MATH 552: Section 001
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MATH 552 Homework 4*

Problem 20.3b Using results in Sec. 20, show that the coefficients in the polynomial $P(z)$ in part (a) can be written

$$a_0 = P(0), \quad a_1 = \frac{P'(0)}{1!}, \quad a_2 = \frac{P''(0)}{2!}, \quad \cdots, \quad a_n = \frac{P^{(n)}(0)}{n!}.$$

Solution.

We claim that for any $m \in \mathbb{Z}_{\geq 0}$,

$$P^{(m)}(z) = \sum_{i=0}^{n-m} \frac{(m+i)!}{i!} a_{(m+i)} z^i.$$

First, let $m = 0$. Then,

$$P^{(0)}(z) = \sum_{i=0}^{n-0} \frac{(0+i)!}{i!} a_{(0+i)} z^i = \sum_{i=0}^{n} a_{(i)} z^i.$$

This is what we expect for $P(z)$, so the claim holds for $m = 0$. Then, let $k \in \mathbb{Z}_{\geq 0}$ be given and suppose the claim is true for $m = k$.

$$\begin{aligned}
P^{(k+1)}(z) &= \frac{dP}{dz}[P^{(k)}(z)] && \text{(using repeated differentiation)} \\
&= \frac{dP}{dz}\left[\sum_{i=0}^{n-k} \frac{(k+i)!}{i!} a_{(k+i)} z^i\right] && \text{(using the claim)} \\
&= \frac{dP}{dz}\left[\sum_{i=1}^{n-k} \frac{(k+i)!}{i!} a_{(k+i)} z^i\right] && (i = 0 \text{ will yield constant with derivative} = 0) \\
&= \sum_{i=1}^{n-k} \frac{i(k+i)!}{i!} a_{(k+i)} z^{i-1} && \text{(differentiating each term using power rule)} \\
&= \sum_{i=1}^{n-k} \frac{(k+i)!}{(i-1)!} a_{(k+i)} z^{i-1} && \text{(rewriting factorial)} \\
&= \sum_{i=0}^{n-k-1} \frac{(k+i+1)!}{i!} a_{(k+i+1)} z^i && \text{(shifting index)}
\end{aligned}$$

If the claim is valid for $k+1$, then we can replace k by $k+1$ on both sides:

$$P^{(k+1)}(z) = \sum_{i=0}^{n-k-1} \frac{(k+1+i)!}{i!} a_{(k+1+i)} z^i$$

This exactly matches the result from differentiating $P^{(k)}(z)$, so the claim must be valid for $m = k+1$. By induction, then, the claim is true for all $m \in \mathbb{Z}_{\geq 0}$.

Using this result, $P^{(m)}(0) = \frac{(m+0)!}{0!}a_{(m+0)}$, because the sum contains $(0)^i$ which is 0 for all $i > 0$. Thus, for $n \in \mathbb{Z}_{\geq 0}$:

$$\begin{aligned} P^{(n)}(0) &= n! a_n \\ a_n &= \frac{P^{(n)}(0)}{n!}. \end{aligned}$$

Problem 20.8b Use the method in Example 2, Sec. 19, to show that $f'(z)$ does not exist at any point z when $f(z) = \operatorname{Im} z$.

Solution.

If $f(z) = \operatorname{Im} z$, then

$$\frac{\Delta w}{\Delta z} = \frac{\operatorname{Im} z + \operatorname{Im} \Delta z - \operatorname{Im} z}{\Delta z} = \frac{\operatorname{Im} \Delta z}{\Delta z}.$$

If the limit of $\Delta w / \Delta z$ exists, it can be found by letting the point $\Delta z = (\Delta x, \Delta y)$ approach the origin $(0, 0)$ in the Δz plane in any manner. In particular, as Δz approaches $(0, 0)$ horizontally through the points $(\Delta x, 0)$ on the real axis,

$$\operatorname{Im} z = \operatorname{Im}(\Delta x + i0) = 0.$$

In that case,

$$\frac{\Delta w}{\Delta z} = \frac{0}{\Delta z} = 0.$$

Hence if the limit of $\Delta w / \Delta z$ exists, its value must be unity. However, when Δz approaches (0,0) vertically through the points $(0, \Delta y)$ on the imaginary axis, so that

$$\operatorname{Im} z = \operatorname{Im}(0 + i\Delta y) = \Delta y = \frac{\Delta z}{i},$$

we find that

$$\frac{\Delta w}{\Delta z} = \frac{\Delta z}{i \Delta z} = -i.$$

Hence the limit must be $-i$ if it exists. Since limits are unique, it follows that dw/dz does not exist anywhere.

