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MATH 555: Section H01
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March 31, 2024

MATH 555 Homework 10

Problem 1 Prove that:

(a) The two series

$$\mathrm{c}(x) := \sum_{k=0}^{\infty} \frac{(-1)^k x^{2k}}{(2k)!} = 1 - \frac{x^2}{2} + \frac{x^4}{4!} - \frac{x^6}{6!} + \frac{x^8}{8!} - \frac{x^{10}}{10!} + \dots$$
$$\mathrm{s}(x) := \sum_{k=0}^{\infty} \frac{(-1)^k x^{2k+1}}{(2k+1)!} = x - \frac{x^3}{3!} + \frac{x^5}{5!} - \frac{x^7}{7!} + \frac{x^9}{9!} - \frac{x^{11}}{11!} + \dots$$

converge absolutely for all $x \in \mathbb{R}$ and therefore these series are absolutely convergent and differentiable for all $x \in \mathbb{R}$.

(b) The derivatives satisfy

$$\mathrm{c}'(x) = -\mathrm{s}(x), \quad \mathrm{s}'(x) = \mathrm{c}(x).$$

(c) The values at $x = 0$ are

$$\mathrm{c}(0) = 1, \quad \mathrm{s}(0) = 0.$$

(d) Also

$$\mathrm{c}'' = -\mathrm{c}, \quad \mathrm{s}'' = -\mathrm{s}\,.$$

(a) We will use the ratio test to show that both series converge. Let $\langle c_n \rangle_{n=1}^{\infty}$ be the terms of $\mathrm{c}(x)$ and $\langle s_n \rangle_{n=1}^{\infty}$ be the terms of $\mathrm{s}(x)$. Then, we can write

$$\begin{aligned}
\lim_{n\to\infty} \left| \frac{c_{n+1}}{c_n} \right| &= \lim_{n\to\infty} \frac{\frac{x^{2(n+1)}}{(2(n+1))!}}{\frac{x^{2n}}{(2n)!}} \\
&= \lim_{n\to\infty} \frac{x^{2n+2}(2n)!}{x^{2n}(2n+2)!} \\
&= \lim_{n\to\infty} \frac{x^2}{(2n+2)(2n+1)} \\
&= x^2 \lim_{n\to\infty} \frac{1}{4n^2+6n+2} \\
&= 0 < 1
\end{aligned}$$

for all $x \in \mathbb{R}$, so $\mathrm{c}(x)$ converges absolutely. Similarly, we can write

$$\begin{aligned}
\lim_{n\to\infty} \left| \frac{s_{n+1}}{s_n} \right| &= \lim_{n\to\infty} \frac{\frac{x^{2(n+1)+1}}{(2(n+1)+1)!}}{\frac{x^{2n+1}}{(2n+1)!}} \\
&= \lim_{n\to\infty} \frac{x^{2n+3}(2n+1)!}{x^{2n+1}(2n+3)!}
\end{aligned}$$

$$
\begin{aligned}
&= \lim_{n\to\infty} \frac{x^2}{(2n+3)(2n+2)} \\
&= x^2 \lim_{n\to\infty} \frac{1}{4n^2+10n+6} \\
&= 0 < 1
\end{aligned}
$$

for all $x \in \mathbb{R}$, so $\mathrm{s}(x)$ converges absolutely.

(b) We have shown in class that since the series converge, we can differentiate term by term. So for all $x \in \mathbb{R}$, we have

$$
\begin{aligned}
\mathrm{c}'(x) &= \sum_{k=0}^{\infty} \frac{d}{dx}\left[\frac{(-1)^k x^{2k}}{(2k)!}\right] \\
&= -\sum_{k=0}^{\infty} \frac{(-1)^{k-1} 2k x^{2k-1}}{(2k)!} \\
&= -\sum_{k=1}^{\infty} \frac{(-1)^{k-1} 2k x^{2k-1}}{(2k)!} && \text{(first term is 0)} \\
&= -\sum_{k=0}^{\infty} \frac{(-1)^{k} 2(k+1) x^{2(k+1)-1}}{(2(k+1))!} && \text{(adjusting bounds)} \\
&= -\sum_{k=0}^{\infty} \frac{(-1)^{k} (2k+2) x^{2k-1}}{(2k+2)!} \\
&= -\sum_{k=0}^{\infty} \frac{(-1)^{k} x^{2k-1}}{(2k+1)!} \\
&= -\,\mathrm{s}(x)
\end{aligned}
$$

