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CSCE 355: Section 001
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CSCE 355 Homework 1

Problem 1 let $A := \{1, 2, 3, 4\}$ and $B := \{2, 5\}$. What are (a) $A \cup B$, (b) $A \cap B$, (c) $A - B$, and (d) $A \triangle B$? What are (e) $A \times B$ and (f) 2^B? In each case, also give the cardinality of the set.

(a) We have $A \cup B = \{1, 2, 3, 4, 5\}$ with cardinality 5.

(b) We have $A \cap B = \{2\}$ with cardinality 1.

(c) We have $A - B = \{1, 3, 4\}$ with cardinality 3.

(d) We have $A \triangle B = \{1, 3, 4, 5\}$ with cardinality 4.

(e) We have $A \times B = \big\{(1, 2), (1, 5), (2, 2), (2, 5), (3, 2), (3, 5), (4, 2), (4, 5)\big\}$ with cardinality 8.

(f) We have $2^B = \big\{\emptyset, \{2\}, \{5\}, \{2, 5\}\big\}$ with cardinality 4.

Problem 2 True or false: $2^\emptyset = \emptyset$. Explain.

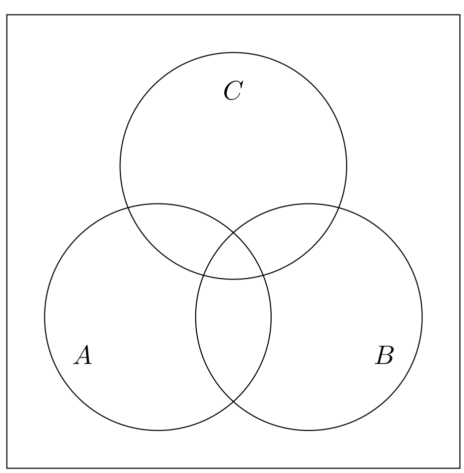
This is false. We have $2^\emptyset = \{\emptyset\}$, which has cardinality 1 while $\emptyset$ has cardinality 0.

Problem 3 Using just braces and commas, write the set $2^{2^{\{\emptyset\}}}$ in "long hand."

We have $2^{\{\emptyset\}} = \{\emptyset, \{\emptyset\}\}$. Using this, we have $2^{2^{\{\emptyset\}}} = \big\{\emptyset, \{\emptyset\}, \{\{\emptyset\}\}, \{\emptyset, \{\emptyset\}\}\big\}$. Now using only brackets and commas, we can write

$$2^{2^{\{\emptyset\}}} = \Bigg\{\{\}, \{\{\}\}, \Big\{\{\{\}\}\Big\}, \Big\{\{\}, \{\{\}\}\Big\}\Bigg\}.$$

Problem 4 Using the figure shown below as a template, draw and fill in a Venn diagram to illustrate each of the expressions below involving sets A, B, C. That is, shade the regions that are part of the expression (one Venn diagram per expression):



