CSE 15
Discrete Mathematics

Lecture 23– Inclusion-Exclusion & Relations

Inclusion-Exclusion (Ch. 8.5)

- The Principle of Inclusion-Exclusion
- Examples

Principle of Inclusion-Exclusion

- In Section 2.2, we developed the following formula for the number of elements in the union of two finite sets:

$$|A \cup B| = |A| + |B| - |A \cap B|$$

- For three sets:

$$|A \cup B \cup C| =$$

$$|A| + |B| + |C| - |A \cap B| - |A \cap C| - |B \cap C| + |A \cap B \cap C|$$

Two Finite Sets

Example: In a discrete mathematics class, every student is a major in computer science or mathematics or both.

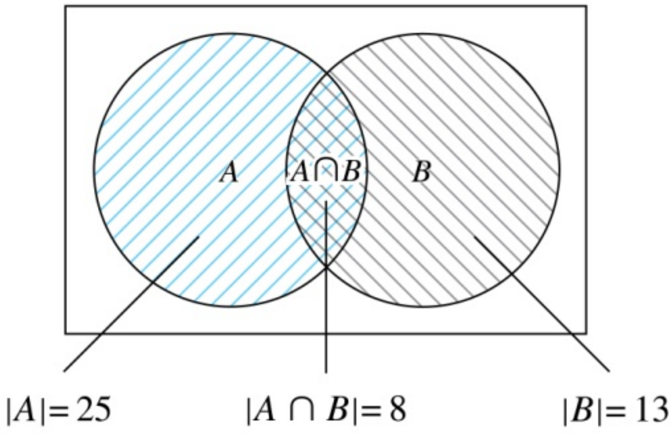
- The number of students having computer science as a major (possibly along with mathematics) is 25.
- The number of students having mathematics as a major (possibly along with computer science) is 13.
- The number of students majoring in both computer science and mathematics is 8.

How many students are in the class?

Solution: $|A \cup B| = |A| + |B| - |A \cap B|$

$= 25 + 13 - 8 = 30$

$|A \cup B| = |A| + |B| - |A \cap B| = 25 + 13 - 8 = 30$



Three Finite Sets Example

Example:

- A total of 1232 students have taken a course in Spanish,
- 879 have taken a course in French,
- and 114 have taken a course in Russian.
- Further, 103 have taken courses in both Spanish and French,
- 23 have taken courses in both Spanish and Russian,
- and 14 have taken courses in both French and Russian.

If 2092 students have taken a course in at least one of Spanish French and Russian, how many students have taken a course in all 3 languages.

Three Finite Sets Continued

Solution: Let

- S be the set of students who have taken a course in Spanish,
- F the set of students who have taken a course in French,
- and R the set of students who have taken a course in Russian.

Then, we have

- $|S| = 1232$, $|F| = 879$, $|R| = 114$,
- $|S \cap F| = 103$, $|S \cap R| = 23$, $|F \cap R| = 14$,
- and $|S \cup F \cup R| = 2092$.

Using the equation

$$|S \cup F \cup R| = |S| + |F| + |R| - |S \cap F| - |S \cap R| - |F \cap R| + |S \cap F \cap R|,$$

$$2092 = 1232 + 879 + 114 - 103 - 23 - 14 + |S \cap F \cap R|.$$

Solving for $|S \cap F \cap R|$ yields 7.

Relations and Their Properties (Ch. 9.1)

- Relations
- Properties of Relations
  - Reflexive Relations
  - Symmetric and Antisymmetric Relations
  - Transitive Relations
- Combining Relations

Binary Relation on a Set

Definition: A binary relation R *on a set* A is a subset of $A \times A$ or a relation from A to A.

Example:

- Suppose that $A = \{a,b,c\}$. Then $R = \{(a,a),(a,b),(a,c)\}$ is a relation on A.
- Let $A = \{1, 2, 3, 4\}$. The ordered pairs in the relation $R = \{(a,b) \mid a \text{ divides } b\}$ are

$(1,1)$, $(1, 2)$, $(1,3)$, $(1, 4)$, $(2, 2)$, $(2, 4)$, $(3, 3)$, and $(4, 4)$.

Reflexive Relations

Definition: R is *reflexive* iff $(a,a) \in R$ for every element $a \in A$.

Written symbolically, R is reflexive if and only if

$$\forall x[x \in A \longrightarrow (x,x) \in R]$$

Example: Which of these relations are reflexive over the set of integers?

$R_1 = \{(a,b) \mid a \leq b\}$,

$R_2 = \{(a,b) \mid a > b\}$,

$R_3 = \{(a,b) \mid a = b \text{ or } a = -b\}$,

$R_4 = \{(a,b) \mid a = b\}$,

$R_5 = \{(a,b) \mid a = b + 1\}$,

$R_6 = \{(a,b) \mid a + b \leq 3\}$.

Reflexive Relations

Example: The following relations on the integers are reflexive:

$R_1 = \{(a,b) \mid a \leq b\}$,

$R_3 = \{(a,b) \mid a = b \text{ or } a = -b\}$,

$R_4 = \{(a,b) \mid a = b\}$.

