CSE 15
Discrete Mathematics

Lecture 10 – Sequences, Summations, Cardinality of Sets

Announcement

- HW #5 to be assigned on 10/11.
  - Due **5pm** 10/19 (Fri) with 1 extra day of re-submission.
- Midterm #1 on 10/9 (Tuesday)
  - During lecture: 75 minutes.
  - CLOSED BOOK & CLOSED NOTES.
  - Covers sections 1.1-1.7:
    - Lectures 1 through 6.
    - HWs # 1, 2, 3.
- Midterm review: on October 4 in class.
- Reading assignment
  - Ch. 3.1 – 3.3 of textbook

Sequences and Summations (Ch. 2.4)

- Sequences
  - Examples: Geometric Progression, Arithmetic Progression
- Recurrence Relations
  - Example: Fibonacci Sequence
- Summations

Introduction

- Sequences are **ordered** lists of elements.
  - 1, 2, 3, 5, 8
  - 1, 3, 9, 27, 81, …….
- Sequences arise throughout mathematics, computer science, and in many other disciplines, ranging from botany to music.
- We will introduce the terminology to represent sequences and sums of the terms in the sequences.

Sequences

Definition: A *sequence* is a **function** from a subset of the integers (usually either the set $\{0, 1, 2, 3, 4, \ldots..\}$ or $\{1, 2, 3, 4, \ldots.\}$) to a set *S*.

- The notation a_n is used to denote the image of the integer *n*.
- We can think of a_n as the equivalent of *f(n)* where *f* is a function from $\{0,1,2,\ldots..\}$ to *S*.
- We call a_n a *term* of the sequence.

Sequences

Example: Consider the sequence $\{a_n\}$ where

$$a_n = \frac{1}{n} \qquad \{a_n\} = \{a_1, a_2, a_3, \ldots\}$$

$$1, \frac{1}{2}, \frac{1}{3}, \frac{1}{4} \ldots$$

Geometric Progression

Definition: A *geometric progression* is a sequence of the form:

$$a, ar, ar^2, \ldots, ar^n, \ldots$$

where the *initial term a* and the *common ratio r* are real numbers.

Examples:

1. Let $a = 1$ and $r = -1$. Then:
$$\{b_n\} = \{b_0, b_1, b_2, b_3, b_4, \ldots\} = \{1, -1, 1, -1, 1, \ldots\}$$
2. Let $a = 2$ and $r = 5$. Then:
$$\{c_n\} = \{c_0, c_1, c_2, c_3, c_4, \ldots\} = \{2, 10, 50, 250, 1250, \ldots\}$$
3. Let $a = 6$ and $r = 1/3$. Then:
$$\{d_n\} = \{d_0, d_1, d_2, d_3, d_4, \ldots\} = \{6, 2, \frac{2}{3}, \frac{2}{9}, \frac{2}{27}, \ldots\}$$

Arithmetic Progression

Definition: A *arithmetic progression* is a sequence of the form:

$$a, a+d, a+2d, \ldots, a+nd, \ldots$$

where the *initial term a* and the *common difference d* are real numbers.

Examples:

1. Let $a = -1$ and $d = 4$:

$$\{s_n\} = \{s_0, s_1, s_2, s_3, s_4, \ldots\} = \{-1, 3, 7, 11, 15, \ldots\}$$

2. Let $a = 7$ and $d = -3$:

$$\{t_n\} = \{t_0, t_1, t_2, t_3, t_4, \ldots\} = \{7, 4, 1, -2, -5, \ldots\}$$

3. Let $a = 1$ and d = 2:

$$\{u_n\} = \{u_0, u_1, u_2, u_3, u_4, \ldots\} = \{1, 3, 5, 7, 9, \ldots\}$$

Strings

Definition: A *string* is a finite sequence of characters from a finite set (an alphabet).

- Sequences of characters or bits are important strings in computer science.
- The *empty string* is represented by λ.
- The string *abcde* has *length* 5.

