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MATH 300 Homework 5

Problem 1

First, let n be an even natural number. Then, $n = 2k$ for some natural number k. Then,

$$\begin{aligned} n^2 + n + 3 &= (2k)^2 + (2k) + 3 \\ &= 4k^2 + 2k + 3 \\ &= 2(2k^2 + k + 1) + 1 \end{aligned}$$

Since $2k^2 + k + 1$ is the product and sum of integers, it must also be an integer. Therefore, $n^2 + n + 3$ can be written as $2j + 1, j \in \mathbb{Z}$ by choosing $j = 2k^2 + k + 1$, so it is odd.

Then, let n be an odd natural number. Then, $n = 2k + 1$ for some natural number k. Then,

$$\begin{aligned} n^2 + n + 3 &= (2k+1)^2 + (2k+1) + 3 \\ &= 4k^2 + 4k + 1 + 2k + 1 + 3 \\ &= 4k^2 + 6k + 5 \\ &= 2(2k^2 + 3k + 2) + 1 \end{aligned}$$

Since $2k^2 + 3k + 2$ is the product and sum of integers, it must also be an integer. Therefore, $n^2 + n + 3$ can be written as $2m + 1, m \in \mathbb{Z}$ by choosing $m = 2k^2 + 3k + 2$, so it is odd.

Since $n^2 + n + 3$ is odd for both even and odd natural numbers, it is odd for natural numbers. $\square$

Problem 2

(a) Let a, b, c be integers such that a divides b and a divides $b + c$. Then, there exist integers k and m such that $ak = b$ and $am = b + c$. Using some algebra,

$$\begin{aligned} 3ak &= 3b && \text{(multiplying LHS/RHS by 3)} \\ 3am &= 3b + 3c && \text{(multiplying LHS/RHS by 3)} \\ 3am - 3ak &= 3b + 3c - 3b \\ a(3m - 3k) &= 3c \end{aligned}$$

Since $3m - 3k$ is the product and difference of integers, it is an integer. Therefore, $3c$ can be written as $an, n \in \mathbb{Z}$ by choosing $n = 3m - 3k$, so a divides $3c$. $\square$

(b) We first write $ax^2 + bx + c = 0$ and solve for x in terms of a, b, c:

$$\begin{aligned}
\left(\sqrt{a}x + \frac{b}{2\sqrt{a}}\right)^2 + c - \frac{b^2}{4a} &= 0 && \text{(completing the square of the LHS)} \\
\left(\sqrt{a}x + \frac{b}{2\sqrt{a}}\right)^2 &= \frac{b^2}{4a} - c \\
\sqrt{a}x + \frac{b}{2\sqrt{a}} &= \pm\sqrt{\frac{b^2}{4a} - c} && \text{(taking square root of both sides)} \\
\sqrt{a}x + \frac{b}{2\sqrt{a}} &= \frac{\pm\sqrt{b^2 - 4ac}}{2\sqrt{a}} && \text{(rearranging)} \\
\sqrt{a}x &= \frac{-b \pm \sqrt{b^2 - 4ac}}{2\sqrt{a}} \\
x &= \frac{-b \pm \sqrt{b^2 - 4ac}}{2a}
\end{aligned}$$

For x to have 2 real solutions, $b^2 - 4ac$ must be positive since $b^2 - 4ac = 0$ would yield a single solution and $b^2 - 4ac < 0$ would yield complex solutions.

Let a, b, c be real numbers such that $ab > 0$ and $bc < 0$. Then, a, b, c are all non-zero, and:

If b is positive, a must be positive to yield $ab > 0$ and c must be negative to yield $bc < 0$.

If b is negative, a must be negative to yield $ab > 0$ and c must be positive to yield $bc < 0$.

Then, ac will either be a positive a times a negative c or a negative a times a positive c, so $ac < 0$.

Since b^2 must be positive if b is real, and $4ac$ must be negative, subtracting $4ac$ from b^2 must yield a positive number.

Therefore, since b^2-4ac (the value under the square root or the discriminant) is greater than 0, $ax^2+bx+c = 0$ has two different real solutions. $\square$

Problem 3

The distance from $(2, 4)$ to $(-1, 5)$ is $\sqrt{(-1-2)^2 + (5-4)^2} = \sqrt{10}$, while the distance from $(2, 4)$ to $(5, 1)$ is $\sqrt{(5-2)^2 + (1-4)^2} = 3\sqrt{2}$.

