CSE 15
Discrete Mathematics

Lecture 21– Counting

Announcement

- HW #9
  - Due **5pm** 12/5 with 1 extra day of re-submission.

- Reading assignment
  - Ch.8.1 and 8.5 of textbook

The Basics of Counting (Ch. 6.1)

- The Product Rule
- The Sum Rule
- The Subtraction Rule
- The Division Rule
- Examples,
- Tree Diagrams

Basic Counting Principles: The Product Rule

The Product Rule: A procedure can be broken down into a sequence of two tasks. There are n_1 ways to do the first task and n_2 ways to do the second task. Then there are $n_1 \cdot n_2$ ways to do the procedure.

Example: How many bit strings of length seven are there?

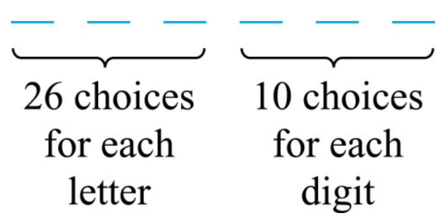
Solution: Since each of the seven bits is either a 0 or a 1, the answer is $2^7 = 128$.

The Product Rule

Example: How many different license plates can be made if each plate contains a sequence of three uppercase English letters followed by three digits?

Solution: By the product rule,

there are $26 \cdot 26 \cdot 26 \cdot 10 \cdot 10 \cdot 10 = 17{,}576{,}000$ different possible license plates.



Counting Functions

Counting Functions: How many functions are there from a set with *m* elements to a set with *n* elements?

Solution: Since a function represents a choice of one of the n elements of the codomain for each of the m elements in the domain, the product rule tells us that there are $n \cdot n \cdots n = n^m$ such functions.

Counting One-to-One Functions: How many one-to-one functions are there from a set with *m* elements to one with *n* elements? //no repetition//

Solution: Suppose the elements in the domain are $a_1, a_2, \ldots, a_m$. There are n ways to choose the value of a_1 and $n-1$ ways to choose a_2, etc. The product rule tells us that there are $n(n-1)(n-2)\cdots(n-m+1)$ such functions.

Basic Counting Principles: The Sum Rule

The Sum Rule: If a task can be done either in one of n_1 ways or in one of n_2 ways, where none of the set of n_1 ways is the same as any of the n_2 ways, then there are $n_1 + n_2$ ways to do the task.

Example: The mathematics department must choose either a student or a faculty member as a representative for a university committee. How many choices are there for this representative if there are 37 members of the mathematics faculty and 83 mathematics majors and no one is both a faculty member and a student.

Solution: By the sum rule it follows that there are $37 + 83 = 120$ possible ways to pick a representative.

The Sum Rule in terms of sets

- The sum rule can be phrased in terms of sets.

  $|A \cup B| = |A| + |B|$ as long as A and B are disjoint sets.
- Or more generally,

$$|A_1 \cup A_2 \cup \cdots \cup A_m| = |A_1| + |A_2| + \cdots + |A_m|$$

when $A_i \cap A_j = \emptyset$ for all i, j.

- The case where the sets have elements in common will be discussed when we consider the subtraction rule.

Combining the Sum and Product Rules

Example: Suppose statement labels in a programming language can be either a single letter or a letter followed by a digit. Find the number of possible labels.

Solution: Use the sum and product rules:

$$26 + 26 \cdot 10 = 286$$

Counting Passwords

Example: A password consist of six to eight characters long, where each character is an uppercase letter or a digit. Each password must contain at least one digit. How many possible passwords are there?

Solution: Let P be the total number of passwords, and let P_6, P_7, and P_8 be the passwords of length 6, 7, and 8.

- By the sum rule $P = P_6 + P_7 + P_8$.
- To find each of P_6, P_7, and P_8 , we find the number of passwords of the specified length composed of letters and digits and subtract the number composed only of letters. We find that:

$P_6 = 36^6 - 26^6 = 2{,}176{,}782{,}336 - 308{,}915{,}776 = 1{,}867{,}866{,}560.$

$P_7 = 36^7 - 26^7 = 78{,}364{,}164{,}096 - 8{,}031{,}810{,}176 = 70{,}332{,}353{,}920.$

$P_8 = 36^8 - 26^8 = 2{,}821{,}109{,}907{,}456 - 208{,}827{,}064{,}576 = 2{,}612{,}282{,}842{,}880.$

Consequently, $P = P_6 + P_7 + P_8 = 2{,}684{,}483{,}063{,}360.$

Basic Counting Principles: Subtraction Rule

Subtraction Rule: If a task can be done either in one of n_1 ways or in one of n_2 ways then the total number of ways to do the task is $n_1 + n_2$ minus the number of ways to do the task that are common to the two different ways.