(a) $A \cap B \cap C$

(b) $A \cup B \cup C$

(c) $A \cap (B \cup C)$

(d) $A - (B \cap C)$

(e) $(A \cup B) - C$

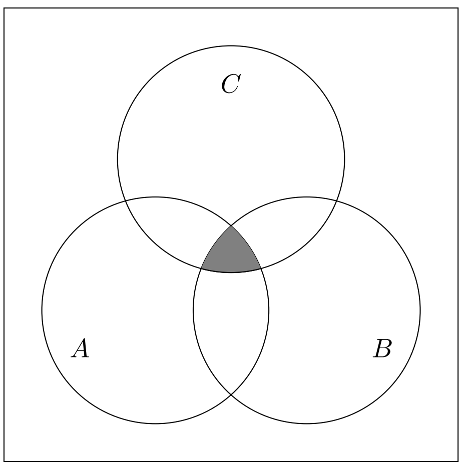
(f) $A - (B - C)$

(g) $(A \triangle B) \triangle C$

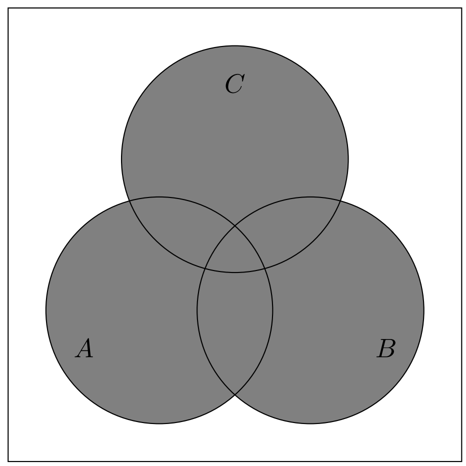
(h) $(A \cap B) \cup (A \cap C)$

(i) $A \triangle (B \triangle C)$

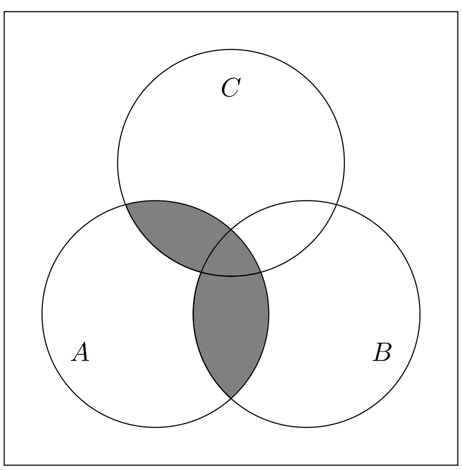
(j) $A \triangle (B \cap C)$



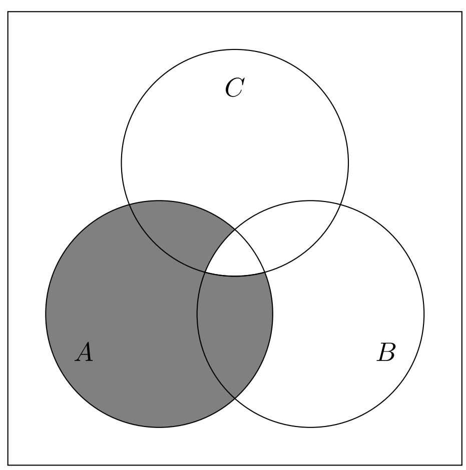
(a)



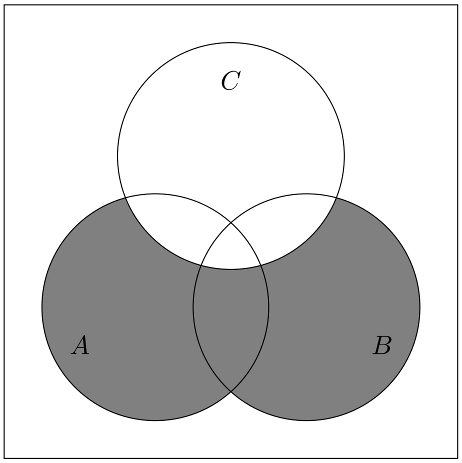
(b)



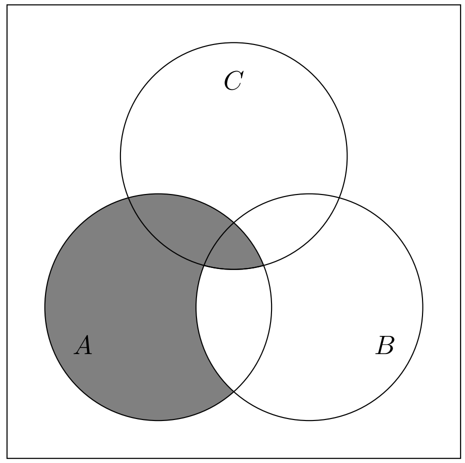
(c)



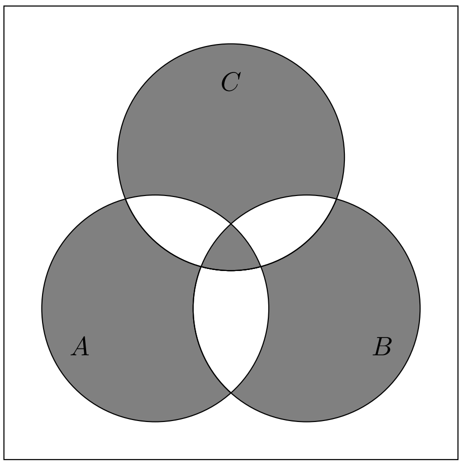
(d)



