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MATH 300: Section H01
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MATH 300 Homework 10

Problem 1

(a) We claim that for each $n \in \mathbb{Z}^+$ we have that $\sum_{i=1}^{n}(i^3) = \frac{n^2(n+1)^2}{4}$.

First, let $n = 1$. Since $\sum_{i=1}^{1}(i^3) = 1^3 = 1 = \frac{2^2}{4} = \frac{1^2(1+1)^2}{4}$, the claim holds for $n = 1$.

Then, let $n \in \mathbb{Z}$, $n \geq 1$. Assume $\sum_{i=1}^{n}(i^3) = \frac{n^2(n+1)^2}{4}$. Then,

$$
\begin{aligned}
\sum_{i=1}^{n}(i^3) &= \frac{n^2(n+1)^2}{4} \\
\sum_{i=1}^{n}(i^3) + (n+1)^3 &= \frac{n^2(n+1)^2}{4} + (n+1)^3 \\
\sum_{i=1}^{n+1}(i^3) &= \frac{n^2(n+1)^2 + 4(n+1)^3}{4} \\
&= \frac{(n+1)^2(n^2+4n+4)}{4} && \text{(factoring out } (n+1)^2\text{)} \\
&= \frac{(n+1)^2(n+2)^2}{4}.
\end{aligned}
$$

Therefore $\sum_{i=1}^{n+1}(i^3) = \frac{(n+1)^2((n+1)+1)^2}{4}$. So if the claim is true for n, it is also true for $n+1$. $\square$

(b) We claim that for each $n \in \mathbb{N}$ we have that $\sum_{i=0}^{n}(3(5)^i) = \frac{3(5^{n+1}-1)}{4}$.

First, let $n = 0$. Since $\sum_{i=0}^{0}(3(5)^i) = 3(5)^0 = 3 = \frac{3(4)}{4} = \frac{3(5^1-1)}{4}$, the claim holds for $n = 0$.

Then, let $n \in \mathbb{N}$. Assume $\displaystyle\sum_{i=0}^{n}(3(5)^i) = \frac{3(5^{n+1}-1)}{4}$. Then,

$$\begin{aligned}
\sum_{i=0}^{n}(3(5)^i) &= \frac{3(5^{n+1}-1)}{4} \\
\sum_{i=0}^{n}(3(5)^i) + 3(5)^{n+1} &= \frac{3(5^{n+1}-1)}{4} + 3(5)^{n+1} \\
\sum_{i=0}^{n+1}(3(5)^i) &= \frac{3(5)^{n+1} - 3 + 12(5)^{n+1}}{4} \\
&= \frac{15(5)^{n+1} - 3}{4} \\
&= \frac{3(5(5)^{n+1} - 1)}{4}.
\end{aligned}$$

Therefore $\displaystyle\sum_{i=0}^{n+1}(3(5)^i) = \frac{3(5^{(n+1)+1}-1)}{4}$. So if the claim is true for n, it is also true for $n+1$. $\square$

(c) We claim that for each $n \in \mathbb{Z}^+$ we have that $\displaystyle\sum_{i=1}^{n}\left(\frac{1}{i(i+1)}\right) = \frac{n}{n+1}$.

First, let $n = 1$. Since $\displaystyle\sum_{i=1}^{1}\left(\frac{1}{i(i+1)}\right) = \frac{1}{1(1+1)} = \frac{1}{2}$, the claim holds for $n = 1$.

Then, let $n \in \mathbb{Z}^+$. Assume $\displaystyle\sum_{i=1}^{n}\left(\frac{1}{i(i+1)}\right) = \frac{n}{n+1}$. Then,

$$\begin{aligned}
\sum_{i=1}^{n}\left(\frac{1}{i(i+1)}\right) &= \frac{n}{n+1} \\
\sum_{i=1}^{n}\left(\frac{1}{i(i+1)}\right) + \frac{1}{(n+1)(n+2)} &= \frac{n}{n+1} + \frac{1}{(n+1)(n+2)} \\
\sum_{i=1}^{n+1}\left(\frac{1}{i(i+1)}\right) &= \frac{n(n+2)+1}{(n+1)(n+2)} \\
&= \frac{n^2+2n+1}{(n+1)(n+2)} \\
&= \frac{(n+1)^2}{(n+1)(n+2)}.
\end{aligned}$$

Therefore $\displaystyle\sum_{i=1}^{n+1}\left(\frac{1}{i(i+1)}\right) = \frac{n+1}{(n+1)+1}$. So if the claim is true for n, it is also true for $n+1$. $\square$

(d) We claim that for each $n \in \mathbb{N}$ we have that $\displaystyle\sum_{i=0}^{n}(q^i) = \frac{q^{n+1}-1}{q-1}$.

