Homework5 Solution

0.5 point for each problem/sub-problem (total 20 points)

Section 2.3

8 (e, f, g, h)

e) 3 **f)** -2 **g)** $\lfloor \frac{1}{2} + 1 \rfloor = \lfloor \frac{3}{2} \rfloor = 1$ **h)** $\lceil 0 + 1 + \frac{1}{2} \rceil = \lceil \frac{3}{2} \rceil = 2$

10 (a, b)

a) This is one-to-one. **b)** This is not one-to-one, since b is the image of both a and b.

12 (c, d)

b) This is not one-to-one, since, for example, $f(3) = f(-3) = 10$.
c) This is one-to-one, since if $n_1^3 = n_2^3$, then $n_1 = n_2$ (take the cube root of each side).

16 (a, b, c, d)

a) This would normally be one-to-one, unless somehow two students in the class had a strange mobile phone service in which they shared the same phone number.
b) This is surely one-to-one; otherwise the student identification number would not "identify" students very well!
c) This is almost surely not one-to-one; unless the class is very small, it is very likely that two students will receive the same grade.
d) This function will be one-to-one as long as no two students in the class hale from the same town (which is rather unlikely, so the function is probably not one-to-one).

22 (c, d)

c) This function is a bijection, but not from $\mathbf{R}$ to $\mathbf{R}$. To see that the domain and range are not $\mathbf{R}$, note that $x = -2$ is not in the domain, and $x = 1$ is not in the range. On the other hand, f is a bijection from $\mathbf{R} - \{-2\}$ to $\mathbf{R} - \{1\}$, since its inverse is $f^{-1}(x) = (1 - 2x)/(x - 1)$.
d) It is clear that this continuous function is increasing throughout its entire domain ($\mathbf{R}$) and it takes on both arbitrarily large values and arbitrarily small (large negative) ones. So it is a bijection. Its inverse is clearly $f^{-1}(x) = \sqrt[5]{x-1}$.

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The key here is that larger denominators make smaller fractions, and smaller denominators make larger fractions. We have two things to prove, since this is an "if and only if" statement. First, suppose that f is strictly increasing. This means that $f(x) < f(y)$ whenever $x < y$. To show that g is strictly decreasing, suppose that $x < y$. Then $g(x) = 1/f(x) > 1/f(y) = g(y)$. Conversely, suppose that g is strictly decreasing. This means that $g(x) > g(y)$ whenever $x < y$. To show that f is strictly increasing, suppose that $x < y$. Then $f(x) = 1/g(x) < 1/g(y) = f(y)$.

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We have $(f \circ g)(x) = f(g(x)) = f(x+2) = (x+2)^2 + 1 = x^2 + 4x + 5$, whereas $(g \circ f)(x) = g(f(x)) = g(x^2+1) = x^2 + 1 + 2 = x^2 + 3$. Note that they are not equal.

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There are three cases. Define the "fractional part" of x to be $f(x) = x - \lfloor x \rfloor$. Clearly $f(x)$ is always between 0 and 1 (inclusive at 0, exclusive at 1), and $x = \lfloor x \rfloor + f(x)$. If $f(x)$ is less than $\frac{1}{2}$, then $x + \frac{1}{2}$ will have a value slightly less than $\lfloor x \rfloor + 1$, so when we round down, we get $\lfloor x \rfloor$. In other words, in this case $\lfloor x + \frac{1}{2} \rfloor = \lfloor x \rfloor$, and indeed that is the integer closest to x. If $f(x)$ is greater than $\frac{1}{2}$, then $x + \frac{1}{2}$ will have a value slightly greater than $\lfloor x \rfloor + 1$, so when we round down, we get $\lfloor x \rfloor + 1$. In other words, in this case $\lfloor x + \frac{1}{2} \rfloor = \lfloor x \rfloor + 1$, and indeed that is the integer closest to x in this case. Finally, if the fractional part is exactly $\frac{1}{2}$, then x is midway between two integers, and $\lfloor x + \frac{1}{2} \rfloor = \lfloor x \rfloor + 1$, which is the larger of these two integers.

