CSE 15
Discrete Mathematics

Lecture 8 – Set Operations & Functions (1)

Announcement

- HW #4: to be assigned on 9/25/18
  - Due **5pm** 10/3 (Wed) with 1 extra day of re-submission.
- Reading assignment
  - Ch. 2.2-2.3 of textbook

Set Operations (Ch. 2.2)

- Set Operations
  - Union
  - Intersection
  - Complementation
  - Difference
- More on Set Cardinality
- Set Identities
- Proving Identities
- Membership Tables

Union

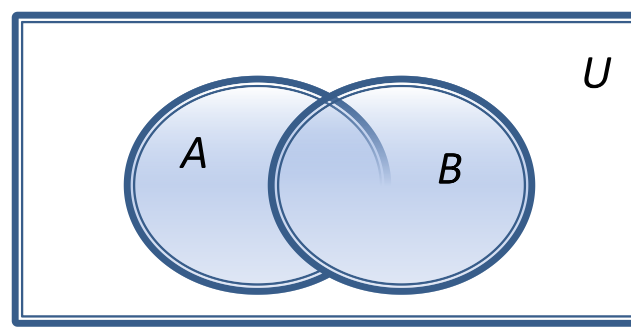
- **Definition**: Let *A* and *B* be sets. The *union* of the sets *A* and *B*, denoted by $A \cup B$, is the set:

$$\{x | x \in A \lor x \in B\}$$

- **Example**: What is $\{1,2,3\} \cup \{3, 4, 5\}$?

Solution: $\{1,2,3,4,5\}$

Venn Diagram for $A \cup B$



Intersection

- **Definition**: The *intersection* of sets *A* and *B*, denoted by $A \cap B$, is

$$\{x | x \in A \wedge x \in B\}$$

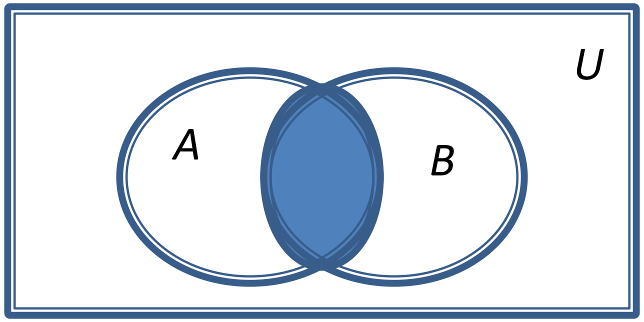
- Note if the intersection is empty, then *A* and *B* are said to be *disjoint*.
- **Example**: What is? $\{1,2,3\} \cap \{3,4,5\}$?

  Solution: $\{3\}$
- **Example:** What is?

  $\{1,2,3\} \cap \{4,5,6\}$?

  Solution: $\emptyset$

Venn Diagram for $A \cap B$



Complement

Definition: If A is a set, then the complement of the A (with respect to U), denoted by $\bar{A}$ is the set U - A

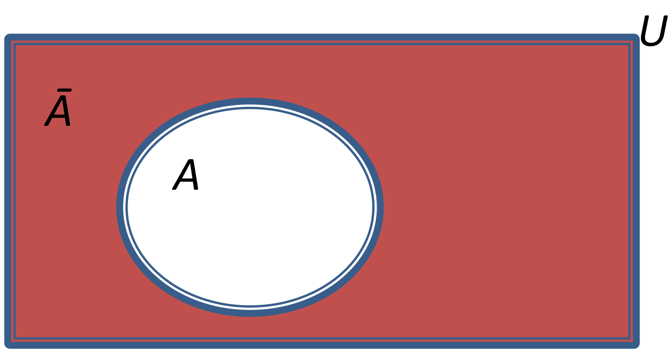
$$\bar{A} = \{x \in U \mid x \notin A\}$$

(The complement of A is sometimes denoted by A^c .)

