CSE 15
Discrete Mathematics

Lecture 2 – Proposition Logic (2)

Announcement

- HW #1
  - Due by **5pm** 9/5 (Wed) with 1 extra day for late-submission.
  - Type your answers in a text file and submit it in CatCourses.
  - Or write your answers on papers and scan them into image files for submission.
  - Ask questions from your TA during lab hours

- Reading assignment
  - Ch. 1.4 – 1.5 of textbook

Propositional Logic

- Constructing Propositions
  - Propositional Variables: *p, q, r, s*, …
  - The proposition that is always true is denoted by **T** and the proposition that is always false is denoted by **F**.
  - Compound Propositions; constructed from logical connectives and other propositions
    - Negation $\neg$
    - Conjunction $\wedge$
    - Disjunction $\vee$
    - Implication $\rightarrow$
    - Biconditional $\leftrightarrow$

Implication (Conditional statement)

- If p and q are propositions, then $p \rightarrow q$ is a *conditional statement* or *implication* which is read as "if p, then q" and has this truth table:

p	q	$p \rightarrow q$
T	T	T
T	F	F
F	T	T
F	F	T

- **Example**: If p denotes "I am at home." and q denotes "I am not driving." then $p \rightarrow q$ denotes
  - "If I am at home then I am not driving."
- In $p \rightarrow q$, p is the *hypothesis* (*antecedent* or *premise*) and q is the *conclusion* (or *consequence*).

Understanding Implications

- One way to view the logical conditional is to think of an obligation or contract.
  - "If I am elected, then I will lower taxes."
  - If score between 95 and 100, then you will get an A.
  - "If you get 100% in the final, then you will get an A."
- If the politician is elected and does not lower taxes, then the voters can say that he or she has broken the campaign pledge. Something similar holds for the professor. This corresponds to the case where p is true and q is false.
- May represent violation of a contract.

Implication: example

- "If today is Friday, then 2+3=6"
  - The statement is true every day except Friday even though 2+3=6 is false

TABLE 5 **The Truth Table for the Conditional Statement $p \rightarrow q$.**

p	q	$p \rightarrow q$
T	T	T
T	F	F
F	T	T
F	F	T

Different Ways of Expressing $p \rightarrow q$

- if p, then q
- if p, q
- q unless $\neg p$
- q if p
- q whenever p
- q follows from p
- p implies q
- p only if q
- q when p
- q is necessary for p
- p is sufficient for q

- a necessary condition for p is q
- a sufficient condition for q is p

$p \to q$

- p only if q:
  - p cannot be true when q is not true
  - The statement is false if p is true but q is false
  - When p is false, q may be either true or false
  - Not to use "q only if p" to express $p \to q$
- q unless $\neg p$
  - If $\neg p$ is false, then q must be true
  - The statement is false when p is true but q is false, but the statement is true otherwise

TABLE 5 The Truth Table for the Conditional Statement $p \to q$.

p	q	$p \to q$
T	T	T
T	F	F
F	T	T
F	F	T

An Example

- If Maria learns discrete mathematics, then she will find a good job
  - Maria will find a good job when she learns discrete mathematics (q when p)
  - For Maria to get a good job, it is sufficient for her to learn discrete mathematics (sufficient condition for q is p)
  - Maria will find a good job unless she does not learn discrete mathematics (q unless not p)

Common mistake for $p \rightarrow q$

- Correct: p only if q
- Mistake to think "q only if p"

Converse, Contrapositive, and Inverse

- From $p \to q$ we can form new conditional statements .
  - $q \to p$ is the **converse** of $p \to q$
  - $\neg q \to \neg p$ is the **contrapositive** of $p \to q$
  - $\neg p \to \neg q$ is the **inverse** of $p \to q$

Example: Find the converse, inverse, and contrapositive of "It is raining is a sufficient condition for my not going to town." (raining → not going to town)

Solution:

converse: If I do not go to town, then it is raining.

contrapositive: If I go to town, then it is not raining.

inverse: If it is not raining, then I will go to town.

- **Contrapositive** and **conditional statements** are equivalent

Biconditional

- If p and q are propositions, then we can form the *biconditional* proposition $p \leftrightarrow q$, read as "p if and only if q." The biconditional $p \leftrightarrow q$ denotes the proposition with this truth table:

p	q	$p \leftrightarrow q$
T	T	T
T	F	F
F	T	F
F	F	T

- If p denotes "Your final score is between 95 and 100" and q denotes "Your grade is A" then $p \leftrightarrow q$ denotes
  - "Your grade is A if and only if your final score is between 95 and 100."