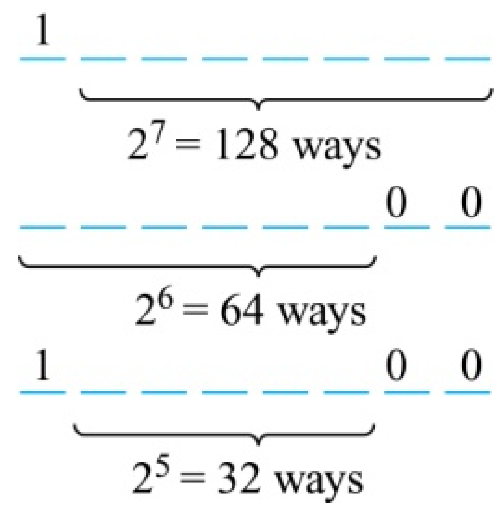
- Also known as, the *principle of inclusion-exclusion*:

$$|A \cup B| = |A| + |B| - |A \cap B|$$

Counting Bit Strings

Example: How many bit strings of length eight either start with a 1 bit or end with the two bits 00? (or=Ex-OR)

Solution: Use the subtraction rule.



- Number of bit strings of length eight that start with a 1 bit: $2^7 = 128$
- Number of bit strings of length eight that end with bits 00: $2^6 = 64$
- Number of bit strings of length eight start with a 1 bit and end with bits 00 : $2^5 = 32$

Hence, the number is $128 + 64 - 32 = 160$.

Tree Diagrams

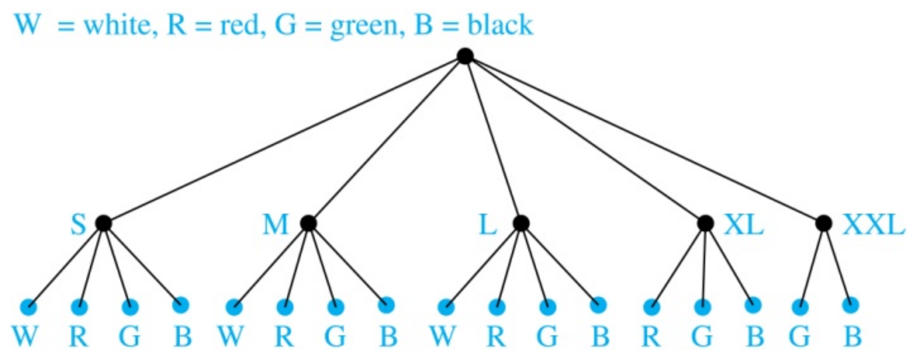
- **Tree Diagrams**: We can solve many counting problems through the use of *tree diagrams*, where a branch represents a possible choice and the leaves represent possible outcomes.

Tree Diagrams

- **Example**: Suppose that "I Love Discrete Math" T-shirts come in five different sizes: S,M,L,XL, and XXL.

  Each size comes in four colors (white, red, green, and black), except XL, which comes only in red, green, and black, and XXL, which comes only in green and black.

  What is the minimum number of shirts that the campus store must stock to have one of each size and color available?
- **Solution**: Draw the tree diagram.

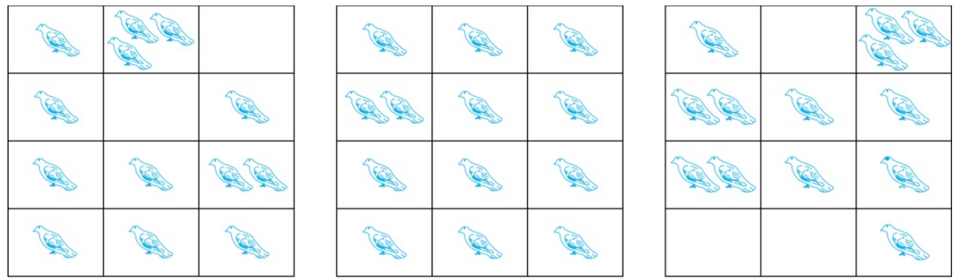


- The store must stock at least 17 T-shirts.

The Pigeonhole Principle (Ch. 6.2)

- The Pigeonhole Principle
- The Generalized Pigeonhole Principle

The Pigeonhole Principle



(a) (b) (c)

Pigeonhole Principle: If *k* is a positive integer and *k* + 1 objects are placed into *k* boxes, then at least one box contains two or more objects.

Corollary 1: A function *f* from a set with *k* + 1 elements to a set with *k* elements is not one-to-one.

Example: Among any group of 367 people, there must be at least two with the same birthday, because there are only 366 possible birthdays.

The Generalized Pigeonhole Principle

The Generalized Pigeonhole Principle: If N objects are placed into k boxes, then there is at least one box containing at least $\lceil N/k \rceil$ objects.

Example: Among 100 people there are at least $\lceil 100/12 \rceil = 9$ who were born in the same month.

1/8, 2/8,3/8,4/8,5/8,6/8,7/8,8/8……12/8 =96 people (4 left)

The Generalized Pigeonhole Principle

Example: How many cards must be selected from a standard deck of 52 cards to guarantee that at least three cards of the same suit are chosen?

Solution: We assume four boxes; one for each suit.

Using the generalized pigeonhole principle, at least one box contains at least $\lceil N/4 \rceil$ cards.

At least three cards of one suit are selected if $\lceil N/4 \rceil \geq 3$.

The smallest integer N such that $\lceil N/4 \rceil \geq 3$ is $N = 2 \cdot 4 + 1 = 9$.