Example: If U is the positive integers less than 100, what is the complement of $\{x \mid x > 70\}$

Solution: $\{x \mid x \leq 70\}$

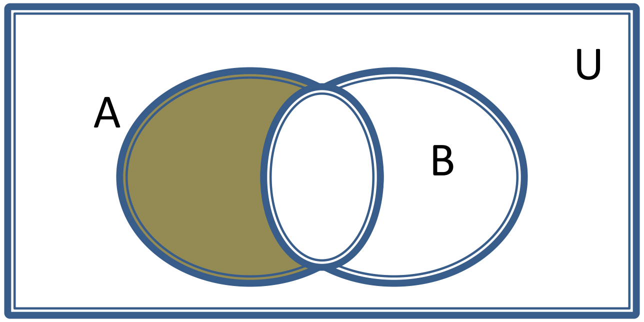
Venn Diagram for Complement



Difference

- **Definition**: Let A and B be sets. The *difference* of A and B, denoted by $A - B$, is the set containing the elements of A that are not in B.

$$A - B = \{x \mid x \in A \wedge x \notin B\} = A \cap \overline{B}$$

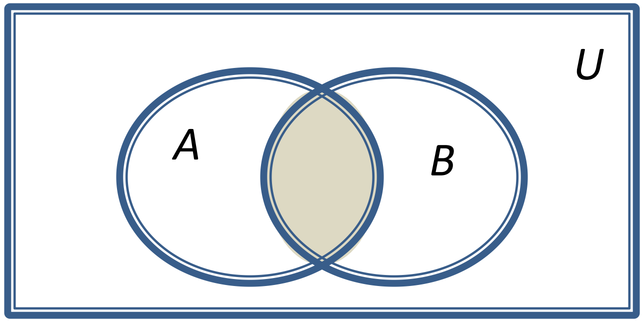


Venn Diagram for $A - B$

The Cardinality of the Union of Two Sets

- Inclusion-Exclusion

$|A \cup B| = |A| + |B| - |A \cap B|$



Venn Diagram for $A \cap B$

- **Example**: Let A be the math majors in your class and B be the CS majors. To count the number of students who are either math majors or CS majors, add the number of math majors and the number of CS majors, and subtract the number of joint CS/math majors.

Review Questions

Example: $U = \{0,1,2,3,4,5,6,7,8,9,10\}$ $A = \{1,2,3,4,5\}$ $B = \{4,5,6,7,8\}$

1. $A \cup B$

   Solution: $\{1,2,3,4,5,6,7,8\}$

2. $A \cap B$

   Solution: $\{4,5\}$

3. $\bar{A}$

   Solution: $\{0,6,7,8,9,10\}$

4. $\bar{B}$

   Solution: $\{0,1,2,3,9,10\}$

5. $A - B$

   Solution: $\{1,2,3\}$

6. $B - A$

   Solution: $\{6,7,8\}$

Set Identities

- Identity laws $A \cup \emptyset = A$ $A \cap U = A$
- Domination laws $A \cup U = U$ $A \cap \emptyset = \emptyset$
- Idempotent laws $A \cup A = A$ $A \cap A = A$
- Complementation law $\overline{(\overline{A})} = A$

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Set Identities

- Commutative laws

$$A \cap B = B \cap A \qquad A \cup B = B \cup A$$

- Associative laws

$$A \cup (B \cup C) = (A \cup B) \cup C$$
$$A \cap (B \cap C) = (A \cap B) \cap C$$

- Distributive laws

$$A \cap (B \cup C) = (A \cap B) \cup (A \cap C)$$

$$A \cup (B \cap C) = (A \cup B) \cap (A \cup C)$$

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Set Identities

- De Morgan's laws

$$\overline{A \cup B} = \overline{A} \cap \overline{B} \qquad \overline{A \cap B} = \overline{A} \cup \overline{B}$$

- Absorption laws

$$A \cup (A \cap B) = A \qquad A \cap (A \cup B) = A$$

- Complement laws

$$A \cup \overline{A} = U \qquad A \cap \overline{A} = \emptyset$$

Proving Set Identities

- Different ways to prove set identities:
  1. Prove that each set (side of the identity) is a subset of the other.
  2. Membership Tables: Verify that elements in the same combination of sets always either belong or do not belong to the same side of the identity. Use 1 to indicate it is in the set and a 0 to indicate that it is not.