A circle is defined as the set of points equidistant from the center, so points with different distances from the center cannot possibly be on the same circle, no matter what the radius is. $\square$

Problem 4

Assume there is a positive integer n such that $\frac{n}{n+1} \leq \frac{n}{n+2}$. Then,

$$\begin{aligned}
n(n+2) &\leq n(n+1) \\
n^2 + 2 &\leq n^2 + 1 \\
2 &\leq 1.
\end{aligned}$$

Since $2 \leq 1$ is a contradiction, there cannot be a positive integer n such that $\frac{n}{n+1} \leq \frac{n}{n+2}$. Therefore, for all positive integers n, $\frac{n}{n+1} > \frac{n}{n+2}$. $\square$

Problem 5

(a) Choose $m = -3$ and $n = 1$. Then, $2(-3) + 7(1) = 1$, so by existential generalization there exists integers m and n such that $2m + 7n = 1$. $\square$

(b) We rewrite $15m + 12n$ as $3(5m + 4n)$. Since $5m + 4n$ is the sum and product of integers, it is an integer.

Thus, $3|(15m + 12n)$ because there is always an integer k such that $3k = 15m + 12n$ (choose $k = 5m + 4n$). However, it is not true that $3|2$ ($\frac{2}{3}$ is not an integer), so $15m + 12n$ cannot equal 2. $\square$

(c) Choose $m = -t$ and $n = t$. Then, $15(-t) + 16(t) = 16t - 15t = t$. Therefore, since m and n can always be chosen this way since t is an integer, the claim holds for all t. $\square$

Problem 6

(a) Proof by contradiction:

Assume there is a nonsingular matrix A with a determinant of 0.

$\vdots$

There is a contradiction, so there cannot be a nonsingular matrix with a determinant of 0. Therefore, all nonsingular matrices have a nonzero determinant.

(b) Direct proof:

Assume there are sets A, B, C such that A is a subset of B and B is a subset of C.

$\vdots$

Thus, A is a subset of C. Therefore, for all sets A, B, C, if A is a subset of B and B is a subset of C then A is a subset of C.

(c) Direct proof:

Assume there are matrices A, B such that A and B are invertible.

$\vdots$

Thus, AB is invertible. Therefore, for all matrices A, B, if A and B are invertible, then AB is also invertible.

Problem 7

(a) Let x, y be real numbers.

Assume x, y are even. Then, $x = 2m$ and $y = 2n$ for some $m \in \mathbb{Z}, n \in \mathbb{Z}$.

Then, $x + y = 2m + 2n = 2(m + n)$. Since $m + n$ is the sum of integers, it is an integer.

Since $2m+2n = 2k$ for some $k \in \mathbb{Z}$ by choosing $k = m+n$, $2m+2n$ is even. Therefore, since $x+y = 2m+2n$, $x + y$ is even. $\square$

(b) Let x, y be real numbers.

Assume x, y are even. Then, $x = 2m$ and $y = 2n$ for some $m \in \mathbb{Z}, n \in \mathbb{Z}$.

Then, $xy = (2m)(2n) = 4mn$. Since mn is the product of integers, it is an integer.

Since $4mn = 4k$ for some $k \in \mathbb{Z}$ by choosing $k = mn$, $4|4mn$. Therefore, since $xy = 4mn$, $4|xy$. $\square$

(c) Let x, y be real numbers.

Assume x, y are odd. Then, $x = 2m + 1$ and $y = 2n + 1$ for some $m \in \mathbb{Z}, n \in \mathbb{Z}$.

Then, $x + y = 2m + 1 + 2n + 1 = 2m + 2n + 2 = 2(m + n + 1)$. Since $m + n + 1$ is the sum of integers, it is an integer.

Since $2m + 2n + 2 = 2k$ for some $k \in \mathbb{Z}$ by choosing $k = m + n + 1$, $2m + 2n + 2$ is even. Therefore, since $x + y = 2m + 2n + 2$, $x + y$ is even. $\square$

(d) Let x, y be real numbers.

Assume x is even and y is odd. Then, $x = 2m$ and $y = 2n + 1$ for some $m \in \mathbb{Z}, n \in \mathbb{Z}$.

Then, $xy = (2m)(2n + 1) = 4mn + 2m = 2(2mn + m)$. Since $2mn + m$ is the product of integers, it is an integer.

Since $4mn + 2m = 2k$ for some $k \in \mathbb{Z}$ by choosing $k = 2mn + m$, $4mn + 2m$ is even. Therefore, since $xy = 4mn + 2m$, xy is even. $\square$

(e) Let a be an integer.

Then, there exists an $m \in \mathbb{Z}$ such that $1m = a$ by choosing $m = a$, and there exists an $n \in \mathbb{Z}$ such that $an = a$ by choosing $n = 1$.

Therefore, $1|a$ and $a|a$ for all a. $\square$

(f) Let x be an integer.

Assume x is even. Then, $x = 2k$ for some $k \in \mathbb{Z}$.