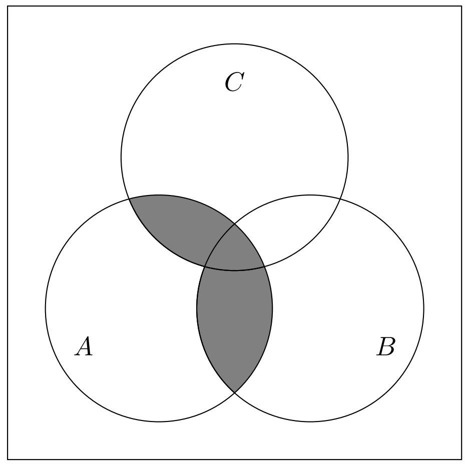
(e)



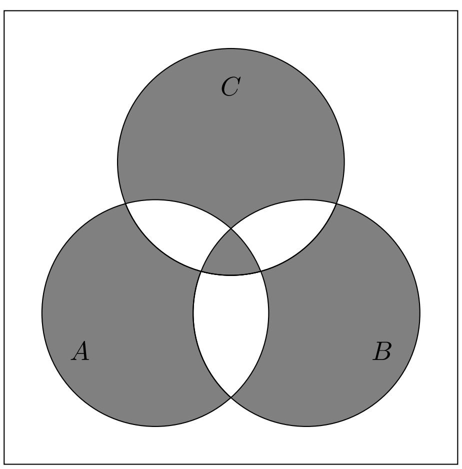
(f)



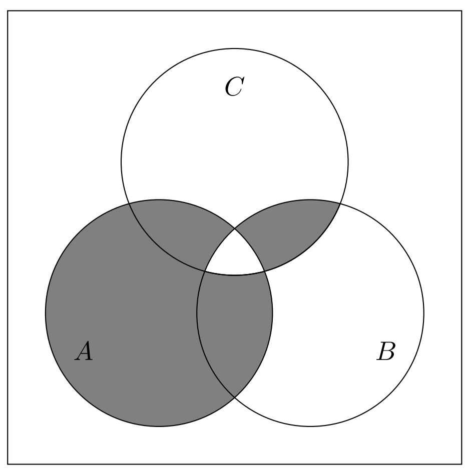
(g)



(h)



(i)



(j)

Problem 5 What set theoretic identities holding for all A, B, C are shown by your Venn diagrams in the last problem?

We can see that the diagrams for **(c)** and **(h)** are the same, so we have

$$A \cap (B \cup C) = (A \cap B) \cup (A \cap C).$$

Also, we can see that the diagrams from **(g)** and **(i)** are the same, so we have

$$(A \triangle B) \triangle C = A \triangle (B \triangle C).$$

Problem 6 Show that the symmetric difference operation $\triangle$ on sets is commutative and associative, and that $A \triangle A = \emptyset$ for all sets A.

(a) Let A, B, C be sets. Then, since union and intersection are commutative, by definition we have

$$A \triangle B = (A \cup B) - (A \cap B) = (B \cup A) - (B \cap A) = B \triangle A,$$

which shows commutativity.

We now show associativity. We note that we have $C - D = C \cap D^c$ essentially by definition for any sets C, D, which we can use to write