First, let $n = 0$. Since $\displaystyle\sum_{i=0}^{0}(q^i) = q^0 = 1 = \frac{q-1}{q-1} = \frac{q^{0+1}-1}{q-1}$, the claim holds for $n = 0$. Multiplying by $\dfrac{q-1}{q-1}$ is valid since $1 \notin \mathbb{R}\backslash\{1\}$ so $q - 1 \neq 0$ for all q.

Then, let $n \in \mathbb{N}$. Assume $\displaystyle\sum_{i=0}^{n}(q^i) = \frac{q^{n+1}-1}{q-1}$. Then,

$$\begin{aligned}
\sum_{i=0}^{n}(q^i) &= \frac{q^{n+1}-1}{q-1} \\
\sum_{i=0}^{n}(q^i) + q^{n+1} &= \frac{q^{n+1}-1}{q-1} + q^{n+1} \\
\sum_{i=0}^{n+1}(q^i) &= \frac{q^{n+1}-1+(q-1)q^{n+1}}{q-1} \\
&= \frac{q^{n+1}(1+q-1)-1}{q-1} \\
&= \frac{q(q^{n+1})-1}{q-1}.
\end{aligned}$$

Therefore $\displaystyle\sum_{i=0}^{n+1}(q^i) = \frac{q^{(n+1)+1}-1}{q-1}$. So if the claim is true for n, it is also true for $n+1$. $\square$

(e) We claim that for each $n \in \mathbb{Z}$, $n > 6$ we have that $3^n < n!$.

First, let $n = 7$. Since $3^7 = 2187 < 5040 = 7!$, the claim holds for $n = 7$.

Then, let $n \in \mathbb{Z}$, $n \geq 7$. Assume $3^n < n!$. Then,

$$\begin{aligned}
3^n &< n! \\
3(3^n) &< 3n! \\
3^{n+1} &< 3n! && \text{(rewriting } 3(3^n) \text{ as } 3^{n+1}) \\
&< (n+1)n! && \text{(since } n > 6,\ n+1 > 3)
\end{aligned}$$

Therefore $3^{n+1} < (n+1)!$, so if the claim is true for n, it is also true for $n+1$. $\square$

(f) We claim that for each $n \in \mathbb{Z}$, $n > 1$ we have that $n! < n^n$.

First, let $n = 2$. Since $2! = 2 < 4 = 2^2$, the claim holds for $n = 2$.

Then, let $n \in \mathbb{Z}$, $n \geq 2$. Assume $n! < n^n$. Then,

$$\begin{aligned}
n! &< n^n \\
(n+1)n! &< (n+1)n^n && \text{(multiplying both sides by } n+1 \text{ (which is positive))} \\
(n+1)! &< (n+1)n^n && \text{(rewriting LHS as factorial)} \\
&< (n+1)(n+1)^n && \text{(since } n+1 > n)
\end{aligned}$$

Therefore $(n+1)! < (n+1)^{n+1}$, so if the claim is true for n, it is also true for $n+1$. $\square$

(g) We claim that for each $n \in \mathbb{Z}^+$ we have that $\displaystyle B \cup \left(\bigcap_{i=1}^{n} A_i\right) = \bigcap_{i=1}^{n}(B \cup A_i)$.

First, let $n = 1$. Since $\displaystyle B \cup \bigcap_{i=1}^{1} A_i = B \cup A_1 = \bigcap_{i=1}^{1}(B \cup A_i)$, the claim holds for $n = 1$.