Recurrence Relations

Definition: A ***recurrence relation*** for the sequence $\{a_n\}$ is an **equation** that expresses a_n in terms of one or more of the previous terms of the sequence, namely, a_0, a_1, …, a_{n-1}, for all integers n with $n \geq n_0$, where n_0 is a nonnegative integer.

- A sequence is called a ***solution*** of a recurrence relation if its terms satisfy the recurrence relation.
- The ***initial conditions*** for a sequence specify the terms that precede the first term where the recurrence relation takes effect.

Questions about Recurrence Relations

Example 1: Let $\{a_n\}$ be a sequence that satisfies the recurrence relation $a_n = a_{n-1} + 3$ for $n = 1,2,3,4,\ldots.$ and suppose that $a_0 = 2$.

What are a_1, a_2 and a_3?

Solution: We see from the recurrence relation that

$$a_1 = a_0 + 3 = 2 + 3 = 5$$

$$a_2 = 5 + 3 = 8$$

$$a_3 = 8 + 3 = 11$$

Questions about Recurrence Relations

Example 2: Let $\{a_n\}$ be a sequence that satisfies the recurrence relation $a_n = a_{n-1} - a_{n-2}$ for $n = 2,3,4,\ldots.$ and suppose that $a_0 = 3$ and $a_1 = 5$.

What are a_2 and a_3?

[Here the initial conditions are $a_0 = 3$ **and** $a_1 = 5$.]

Solution: We see from the recurrence relation that

$$a_2 = a_1 - a_0 = 5 - 3 = 2$$

$$a_3 = a_2 - a_1 = 2 - 5 = -3$$

Fibonacci Sequence

Definition: Define the *Fibonacci sequence*, $f_0, f_1, f_2, \ldots,$ by:

- Initial Conditions: $f_0 = 0, f_1 = 1$
- Recurrence Relation: $f_n = f_{n-1} + f_{n-2}$

Example: Find f_2, f_3, f_4, f_5 and f_6 .

Answer:

$$f_2 = f_1 + f_0 = 1 + 0 = 1,$$
$$f_3 = f_2 + f_1 = 1 + 1 = 2,$$
$$f_4 = f_3 + f_2 = 2 + 1 = 3,$$
$$f_5 = f_4 + f_3 = 3 + 2 = 5,$$
$$f_6 = f_5 + f_4 = 5 + 3 = 8.$$

Solving Recurrence Relations

- Finding a formula for the *n*th term of the sequence generated by a recurrence relation **in terms of *n*** is called ***solving the recurrence relation***.
- Such a formula is called a ***closed formula***.
- We illustrate by example the method of iteration in which we need to guess the formula.

Iterative Solution Example

Method 1: Working upward, forward substitution.

Example: Let $\{a_n\}$ be a sequence that satisfies the recurrence relation $a_n = a_{n-1} + 3$ for $n = 2,3,4,\ldots.$ and suppose that $a_1 = 2$.

Specify a_n in terms of n.

$$a_2 = 2 + 3$$
$$a_3 = (2 + 3) + 3 = 2 + 3 \cdot 2$$
$$a_4 = (2 + 2 \cdot 3) + 3 = 2 + 3 \cdot 3$$
$$\ldots$$
$$a_n = a_{n-1} + 3 = (2 + 3 \cdot (n - 2)) + 3 = 2 + 3(n - 1)$$

Iterative Solution Example

Method 2: Working downward, backward substitution.

Example: Let $\{a_n\}$ be a sequence that satisfies the recurrence relation $a_n = a_{n-1} + 3$ for $n = 2,3,4,\ldots.$ and suppose that $a_1 = 2$.

