yxy^{-1} = x^2$. Since $x^{-1} = x^2 = yxy^{-1}$, we have $x^{-1}y = yx$. Consider yx. We cannot have $yx = e$ since $|y| \neq |x|$. Also, $(yx)^2 = yxx^{-1}y = y^2 = e$, so $|yx| = 2$. Now consider yx^2. We cannot have $yx = e$ since $|y| \neq |x|$. Also,

$$(yx^2)^2 = yx^2yx^2 = y(yxy^{-1})y(yxy^{-1}) = y^2xyxy^{-1} = xyxy = xx^{-1}yy = e,$$

so $|yx^2| = 2$. We cannot have $y = yx$ or $y = yx^2$ since $x \neq e \neq x^2$, and we cannot have $yx = yx^2$ since $x \neq e$. Thus, we have that e, x, x^2, y, yx, and yx^2 are all distinct and thus are the six elements of G. We can map

$$e \mapsto (1),\ x \mapsto (1\ 2\ 3),\ x^2 \mapsto (1\ 3\ 2),\ y \mapsto (1\ 2),\ yx \mapsto (2\ 3),\ yx^2 \mapsto (1\ 3),$$

so $G \cong S_3$ in this case.

Therefore, the two groups of order 6 up to isomorphism are Z_6 and S_3. If G is an order 6 group and has an element of order 6, then $G \cong Z_6$, and otherwise $G \cong S_3$. $\square$

(ii) Let G be a group with $|G| = 9$.

Case 1: G is cyclic. Then $G \cong Z_9$.

Case 2: G is not cyclic. By Lagrange, every element has order 1, 3, or 9, so since G is not cyclic every element has order 3 or is the identity. Let $x \in G$ and let $y \in G \setminus \langle x \rangle$. Then x^iy^j are pairwise-distinct for all $i, j \in \{0, 1, 2\}$, so every element in G can be written in this form. We determine how to write

yx in this form. Since $|x| = |y| = 3$, we have must have $yx = x^i y^j$ with i, j non-zero. So we have $yx \in \{xy, x^2y, xy^2, x^2y^2\}$. If $yx = x^2y$, then $yxy^{-1} = x^2$, so we would have

$$x = y^3xy^{-3} = y^2(yxy^{-1})y^{-2} = y^2x^2y^{-2} = y(yxy^{-1})(yxy^{-1})y^{-1} = yx^4y^{-1} = (x^2)^4 = x^8 = x^2,$$

a contradiction. If $yx = xy^2$, then we would have $x^{-1}yx = y^2$, which by the same reasoning would imply $y = y^2$, a contradiction. If $yx = x^2y^2$, then $yx = x^{-1}y^{-1} = (yx)^{-1}$, contradicting $|yx| = 3$. So we must have $yx = xy$, and we can conclude that G is abelian. (More generally, we proved in class that any group of order p^2 where p is prime is abelian).

Thus, we can apply the fundamental theorem of finitely generated abelian groups. The only possibility is $G \cong Z_3 \times Z_3$ since $9 = 3^2$.

Therefore, the two groups of order 9 up to isomorphism are Z_9 and $Z_3 \times Z_3$. If G is an order 9 group and has an element of order 9, then $G \cong Z_9$, and otherwise $G \cong Z_3 \times Z_3$. $\square$

(iii) Let G be a group with $|G| = 15$. Since $15 = 3 \cdot 5$, there exists a Sylow 3-subgroup H and a Sylow 5-subgroup K, and we have $n_3 \equiv 1 \pmod 3$ and $n_5 \equiv 1 \pmod 5$. So $n_3 \in \{1, 4, 7, 11, 14\}$, and $n_5 \in \{1, 6, 11\}$. None of these numbers divide 15 other than 1, so $n_3 = n_5 = 1$. So by Corollary 4.20, H and K are both normal in G. Since H and K are both cyclic, every non-identity element in H will have order 3 and every non-identity element in K will have order 5. So $H \cap K = \{e\}$. By Theorem 5.9, then $HK \cong H \times K$.

Since $|H \times K| = 15$, we have $G = HK$, so $G \cong H \times K$. Since $H \cong Z_3$ and $K \cong Z_5$ (both are prime and thus cyclic), we have $G \cong Z_3 \times Z_5$. Since 3 and 5 are coprime, by the Chinese Remainder Theorem we have $G \cong Z_{15}$. Therefore, the only group of order 15 up to isomorphism is Z_{15}. $\square$

Problem 5 Find a conjugacy class C of the alternating group A_5 such that $A_5 = C^2$. Please explain why the class you chose satisfies this property.

Let C be the conjugacy class with representative $(1\ 2\ 3)$. Since $C_{A_5}\big((1\ 2\ 3)\big) = \langle (1\ 2\ 3) \rangle$ has order 3, and $|A_5| = 60$, C has 20 elements. Since conjugating a 3-cycle by any permutation yields a 3-cycle, and there are 20 3-cycles in A_5, C contains all 3-cycles in A_5. We claim that $A_5 = C^2$.

Let $\tau \in A_5$. We show that τ can be written as the product of two cycles in C.

Case 1: $\tau = (1)$. Then $\tau = (1\ 2\ 3)(1\ 3\ 2)$.

Case 2: τ is a 3-cycle $(a\ b\ c)$. Then $\tau = (a\ c\ b)(a\ c\ b)$.

Case 3: τ is the product of two disjoint transpositions $(a\ b)$ and $(c\ d)$. Then $\tau = (a\ b\ c)(b\ c\ d)$.

Case 4: τ is a 5-cycle $(a\ b\ c\ d\ e)$. Then $\tau = (a\ b\ c)(c\ d\ e)$.

Therefore, $A_5 \subseteq C^2$. Clearly $C^2 \subseteq A_5$, so $A_5 = C^2$. $\square$