CSE 15 Homework 10

Solution (1pt each problem/sub-problem, total = 20)

Type your answers in a text file and submit it in UCMCROPS.
You can also write your answers on papers and scan them into image files for submission.

Section 5.4

6.

With this input, the algorithm uses the **else** clause to find that $\gcd(12,17) = \gcd(17 \textbf{ mod } 12, 12) = \gcd(5,12)$. It uses this clause again to find that $\gcd(5,12) = \gcd(12 \textbf{ mod } 5, 5) = \gcd(2,5)$, then to get $\gcd(2,5) = \gcd(5 \textbf{ mod } 2, 2) = \gcd(1,2)$, and once more to get $\gcd(1,2) = \gcd(2 \textbf{ mod } 1, 1) = \gcd(0,1)$. Finally, to find $\gcd(0,1)$ it uses the first step with $a = 0$ to find that $\gcd(0,1) = 1$. Consequently, the algorithm finds that $\gcd(12,17) = 1$.

10.

The recursive algorithm works by comparing the last element with the maximum of all but the last. We assume that the input is given as a sequence.

procedure $max(a_1, a_2, \ldots, a_n$: integers)
if $n = 1$ **then return** a_1
else
  $m := max(a_1, a_2, \ldots, a_{n-1})$
  if $m > a_n$ **then return** m
  else return a_n

22.

The largest in a list of one integer is that one integer, and that is the answer the recursive algorithm gives when $n = 1$, so the basis step is correct. Now assume that the algorithm works correctly for $n = k$. If $n = k+1$, then the else clause of the algorithm is executed. First, by the inductive hypothesis, the algorithm correctly sets m to be the largest among the first k integers in the list. Next it returns as the answer either that value or the $(k+1)$st element, whichever is larger. This is clearly the largest element in the entire list. Thus the algorithm correctly finds the maximum of a given list of integers.

28.

To compute f_7, Algorithm 7 requires $f_8 - 1 = 20$ additions, and Algorithm 8 requires $7 - 1 = 6$ additions.

Section 6.1

8.

There are 26 choices for the first initial, then 25 choices for the second, if no letter is to be repeated, then 24 choices for the third. (We interpret "repeated" broadly, so that a string like RWR, for example, is prohibited, as well as a string like RRW.) Therefore by the product rule the answer is $26 \cdot 25 \cdot 24 = 15{,}600$.

10.

We have two choices for each bit, so there are $2^8 = 256$ bit strings.

12.

We use the sum rule, adding the number of bit strings of each length up to 6. If we include the empty string, then we get $2^0 + 2^1 + 2^2 + 2^3 + 2^4 + 2^5 + 2^6 = 2^7 - 1 = 127$ (using the formula for the sum of a geometric progression—see Theorem 1 in Section 2.4).

14.

If $n = 0$, then the empty string—vacuously—satisfies the condition (or does not, depending on how one views it). If $n = 1$, then there is one, namely the string 1. If $n \geq 2$, then such a string is determined by specifying the $n - 2$ bits between the first bit and the last, so there are 2^{n-2} such strings.

28.

$10^3 26^3 + 26^3 10^3 = 35{,}152{,}000$

36.

There are 2^n such functions, since there is a choice of 2 function values for each element of the domain.

44.

If we ignore the fact that the table is round and just count ordered arrangements of length 4 from the 10 people, then we get $10 \cdot 9 \cdot 8 \cdot 7 = 5040$ arrangements. However, we can rotate the people around the table in 4 ways and get the same seating arrangement, so this overcounts by a factor of 4. (For example, the sequence Mary–Debra–Cristina–Julie gives the same circular seating as the sequence Julie–Mary–Debra–Cristina.) Therefore the answer is $5040/4 = 1260$.

46.

a) We first place the bride in any of the 6 positions. Then, from left to right in the remaining positions, we choose the other five people to be in the picture; this can be done in $9 \cdot 8 \cdot 7 \cdot 6 \cdot 5 = 15120$ ways. Therefore the answer is $6 \cdot 15120 = 90{,}720$.

b) We first place the bride in any of the 6 positions, and then place the groom in any of the 5 remaining positions. Then, from left to right in the remaining positions, we choose the other four people to be in the picture; this can be done in $8 \cdot 7 \cdot 6 \cdot 5 = 1680$ ways. Therefore the answer is $6 \cdot 5 \cdot 1680 = 50{,}400$.

Section 6.2

4.

We assume that the woman does not replace the balls after drawing them.

a) There are two colors: these are the pigeonholes. We want to know the least number of pigeons needed to insure that at least one of the pigeonholes contains three pigeons. By the generalized pigeonhole principle, the answer is 5. If five balls are selected, at least $\lceil 5/2 \rceil = 3$ must have the same color. On the other hand four balls is not enough, because two might be red and two might be blue. Note that the number of balls was irrelevant (assuming that it was at least 5).

b) She needs to select 13 balls in order to insure at least three blue ones. If she does so, then at most 10 of them are red, so at least three are blue. On the other hand, if she selects 12 or fewer balls, then 10 of them could be red, and she might not get her three blue balls. This time the number of balls did matter.

6.

There are only d possible remainders when an integer is divided by d, namely 0, 1, $\ldots$, $d-1$. By the pigeonhole principle, if we have $d+1$ remainders, then at least two must be the same.

8.

This is just a restatement of the pigeonhole principle, with $k = |T|$.

18.

a) If not, then there would be 4 or fewer male students and 4 or fewer female students, so there would be $4+4=8$ or fewer students in all, contradicting the assumption that there are 9 students in the class.

b) If not, then there would be 2 or fewer male students and 6 or fewer female students, so there would be $2+6=8$ or fewer students in all, contradicting the assumption that there are 9 students in the class.

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This is similar to Example 9. Label the computers C_1 through C_8, and label the printers P_1 through P_4. If we connect C_k to P_k for $k = 1, 2, 3, 4$ and connect each of the computers C_5 through C_8 to *all* the printers, then we have used a total of $4 + 4 \cdot 4 = 20$ cables. Clearly this is sufficient, because if computers C_1 through C_4 need printers, then they can use the printers with the same subscripts, and if any computers with higher subscripts need a printer instead of one or more of these, then they can use the printers that are not being used, since they are connected to all the printers. Now we must show that 19 cables are not enough. Since there are 19 cables and 4 printers, the average number of computers per printer is $19/4$, which is less than 5. Therefore some printer must be connected to fewer than 5 computers (the average of a set of numbers cannot be bigger than each of the numbers in the set). That means it is connected to 4 or fewer computers, so there are at least 4 computers that are not connected to it. If those 4 computers all needed a printer simultaneously, then they would be out of luck, since they are connected to at most the 3 other printers.