Then, let $n \in \mathbb{Z}^+$. Assume $B \cup \left(\bigcap_{i=1}^{n} A_i \right) \equiv \bigcap_{i=1}^{n} (B \cup A_i)$. Then,

$$\begin{aligned}
B \cup \left(\bigcap_{i=1}^{n+1} A_i \right) &= B \cup \left(\bigcap_{i=1}^{n} (A_i) \cap A_{n+1} \right) && \text{(rewriting intersection)} \\
&\equiv \left(B \cup \bigcap_{i=1}^{n} A_i \right) \cap (B \cup A_{n+1}) && \text{(distributive law of sets)} \\
&\equiv \left(\bigcap_{i=1}^{n} (B \cup A_i) \right) \cap (B \cup A_{n+1}) && \text{(using induction hypothesis)} \\
&= \bigcap_{i=1}^{n+1} (B \cup A_i). && \text{(rewriting intersection)}
\end{aligned}$$

Therefore $B \cup \left(\bigcap_{i=1}^{n+1} A_i \right) \equiv \bigcap_{i=1}^{n+1} (B \cup A_i)$, so if the claim is true for n, it is also true for $n+1$. $\square$

(h) We claim that for each $n \in \mathbb{Z}^+$ we have that $\neg \left(\bigwedge_{i=1}^{n} p_i \right) \equiv \bigvee_{i=1}^{n} (\neg p_i)$.

First, let $n = 1$. Since $\neg \left(\bigwedge_{i=1}^{1} p_i \right) = \neg(p_1) \equiv \neg p_1 = \bigvee_{i=1}^{1} (\neg p_1)$, the claim holds for $n = 1$.

Then, let $n \in \mathbb{Z}^+$. Assume $\neg \left(\bigwedge_{i=1}^{n} p_i \right) \equiv \bigvee_{i=1}^{n} (\neg p_i)$. Then,

$$\begin{aligned}
\neg \left(\bigwedge_{i=1}^{n+1} p_i \right) &= \neg \left(\bigwedge_{i=1}^{n} (p_i) \wedge p_{n+1} \right) && \text{(rewriting conjunction)} \\
&\equiv \neg \left(\bigwedge_{i=1}^{n} p_i \right) \vee \neg p_{n+1} && \text{(De Morgan's law)} \\
&\equiv \left(\bigvee_{i=1}^{n} (\neg p_i) \right) \vee \neg p_{n+1} && \text{(using induction hypothesis)} \\
&= \bigvee_{i=1}^{n+1} (\neg p_i) && \text{(rewriting disjunction)}
\end{aligned}$$

Therefore $\neg \left(\bigwedge_{i=1}^{n+1} p_i \right) \equiv \bigvee_{i=1}^{n+1} (\neg p_i)$, so if the claim is true for n, it is also true for $n+1$. $\square$

(i) We claim that for each $n \in \mathbb{Z}$, $n \geq 2$ we have that $\sum_{i=1}^{n} \frac{1}{i^2} < 2 - \frac{1}{n}$.

First, let $n = 2$. Since $\sum_{i=1}^{2} \frac{1}{i^2} = \frac{1}{1^2} + \frac{1}{2^2} = \frac{5}{4} < \frac{3}{2} = 2 - \frac{1}{2}$, the claim holds for $n = 2$.

Then, let $n \in \mathbb{Z}$, $n \geq 2$. Assume $\displaystyle\sum_{i=1}^{n} \frac{1}{i^2} < 2 - \frac{1}{n}$. Then,

$$
\begin{aligned}
\sum_{i=1}^{n} \frac{1}{i^2} &< 2 - \frac{1}{n} \\
\sum_{i=1}^{n} \frac{1}{i^2} + \frac{1}{(n+1)^2} &< 2 - \frac{1}{n} + \frac{1}{(n+1)^2} \\
\sum_{i=1}^{n+1} \frac{1}{i^2} &< \frac{2n(n+1)^2 - (n+1)^2 + n}{n(n+1)^2} \\
&= \frac{(n+1)^2(2n-1) + n}{n(n+1)^2} \\
&= \frac{(n^2+2n+1)(2n-1) + n}{n(n+1)^2} \\
&= \frac{2n^3 + 4n^2 + 2n - n^2 - 2n - 1 + n}{n(n+1)^2} \\
&= \frac{2n^3 + 3n^2 + n - 1}{n(n+1)^2} \\
&= \frac{2n^2 + 3n + 1}{(n+1)^2} - \frac{1}{n(n+1)^2} \\
&= \frac{(2n+1)(n+1)}{(n+1)^2} - \frac{1}{n(n+1)^2} \\
&= \frac{2n+1}{n+1} - \frac{1}{n(n+1)^2} \\
&< \frac{2n+1}{n+1} && \text{(since } n \text{ is guaranteed to be positive)} \\
&= \frac{2(n+1) - 1}{n+1} && \text{(rewriting numerator)} \\
&= 2 - \frac{1}{n+1}.
\end{aligned}
$$

Therefore, $\displaystyle\sum_{i=1}^{n+1} \frac{1}{i^2} < 2 - \frac{1}{n+1}$, so if the claim is true for n, it is also true for $n+1$. $\square$

(j) We claim that for each $n \in \mathbb{N}$ we have that $3|(n^3 + 2n)$.