Specify a_n in terms of n.

$$
\begin{aligned}
a_n &= a_{n-1} + 3 \\
&= (a_{n-2} + 3) + 3 = a_{n-2} + 3 \cdot 2 \\
&= (a_{n-3} + 3\,) + 3 \cdot 2 \;= a_{n-3} + 3 \cdot 3 \\
&\qquad\qquad \ldots \\
&= a_2 + 3(n-2) = (a_1 + 3) + 3(n-2) = 2 + 3(n-1)
\end{aligned}
$$

Financial Application

Example: Suppose that a person deposits \$10,000.00 in a savings account at a bank yielding 11% per year with interest compounded annually.

How much will be in the account after 30 years?

Let P_n denote the amount in the account after 30 years.

P_n satisfies the following recurrence relation:

$$P_n = P_{n-1} + 0.11P_{n-1} = (1.11)\, P_{n-1}$$

with the initial condition $P_0 = 10{,}000$

Continued on next slide →

Financial Application

$$P_n = P_{n-1} + 0.11P_{n-1} = (1.11)\, P_{n-1}$$

with the initial condition $P_0 = 10{,}000$

Solution: Forward substitution:

$$P_1 = (1.11)P_0$$

$$P_2 = (1.11)P_1 = (1.11)^2 P_0$$

$$P_3 = (1.11)P_2 = (1.11)^3 P_0$$

:

$$P_n = (1.11)P_{n-1} = (1.11)^n \mathrm{P}_0$$

$$P_n = (1.11)^n\, 10{,}000$$

$$P_{30} = (1.11)^{30}\, 10{,}000 = \$228{,}992.97$$

Summations

- Sum of the terms $a_m, a_{m+1}, \ldots, a_n$ from the sequence $\{a_n\}$.
- The notation:

$$\sum_{j=m}^{n} a_j \qquad \textstyle\sum_{j=m}^{n} a_j \qquad \textstyle\sum_{m \le j \le n} a_j$$

  represents $a_m + a_{m+1} + \cdots + a_n$.
- The variable *j* is called the *index of summation*.
- It runs through all the integers starting with its *lower limit m* and ending with its *upper limit n*.

Summations

- More generally for a set *S*: $\sum_{j \in S} a_j$
- **Examples**:

$$r^0 + r^1 + r^2 + r^3 + \cdots + r^n = \sum_{0}^{n} r^j$$

$$1 + \frac{1}{2} + \frac{1}{3} + \frac{1}{4} + \cdots = \sum_{1}^{\infty} \frac{1}{i}$$

$$\text{If } S = \{2, 5, 7, 10\} \text{ then } \sum_{j \in S} a_j = a_2 + a_5 + a_7 + a_{10}$$

Product Notation

- Product of the terms $a_m, a_{m+1}, \ldots, a_n$ from the sequence $\{a_n\}$.

- The notation:

$$\prod_{j=m}^{n} a_j \qquad \prod\nolimits_{j=m}^{n} a_j \qquad \prod\nolimits_{m \le j \le n} a_j$$

represents

$$a_m \times a_{m+1} \times \cdots \times a_n$$

Geometric Series

Sums of terms of geometric progressions:

$$\sum_{j=0}^{n} ar^j = \begin{cases} \frac{ar^{n+1}-a}{r-1} & r \neq 1 \\ (n+1)a & r = 1 \end{cases}$$

Proof: Let $S_n = \sum_{j=0}^{n} ar^j$

To compute S_n, first multiply both sides of the equality by *r* and then manipulate the resulting sum as follows:

$$rS_n = r\sum_{j=0}^{n} ar^j$$

$$= \sum_{j=0}^{n} ar^{j+1}$$

Continued on next slide →

Geometric Series

$$= \sum_{j=0}^{n} ar^{j+1}$$

From previous slide.

$$= \sum_{k=1}^{n+1} ar^{k}$$

Shifting the index of summation with $k = j + 1$.

$$= \left(\sum_{k=0}^{n} ar^{k}\right) + (ar^{n+1} - a)$$

Removing $k = n + 1$ term and adding $k = 0$ term.

$$= S_n + (ar^{n+1} - a)$$

Substituting S for summation formula

$$\therefore \quad rS_n = S_n + (ar^{n+1} - a)$$

$$S_n = \frac{ar^{n+1} - a}{r - 1}$$

if $r \neq 1$

$$S_n = \sum_{j=0}^{n} ar^{j} = \sum_{j=0}^{n} a = (n+1)a$$

if $r = 1$

Some Useful Summation Formulae

TABLE 2 Some Useful Summation Formulae.