and

$$
\begin{aligned}
\mathrm{s}'(x) &= \sum_{k=0}^{\infty} \frac{d}{dx}\left[\frac{(-1)^k x^{2k+1}}{(2k+1)!}\right] \\
&= \sum_{k=0}^{\infty} \frac{(-1)^k (2k+1) x^{2k}}{(2k+1)!} \\
&= \sum_{k=0}^{\infty} \frac{(-1)^k x^{2k}}{(2k)!} \\
&= \mathrm{c}(x).
\end{aligned}
$$

□

(c) Since an infinite series of 0s is 0, it follows immediately that $\mathrm{c}(0) = 1$ and $\mathrm{s}(0) = 0$ by substituting 0 into each term.

(d) For all $x \in \mathbb{R}$, we can use (b) to write

$$\mathrm{c}''(x) = (\mathrm{c}')'(x) = (-\,\mathrm{s})'(x) = -\,\mathrm{c}(x)$$

and

$$\mathrm{s}''(x) = (\mathrm{s}')'(x) = \mathrm{c}'(x) = -\,\mathrm{s}(x).$$

Problem 2 Prove that these functions satisfy

$$\mathrm{c}(x)^2 + \mathrm{s}(x)^2 = 1.$$

Let $f(x) = \mathrm{c}(x)^2 + \mathrm{s}(x)^2$. Then, we can use the chain and power rule to write, for all $x \in \mathbb{R}$,

$$f'(x) = -2\,\mathrm{c}(x)\,\mathrm{s}(x) + 2\,\mathrm{s}(x)\,\mathrm{c}(x) = 0.$$

So f is constant on $\mathbb{R}$. From Problem 1d,

$$f(0) = \mathrm{c}(0)^2 + \mathrm{s}(0)^2 = 1^2 + 0^2 = 1.$$

So $f \equiv 1$, and thus $\mathrm{c}(x)^2 + \mathrm{s}(x)^2 = 1$. $\square$

Problem 3 Prove that if g is two times differentiable on $\mathbb{R}$ and

$$g'' = -g, \quad g(0) = 0, \quad g'(0) = 0,$$

then $g(x) = 0$ for all x.

Define $E : \mathbb{R} \to \mathbb{R}$ by $E(x) := g(x)^2 + (g'(x))^2$. Then, we have

$$\begin{aligned} E'(x) &= 2g(x)g'(x) + 2g'(x)g''(x) && \text{(power/chain rules)} \\ &= 2g(x)g'(x) - 2g'(x)g(x) && (g'' = -g) \\ &= 0. \end{aligned}$$

So E is constant, and since

$$E(0) = g(0)^2 + (g'(0))^2 = 0^2 + 0^2 = 0,$$

we have $E(x) = 0$ for all x. Since E is the sum of squares, this is only possible if $g(x) \equiv 0 \equiv g'(x)$, so in particular $g(x) = 0$ for all x. $\square$

Problem 4 Prove that if f is twice differentiable on $\mathbb{R}$ and

$$f'' = -f,$$

then f is a linear combination of c and s. In particular,

$$f(x) = f(0)\,\mathrm{c}(x) + f'(0)\,\mathrm{s}(x).$$

Consider $g : \mathbb{R} \to \mathbb{R}$ defined by $g(x) := f(x) - f(0)\,\mathrm{c}(x) - f'(0)\,\mathrm{s}(x)$. We claim that g satisfies the assumptions in Problem 3. To see this, observe that using Problem 1b, we have

$$g'(x) = f'(x) - f(0)\,\mathrm{c}'(x) - f'(0)\,\mathrm{s}'(x) = f'(x) + f(0)\,\mathrm{s}(x) - f'(0)\,\mathrm{c}(x),$$

and using Problem 1d and that $f'' = -f$, we have

$$g''(x) = f''(x) - f(0)\,\mathrm{c}''(x) - f'(0)\,\mathrm{s}''(x) = -f(x) + f(0)\,\mathrm{c}(x) + f'(0)\,\mathrm{s}(x) = -g(x). \tag{1}$$