$$\begin{aligned}
(A \triangle B) \triangle C &= \big[(A \triangle B) - C\big] \cup \big[C - (A \triangle B)\big] && \text{(definition)}\\
&= \Big(\big[(A - B) \cup (B - A)\big] - C\Big) \cup \Big(C - \big[(A \cup B) - (A \cap B)\big]\Big) && \text{(both definitions)}\\
&= \Big(\big[(A \cap B^c) \cup (B \cap A^c)\big] \cap C^c\Big) \cup \Big(C \cap \big[(A \cup B) \cap (A \cap B)^c\big]^c\Big) && \text{(from above)}\\
&= \Big(\big[(A \cap B^c) \cup (B \cap A^c)\big] \cap C^c\Big) \cup \Big(C \cap \big[(A \cup B)^c \cup (A \cap B)\big]\Big) && \text{(de Morgan)}\\
&= \Big(\big[(A \cap B^c) \cup (B \cap A^c)\big] \cap C^c\Big) \cup \Big(C \cap \big[(A^c \cap B^c) \cup (A \cap B)\big]\Big) && \text{(de Morgan)}\\
&= \big[(A \cap B^c \cap C^c) \cup (B \cap A^c \cap C^c)\big] \cup \big[(C \cap A^c \cap B^c) \cup (C \cap A \cap B)\big] && \text{(distributivity)}\\
&= (A \cap B^c \cap C^c) \cup (B \cap C^c \cap A^c) \cup (C \cap B^c \cap A^c) \cup (A \cap B \cap C) && \text{(commutativity)}\\
&= (A \cap B^c \cap C^c) \cup (A \cap B \cap C) \cup (B \cap C^c \cap A^c) \cup (C \cap B^c \cap A^c) && \text{(commutativity)}\\
&= \Big(A \cap \big[(B^c \cap C^c) \cup (B \cap C)\big]\Big) \cup \Big(\big[(B \cap C^c) \cup (C \cap B^c)\big] \cap A^c\Big) && \text{(distrubitivity)}\\
&= \Big(A \cap \big[(B \cup C)^c \cup (B \cap C)\big]\Big) \cup \Big(\big[(B \cap C^c) \cup (C \cap B^c)\big] \cap A^c\Big) && \text{(de Morgan)}\\
&= \Big(A \cap \big[(B \cup C) \cap (B \cap C)^c\big]^c\Big) \cup \Big(\big[(B \cap C^c) \cup (C \cap B^c)\big] \cap A^c\Big) && \text{(de Morgan)}\\
&= \Big(A - \big[(B \cup C) - (B \cap C)\big]\Big) \cup \Big(\big[(B - C) \cup (C - B)\big] - A\Big) && \text{(from above)}\\
&= \big[A - (B \triangle C)\big] \cup \big[(B \triangle C) - A\big] && \text{(both definitions)}\\
&= A \triangle (B \triangle C), && \text{(definition)}
\end{aligned}$$

which shows associativity. This matches what we expected from the Venn diagram. □

(b) Let A be a set. We have $A \cup A = A \cap A = A$, so $A \triangle A = (A \cup A) - (A \cap A) = A - A = \emptyset$. □

Problem 7 Let $A := \{1, 2, 3, 4\}$ as in the first problem, above. Let $R := \{(1, 2), (2, 3), (3, 4)\}$. ($R$ is a relation on A.)

(a) Add the fewest possible ordered pairs to R to make a reflexive relation (on A).

(b) Add the fewest possible ordered pairs to R to make a symmetric relation.

(c) Add the fewest possible ordered pairs to R to make a transitive relation.

(d) Add the fewest possible ordered pairs to R to make an equivalence relation on A. How many equivalence classes are there?

(a) We will need to add $(1, 1)$, $(2, 2)$, $(3, 3)$, and $(4, 4)$.

(b) We will need to add $(2, 1)$, $(3, 2)$, and $(4, 3)$.

(c) We will need to add $(1, 3)$, $(1, 4)$, and $(2, 4)$.

(d) To extend R to an equivalence relation, R must be all of $A \times A$. So we will need to add $(4)(4) - 3 = 13$ pairs, and there will only be one equivalence class.

Problem 8 Same as the problem 7, but now let $R := \{(1, 2), (2, 3), (3, 1), (4, 4)\}$.

(a) We will need to add $(1, 1)$, $(2, 2)$, and $(3, 3)$.

(b) We will need to add $(2, 1)$, $(3, 2)$, and $(1, 3)$.

(c) We will need to add $(1, 3)$, $(2, 1)$, and $(3, 2)$. Then, we will need to add $(3, 3)$, $(2, 2)$, and $(1, 1)$.

(d) We will need to add the six pairs from **(c)**. Then, we will have two equivalence classes: $\{1, 2, 3\}$ and $\{4\}$.

Problem 9 Give an example of a nonempty binary relation on A that is symmetric and transitive but not reflexive.

This is satisfied by $\big\{(1, 1), (1, 2), (2, 1), (2, 2)\big\}$.

Problem 10 Suppose $\leq$ is a quasiorder on some set A. For every $a, b \in A$, define

$$a \equiv b \iff (a \leq b \text{ and } b \leq a) .$$

Show that $\equiv$ is an equivalence relation on A.

