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STAT 509: Section 003
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# STAT 509 Homework Extra Credit

(We assume that $\mathbb{N}$ excludes 0 for this problem.)

**Problem** We have a game with an attacker and a defender. The attacker has $m$ fair $s$-sided dice, and the defender has $n$ fair $s$-sided dice, where $m, n, s \in \mathbb{N}$. Both players roll all their dice at once, and both choose the die with the highest value from their hand. The attacker wins if their die has a value greater than or equal to the value of the defender's die, and the defender wins otherwise. In terms of $m$, $n$, and $s$, what is the probability the attacker wins?

Solution.

Since both players always choose one die after rolling, we can simply model each player as rolling a sort of weighted $s$-sided die, and then use the probability mass distribution of each player's weighted die to calculate the probability that the attacker's is higher in value.

We first calculate the PMF of a hand. Let $k, s \in \mathbb{N}$, and $X_{k,s}$ be a random variable that counts the greatest value rolled from a hand of $k$ fair $s$-sided dice. Then, the support of $X_{k,s}$ is $\{1, 2, \dots, s\}$. For example, $X_{2,6}$ takes values as shown in the chart below, with the columns representing the values from the first die and the rows representing the values from the second die:

| **D2\D1** | **1** | **2** | **3** | **4** | **5** | **6** |
|---|---|---|---|---|---|---|
| **1** | 1 | 2 | 3 | 4 | 5 | 6 |
| **2** | 2 | 2 | 3 | 4 | 5 | 6 |
| **3** | 3 | 3 | 3 | 4 | 5 | 6 |
| **4** | 4 | 4 | 4 | 4 | 5 | 6 |
| **5** | 5 | 5 | 5 | 5 | 5 | 6 |
| **6** | 6 | 6 | 6 | 6 | 6 | 6 |

As one can see in the table, the values of $X$ are partitioned in "shells" centered at the top left. This is because we are interested in the highest of the $k$ dice in general, so any group of dice will be assigned to the maximum value of the hand (this is similar to the notion of Chebyshev distance, which assigns distance in shells in the same way).

Choose $x$ in the values of $X$. Then, there are $x^k$ hands that have greatest value at most $x$, and if we subtract the $x^{k-1}$ hands that have greatest value at most $x-1$, we have the number of hands with greatest value exactly $x$. Since there are $s^k$ possible hands, we have

$$p_{X_{k,s}}(x) = \frac{x^k - (x-1)^k}{s^k}.$$

Each value in $x$ represents a "shell", so it is clear that we have a valid probability distribution because

$$(1^k - 0^k) + (2^k - 1^k) + \ldots + (s^k - (s-1)^k) = s^k.$$

Thus, we can model the game as the attacker rolling a weighted $s$-sided die with PMF $p_{X_{m,s}}$ and the defender rolling a weighted $s$-sided die with PMF $p_{X_{n,s}}$.

Now, let $Y_{m,n,s}$ be the difference between the attacker's weighted die and the defender's weighted die. For example, the $s = 6$ case takes values as shown in the chart below, with the columns representing the values from the attacker's die and the rows representing the values from the defender's die:

| **D\A** | **1** | **2** | **3** | **4** | **5** | **6** |
|---|---|---|---|---|---|---|
| **1** | 0 | 1 | 2 | 3 | 4 | 5 |
| **2** | -1 | 0 | 1 | 2 | 3 | 4 |
| **3** | -2 | -1 | 0 | 1 | 2 | 3 |
| **4** | -3 | -2 | -1 | 0 | 1 | 2 |
| **5** | -4 | -3 | -2 | -1 | 0 | 1 |
| **6** | -5 | -4 | -3 | -2 | -1 | 0 |

Since we are interested in the odds of the attacker winning, we are trying to calculate $P(Y \geq 0)$. To do this, we can sum over all the probabilities in the sample space that satisfy $Y \geq 0$. We first need to consider all the values the attacking die can take, which is $\{1, 2, \dots, s\}$. For each value $a$, the attacker wins if the defending die takes any value from 1 to $a$, inclusive. Since the two dice are independent, we can write

$$P(Y \geq 0) = \sum_{a=1}^{s} \sum_{d=1}^{a} p_{X_{m,s}}(a) p_{X_{n,s}}(d).$$

Using the PMF we calculated, we have that

$$P(Y \geq 0) = \sum_{a=1}^{s} \sum_{d=1}^{a} \frac{a^m - (a-1)^m}{s^m} \cdot \frac{d^n - (d-1)^n}{s^n}.$$

Therefore, the probability that the attacker wins is

$$\frac{1}{s^{m+n}} \sum_{a=1}^{s} \sum_{d=1}^{a} (a^m - (a-1)^m)(d^n - (d-1)^n).$$

We can represent this in R with a function:

```
> attacker_wins <- function(m,n,s) {
    sum = 0
    for (a in 1:s) {
        for (d in 1:a) {
            sum = sum + (a^m-(a-1)^m)*(d^n-(d-1)^n)
        }
    }
    return (sum/s^(m+n))
 }
> attacker_wins(2,1,6)
[1] 0.7453704
```

As expected from the simulation showed in class, R calculates the probability of an attacker winning with 2 6-sided dice against a defender with 1 6-sided die as $\approx 0.745$.

If we are interested in the $m = 2, n = 1$ case, we can get a closed form:

$$\begin{aligned}
&\frac{1}{s^{2+1}} \sum_{a=1}^{s} \sum_{d=1}^{a} (a^2 - (a-1)^2)(d^1 - (d-1)^1) && \text{(substituting } m = 2 \text{ and } n = 1\text{)} \\
&= \frac{1}{s^3} \sum_{a=1}^{s} \sum_{d=1}^{a} (a^2 - (a^2 - 2a + 1))(d - (d-1)) \\
&= \frac{1}{s^3} \sum_{a=1}^{s} \left[ (2a-1) \sum_{d=1}^{a} (1) \right] && \text{(} 2a - 1 \text{ is constant with respect to } d\text{)} \\
&= \frac{1}{s^3} \sum_{a=1}^{s} \left[ a(2a-1) \right] && \text{(replacing sum by } a\text{)} \\
&= \frac{1}{s^3} \left[ 2 \sum_{a=1}^{s} (a^2) - \sum_{a=1}^{s} (a) \right] && \text{(splitting sum)} \\
&= \frac{1}{s^3} \left[ 2 \left( \frac{s(s+1)(2s+1)}{6} \right) + \frac{s(s+1)}{2} \right] && \text{(evaluating sums)} \\
&= \frac{(s+1)(2s+1)}{3s^2} - \frac{s+1}{2s^2} \\
&= \frac{2(2s^2 + s + 2s + 1) - 3(s+1)}{6s^2} \\
&= \frac{4s^2 + 6s + 2 - 3s - 3}{6s^2} \\
&= \frac{4s^2 + 3s - 1}{6s^2}.
\end{aligned}$$

If we let $s = 6$, we get

$$\frac{4(6)^2 + 3(6) - 1}{6(6)^2} \approx 0.745,$$

which is consistent with our result from R.