We can use problem 1c to write

$$g(0) = f(0) - f(0)\,\mathrm{c}(0) - f'(0)\,\mathrm{s}(0) = f(0) - f(0) = 0 \tag{2}$$

and our computation of g' to write

$$g'(0) = f'(0) + f(0)\,\mathrm{s}(0) - f'(0)\,\mathrm{c}(0) = f'(0) - f'(0) = 0. \tag{3}$$

So $g'' = -g$ from (1), $g(0) = 0$ from (2), and $g'(0) = 0$ from (3), so we can apply Problem 3 and conclude that $g(x) = 0$ for all x. So $0 = f(x) - f(0)\,\mathrm{c}(x) - f'(0)\,\mathrm{s}(x)$, and therefore

$$f(x) = f(0)\,\mathrm{c}(x) + f'(0)\,\mathrm{s}(x).$$

$\square$

Problem 5 Prove that the functions c and s satisfy

(a) $\mathrm{c}(x+a) = \mathrm{c}(a)\,\mathrm{c}(x) - \mathrm{s}(a)\,\mathrm{s}(x)$.

(b) $\mathrm{s}(x+a) = \mathrm{s}(a)\,\mathrm{c}(x) + \mathrm{c}(a)\,\mathrm{s}(x)$.

(a) We can use the chain rule and Problem 1d to write

$$\mathrm{c}''(x+a) = (-\,\mathrm{s})'(x+a) = -\,\mathrm{c}(x+a),$$

so we can apply Problem 4 to write

$$\mathrm{c}(x+a) = \mathrm{c}(0+a)\,\mathrm{c}(x) + \mathrm{c}'(0+a)\,\mathrm{s}(x) = \mathrm{c}(a)\,\mathrm{c}(x) - \mathrm{s}(a)\,\mathrm{s}(x).$$

(b) Similarly, we can write

$$\mathrm{s}''(x+a) = \mathrm{c}'(x+a) = -\,\mathrm{s}(x+a),$$

so Problem 4 implies

$$\mathrm{s}(x+a) = \mathrm{s}(0+a)\,\mathrm{c}(x) + \mathrm{s}'(0+a)\,\mathrm{s}(x) = \mathrm{s}(a)\,\mathrm{c}(x) + \mathrm{c}(a)\,\mathrm{s}(x).$$

$\square$

Problem 6 Prove that if $0 < x < 7$, the inequality

$$\mathrm{c}(x) < 1 - \frac{x^2}{2} + \frac{x^4}{24}$$

holds.