Example

- P: “you can take the flight”, q: “you buy a ticket”
- P $\leftrightarrow$ q:
  - “You can take the flight if and only if you buy a ticket”
  - This statement is true
    - If you buy a ticket and take the flight
    - If you do not buy a ticket and you cannot take the flight

Expressing the Biconditional

- Some alternative ways "p if and only if q" is expressed in English:
  - **p is necessary and sufficient for q**
  - **if p then q , and conversely**
  - **p iff q**

Truth Tables For Compound Propositions

- Construction of a truth table:
- Rows
  - Need a row for every possible combination of values for the atomic propositions.
- Columns
  - Need a column for the compound proposition (usually at far right)
  - Need a column for the truth value of each expression that occurs in the compound proposition as it is built up.
    - This includes the atomic propositions

Example Truth Table

- Construct a truth table for

$$p \lor q \to \neg r$$

p	q	r	¬r	p ∨ q	p ∨ q → ¬r
T	T	T	F	T	F
T	T	F	T	T	T
T	F	T	F	T	F
T	F	F	T	T	T
F	T	T	F	T	F
F	T	F	T	T	T
F	F	T	F	F	T
F	F	F	T	F	T

Truth table of compound propositions

TABLE 7 The Truth Table of $(p \vee \neg q) \rightarrow (p \wedge q)$.

p	q	$\neg q$	$p \vee \neg q$	$p \wedge q$	$(p \vee \neg q) \rightarrow (p \wedge q)$
T	T	F	T	T	T
T	F	T	T	F	F
F	T	F	F	F	T
F	F	T	T	F	F

Equivalent Propositions

- Two propositions are *equivalent* if they always have the same truth value.
- **Example**: Show using a truth table that the **conditional** is equivalent to the **contrapositive**.

Solution:

p	q	$\neg p$	$\neg q$	$p \rightarrow q$	$\neg q \rightarrow \neg p$
T	T	F	F	T	T
T	F	F	T	F	F
F	T	T	F	T	T
F	F	T	T	T	T

Using a Truth Table to Show Non-Equivalence

Example: Show using truth tables that neither the **converse** nor **inverse** of an implication are equivalent to the implication.

Solution:

p	q	$\neg p$	$\neg q$	$p \to q$	$\neg p \to \neg q$	$q \to p$
T	T	F	F	T	T	T
T	F	F	T	F	T	T
F	T	T	F	T	F	F
F	F	T	T	T	T	T

Problem

- How many rows are there in a truth table with *n* propositional variables?

  Solution: 2^n We will see how to prove this in Chapter 6.

- Note that this means that with n propositional variables, we can construct 2^n distinct (i.e., not equivalent) propositions.

Precedence of Logical Operators

Operator	Precedence
$\neg$	1
$\wedge$ $\vee$	2 3
$\rightarrow$ $\leftrightarrow$	4 5

$p \vee q \rightarrow r$ $\quad$ $(p \vee q) \rightarrow r$

$\neg p \wedge q$ $\quad$ $(\neg p) \wedge q$

$p \vee q \rightarrow \neg r$ is equivalent to $(p \vee q) \rightarrow \neg r$
If the intended meaning is $p \vee (q \rightarrow \neg r)$
then parentheses must be used.

Bit operations

TABLE 9 **Table for the Bit Operators *OR*, *AND*, and *XOR*.**

x	y	$x \vee y$	$x \wedge y$	$x \oplus y$
0	0	0	0	0
0	1	1	0	1
1	0	1	0	1
1	1	1	1	0

Applications of Propositional Logic (Ch.1.2)

- Translating English to Propositional Logic
- System Specifications
- Boolean Searching
- Logic Puzzles
- Logic Circuits

Translating English to logical expressions

Why?

- English is often ambiguous and translating sentences into compound propositions removes the ambiguity.
- Using logical expressions, we can analyze them and determine their truth values.
- We can use rules of inference to reason about them.

Translating English Sentences

- Steps to convert an English sentence to a statement in propositional logic
  - Identify atomic propositions and represent using propositional variables.
  - Determine appropriate logical connectives
- "If I go to Harry's or to the country, I will not go shopping."
  - *p*: I go to Harry's
  - q: I go to the country.
  - *r*: I will go shopping.