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Write $x = n - \epsilon$, where n is an integer and $0 \le \epsilon < 1$; thus $\lceil x \rceil = n$. Then $\lceil x + m \rceil = \lceil n - \epsilon + m \rceil = n + m = \lceil x \rceil + m$. Alternatively, we could proceed along the lines of the proof of property 4a of Table 1, shown in the text.

52 (a, b)

a) The "if" direction is trivial, since $x \le \lceil x \rceil$. For the other direction, suppose that $x \le n$. Since n is an integer no smaller than x, and $\lceil x \rceil$ is by definition the smallest such integer, clearly $\lceil x \rceil \le n$.
b) The "if" direction is trivial, since $\lfloor x \rfloor \le x$. For the other direction, suppose that $n \le x$. Since n is an integer not exceeding x, and $\lfloor x \rfloor$ is by definition the largest such integer, clearly $n \le \lfloor x \rfloor$.

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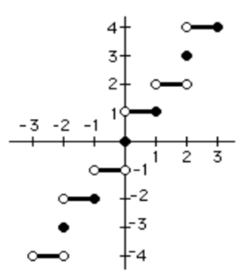
The function values for this step function change only at integer values of x, and different things happen for odd x and for even x because of the $x/2$ term. Whatever jump pattern is established on the closed interval $[0, 2]$ must repeat indefinitely in both directions. A thoughtful analysis then yields the following graph.



Section 2.4

10 (b, c, d)

b) $a_0 = 2,\ a_1 = -1,\ a_2 = a_1 - a_0 = -3,\ a_3 = a_2 - a_1 = -2,\ a_4 = a_3 - a_2 = 1,\ a_5 = a_4 - a_3 = 3$

c) $a_0 = 1,\ a_1 = 3a_0^2 = 3,\ a_2 = 3a_1^2 = 27 = 3^3,\ a_3 = 3a_2^2 = 2187 = 3^7,\ a_4 = 3a_3^2 = 14348907 = 3^{15},$ $a_5 = 3a_4^2 = 617673396283947 = 3^{31}$

d) $a_0 = -1,\ a_1 = 0,\ a_2 = 2a_1 + a_0^2 = 1,\ a_3 = 3a_2 + a_1^2 = 3,\ a_4 = 4a_3 + a_2^2 = 13,\ a_5 = 5a_4 + a_3^2 = 74$

12 (b, c)

b) $-3a_{n-1} + 4a_{n-2} = -3 \cdot 1 + 4 \cdot 1 = 1 = a_n$

c) $-3a_{n-1} + 4a_{n-2} = -3 \cdot (-4)^{n-1} + 4 \cdot (-4)^{n-2} = (-4)^{n-2}\big((-3)(-4) + 4\big) = (-4)^{n-2} \cdot 16 = (-4)^{n-2}(-4)^2 = (-4)^n = a_n$

16 (c, d, e, f)

c)

$$\begin{aligned}
a_n &= -n + a_{n-1} \\
&= -n + \big(-(n-1) + a_{n-2}\big) = -\big(n + (n-1)\big) + a_{n-2} \\
&= -\big(n + (n-1)\big) + \big(-(n-2) + a_{n-3}\big) = -\big(n + (n-1) + (n-2)\big) + a_{n-3} \\
&\ \ \vdots \\
&= -\big(n + (n-1) + (n-2) + \cdots + (n-(n-1))\big) + a_{n-n} \\
&= -\big(n + (n-1) + (n-2) + \cdots + 1\big) + a_0 \\
&= -\frac{n(n+1)}{2} + 4 = \frac{-n^2 - n + 8}{2}
\end{aligned}$$

d)

$$\begin{aligned}
a_n &= -3 + 2a_{n-1} \\
&= -3 + 2(-3 + 2a_{n-2}) = -3 + 2(-3) + 4a_{n-2} \\
&= -3 + 2(-3) + 4(-3 + 2a_{n-3}) = -3 + 2(-3) + 4(-3) + 8a_{n-3} \\
&= -3 + 2(-3) + 4(-3) + 8(-3 + 2a_{n-4}) = -3 + 2(-3) + 4(-3) + 8(-3) + 16a_{n-4} \\
&\ \ \vdots \\
&= -3(1 + 2 + 4 + \cdots + 2^{n-1}) + 2^n a_{n-n} = -3(2^n - 1) + 2^n(-1) = -2^{n+2} + 3
\end{aligned}$$