First, let $n = 0$. Since $0^3 + 2(0) = 0$ and $3(0) = 0$, $3|(0^3 + 2(0))$, so the claim holds for $n = 0$.

Then, let $n \in \mathbb{N}$. Assume $3|(n^3 + 2n)$. Then, $(\exists k)[3k = n^3 + 2n]$. We observe that

$$
\begin{aligned}
(n+1)^3 + 2(n+1) &= n^3 + 3n^2 + 3n + 1 + 2n + 2 \\
&= (n^3 + 2n) + (3n^2 + 3n + 3) \\
&= (3k) + 3(n^2 + n + 1). && \text{(by the induction hypothesis)}
\end{aligned}
$$

So by choosing $m = k + n^2 + n + 1$, $3m = (n+1)^3 + 2(n+1)$. Since m is the sum and product of integers, it is an integer, and therefore $3|(n+1)^3 + 2(n+1)$. So if the claim is true for n, it is also true for $n+1$. $\square$

(k) We claim that for each $n \in \mathbb{N}$ we have that $2|n(n+1)$.

First, let $n = 0$. Since $0(0+1) = 0$ and $2(0) = 0$, $2|0(0+1)$ and the claim holds for $n = 0$.

Then, let $n \in \mathbb{N}$. Assume $2|n(n+1)$. Then, $(\exists k)[2k = n(n+1)]$. We observe that

$$\begin{aligned}(n+1)(n+2) &= n^2 + 3n + 2 \\ &= (n^2 + n) + (2n + 2) \\ &= (2k) + 2(n+1). && \text{(by the induction hypothesis)}\end{aligned}$$

So by choosing $m = k + n + 1$, $2m = (n+1)(n+2)$. Since m is the sum and product of integers, it is an integer, and therefore $2|(n+1)((n+1)+1)$. So if the claim is true for n, it is also true for $n+1$. $\square$

(l) We claim that for each $n \in \mathbb{N}$ we have that $6|n(n+1)(n+2)$.

First, let $n = 0$. Since $0(0+1)(0+2) = 0$ and $6(0) = 0$, $6|0(0+1)(0+2)$ and the claim holds for $n = 0$.

Then, let $n \in \mathbb{N}$. Assume $6|n(n+1)(n+2)$. Then, $(\exists k)[6k = n(n+1)(n+2) = n^3 + 3n^2 + 2n]$. We observe that

$$\begin{aligned}(n+1)(n+2)(n+3) &= (n^2 + 3n + 2)(n+3) \\ &= (n^3 + 3n^2 + 2n) + (3n^2 + 9n + 6) \\ &= 6k + 3(n^2 + 3n + 2) && \text{(by the induction hypothesis)} \\ &= 6k + 3(n+1)(n+2) && \text{(refactoring)} \\ &= 6\left(k + \frac{(n+1)(n+2)}{2}\right).\end{aligned}$$

So by choosing $m = k + \dfrac{(n+1)(n+2)}{2}$, $6m = (n+1)(n+2)(n+3)$. By the result from **(k)**, we know that an integer times its successor is divisible by 2, so $\dfrac{(n+1)(n+2)}{2}$ is an integer. Thus, since m is the sum and product of integers, it is an integer, and therefore $6|(n+1)((n+1)+1)((n+1)+2)$. So if the claim is true for n, it is also true for $n+1$. $\square$

Problem 2

(a) The formula is not true for $n = 1$. $\sum_{i=1}^{1}(i) = 1$, but $\dfrac{(1+\frac{1}{2})^2}{2} = \dfrac{9}{8}$.

(b) The step from $\max(x, y) = n + 1$ to $\max(x - 1, y - 1) = n$ and then applying the induction hypothesis is not valid, because there is no guarantee that x and y are greater than 1. If either equals 1, then it will no longer be a positive integer after subtracting 1. But the induction hypothesis requires that x^*, y^* be positive integers, so applying it is not valid.