The following relations are not reflexive:

$R_2 = \{(a,b) \mid a > b\}$ (note that $3 \ngtr 3$),

$R_5 = \{(a,b) \mid a = b + 1\}$ (note that $3 \neq 3 + 1$),

$R_6 = \{(a,b) \mid a + b \leq 3\}$ (note that $4 + 4 \nleq 3$).

Symmetric Relations

Definition: R is *symmetric* iff $(b,a) \in R$ whenever $(a,b) \in R$ for all $a,b \in A$.

Written symbolically, R is symmetric if and only if

$$\forall x \forall y \, [(x,y) \in R \longrightarrow (y,x) \in R]$$

Example: Which of these relations are symmetric over the set of integers?

$R_1 = \{(a,b) \mid a \leq b\}$,

$R_2 = \{(a,b) \mid a > b\}$,

$R_3 = \{(a,b) \mid a = b \text{ or } a = -b\}$,

$R_4 = \{(a,b) \mid a = b\}$,

$R_5 = \{(a,b) \mid a = b + 1\}$,

$R_6 = \{(a,b) \mid a + b \leq 3\}$.

Symmetric Relations

Example: The following relations on the integers are symmetric:

$R_3 = \{(a,b) \mid a = b \text{ or } a = -b\}$,

$R_4 = \{(a,b) \mid a = b\}$,

$R_6 = \{(a,b) \mid a + b \leq 3\}$.

The following are not symmetric:

$R_1 = \{(a,b) \mid a \leq b\}$ (note that $3 \leq 4$, but $4 \nleq 3$),

$R_2 = \{(a,b) \mid a > b\}$ (note that $4 > 3$, but $3 \ngtr 4$),

$R_5 = \{(a,b) \mid a = b + 1\}$ (note that $4 = 3 + 1$, but $3 \neq 4 + 1$).

Antisymmetric Relations

Definition: A relation R on a set A such that for all $a,b \in A$ if $(a,b) \in R$ and $(b,a) \in R$ then $a = b$ is called a*ntisymmetric*.

Written symbolically, R is antisymmetric if and only if

$$\forall x \forall y \, [(x,y) \in R \wedge (y,x) \in R \longrightarrow x = y]$$

Example: Which of these relations are antisymmetric on the set of integers?

$R_1 = \{(a,b) \mid a \leq b\}$,

$R_2 = \{(a,b) \mid a > b\}$,

$R_3 = \{(a,b) \mid a = b \text{ or } a = -b\}$,

$R_4 = \{(a,b) \mid a = b\}$,

$R_5 = \{(a,b) \mid a = b + 1\}$,

$R_6 = \{(a,b) \mid a + b \leq 3\}$.

Antisymmetric Relations

Example: The following relations on the integers are antisymmetric:

$R_1 = \{(a,b) \mid a \leq b\}$,

$R_2 = \{(a,b) \mid a > b\}$,

$R_4 = \{(a,b) \mid a = b\}$,

$R_5 = \{(a,b) \mid a = b + 1\}$.

For any integer, if a $a \leq b$ and $b \leq a$, then $a = b$.

The following relations are not antisymmetric:

$R_3 = \{(a,b) \mid a = b \text{ or } a = -b\}$

(note that both $(1,-1)$ and $(-1,1)$ belong to R_3),

$R_6 = \{(a,b) \mid a + b \leq 3\}$

(note that both $(1,2)$ and $(2,1)$ belong to R_6).

Transitive Relations

Definition: A relation R on a set A is called transitive if whenever $(a,b)\in R$ and $(b,c)\in R$, then $(a,c)\in R$, for all $a,b,c \in A$.

Written symbolically, R is transitive if and only if

$$\forall x \forall y \forall z[(x,y) \in R \wedge (y,z) \in \mathrm{R} \longrightarrow (x,z) \in R\,]$$

Example: Which of these relations are transitive on the set of integers?

$R_1 = \{(a,b) \mid a \leq b\}$,

$R_2 = \{(a,b) \mid a > b\}$,

$R_3 = \{(a,b) \mid a = b \text{ or } a = -b\}$,

$R_4 = \{(a,b) \mid a = b\}$,

$R_5 = \{(a,b) \mid a = b + 1\}$,

$R_6 = \{(a,b) \mid a + b \leq 3\}$.

Transitive Relations

Example: The following relations on the integers are transitive:

$R_1 = \{(a,b) \mid a \leq b\}$,

$R_2 = \{(a,b) \mid a > b\}$,

$R_3 = \{(a,b) \mid a = b \text{ or } a = -b\}$,

$R_4 = \{(a,b) \mid a = b\}$.

For every integer, $a \leq b$ and $b \leq c$, then $a \leq c$.

The following are not transitive:

$R_5 = \{(a,b) \mid a = b + 1\}$

(both (3,2) and (4,3) belong to R_5, but not (3,3)),

$R_6 = \{(a,b) \mid a + b \leq 3\}$

(both (2,1) and (1,2) belong to R_6, but not (2,2)).