If *p* or *q* then not *r*.

$$(p \vee q) \rightarrow \neg r$$

Example

Problem: Translate the following sentence into propositional logic:

"You can access the Internet from campus only if you are a computer science major or you are not a freshman."

One Solution: Let a, c, and f represent respectively "You can access the internet from campus," "You are a computer science major," and "You are a freshman."

$$a \rightarrow (c \lor \neg f)$$

System Specifications

- System and Software engineers take requirements in English and express them in a precise specification language based on logic.

Example: Express in propositional logic:

"The automated reply cannot be sent when the file system is full"

Solution: One possible solution: Let *p* denote "The automated reply can be sent" and *q* denote "The file system is full."

$$q \rightarrow \neg p$$

Consistent System Specifications

Definition: A list of propositions is *consistent* if it is possible to assign truth values to the proposition variables so that each proposition is true.

Exercise: Are these specifications consistent?

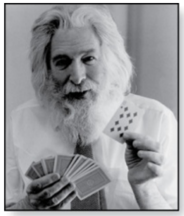
- "The diagnostic message is stored in the buffer or it is retransmitted."
- "The diagnostic message is not stored in the buffer."
- "If the diagnostic message is stored in the buffer, then it is retransmitted."

Solution: Let p denote "The diagnostic message is stored in the buffer." Let q denote "The diagnostic message is retransmitted" The specification can be written as: $p \vee q, \neg p, p \rightarrow q$. When p is false and q is true all three statements are true. So the specification is consistent.

- What if "The diagnostic message is not retransmitted is added."

  Solution: Now we are adding $\neg q$ and there is no satisfying asignment. So the specification is not consistent.

Logic Puzzles



Raymond Smullyan (Born 1919)

- An island has two kinds of inhabitants, *knights*, who always tell the truth, and *knaves*, who always lie.
- You go to the island and meet A and B.
  - A says "B is a knight."
  - B says "The two of us are of opposite types."

Example: What are the types of A and B?

Solution: Let p and q be the statements that A is a knight and B is a knight, respectively. So, then $\neg p$ represents the proposition that A is a knave and $\neg q$ that B is a knave.

- If A is a knight, then p is true. Since knights tell the truth, q must also be true. Then $(p \wedge \neg q) \vee (\neg p \wedge q)$ would have to be true, but it is not. So, A is not a knight and therefore $\neg p$ must be true.
- If A is a knave, then B must not be a knight since knaves always lie. So, then both $\neg p$ and $\neg q$ hold since both are knaves.

Propositional Equivalences (Ch. 1.3)

- Tautologies, Contradictions, and Contingencies.
- Logical Equivalence
  - Important Logical Equivalences
  - Showing Logical Equivalence

Tautologies, Contradictions, and Contingencies

- A tautology is a proposition which is **always true**.
  - Example: $p \vee \neg p$
- A *contradiction* is a proposition which is **always false**.
  - Example: $p \wedge \neg p$
- A *contingency* is a proposition which is neither a tautology nor a contradiction, such as *p*

P	$\neg p$	$p \vee \neg p$	$p \wedge \neg p$
T	F	T	F
F	T	T	F

Logically Equivalent

- Two compound propositions *p* and *q* are logically equivalent if $p \leftrightarrow q$ is a tautology.
- We write this as $p \Leftrightarrow q$ or as $p \equiv q$ where p and q are compound propositions.
- Two compound propositions p and q are equivalent if and only if the columns in a truth table giving their truth values agree.
- This truth table shows $\neg p \vee q$ is equivalent to $p \rightarrow q$.

p	q	$\neg p$	$\neg p \vee q$	$p \rightarrow q$
T	T	F	T	T
T	F	F	F	F
F	T	T	T	T
F	F	T	T	T

De Morgan's Laws

$$\neg(p \wedge q) \equiv \neg p \vee \neg q$$

$$\neg(p \vee q) \equiv \neg p \wedge \neg q$$

Augustus De Morgan

1806-1871

This truth table shows that De Morgan's Second Law holds.

p	q	$\neg p$	$\neg q$	$(p \vee q)$	$\neg(p \vee q)$	$\neg p \wedge \neg q$
T	T	F	F	T	F	F
T	F	F	T	T	F	F
F	T	T	F	T	F	F
F	F	T	T	F	T	T

Key Logical Equivalences

- Identity Laws: $p \wedge T \equiv p$, $p \vee F \equiv p$
- Domination Laws: $p \vee T \equiv T$, $p \wedge F \equiv F$
- Idempotent laws: $p \vee p \equiv p$, $p \wedge p \equiv p$
- Double Negation Law: $\neg(\neg p) \equiv p$
- Negation Laws: $p \vee \neg p \equiv T$, $p \wedge \neg p \equiv F$