Then, $x + 2 = 2k + 2 = 2(k + 1)$. Since $k + 1$ is the sum of integers, it is an integer. As $2k + 2$ can be written as $2m$ for some $m \in \mathbb{Z}$ by choosing $m = k + 1$, $2k + 2$ is even.

Therefore, since $x + 2 = 2k + 2$, $x + 2$ is even. $\square$

(g) Let x be an integer.

Assume x is odd. Then, $x = 2k + 1$ for some $k \in \mathbb{Z}$.

Then, $x^2 - 1 = (2k + 1)^2 - 1 = 4k^2 + 4k = 4(k^2 + k) = 4(k)(k + 1)$.

Since k and $k+1$ are consecutive integers, one must even and the other must be odd (see part (i)). Therefore, one of them can be written as 2 times an integer, and as a result $\frac{k(k+1)}{2}$ is an integer (since the 2 in the numerator and the 2 in the demoninator cancel).

Then, $4(k)(k+1) = 8\frac{k(k+1)}{2}$. As $(2k+1)^2 - 1$ can be written as $8m$ for some $m \in \mathbb{Z}$ by choosing $m = \frac{k(k+1)}{2}$, $(2k + 1)^2 - 1$ is even.

Therefore, since $x^2 - 1 = (2k + 1)^2 - 1$, $8|(x^2 + 1)$. $\square$

(h) Let a, b, c be positive integers.

First, assume $a|b$. Then, $aj = b$ for some $j \in \mathbb{Z}$.

We want to see if there exists a $k \in \mathbb{Z}$ such that $ack = bc$. Since $c \neq 0$, we can divide by c to get $ak = b$. Thus, there does exist a k by choosing $k = j$.

Next, assume $ac|bc$. Then, $acm = bc$ for some $m \in \mathbb{Z}$.

We want to see if there exists an $n \in \mathbb{Z}$ such that $an = b$. Since $c \neq 0$, we can divide $acm = bc$ by c to get $am = b$. Thus, there does exist an n by choosing $n = m$.

Since $ac|bc$ if $a|b$ and $a|b$ if $ac|bc$, a divides b if and only if ac divides bc. $\square$

(i) Let a be an integer.

First, assume a is odd. Then, $a = 2j + 1$ for some $k \in \mathbb{Z}$.

Then, $a+1 = 2j+2 = 2(j+1)$. Since $j+1$ is the sum of integers, it is an integer. As $2j+2$ can be written as $2k$ for some $k \in \mathbb{Z}$ by choosing $k = j+1$, $2j+2$ is even. Since $a+1 = 2j+2$, $a+1$ is even.

Next, assume a is not odd. Then, a is even, and $a = 2m$ for some $m \in \mathbb{Z}$.

Then, $a+1 = 2m+1$. As $2m+1$ can be written as $2n+1$ for some $n \in \mathbb{Z}$ by choosing $m = n$, $2k+1$ is odd. Since $a+1 = 2k+1$, $a+1$ is not even. By the contrapositive, then, if $a+1$ is even, then a is odd.

Therefore, since $a+1$ is even if a is odd and a is odd if $a+1$ is even, a is odd if and only if $a+1$ is even. □

(j) Let a, b be positive integers.

Assume a, b satisfy $(a+1)|b$ and $b|(b+3)$. Then, there are integers m, n such that $(a+1)m = b$ and $bn = b+3$.

Thus, $3 = b(n-1)$, and since $n-1$ is an integer (it is the difference of two integers), $b|3$. Only 1 and 3 divide 3 since it is prime, so b cannot be equal to anything but 1 or 3.

If $b = 1$, then $(a+1)|1$. But this is not possible, because $a > 0$, and nothing can possibly divide something less than itself. So b cannot equal 1.

If $b = 3$, then $(a+1)|3$. Then, the only value that a can take is 2, since a is positive and cannot equal -1. Therefore, a can only equal 2, and b can only equal 3.

Choosing $a = 2$ and $b = 3$ yields $(2+1)|3 \equiv 3|3$ and $3|(3+3) \equiv 3|6$. Since $3(1) = 3$ and $3(2) = 6$, $a = 2$ and $b = 3$ satisfy the claims. Therefore, $(a+1)|b$ and $b|(b+3)$ if and only if $a = 2$ and $b = 3$. □

Problem 8

(a) **True:**

Let x, y be real numbers. Assume x is rational, y is irrational, and $x+y$ is rational.

Then, there are integers a, b, c, d, $b \neq 0$, $d \neq 0$ such that $x = \frac{a}{b}$ and $x+y = \frac{c}{d}$. Then, $x+y-x = y = \frac{c}{d} - \frac{a}{b} = \frac{bc-ad}{bd}$. Since $bc-ad$ is the product and difference of integers, it is an integer, and since bd is the product of nonzero integers, it is a nonzero integer.