For any $x \in (0, 7)$, we have by definition that

$$\mathrm{c}(x) = 1 - \frac{x^2}{2!} + \frac{x^4}{4!} + S(x),$$

where $S(x)$ is an error term given by

$$S = \sum_{k=3}^{\infty} \frac{(-1)^k x^{2k}}{(2k)!}.$$

We can write

$$\begin{aligned}
S &= -\frac{x^6}{6!} + \frac{x^8}{8!} - \frac{x^{10}}{10!} + \frac{x^{12}}{12!} - \frac{x^{14}}{14!} + \frac{x^{16}}{16!} - \dots \\
&= -\frac{x^6}{6!}\left(1 - \frac{x^2}{8\cdot 7}\right) - \frac{x^{10}}{10!}\left(1 - \frac{x^2}{12\cdot 11}\right) - \frac{x^{14}}{14!}\left(1 - \frac{x^2}{16\cdot 15}\right) - \dots \\
&= -\sum_{k=0}^{\infty} \frac{x^{4k+6}}{(4k+6)!}\left(1 - \frac{x^2}{(4k+8)\cdot(4k+7)}\right) \\
&= \sum_{k=0}^{\infty} \frac{x^{4k+6}}{(4k+6)!}\left(\frac{x^2}{(4k+8)\cdot(4k+7)} - 1\right) \\
&< \sum_{k=0}^{\infty} \frac{x^{4k+6}}{(4k+6)!}\left(\frac{x^2}{7\cdot 7} - 1\right) && \text{(because } 4k+8 > 4k+7 > 7 \text{ for all } k \geq 0) \\
&\leq \sum_{k=0}^{\infty} \frac{x^{4k+6}}{(4k+6)!}\left(\frac{7^2}{7\cdot 7} - 1\right) && (0 < x < 7 \implies x^2 \leq 7) \\
&= \sum_{k=0}^{\infty} \frac{x^{4k+6}}{(4k+6)!}(1-1) = 0.
\end{aligned}$$

Therefore, since $S(x) < 0$, we have

$$\mathrm{c}(x) = 1 - \frac{x^2}{2!} + \frac{x^4}{4!} + S(x) < 1 - \frac{x^2}{2!} + \frac{x^4}{4!}.$$

$\square$

Problem 7 Prove that the following hold:

(a) $\mathrm{c}(\pi/2) = 0, \quad \mathrm{s}(\pi/2) = 1.$

(b) $\mathrm{c}(\pi) = -1, \quad \mathrm{s}(\pi) = 0.$

(c) $\mathrm{c}(2\pi) = 1, \quad \mathrm{s}(2\pi) = 0.$

(a) We defined π to be such that $\pi/2$ is the unique zero of c on $[0,2]$, so $\mathrm{c}(\pi/2) = 0$ by definition. From Problem 2, we have

$$\mathrm{c}(\pi/2)^2 + \mathrm{s}(\pi/2)^2 = 1,$$

so $s(\pi/2)^2 = 1$ and thus $\mathrm{s}(\pi/2) \in \{-1, 1\}$. From the notes, $\mathrm{s}(x) \geq x - \frac{x^3}{3!}$, which is non-negative on $[0,2]$, so since $\pi/2 \in [0,2]$ we have $\mathrm{s}(\pi/2) \geq 0$. We can then conclude that $\mathrm{s}(\pi/2) = 1$.

(b) We can use Problem 5 and part (a) to write

$$\mathrm{c}(\pi) = \mathrm{c}(\pi/2 + \pi/2) = \mathrm{c}(\pi/2)\,\mathrm{c}(\pi/2) - \mathrm{s}(\pi/2)\,\mathrm{s}(\pi/2) = 0\cdot 0 - 1\cdot 1 = 0 - 1 = -1$$

and

$$\mathrm{s}(\pi) = \mathrm{s}(\pi/2 + \pi/2) = \mathrm{s}(\pi/2)\,\mathrm{c}(\pi/2) + \mathrm{c}(\pi/2)\,\mathrm{s}(\pi/2) = 1\cdot 0 + 0\cdot 1 = 0 + 0 = 0.$$

(c) We can use Problem 5 and part (b) to write

$$\mathrm{c}(2\pi) = \mathrm{c}(\pi + \pi) = \mathrm{c}(\pi)\,\mathrm{c}(\pi) - \mathrm{s}(\pi)\,\mathrm{s}(\pi) = -1\cdot -1 - 0\cdot 0 = 1 - 0 = 1$$

and

$$\mathrm{s}(2\pi) = \mathrm{s}(\pi + \pi) = \mathrm{s}(\pi)\,\mathrm{c}(\pi) + \mathrm{c}(\pi)\,\mathrm{s}(\pi) = 0\cdot -1 + -1\cdot 0 = 0 + 0 = 0.$$

Problem 9 Define

$$\sin(x) = \mathrm{s}(x),\ \cos(x) = \mathrm{c}(x),\ \tan(x) = \frac{\sin(x)}{\cos(x)}.$$