Proof of Second De Morgan Law

Example: Prove that $\overline{A \cap B} = \overline{A} \cup \overline{B}$

Solution: We prove this identity by showing that:

1) $\overline{A \cap B} \subseteq \overline{A} \cup \overline{B}$ and

2) $\overline{A} \cup \overline{B} \subseteq \overline{A \cap B}$

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Proof of Second De Morgan Law

These steps show that: $\overline{A \cap B} \subseteq \overline{A} \cup \overline{B}$

$x \in \overline{A \cap B}$	by assumption
$x \notin A \cap B$	defn. of complement
$\neg((x \in A) \wedge (x \in B))$	defn. of intersection
$\neg(x \in A) \vee \neg(x \in B)$	1st De Morgan Law for Prop Logic
$x \notin A \vee x \notin B$	defn. of negation
$x \in \overline{A} \vee x \in \overline{B}$	defn. of complement
$x \in \overline{A} \cup \overline{B}$	defn. of union

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Proof of Second De Morgan Law

These steps show that: $\overline{A} \cup \overline{B} \subseteq \overline{A \cap B}$

$x \in \overline{A} \cup \overline{B}$	by assumption
$(x \in \overline{A}) \vee (x \in \overline{B})$	defn. of union
$(x \notin A) \vee (x \notin B)$	defn. of complement
$\neg(x \in A) \vee \neg(x \in B)$	defn. of negation
$\neg((x \in A) \wedge (x \in B))$	by 1st De Morgan Law for Prop Logic
$\neg(x \in A \cap B)$	defn. of intersection
$x \in \overline{A \cap B}$	defn. of complement

Set-Builder Notation: Second De Morgan Law

$$
\begin{array}{rcll}
\overline{A \cap B} & = & \{x \mid x \notin A \cap B\} & \text{by defn. of complement} \\
& = & \{x \mid \neg(x \in (A \cap B))\} & \text{by defn. of does not belong symbol} \\
& = & \{x \mid \neg(x \in A \wedge x \in B\} & \text{by defn. of intersection} \\
& = & \{x \mid \neg(x \in A) \vee \neg(x \in B)\} & \text{by 1st De Morgan law} \\
& & & \text{for Prop Logic} \\
& = & \{x \mid x \notin A \vee x \notin B\} & \text{by defn. of not belong symbol} \\
& = & \{x \mid x \in \overline{A} \vee x \in \overline{B}\} & \text{by defn. of complement} \\
& = & \{x \mid x \in \overline{A} \cup \overline{B}\} & \text{by defn. of union} \\
& = & \overline{A} \cup \overline{B} & \text{by meaning of notation}
\end{array}
$$