Thus, y can be written as $\frac{p}{q}$ for some $p, q \in \mathbb{Z}, q \neq 0$ by choosing $p = bc-ad$ and $q = bd$. As a result, y must be rational, but we assumed y is irrational. A contradiction ensues, so $x+y$ must be irrational if x is rational and y is irrational. Therefore, the sum of every rational and irrational number is irrational. □

(b) **True:**

Let x, y be real numbers. Assume x and y are rational.

Then, there are integers a, b, c, d, $b \neq 0$, $d \neq 0$ such that $x = \frac{a}{b}$ and $y = \frac{c}{d}$. Then, $x+y = \frac{a}{b} + \frac{c}{d} = \frac{ad+bc}{bd}$. Since $ad+bc$ is the product and sum of integers, it is an integer, and since bd is the product of nonzero integers, it is a nonzero integer.

Thus, $x+y$ can be written as $\frac{p}{q}$ for some $p, q \in \mathbb{Z}, q \neq 0$ by choosing $p = ad+bc$ and $q = bd$. As a result, $x+y$ must be rational. Therefore, the sum of two rational numbers is rational. □

(c) **False:**

Let x, y be real numbers. Choose $x = \sqrt{2}$ and $y = -\sqrt{2}$. We know $\pm\sqrt{2}$ are irrational, so x and y are both irrational, but $x+y = \sqrt{2} - \sqrt{2} = 0$ is rational.

Therefore, by existential generalization, the sum of two irrational numbers is not always irrational. □

(d) **False:**

Let x, y be real numbers. Then, choose $x = 0$ and $y = \sqrt{2}$. With these choices, x is rational and y is irrational, and $xy = 0$, so xy is rational. Therefore, by existential generalization, the product of a rational and irrational number is not always irrational. $\square$

(e) **True:**

Let x, y be real numbers. Assume x and y are rational.

Then, there are integers a, b, c, d, $b \neq 0$, $d \neq 0$ such that $x = \frac{a}{b}$ and $y = \frac{c}{d}$. Then, $xy = \frac{ac}{bd}$. Since ac is the product of integers, it is an integer, and since bd is the product of nonzero integers, it is a nonzero integer.

Thus, xy can be written as $\frac{p}{q}$ for some $p, q \in \mathbb{Z}, q \neq 0$ by choosing $p = ac$ and $q = bd$. As a result, xy must be rational. Therefore, the product of two rational numbers is rational. $\square$

(f) **False:**

Let x, y be real numbers. Choose $x = y = \sqrt{2}$. Then, x, y are irrational, and $xy = 2$ so xy is rational.

Therefore, by existential generalization, the product of two irrational numbers is not always irrational. $\square$

(g) **False:**

Let p, q be real numbers. Choose $p = q = 0$. Then, p, q are rational, and $\frac{p}{q} = \frac{0}{0}$ so pq is not defined and thus is neither irrational nor irrational.

Therefore, by existential generalization, the quotient of two rational numbers is not always rational. $\square$

(h) **False:**

Let p, q be real numbers. Choose $p = q = \sqrt{2}$. Then, p, q are irrational, and $\frac{p}{q} = \frac{\sqrt{2}}{\sqrt{2}} = 1$ so pq is rational.

Therefore, by existential generalization, the quotient of two irrational numbers is not always irrational. $\square$

(i) **False:**

Let p, q be real numbers. Then, choose $p = 0$ and $q = \sqrt{2}$. With these choices, p is rational and q is irrational, and $\frac{p}{q} = 0$, so $\frac{p}{q}$ is rational.

Therefore, by existential generalization, a rational number divided by an irrational number is not always irrational. $\square$

(j) **False:**

Let p, q be real numbers. Then, choose $p = \sqrt{2}$ and $q = 0$. With these choices, p is irrational and q is rational, and $\frac{p}{q} = \frac{\sqrt{2}}{0}$, so $\frac{p}{q}$ is undefined and thus is neither rational nor irrational.

Therefore, by existential generalization, an irrational number divided by a rational number is not always irrational. $\square$

(k) **True:**

Assume a, b, c are integers such that $a|b$ and $a|c$. Then, there are integers i, j such that $ai = b$ and $aj = c$.

Thus, $nb + mc = nai + maj = a(ni + mj)$. Thus, there is an integer k such that $ak = nai + maj$ by choosing $k = ni + mj$, so $a|(nai + maj)$.

Therefore, since $nb + mc = nai + maj$, $a|(nb + mc)$ if $a|b$ and $a|c$. $\square$