Sum	*Closed Form*
$\sum_{k=0}^{n} ar^k \ (r \neq 0)$	$\dfrac{ar^{n+1} - a}{r - 1}, r \neq 1$
$\sum_{k=1}^{n} k$	$\dfrac{n(n+1)}{2}$
$\sum_{k=1}^{n} k^2$	$\dfrac{n(n+1)(2n+1)}{6}$
$\sum_{k=1}^{n} k^3$	$\dfrac{n^2(n+1)^2}{4}$
$\sum_{k=0}^{\infty} x^k, \lvert x \rvert < 1$	$\dfrac{1}{1-x}$
$\sum_{k=1}^{\infty} kx^{k-1}, \lvert x \rvert < 1$	$\dfrac{1}{(1-x)^2}$

Geometric series: we just proved this.

Later we will prove some of these by induction.

Proof in text (requires calculus).

Cardinality of Sets (Ch. 2.5)

- Cardinality
- Countable Sets

Cardinality

Definition: The ***cardinality*** of a set A is equal to the cardinality of a set B, denoted

$$|A| = |B|,$$

if and only if there is a one-to-one correspondence (*i.e.*, a bijection) from A to B.

- If there is a one-to-one function (*i.e.*, an injection) from A to B, the cardinality of A is less than or the same as the cardinality of B and we write $|A| \leq |B|$.
- When $|A| \leq |B|$ and A and B have different cardinality, we say that the cardinality of A is less than the cardinality of B and write $|A| < |B|$.

Cardinality

- **Definition**: A set that is either finite or has the same cardinality as the set of positive integers ($\mathbf{Z}^+$) is called ***countable***. A set that is not countable is ***uncountable***.
- The set of real numbers **R** is an uncountable set.
- When an infinite set is countable (*countably infinite*) its cardinality is $\aleph_0$ (where $\aleph$ is aleph, the 1st letter of the Hebrew alphabet). We write $|S| = \aleph_0$ and say that S has cardinality "aleph null."

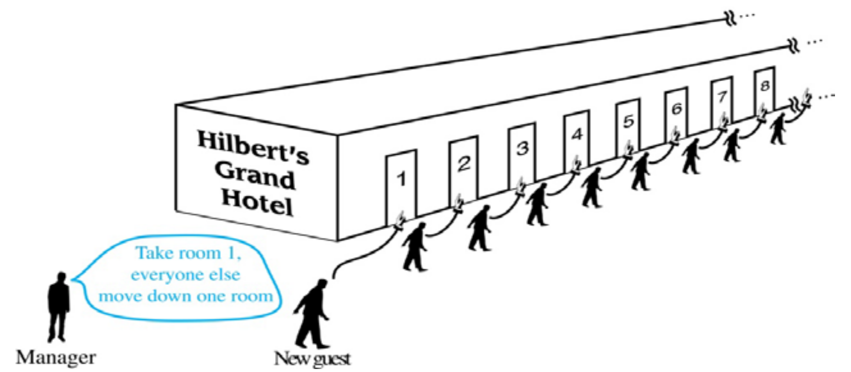
Showing that a Set is Countable

- An infinite set is countable if and only if it is possible to list the elements of the set in a sequence (indexed by the positive integers).
- The reason for this is that a one-to-one correspondence *f* from the set of positive integers to a set *S* can be expressed in terms of a sequence $a_1, a_2, \ldots, a_n, \ldots$ where $a_1 = f(1)$, $a_2 = f(2), \ldots, a_n = f(n), \ldots$

Hilbert's Grand Hotel

The Grand Hotel (example due to David Hilbert) has countably infinite number of rooms, each occupied by a guest. We can always accommodate a new guest at this hotel. How is this possible?