e)

$$\begin{aligned}
a_n &= (n+1)a_{n-1} = (n+1)na_{n-2} \\
&= (n+1)n(n-1)a_{n-3} = (n+1)n(n-1)(n-2)a_{n-4} \\
&\ \ \vdots \\
&= (n+1)n(n-1)(n-2)(n-3)\cdots(n-(n-2))\, a_{n-n} \\
&= (n+1)n(n-1)(n-2)(n-3)\cdots 2 \cdot a_0 \\
&= (n+1)! \cdot 2 = 2(n+1)!
\end{aligned}$$

f)

$$
\begin{aligned}
a_n &= 2na_{n-1} \\
&= 2n\big(2(n-1)a_{n-2}\big) = 2^2\big(n(n-1)\big)a_{n-2} \\
&= 2^2\big(n(n-1)\big)\big(2(n-2)a_{n-3}\big) = 2^3\big(n(n-1)(n-2)\big)a_{n-3} \\
&\ \ \vdots \\
&= 2^n n(n-1)(n-2)(n-3)\cdots\big(n-(n-1)\big)a_{n-n} \\
&= 2^n n(n-1)(n-2)(n-3)\cdots 1 \cdot a_0 \\
&= 3 \cdot 2^n n!
\end{aligned}
$$

22 (a, b, c)

We let a_n be the salary, in thousands of dollars, n years after 2009.

a) $a_n = 1 + 1.05a_{n-1}$, with $a_0 = 50$

b) Here $n = 8$. We can either iterate the recurrence relation 8 times, or we can use the result of part **(c)**. The answer turns out to be approximately $a_8 = 83.4$, i.e., a salary of approximately \$83,400.

c) We use the iterative approach.

$$
\begin{aligned}
a_n &= 1 + 1.05a_{n-1} \\
&= 1 + 1.05(1 + 1.05a_{n-2}) \\
&= 1 + 1.05 + (1.05)^2 a_{n-2} \\
&\ \ \vdots \\
&= 1 + 1.05 + (1.05)^2 + \cdots + (1.05)^{n-1} + (1.05)^n a_0 \\
&= \frac{(1.05)^n - 1}{1.05 - 1} + 50 \cdot (1.05)^n \\
&= 70 \cdot (1.05)^n - 20
\end{aligned}
$$

26 (d, e, f, g)

d) The sequence consists of one 1, followed by three 2's, followed by five 3's, followed by seven 5's, and so on, with the number of copies of the next value increasing by 2 each time, and the values themselves following the rule that the first two values are 1 and 2 and each subsequent value is the sum of the previous two values. Obviously other answers are possible as well. By our rule, the next three terms would be 8, 8, 8.

e) If we stare at this sequence long enough and compare it with Table 1, then we notice that the n^{th} term is $3^n - 1$. Thus the next three terms are 59048, 177146, 531440.

f) We notice that each term evenly divides the next, and the multipliers are successively 3, 5, 7, 9, 11, and so on. That must be the intended pattern. One notation for this is to use $n!!$ to mean $n(n-2)(n-4)\cdots$; thus the n^{th} term is $(2n-1)!!$. Thus the next three terms are 654729075, 13749310575, 316234143225.

g) The sequence consists of one 1, followed by two 0s, then three 1s, four 0s, five 1s, and so on, alternating between 0s and 1s and having one more item in each group than in the previous group. Thus six 0's will follow next, so the next three terms are 0, 0, 0.

34 (a, c)

We will just write out the sums explicitly in each case.

a) $(1-1)+(1-2)+(2-1)+(2-2)+(3-1)+(3-2)=3$

c) $(0+1+2)+(0+1+2)+(0+1+2)=9$

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$$n! = \prod_{i=1}^{n} i$$