Membership Table

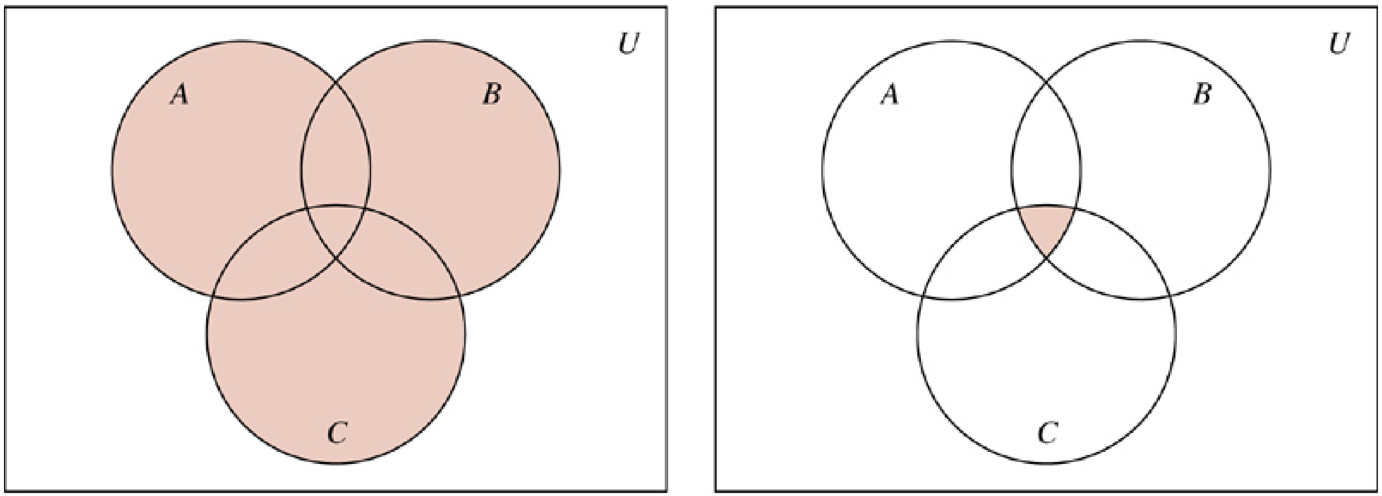
Example: Construct a membership table to show that the distributive law holds.

$$A \cup (B \cap C) = (A \cup B) \cap (A \cup C)$$

Solution:

A	**B**	**C**	$B \cap C$	$A \cup (B \cap C)$	$A \cup B$	$A \cup C$	$(A \cup B) \cap (A \cup C)$
1	1	1	1	1	1	1	1
1	1	0	0	1	1	1	1
1	0	1	0	1	1	1	1
1	0	0	0	1	1	1	1
0	1	1	1	1	1	1	1
0	1	0	0	0	1	0	0
0	0	1	0	0	0	1	0
0	0	0	0	0	0	0	0

Generalized union and intersection



(a) $A \cup B \cup C$ is shaded. (b) $A \cap B \cap C$ is shaded.

- A={0,2,4,6,8}, B={0,1,2,3,4}, C={0,3,6,9}
- $A \cup B \cup C = \{0,1,2,3,4,6,8,9\}$
- $A \cap B \cap C = \{0\}$

General case

- Union: $A_1 \cup A_2 \cup \cdots \cup A_n = \bigcup_{i=1}^{n} A_i$
- Intersection: $A_1 \cap A_2 \cap \cdots \cap A_n = \bigcap_{i=1}^{n} A_i$
- Union: $A_1 \cup A_2 \cup \cdots \cup A_n \cup \cdots = \bigcup_{i=1}^{\infty} A_i$
- Intersection: $A_1 \cap A_2 \cap \cdots \cap A_n \cap \cdots = \bigcap_{i=1}^{\infty} A_i$
- Suppose A_i={1, 2, 3,…, i} for i=1,2,3,…

$$\bigcup_{i=1}^{\infty} A_i = \bigcup_{i=1}^{\infty} \{1,2,3,\ldots,i\} = \{1,2,3,\ldots\} = Z^+$$

$$\bigcap_{i=1}^{\infty} A_i = \bigcap_{i=1}^{\infty} \{1,2,3,\ldots,i\} = \{1\}$$