Explanation: Because the rooms of Grand Hotel are countable, we can list them as Room 1, Room 2, Room 3, and so on. When a new guest arrives, we move the guest in Room 1 to Room 2, the guest in Room 2 to Room 3, and in general the guest in Room n to Room $n + 1$, for all positive integers n. This frees up Room 1, which we assign to the new guest, and all the current guests still have rooms.



The hotel can also accommodate a countable number of new guests, all the guests on a countable number of buses where each bus contains a countable number of guests (see exercises).

Showing that a Set is Countable

Example 1: Show that the set of positive even integers E is countable set.

Solution: Let $f(x) = 2x$.

$$\begin{array}{ccccccc} 1 & 2 & 3 & 4 & 5 & 6 & \ldots\ldots \\ \updownarrow & \updownarrow & \updownarrow & \updownarrow & \updownarrow & \updownarrow & \\ 2 & 4 & 6 & 8 & 10 & 12 & \ldots\ldots \end{array}$$

Then f is a bijection from **N** to E since f is both one-to-one and onto. To show that it is one-to-one, suppose that $f(n) = f(m)$. Then $2n = 2m$, and so $n = m$. To see that it is onto, suppose that t is an even positive integer. Then $t = 2k$ for some positive integer k and $f(k) = t$.

The Positive Rational Numbers are Countable

- **Definition**: A *rational number* can be expressed as the ratio of two integers *p* and *q* such that $q \neq 0$.
  - ¾ is a rational number
  - $\sqrt{2}$ is not a rational number.

Example 3: Show that the positive rational numbers are countable.

Solution: The positive rational numbers are countable since they can be arranged in a sequence:

$$r_1, r_2, r_3, \ldots$$

The next slide shows how this is done. →

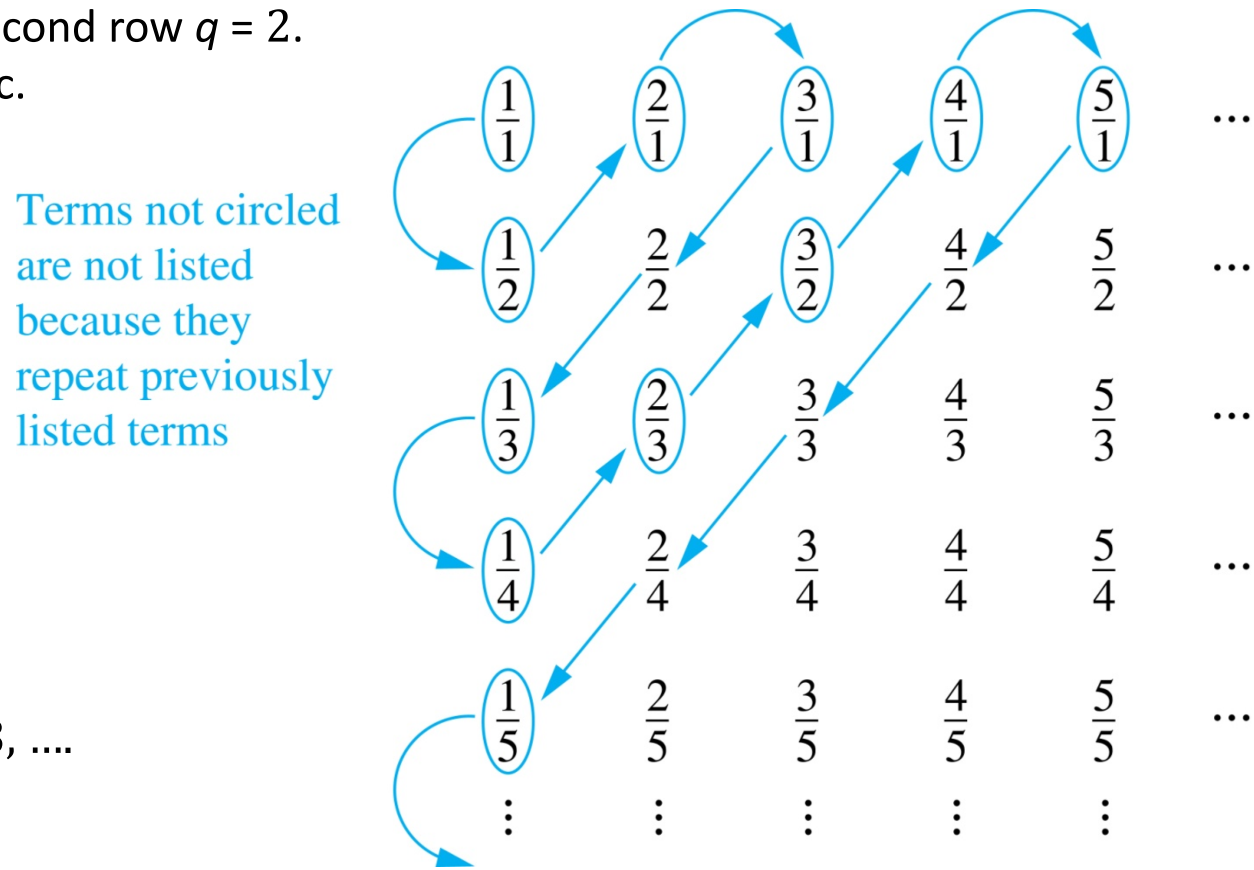
The Positive Rational Numbers are Countable