Combining Relations

Given two relations R_1 and R_2, we can combine them using basic set operations to form new relations such as $R_1 \cup R_2$, $R_1 \cap R_2$, $R_1 - R_2$, and $R_2 - R_1$.

Example: Let $A = \{1,2,3\}$ and $B = \{1,2,3,4\}$.

The relations

- $R_1 = \{(1,1),(2,2),(3,3)\}$ and
- $R_2 = \{(1,1),(1,2),(1,3),(1,4)\}$

can be combined using basic set operations to form new relations:

$$R_1 \cup R_2 = \{(1,1),(1,2),(1,3),(1,4),(2,2),(3,3)\}$$

$$R_1 \cap R_2 = \{(1,1)\}$$

$$R_1 - R_2 = \{(2,2),(3,3)\}$$

$$R_2 - R_1 = \{(1,2),(1,3),(1,4)\}$$

Composition

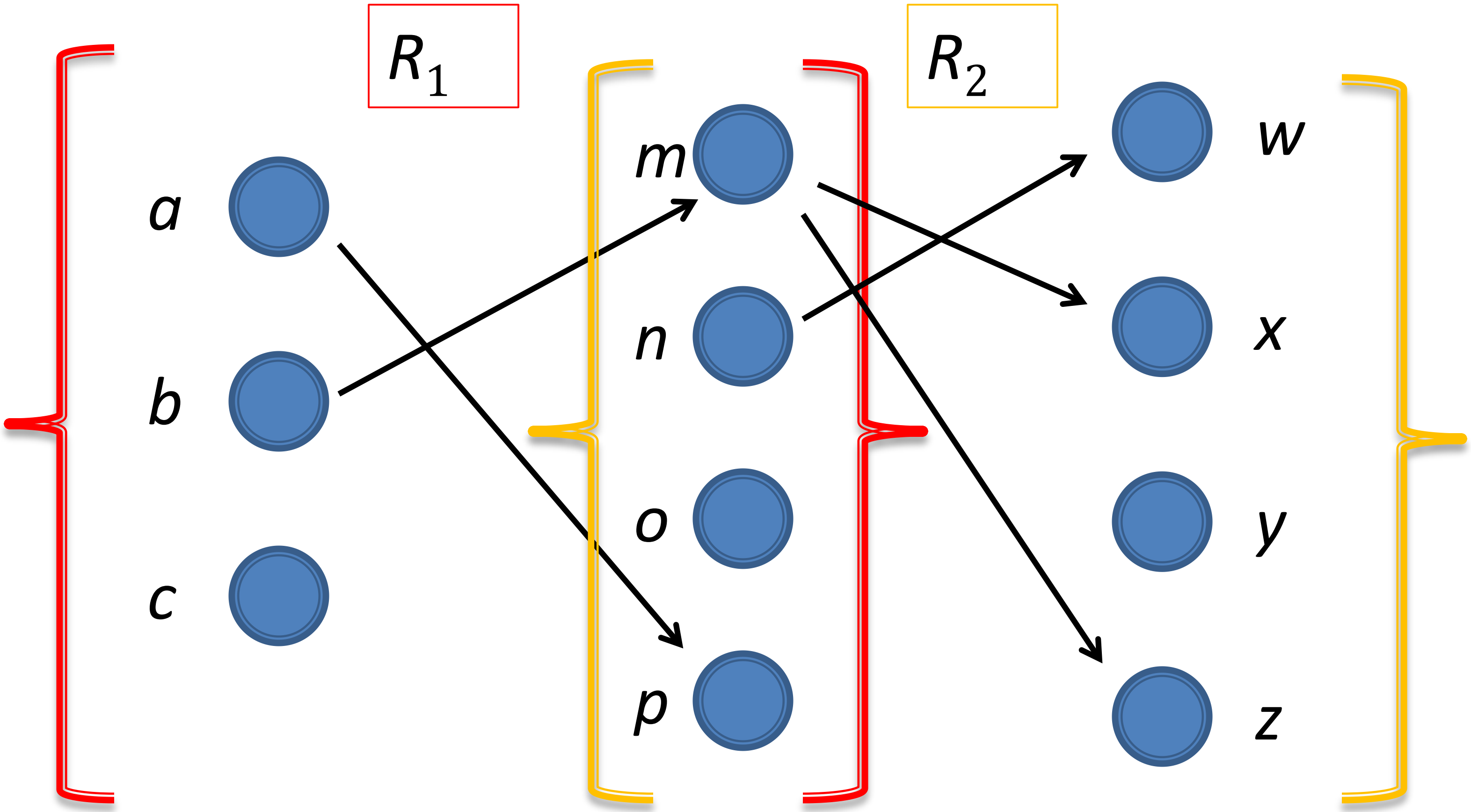
Definition: Suppose

- R_1 is a relation from a set A to a set B.
- R_2 is a relation from B to a set C.

Then the *composition* (or *composite*) of R_2 with R_1, is a relation from A to C where

- If (x,y) is a member of R_1
- and (y,z) is a member of R_2
- then (x,z) is a member of $R_2 \circ R_1$.

Representing the Composition of Relations



$$R_2 \circ R_1 = \{(b,x),(b,z)\}$$