Computer representation of sets

- U={1,2,3,4,5,6,7,8,9,10}
- A={1,3,5,7,9} (odd integer ≤10),B={1,2,3,4,5} (integer ≤5)
- Represent A and B as 1010101010, and 1111100000
- Complement of A: 0101010101
- A⋂B: 1010101010&1111100000=1010100000

  which corresponds to {1,3,5}

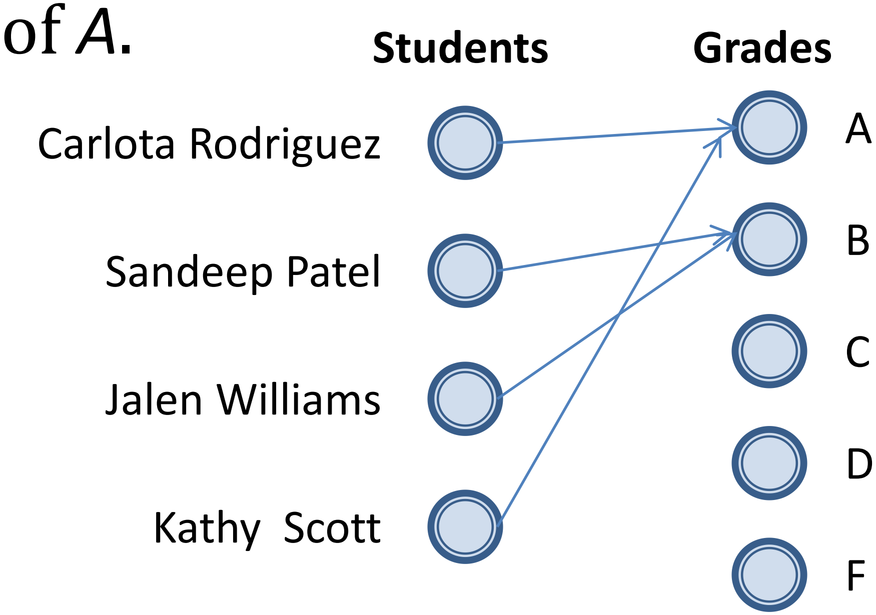
Functions (Ch. 2.3)

- Definition of a Function.
  - Domain, Codomain
  - Image, Preimage
- Injection, Surjection, Bijection
- Inverse Function
- Function Composition
- Graphing Functions
- Floor, Ceiling

Functions

Definition: Let A and B be nonempty sets. A *function* f from A to B, denoted $f: A \to B$ is an assignment of each element of A to exactly one element of B. We write $f(a) = b$ if b is the unique element of B assigned by the function f to the element a of A.

- Functions are sometimes called *mappings* or *transformations*.



Functions

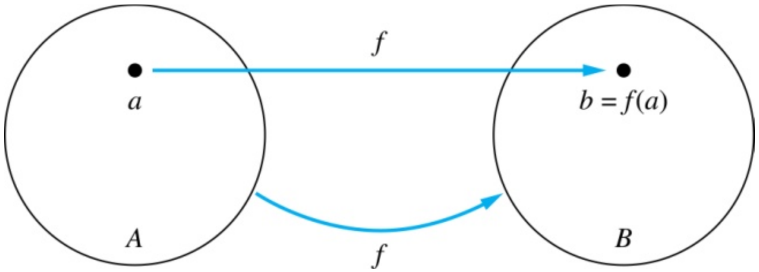
- A function $f : A \rightarrow B$ can also be defined as a subset of $A \times B$ (a relation). This subset is restricted to be a relation where no two elements of the relation have the same first element.
- Specifically, a function *f* from *A* to *B* contains one, and only one ordered pair (*a, b*) for every element $a \in A$.

and $\forall x[x \in A \rightarrow \exists y[y \in B \wedge (x, y) \in f]]$

$$\forall x, y_1, y_2 \left[\left[(x, y_1) \in f \wedge (x, y_2) \in f \right] \rightarrow y_1 = y_2 \right]$$

Functions

Given a function f: $A \to B$:



- We say f *maps* A to B *or*
  f is a *mapping* from A to B.
- A is called the *domain* of f.
- B is called the *codomain* of f.
- If $f(a) = b$,
  - then b is called the *image* of a under f.
  - a is called the *preimage* of b.
- The range of f is the set of all images of points in **A** under f. We denote it by $f(A)$.
- Two functions are *equal* when they have the same domain, the same codomain, and map each element of the domain to the same element of the codomain.