First row $q = 1$.
Second row $q = 2$.
etc.

Constructing the List

First list p/q with $p + q = 2$.
Next list p/q with $p + q = 3$

And so on.



1, ½, 2, 3, 1/3,1/4, 2/3, ….

The Real Numbers are Uncountable

Example: Show that the set of real numbers is uncountable.

Solution: The method is called the ***Cantor diagonalization argument***, and is a proof by contradiction.

1. Suppose **R** is countable. Then the real numbers between 0 and 1 are also countable (any subset of a countable set is countable - an exercise in the text).
2. The real numbers between 0 and 1 can be listed in order $r_1, r_2, r_3, \ldots$.
3. Let the decimal representation of this listing be

$$\begin{aligned} r_1 &= 0.d_{11}d_{12}d_{13}d_{14}d_{15}d_{16}\ldots \\ r_2 &= 0.d_{21}d_{22}d_{23}d_{24}d_{25}d_{26}\ldots \\ r_3 &= 0.d_{31}d_{32}d_{33}d_{34}d_{35}d_{36}\ldots \\ &\vdots \end{aligned}$$

4. Form a new real number with the decimal expansion $r = .r_1r_2r_3r_4\ldots$ where $r_i = 3 \text{ if } d_{ii} \neq 3$ and $r_i = 4 \text{ if } d_{ii} = 3$

The Real Numbers are Uncountable

5. r is not equal to any of the $r_1, r_2, r_3, \ldots$ Because it differs from r_i in its ith position after the decimal point. **Therefore there is a real number between 0 and 1 that is not on the list since every real number has a unique decimal expansion. Hence, all the real numbers between 0 and 1 cannot be listed, so the set of real numbers between 0 and 1 is uncountable.**
6. Since a set with an uncountable subset is uncountable (an exercise), the set of real numbers is uncountable.

Example

$$
\begin{aligned}
r_1 &= 0.23794102\cdots \\
r_2 &= 0.43590138\cdots \\
r_3 &= 0.09118764\cdots \\
r_4 &= 0.80535900\cdots \\
r &= 0.3434\ldots
\end{aligned}
$$

$$r = 0.r_1 r_2 r_3 r_4 \cdots$$

$$r_i = \begin{cases} 3 & \text{if } d_{ii} \neq 3 \\ 4 & \text{if } d_{ii} = 3 \end{cases}$$