Problem 24.1a Use the theorem in Sec. 21 to show that $f'(z)$ does not exist at any point if $f(z) = \overline{z}$.

Solution. Let $f(z) = u(x, y) + iv(x, y)$ and $z = x + iy$. Since $\overline{z} = x - iy$, $u(x, y) = x$ and $v(x, y) = -y$.

$$\begin{aligned} u_x &= 1, u_y = 0 \\ v_x &= 0, v_y = -1 \end{aligned}$$

Since $u_x = 1 \neq v_y = -1$, the Cauchy-Riemann equations are not satisfied and the derivative cannot exist.

Problem 24.2b Use the theorem in Sec. 23 to show that $f'(z)$ and its derivative $f''(z)$ exist everywhere, and find $f''(z)$ when $f(z) = e^{-x}e^{-iy}$.

Solution.

$$\begin{aligned} f(z) &= e^{-x}e^{-iy} \\ f(z) &= e^{-x}(\cos -y + i \sin -y) && \text{(using Euler's formula)} \\ f(z) &= e^{-x}\cos y + e^{-x}(-i \sin y) && \text{(rearranging)} \\ f(z) &= u(x, y) + v(x, y) \text{ where } u(x, y) = e^{-x}\cos y, v(x, y) = -e^{-x}(i \sin y) \end{aligned}$$

We can now check the Cauchy-Riemann equations:

$$u_x = -e^{-x}\cos y, u_y = -e^{-x}\sin y$$
$$v_x = e^{-x}\sin y, v_y = -e^{-x}\cos y$$

Since the Cauchy-Riemann equations hold, and $u(x,y)$, $v(x,y)$, and their partials are continuous everywhere, $f'(z)$ exists everywhere and

$$f'(z) = u_x(x,y) + iv_x(x,y) = -e^{-x}\cos y + ie^{-x}\sin y.$$

To take the second derivative, let

$$f'(z) = s(x,y) + t(x,y) \text{ where } s(x,y) = -e^{-x}\cos y, t(x,y) = e^{-x}(i\sin y).$$

We can now check the Cauchy-Riemann equations again:

$$s_x = e^{-x}\cos y, s_y = -e^{-x}\sin y$$
$$t_x = e^{-x}\sin y, t_y = e^{-x}\cos y$$

Since they hold again, and $s(x,y)$, $t(x,y)$, and their partials are continuous everywhere, $f''(z)$ exists everywhere and

$$f''(z) = s_x(x,y) + it_x(x,y) = e^{-x}\cos y - ie^{-x}\sin y.$$

Problem 24.4b Use the theorem in Sec. 24 to show that $f(z) = e^{-\theta}\cos(\ln r) + ie^{-\theta}\sin(\ln r)$ is differentiable in the domain $(r > 0, 0 < \theta < 2\pi)$, and also to find $f'(z)$.

Solution.

$$f(z) = u(r,\theta) + iv(r,\theta) \text{ where } u(r,\theta) = e^{-\theta}\cos(\ln r), v(r,\theta) = e^{-\theta}\sin(\ln r)$$

We can now check the Cauchy-Riemann equations:

$$ru_r = r\left(-e^{-\theta}\sin(\ln r)\frac{1}{r}\right) = -e^{-\theta}\sin(\ln r)$$
$$u_\theta = -e^{-\theta}\cos(\ln r)$$
$$rv_r = r\left(e^{-\theta}\cos(\ln r)\frac{1}{r}\right) = e^{-\theta}\cos(\ln r)$$
$$v_\theta = -e^{-\theta}\sin(\ln r)$$

Since the Cauchy-Riemann equations hold, and $u(x,y)$, $v(x,y)$, and their partials are continuous throughout the domain, $f'(z)$ exists throughout the domain and

$$f'(z) = e^{-i\theta}(u_r(r,\theta) + iv_r(r,\theta)) = \frac{e^{-i\theta}}{r}(-e^{-\theta}\sin(\ln r) + ie^{-\theta}\cos(\ln r)).$$