Representing Functions

- Functions may be specified in different ways:
  - An explicit statement of the assignment.

    Students and grades example.
  - A formula.

    $f(x) = x + 1$
  - A computer program.
    - A Java program that when given an integer *n*, produces the *n*th Fibonacci Number (covered in the next section and also in Chapter 5).

Questions

f(a) = ?   z

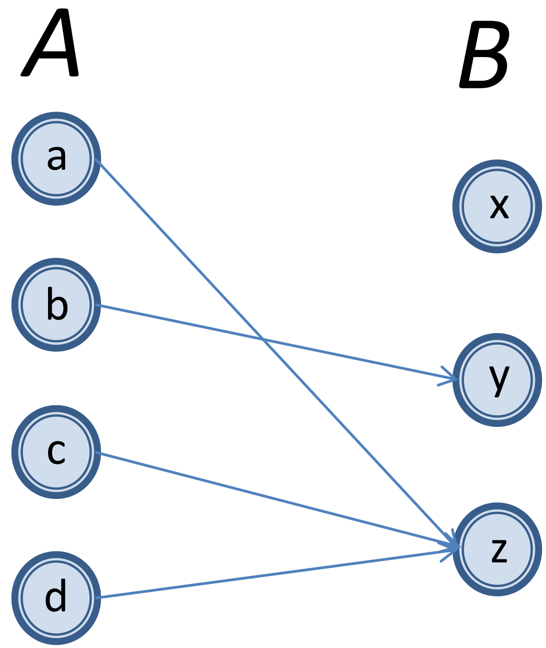
The image of d is ?   z

The domain of f is ?   *A*

The codomain of f is ?   *B*

The preimage of y is ?   b

f(A) = ?   {y,z}



The preimage(s) of z is (are) ?   {a,c,d}

Example

- Let R be the relation consisting of (Abdul, 22), (Brenda, 24), (Carla, 21), (Desire, 22), (Eddie, 24) and (Felicia, 22)
- f: f(Abdul)=22, f(Brenda)=24, f(Carla)=21, f(Desire)=22, f(Eddie)=24, and f(Felicia)=22
- Domain?
  - {Abdul, Brenda, Carla, Desire, Eddie, Felicia}
- Codomain?
  - Set of positive integers.
- Range?
  - {21, 22, 24}

Example

- f: assigns the last two bits of a bit string of length 2 or greater to that string, e.g., f(11010)=10
- Domain?
  - All bit strings of length 2 or greater.
- Codomain?
  - {00, 01, 10, 11}
- Range?
  - {00, 01, 10, 11}

Functions

- Two real-valued functions with the same domain can be added and multiplied.
- Let f_1 and f_2 be functions from A to **R**, then f_1+f_2, and f_1f_2 are also functions from A to **R** defined by
  - $(f_1+f_2)(x)= f_1(x)+f_2(x)$
  - $(f_1f_2)(x)= f_1(x)\, f_2(x)$
- Note that the functions f_1+f_2 and f_1f_2 at x are defined in terms f_1 and f_2 at x.

Example

- $f_1(x) = x^2$ and $f_2(x) = x - x^2$
  - $(f_1+f_2)(x)$?
    - $= f_1(x)+f_2(x) = x^2+x-x^2 = x$
  - $(f_1 f_2)(x)$?
    - $= f_1(x)f_2(x) = x^2(x-x^2) = x^3-x^4$