- Reflexivity: Let $a \in A$. Since $\leq$ is a quasiorder, it is reflexive, so $a \leq a$ (and $a \leq a$). Thus $a \equiv a$.
- Symmetry: Let $a, b \in A$ and suppose that $a \equiv b$. Then by definition, $a \leq b$ and $b \leq a$. So $b \leq a$ and $a \leq b$, and thus $b \equiv a$.
- Transitivity: Let $a, b, c \in A$ and suppose that $a \equiv b$ and $b \equiv c$. So we have $a \leq b$, $b \leq a$, $b \leq c$, and $c \leq b$. Since $\leq$ is a quasiorder, it is transitive, so $a \leq b$ and $b \leq c$ implies $a \leq c$. Also since $c \leq b$ and $b \leq a$, we have $c \leq a$. Therefore, $a \equiv c$.

Since $\equiv$ is reflexive, symmetric, and transitive, it is an equivalence relation. $\square$

Problem 11 Let $A := \{1, 2, 3, 4\}$ and $B := \{2, 5\}$ be as in the first problem, above. Give an example of a one-to-one function $f : B \to A$. How many such functions are there?

One such function is $\big\{(2, 1), (5, 2)\big\}$. There are 4 choices for the 2 to map to, and 3 choices for the 5 to map to since it cannot map to the same as 2. By the product rule of combinatorics, then, there are 12 one-to-one functions from B to A.

Problem 12 Let $A := \{1, 2, 3, 4\}$ and $B := \{2, 5\}$ be as in the first problem, above. Give an example of an onto function $g : A \to B$. How may such functions are there?

One such function is $\{(1,2),(2,2),(3,5),(4,5)\}$. There are $2^4 = 16$ total functions from A to B, because each of the four elements in A have two choices of where to map to. The function that maps all elements to 2 and the other function that maps all elements to 5 are the only functions that are not onto, so there are $16 - 2 = 14$ onto functions.

Problem 1.6.1 Give inductive proofs of the following for all strings x, y, and z. You may assume without proof standard proofs about natural numbers.

(a) $|x| \geq 0$.

(b) $|xy| = |x| + |y|$.

(c) If $xz = yz$, then $x = y$.

(d) If $xy = xz$, then $y = z$.

(a) We will induct on x.

Base Case: Let $x = \varepsilon$. Then $|x| = 0$ by definition, so $|x| \geq 0$.

Induction Step: Let $x \neq \varepsilon$, and let $u \in \Sigma^*$ be the principal prefix of x, which is well-defined since $x \neq \varepsilon$. Suppose that $|u| \geq 0$. By definition, $|x| = |u| + 1$, so we have $|x| \geq 1$. So $|x| \geq 0$. □

(b) We will induct on y. Let x be arbitrary.

Base Case: Let $y = \varepsilon$. Then $xy = x\varepsilon = x$ by the result proved in the notes, so

$$|xy| = |x| = |x| + 0 = |x| + |\varepsilon| = |x| + |y|,$$

so the result holds.

Induction Step: Let $y \neq \varepsilon$, and let $u \in \Sigma^*$, $a \in \Sigma$ such that $y = ua$. Suppose that $|xu| = |x| + |u|$. By definition, we have $|y| = |u| + 1$. We can then use the inductive hypothesis to write

$$\begin{aligned}
|xy| &= \big|x(ua)\big| \\
&= \big|(xu)a\big| && \text{(associativity)} \\
&= |xu| + 1 && (a \text{ is last symbol of } xu) \\
&= (|x| + |u|) + 1 && \text{(inductive hypothesis)} \\
&= (|x| + |y| - 1) + 1 && (|y| = |u| + 1) \\
&= |x| + |y|.
\end{aligned}$$

□

(c) We will induct on z.

Base Case: Let $z = \varepsilon$. Let x, y be such that $xz = yz$. Since $xz = x\varepsilon = x$ and $yz = y\varepsilon = y$ by definition, we have $x = y$.

Induction Step: Let $z \neq \varepsilon$, let $u \in \Sigma^*$, $a \in \Sigma$ such that $z = ua$, and let x, y be such that $xz = yz$. Suppose that if $xu = yu$, then $x = y$. By our assumptions, we can write $xua = yua$. The primary prefix of this string is unique, and since the primary prefix can be be written as either xu or yu, we have $xu = yu$. Then, by the inductive hypothesis, we have $x = y$. □

(d) We will induct on y.