Prove that

(a) The derivative of $\tan(x)$ is given by

$$\frac{d}{dx}\tan(x) = 1 + \tan^2(x).$$

(b) tan is periodic with period π:

$$\tan(x+\pi) = \tan(x).$$

(a) We can write

$$\begin{aligned}
\frac{d}{dx}\left[\tan(x)\right] &= \frac{d}{dx}\left[\frac{\sin(x)}{\cos(x)}\right] \\
&= \frac{\cos(x)\sin'(x) - \sin(x)\cos'(x)}{\cos^2(x)} && \text{(quotient rule)} \\
&= \frac{\cos(x)\cos(x) + \sin(x)\sin(x)}{\cos^2(x)} && \text{(from 1d)} \\
&= \frac{\cos^2(x)}{\cos^2(x)} + \frac{\sin^2(x)}{\cos^2(x)} && \text{(splitting fraction)} \\
&= 1 + \left(\frac{\sin(x)}{\cos(x)}\right)^2 \\
&= 1 + \tan^2(x). && \text{(definition of tan)}
\end{aligned}$$

(b) We first note using Problem 7b that

$$\tan(\pi) = \frac{\sin(\pi)}{\cos(\pi)} = \frac{0}{-1} = 0. \tag{$*$}$$

We can also derive the general identity that

$$\begin{aligned}
\tan(x+a) &= \frac{\sin(x+a)}{\cos(x+a)} && \text{(definition)} \\
&= \frac{\sin(a)\cos(x) + \cos(a)\sin(x)}{\cos(a)\cos(x) - \sin(a)\sin(x)} && \text{(Problem 5)} \\
&= \frac{\frac{\sin(a)\cos(x)+\cos(a)\sin(x)}{\cos(a)\cos(x)}}{\frac{\cos(a)\cos(x)-\sin(a)\sin(x)}{\cos(a)\cos(x)}} && \text{(dividing by same denominator)} \\
&= \frac{\frac{\sin(a)}{\cos(a)} + \frac{\sin(x)}{\cos(x)}}{1 - \left(\frac{\sin(a)}{\cos(a)}\right)\left(\frac{\sin(x)}{\cos(x)}\right)} && \text{(splitting/cancelling)} \\
&= \frac{\tan(a) + \tan(x)}{1 - \tan(a)\tan(x)}. && \text{(definition of tan)}
\end{aligned}$$

We can then apply the identity with $(*)$ to obtain

$$\tan(x+\pi) = \frac{\tan(\pi) + \tan(x)}{1 - \tan(\pi)\tan(x)} = \frac{0 + \tan(x)}{1 - 0\cdot\tan(x)} = \frac{\tan(x)}{1-0} = \tan(x).$$

$\square$

Problem 10 Prove that the one-sided limits

$$\lim_{x\uparrow\pi/2}\tan(x) = +\infty, \qquad \lim_{x\downarrow-\pi/2}\tan(x) = -\infty$$

hold.

We recall the following definitions. Let $f : \mathbb{R} \to \mathbb{R}$ and $a \in \mathbb{R}$. Then,

- $\lim_{x\uparrow a} f(x) = \infty$ if and only if for all $M > 0$, there exists a $\delta > 0$ such that for all $x \in \mathbb{R}$,

$$0 < a - x < \delta \implies f(x) \geq M.$$

- $\lim_{x\downarrow a} f(x) = -\infty$ if and only if for all $M > 0$, there exists a $\delta > 0$ such that for all $x \in \mathbb{R}$,

$$0 < x - a < \delta \implies f(x) \leq M.$$

We[1] first show that

$$\lim_{x\uparrow\pi/2}\tan(x) = +\infty.$$

Let $M > 0$. Since sin and cos are differentiable, they are continuous, and in particular continuous at $x = \pi/2$. Thus, there exist $\delta_s, \delta_c > 0$ such that for all $x \in \mathbb{R}$,

$$\left|x - \pi/2\right| < \delta_s \implies \left|\sin(x) - \sin(\pi/2)\right| < \frac{1}{2} \tag{1}$$

and

$$\left|x - \pi/2\right| < \delta_c \implies \left|\cos(x) - \cos(\pi/2)\right| < \frac{1}{2M}. \tag{2}$$