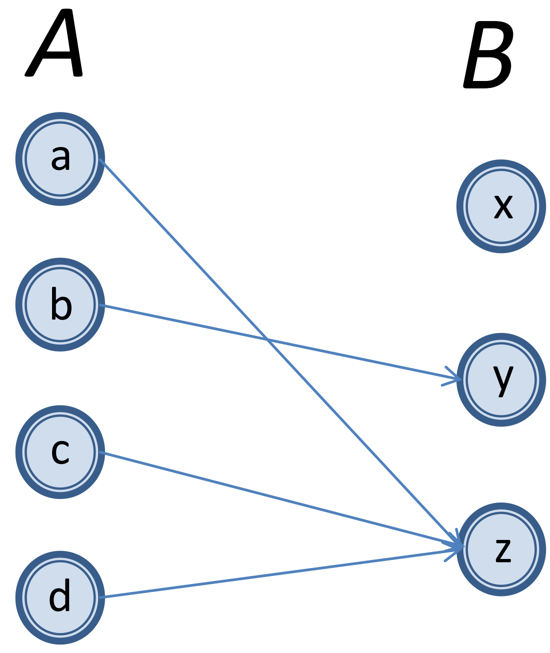
Question on Functions and Sets

- If $f : A \to B$ and S is a subset of A, then

$$f(S) = \{f(s) | s \in S\}$$

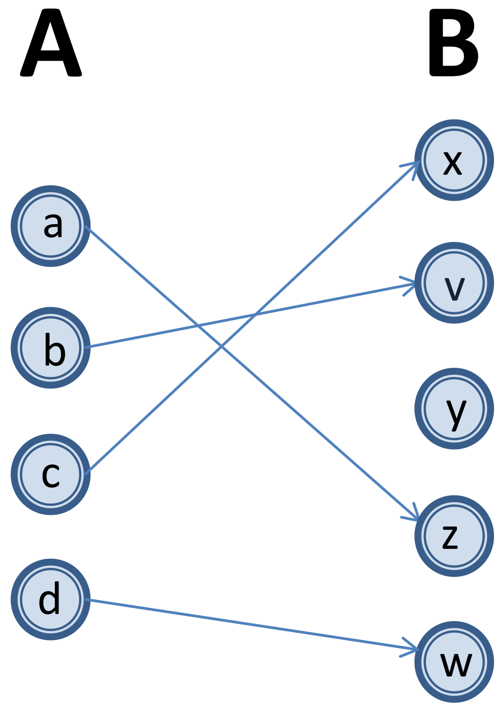
f {a,b,c,} is ?   {y,z}

f {c,d} is ?   {z}



Injections or one-to-one functions

Definition: A function f is said to be *one-to-one* , or *injective*, if and only if $f(a) = f(b)$ implies that $a = b$ for all a and b in the domain of f. A function is said to be an *injection* if it is one-to-one.



One-to-one functions

- A function *f* is **one-to-one** if and only if f(a)=f(b) implies a=b for all a and b in the domain of f.

$$\forall a \forall b (f(a) = f(b) \rightarrow a = b)$$

- A function *f* is **one-to-one** if and only if f(a)≠f(b) whenever a≠b.

$$\forall a \forall b (a \neq b \rightarrow f(a) \neq f(b))$$

- Every element of B is the image of a unique element of A

Example

- Let $f(x)=x^2$, from the set of integers to the set of integers. Is it one-to-one?
- No: $f(1)=1$, $f(-1)=1$, $f(1)=f(-1)$ but $1 \neq -1$
- However, $f(x)=x^2$ is one-to-one for Z^+
- Determine if $f(x)=x+1$ from real numbers to itself is one-to-one or not.
- It is one-to-one. To show this, note that $x+1 \neq y+1$ when $x \neq y$

Increasing/decreasing functions

- Increasing (decreasing): if $f(x) \le f(y)$ ($f(x) \ge f(y)$), whenever $x<y$ and x, y are in the domain of f.

$$\forall x \forall y (x < y \rightarrow f(x) \le f(y))$$

- Strictly increasing (decreasing): if $f(x)<f(y)$ ($f(x) > f(y)$) whenever $x<y$, and x, y are in the domain of f.
- A function that is either strictly increasing or decreasing must be one-to-one.