Base Case: Let $y = \varepsilon$, and let x, z be such that $xy = xz$. From **(b)**, we have $|xy| = |x| + |y|$ and $|xz| = |x| + |z|$, so $|x| + |y| = |x| + |z|$ and thus $|y| = |z|$. Since $y = \varepsilon$ is the only string with length 0, and $0 = |y| = |z|$, we have $z = \varepsilon$ and thus $y = z$.

Induction Step: Let $y \neq \varepsilon$, let $u \in \Sigma^*$, $a \in \Sigma$ such that $y = ua$, and let x, z be such that $xy = xz$. Suppose that for any strings q, r, we have

$$qu = qr \implies u = r.$$

Let $w \in \Sigma^*$, $b \in \Sigma$ be such that $z = wb$. By our assumptions, we can write

$$\begin{aligned} & xy = xz \\ \implies\ & xua = xwb \\ \implies\ & xu = xw && \text{(both primary prefixes of the same string)} \\ \implies\ & u = w && \text{(induction hypothesis)} \end{aligned}$$

Also, since $xua = xwb$, we have $a = b$ since both are the last character of the same string. Therefore, we have

$$y = ua = wb = z.$$

$\square$

Problem 14 The reversal of a string x (denoted x^R) is the string formed by putting the symbols of x in reverse order. (For example, $(abcb)^R = bcba$.)

(a) Give a precise, inductive definition of the reversal x^R of a string.

(b) Using your definition, give proofs by induction that $\left|x^R\right| = |x|$ and that $(x^R)^R = x$ for any string x.

(a) Let x be a string. If $x = \varepsilon$, then $x^R := \varepsilon$. If $x \neq \varepsilon$, then there exists $y \in \Sigma^*$, $a \in \Sigma$ such that $x = ya$, and $x^R := a(y^R)$.

(b)

- Let x be a string. We first prove that $|x| = \left|x^R\right|$. We will induct on x.

  Base Case: Let $x = \varepsilon$. Then $x^R = \varepsilon$, so $|x| = |\varepsilon| = \left|x^R\right|$.

  Induction Step: Let $x \neq \varepsilon$, and let $u \in \Sigma^*, a \in \Sigma$ such that $x = ua$, which then implies that $x^R = au^R$. Suppose that $|u| = \left|u^R\right|$. We proved earlier that $|wz| = |w| + |z|$ for any strings w, z, so we can write

  $$|x| = |ua| = |u| + |a| = |a| + |u| = |a| + \left|u^R\right| = \left|au^R\right| = \left|x^R\right|.$$

  $\square$

- We will now prove a lemma: we claim that for any $a \in \Sigma$, $z \in \Sigma^*$, we have $(az)^R = z^R a$. We will prove this by induction on z. Let $a \in \Sigma$.

  Base Case: Let $z = \varepsilon$. Then $az = a\varepsilon = a$, and we have proved before that $\varepsilon a = a$, so we have

  $$(az)^R = (a\varepsilon)^R = a^R = \varepsilon a = \varepsilon^R a = z^R a.$$

Induction Step: Let $z \neq \varepsilon$, and let $y \in \Sigma^*, a \in \Sigma$ such that $z = yb$. Suppose that $(ay)^R = y^R a$. Then, we have

$$\begin{aligned}
(az)^R &= (ayb)^R \\
&= b(ay)^R && \text{(as defined in } \mathbf{(a)}\text{)} \\
&= by^R a && \text{(induction hypothesis)} \\
&= (yb)^R a && \text{(as defined in } \mathbf{(a)}\text{)} \\
&= z^R a.
\end{aligned}$$

So the lemma holds.

- We now this to prove that $(x^R)^R$. We will induct on x.

  Base Case: Let $x = \varepsilon$. Then $x^R = \varepsilon^R = \varepsilon$ by definition, and so $(x^R)^R = \varepsilon^R = \varepsilon$ as well. Thus $x = \varepsilon = (x^R)^R$.

  Induction Step: Let $x \neq \varepsilon$, and let $u \in \Sigma^*, a \in \Sigma$ such that $x = ua$. Suppose that $(u^R)^R = u$. Then, we have

$$\begin{aligned}
(x^R)^R &= \Big((ua)^R\Big)^R \\
&= (au^R)^R && \text{(as defined in part (a))} \\
&= (u^R)^R a && \text{(lemma)} \\
&= ua && \text{(induction hypothesis)} \\
&= x. && \text{(as defined)}
\end{aligned}$$

$\square$