Key Logical Equivalences (*cont*)

- Commutative Laws: $p \vee q \equiv q \vee p$, $p \wedge q \equiv q \wedge p$

- Associative Laws: $(p \wedge q) \wedge r \equiv p \wedge (q \wedge r)$
  $(p \vee q) \vee r \equiv p \vee (q \vee r)$

- Distributive Laws: $(p \vee (q \wedge r) \equiv (p \vee q)) \wedge (p \vee r)$
  $(p \wedge (q \vee r)) \equiv (p \wedge q) \vee (p \wedge r)$

- Absorption Laws: $p \vee (p \wedge q) \equiv p$, $p \wedge (p \vee q) \equiv p$

More Logical Equivalences

TABLE 7 **Logical Equivalences Involving Conditional Statements.**

$$p \to q \equiv \neg p \vee q$$
$$p \to q \equiv \neg q \to \neg p$$
$$p \vee q \equiv \neg p \to q$$
$$p \wedge q \equiv \neg(p \to \neg q)$$
$$\neg(p \to q) \equiv p \wedge \neg q$$
$$(p \to q) \wedge (p \to r) \equiv p \to (q \wedge r)$$
$$(p \to r) \wedge (q \to r) \equiv (p \vee q) \to r$$
$$(p \to q) \vee (p \to r) \equiv p \to (q \vee r)$$
$$(p \to r) \vee (q \to r) \equiv (p \wedge q) \to r$$

TABLE 8 **Logical Equivalences Involving Biconditional Statements.**

$$p \leftrightarrow q \equiv (p \to q) \wedge (q \to p)$$
$$p \leftrightarrow q \equiv \neg p \leftrightarrow \neg q$$
$$p \leftrightarrow q \equiv (p \wedge q) \vee (\neg p \wedge \neg q)$$
$$\neg(p \leftrightarrow q) \equiv p \leftrightarrow \neg q$$

Constructing New Logical Equivalences

- We can show that two expressions are logically equivalent by developing a series of logically equivalent statements.
- To prove that $A \equiv B$ we produce a series of equivalences beginning with A and ending with B.

$$\begin{gathered} A \equiv A_1 \\ \vdots \\ A_n \equiv B \end{gathered}$$

- Keep in mind that whenever a proposition (represented by a propositional variable) occurs in the equivalences listed earlier, it may be replaced by an arbitrarily complex compound proposition.

Equivalence Proofs

Example: Show that $\neg(p \vee (\neg p \wedge q))$

is logically equivalent to $\neg p \wedge \neg q$

Solution:

$$
\begin{array}{rcll}
\neg(p \vee (\neg p \wedge q)) & \equiv & \neg p \wedge \neg(\neg p \wedge q) & \text{by the second De Morgan law} \\
 & \equiv & \neg p \wedge [\neg(\neg p) \vee \neg q] & \text{by the first De Morgan law} \\
 & \equiv & \neg p \wedge (p \vee \neg q) & \text{by the double negation law} \\
 & \equiv & (\neg p \wedge p) \vee (\neg p \wedge \neg q) & \text{by the second distributive law} \\
 & \equiv & F \vee (\neg p \wedge \neg q) & \text{because } \neg p \wedge p \equiv F \\
 & \equiv & (\neg p \wedge \neg q) \vee F & \text{by the commutative law} \\
 & & & \text{for disjunction} \\
 & \equiv & (\neg p \wedge \neg q) & \text{by the identity law for } \mathbf{F}
\end{array}
$$

Equivalence Proofs

Example: Show that $(p \wedge q) \rightarrow (p \vee q)$ is a tautology.

Solution:

$$
\begin{array}{rcll}
(p \wedge q) \rightarrow (p \vee q) & \equiv & \neg(p \wedge q) \vee (p \vee q) & \text{by truth table for } \rightarrow \\
& \equiv & (\neg p \vee \neg q) \vee (p \vee q) & \text{by the first De Morgan law} \\
& \equiv & (\neg p \vee p) \vee (\neg q \vee q) & \text{by associative and} \\
& & & \text{commutative laws} \\
& & & \text{laws for disjunction} \\
& \equiv & T \vee T & \text{by truth tables} \\
& \equiv & T & \text{by the domination law}
\end{array}
$$