Homework 6 Solution

Type your answers in a text file and submit it in CatCourses.
You can also write your answers on papers and scan them into image files for submission.

Section 2.5

2. (0.5pt each sub-problem, 2 pt total)

a) This set is countably infinite. The integers in the set are 11, 12, 13, 14, and so on. We can list these numbers in that order, thereby establishing the desired correspondence. In other words, the correspondence is given by $1 \leftrightarrow 11$, $2 \leftrightarrow 12$, $3 \leftrightarrow 13$, and so on; in general $n \leftrightarrow (n+10)$.
b) This set is countably infinite. The integers in the set are -1, -3, -5, -7, and so on. We can list these numbers in that order, thereby establishing the desired correspondence. In other words, the correspondence is given by $1 \leftrightarrow -1$, $2 \leftrightarrow -3$, $3 \leftrightarrow -5$, and so on; in general $n \leftrightarrow -(2n-1)$.
c) This set is $\{-999{,}999, -999{,}998, \ldots, -1, 0, 1, \ldots, 999{,}999\}$. It is finite, with cardinality 1,999,999.

d) This set is uncountable. We can prove it by the same diagonalization argument as was used to prove that the set of all reals is uncountable in Example 5.

4. (0.5pt each sub-problem, 2 pt total)
a) This set is countable. The integers in the set are ±1, ±2, ±4, ±5, ±7, and so on. We can list these numbers in the order $1, -1, 2, -2, 4, -4, 5, -5, 7, -7, \ldots$, thereby establishing the desired correspondence. In other words, the correspondence is given by $1 \leftrightarrow 1$, $2 \leftrightarrow -1$, $3 \leftrightarrow 2$, $4 \leftrightarrow -2$, $5 \leftrightarrow 4$, and so on.
b) This is similar to part (a); we can simply list the elements of the set in order of increasing absolute value, listing each positive term before its corresponding negative: $5, -5, 10, -10, 15, -15, 20, -20, 25, -25, 30, -30, 40, -40, 45, -45, 50, -50, \ldots$.
c) This set is countable but a little tricky. We can arrange the numbers in a 2-dimensional table as follows:

$$\begin{array}{llllllll}
.\overline{1} & .1 & .11 & .111 & .1111 & .11111 & .111111 & \ldots \\
1.\overline{1} & 1 & 1.1 & 1.11 & 1.111 & 1.1111 & 1.11111 & \ldots \\
11.\overline{1} & 11 & 11.1 & 11.11 & 11.111 & 11.1111 & 11.11111 & \ldots \\
111.\overline{1} & 111 & 111.1 & 111.11 & 111.111 & 111.1111 & 111.11111 & \ldots \\
\vdots & \vdots & \vdots & \vdots & \vdots & \vdots & \vdots &
\end{array}$$

Thus we have shown that our set is the countable union of countable sets (each of the countable sets is one row of this table). Therefore by Exercise 27, the entire set is countable. For an explicit correspondence with the positive integers, we can zigzag along the positive-sloping diagonals as in Figure 3: $1 \leftrightarrow .\overline{1}$, $2 \leftrightarrow 1.\overline{1}$, $3 \leftrightarrow .1$, $4 \leftrightarrow .11$, $5 \leftrightarrow 1$, and so on.

d) This set is not countable. We can prove it by the same diagonalization argument as was used to prove that the set of all reals is uncountable in Example 5. All we need to do is choose $d_i = 1$ when $d_{ii} = 9$ and choose $d_i = 9$ when $d_{ii} = 1$ or d_{ii} is blank (if the decimal expansion is finite).

12. (1pt)
The definition of $|A| \le |B|$ is that there is a one-to-one function from A to B. In this case the desired function is just $f(x) = x$ for each $x \in A$.

16. (1pt)

If a set A is countable, then we can list its elements, $a_1, a_2, a_3, \ldots, a_n, \ldots$ (possibly ending after a finite number of terms). Every subset of A consists of some (or none or all) of the items in this sequence, and we can list them in the same order in which they appear in the sequence. This gives us a sequence (again, infinite or finite) listing all the elements of the subset. Thus the subset is also countable.

20. (1pt)

By definition, we have one-to-one onto functions $f: A \to B$ and $g: B \to C$. Then $g \circ f$ is a one-to-one onto function from A to C, so $|A| = |C|$.

Section 2.6

14. (1pt)

Let $\mathbf{A}$ and $\mathbf{B}$ be two diagonal $n \times n$ matrices. Let $\mathbf{C} = [c_{ij}]$ be the product $\mathbf{AB}$. From the definition of matrix multiplication, $c_{ij} = \sum a_{iq}b_{qj}$. Now all the terms a_{iq} in this expression are 0 except for $q = i$, so $c_{ij} = a_{ii}b_{ij}$. But $b_{ij} = 0$ unless $i = j$, so the only nonzero entries of $\mathbf{C}$ are the diagonal entries $c_{ii} = a_{ii}b_{ii}$.