Let $\delta := \min\{\delta_1, \delta_2, \frac{\pi}{2}\}$ and let $x \in \mathbb{R}$ such that $0 < \pi/2 - x < \delta$. Since $\left|x - \pi/2\right| < \delta \leq \delta_s$, we have

$$\begin{aligned}
\left|\sin(x) - 1\right| &= \left|\sin(x) - \sin(\pi/2)\right| && \text{(from Problem 1c)}\\
&< \frac{1}{2} && \text{(from (1))}\\
\implies \sin(x) &> \frac{1}{2}, && \text{(adding and subtracting)}
\end{aligned}$$

and since $\left|x - \pi/2\right| < \delta \leq \delta_c$, we have

$$\begin{aligned}
\cos(x) &= \left|\cos(x)\right| && \text{(cos positive on } [0, \pi/2])\\
&= \left|\cos(x) - \cos(\pi/2)\right| && \text{(from Problem 1c)}\\
&< \frac{1}{2M} && \text{(from (2))}\\
\implies \frac{1}{\cos(x)} &> 2M. && \text{(inequality property)}
\end{aligned}$$

We can then use these results to write

$$\tan(x) = \frac{\sin(x)}{\cos(x)} = \sin(x)\left(\frac{1}{\cos(x)}\right) > \left(\frac{1}{2}\right)2M = M.$$

Therefore, by definition

$$\lim_{x\uparrow\pi/2}\tan(x) = +\infty.$$

[1] Credit to https://math.stackexchange.com/questions/2782609/proving-lim-x-to-pi-2-tanx-infty-using-epsilon-delta

Showing the other limit holds can be done with nearly identical reasoning. Thus, tan is surjective by definition. It is injective on $[-\pi/2, \pi/2]$ since its derivative $\tan'(x) = 1 + \tan^2(x)$ is positive on the interval, which implies tan is strictly increasing on the interval. Therefore, the restriction of tan to the interval is bijective and has an inverse function $\arctan : \mathbb{R} \to [-\pi/2, \pi/2]$. $\square$

Problem 11 Use that $\tan'(x) = 1 + \tan^2(x)$ to show

$$\frac{d}{dx}\arctan(x) = \frac{1}{1+x^2} = \sum_{k=0}^{\infty}(-1)^k x^{2k} = 1 - x^2 + x^4 - x^6 + x^8 - \dots,$$

which has radius of convergence $\mathbb{R} = 1$. Integrate this to show

$$\arctan(x) = \sum_{k=0}^{\infty}\frac{(-1)^k x^{2k+1}}{2k+1} = x - \frac{x^3}{3} + \frac{x^5}{5} - \frac{x^7}{7} + \dots$$

which will hold for $|x| < 1$.

Let $x_0 \in \mathbb{R}$, and $y_0 := \tan(x_0)$. We can use the theorem from class about derivatives of inverse functions along with Problem 9 to write

$$\arctan'(y_0) = \frac{1}{\tan'(x_0)} = \frac{1}{1+\tan^2(x_0)} = \frac{1}{1+\tan^2(\arctan(y_0))} = \frac{1}{1+y_0^2}.$$

Therefore, we have

$$\frac{d}{dx}\arctan(x) = \frac{1}{1+x^2} = \sum_{k=0}^{\infty}(-1)^k x^{2k}.$$

We can then integrate this to obtain

$$\arctan(x) = \int_0^x \frac{d}{dx}\arctan(t)\,dt = \int_0^x \sum_{k=0}^{\infty}(-1)^k t^{2k}\,dx = \sum_{k=0}^{\infty}\frac{(-1)^k x^{2k+1}}{2k+1}.$$

$\square$