22. (1pt)

A matrix is symmetric if and only if it equals its transpose. So let us compute the transpose of $\mathbf{AA}^t$ and see if we get this matrix back. Using Exercise 17b and then Exercise 16, we have $(\mathbf{AA}^t)^t = ((\mathbf{A}^t)^t)\mathbf{A}^t = \mathbf{AA}^t$, as desired.

28. (1pt)

We follow the definition and obtain $\begin{bmatrix} 1 & 0 \\ 1 & 1 \\ 1 & 1 \end{bmatrix}$.

30. (0.5 pt each sub-problem, 1pt total)

a) $\mathbf{A} \vee \mathbf{A} = [a_{ij} \vee a_{ij}] = [a_{ij}] = \mathbf{A}$ **b)** $\mathbf{A} \wedge \mathbf{A} = [a_{ij} \wedge a_{ij}] = [a_{ij}] = \mathbf{A}$

Section 3.1

6. (1pt)

We need to go through the list and count the negative entries.

procedure *negatives*($a_1, a_2, \ldots, a_n$: integers)
$k := 0$
for $i := 1$ **to** n
    if $a_i < 0$ **then** $k := k + 1$
return k {the number of negative integers in the list}

10. (1pt)

We assume that if the input $x = 0$, then $n > 0$, since otherwise x^n is not defined. In our procedure, we let $m = |n|$ and compute x^m in the obvious way. Then if n is negative, we replace the answer by its reciprocal.

procedure *power*(x : real number, n : integer)
$m := |n|$
$power := 1$
for $i := 1$ **to** m
    $power := power \cdot x$
if $n < 0$ **then** $power := 1/power$
return $power$ $\{ power = x^n \}$

36. (1pt)

The procedure is the same as that given in the solution to Exercise 35. We will exhibit the lists obtained after each step, with all the lists obtained during one pass on the same line.

$dfkmab$, $dfkmab$, $dfkmab$, $dfkamb$, $dfkabm$
$dfkabm$, $dfkabm$, $dfakbm$, $dfabkm$
$dfabkm$, $dafbkm$, $dabfkm$
$adbfkm$, $abdfkm$
$abdfkm$

40. (1pt)

We start with d, f, k, m, a, b. The first step inserts f correctly into the sorted list d, producing no change. Similarly, no change results when k and m are inserted into the sorted lists d, f and d, f, k, respectively. Next a is inserted into d, f, k, m, and the list reads a, d, f, k, m, b. Finally b is inserted into a, d, f, k, m, and the list reads a, b, d, f, k, m. At each insertion, the element to be inserted is compared with the elements already sorted, starting from the beginning, until its correct spot is found, and then the previously sorted elements beyond that spot are each moved one position toward the back of the list.

52. (0.5pt each sub-problem, 2 pt total)

In each case we use as many quarters as we can, then as many dimes to achieve the remaining amount, then as many nickels, then as many pennies.

a) The algorithm uses the maximum number of quarters, three, leaving 12 cents. It then uses the maximum number of dimes (one) and nickels (none), before using two pennies.

b) one quarter, leaving 24 cents, then two dimes, leaving 4 cents, then four pennies

c) three quarters, leaving 24 cents, then two dimes, leaving 4 cents, then four pennies

d) one quarter, leaving 8 cents, then one nickel and three pennies

54. (0.5pt each sub-problem, 2 pt total)

a) The algorithm uses the maximum number of quarters, three, leaving 12 cents. It then uses the maximum number of dimes (one), and then two pennies. The greedy algorithm worked, since we got the same answer as in Exercise 52.

b) one quarter, leaving 24 cents, then two dimes, leaving 4 cents, then four pennies (the greedy algorithm worked, since we got the same answer as in Exercise 52)

c) three quarters, leaving 24 cents, then two dimes, leaving 4 cents, then four pennies (the greedy algorithm worked, since we got the same answer as in Exercise 52)

d) The greedy algorithm would have us use one quarter, leaving 8 cents, then eight pennies, a total of nine coins. However, we could have used three dimes and three pennies, a total of six coins. Thus the greedy algorithm is not correct for this set of coins.

56. (1pt)

One approach is to come up with an example in which using the 12-cent coin before using dimes or nickels would be inefficient. A dime and a nickel together are worth 15 cents, but the greedy algorithm would have us use four coins (a 12-cent coin and three pennies) rather than two. An alternative example would be 29 cents, in which case the greedy algorithm would use a quarter and four pennies, but we could have done better using two 12-cent coins and a nickel.