CSE 15
Discrete Mathematics

Lecture 5 – Proposition Logic (5)

Announcement

- HW #2 out (9/6)
  - Due **5pm** 9/14 (Fri) with 1 extra day of re-submission.
  - Type your answers in a text file and submit it on Catcourses.
  - Or write your answers on papers and scan them into image files for submission.
  - Work on it during and outside lab hours.

- Reading assignment
  - Ch. 1.6 – 1.8 of textbook

Valid Arguments

Example 1: From the single proposition

$$p \land (p \to q)$$

Show that *q* is a conclusion.

Solution:

Step	**Reason**
1. $p \land (p \to q)$	Premise
2. p	Conjunction using (1)
3. $p \to q$	Conjunction using (1)
4. q	Modus Ponens using (2) and (3)

Valid Arguments

Example 2:

- With these hypotheses:

  "It is not sunny this afternoon and it is colder than yesterday."

  "We will go swimming only if it is sunny."

  "If we do not go swimming, then we will take a canoe trip."

  "If we take a canoe trip, then we will be home by sunset."

- Using the inference rules, construct a valid argument for the conclusion:

  "We will be home by sunset."

Valid Arguments

Solution:

1. Choose propositional variables:

   p : "It is sunny this afternoon."

   r : "We will go swimming."

   t : "We will be home by sunset."

   q : "It is colder than yesterday."

   s : "We will take a canoe trip."

2. Translation into propositional logic:

$$\text{Hypotheses: } \neg p \wedge q,\ r \rightarrow p,\ \neg r \rightarrow s,\ s \rightarrow t$$
$$\text{Conclusion: } t$$

Valid Arguments (Solution cont.)

3. Construct the Valid Argument

Step	**Reason**
1. $\neg p \wedge q$	Premise
2. $\neg p$	Simplification using (1)
3. $r \rightarrow p$	Premise
4. $\neg r$	Modus tollens using (2) and (3)
5. $\neg r \rightarrow s$	Premise
6. s	Modus ponens using (4) and (5)
7. $s \rightarrow t$	Premise
8. t	Modus ponens using (6) and (7)

Note that a truth table would have 32 rows since we have 5 propositional variables.

Be careful if one or more of the premises is false

If $\sqrt{2} > \frac{3}{2}$ then $(\sqrt{2})^2 > (\frac{3}{2})^2$. We know that $\sqrt{2} > \frac{3}{2}$

Consequently, $(\sqrt{2})^2 = 2 > (\frac{3}{2})^2 = \frac{9}{4}$

Is it a valid argument? Is conclusion true?

- The premises of the argument are $p \rightarrow q$ and p, while q is the conclusion
- This argument is valid by using modus ponens
- But one of the premises is false, consequently we cannot conclude the conclusion is true
- Furthermore, the conclusion is not true

Handling Quantified Statements

- In a valid argument, each statement is either a premise or follows from previous statements by rules of inference which include:
  - Rules of Inference for Propositional Logic
  - Rules of Inference for Quantified Statements
- The rules of inference for quantified statements are introduced in the next several slides.

Universal Instantiation (UI)

$$\frac{\forall x P(x)}{\therefore P(c)}$$

Example:

Our domain consists of all dogs and Fido is a dog.

“All dogs are cuddly.”

“Therefore, Fido is cuddly.”

Universal Generalization (UG)

$$\frac{P(c) \text{ for an arbitrary } c}{\therefore \forall x P(x)}$$

c is randomly picked from the domain, without other assumptions.
Used often implicitly in Mathematical Proofs.

Existential Instantiation (EI)

$$\frac{\exists x P(x)}{\therefore P(c) \text{ for some element } c}$$

Example:

"There is someone who got an A in the course."
"Let's call her *a* and say that *a* got an A"

Existential Generalization (EG)

$$\frac{P(c) \text{ for some element } c}{\therefore \exists x P(x)}$$

Example:

"Michelle got an A in the class."
"Therefore, someone got an A in the class."

Using Rules of Inference

Example 1: Using the rules of inference, construct a valid argument to show that

"John Smith has two legs"

is a consequence of the premises:

"Every man has two legs." "John Smith is a man."

Solution: Let $M(x)$ denote *"x is a man"* and $L(x)$ *" x has two legs"* and let John Smith be a member of the domain.

Valid Argument:

Step	**Reason**
1. $\forall x(M(x) \rightarrow L(x))$	Premise
2. $M(J) \rightarrow L(J)$	UI from (1)
3. $M(J)$	Premise
4. $L(J)$	Modus Ponens using (2) and (3)

Using Rules of Inference

Example 2: Use the rules of inference to construct a valid argument showing that the conclusion

"Someone who passed the first exam has not read the book."

follows from the premises

"A student in this class has not read the book."

"Everyone in this class passed the first exam."

Solution: Let *C*(*x*) denote *"x is in this class," B(x) denote " x has read the book,"* and *P*(*x*) denote *"x passed the first exam."*

First we translate the premises and conclusion into symbolic form.

$$\begin{array}{l} \exists x(C(x) \land \neg B(x)) \\ \forall x(C(x) \rightarrow P(x)) \\ \hline \therefore\ \exists x(P(x) \land \neg B(x)) \end{array}$$

Continued on next slide →

Using Rules of Inference

Valid Argument:

Step	**Reason**
1. $\exists x(C(x) \wedge \neg B(x))$	Premise
2. $C(a) \wedge \neg B(a)$	EI from (1)
3. $C(a)$	Simplification from (2)
4. $\forall x(C(x) \rightarrow P(x))$	Premise
5. $C(a) \rightarrow P(a)$	UI from (4)
6. $P(a)$	MP from (3) and (5)
7. $\neg B(a)$	Simplification from (2)
8. $P(a) \wedge \neg B(a)$	Conj from (6) and (7)
9. $\exists x(P(x) \wedge \neg B(x))$	EG from (8)

Returning to the Socrates Example

$$\forall x(Man(x) \rightarrow Mortal(x))$$

$$Man(Socrates)$$

$$\therefore \quad Mortal(Socrates)$$

Solution for Socrates Example

Valid Argument

Step	**Reason**
1. $\forall x(Man(x) \rightarrow Mortal(x))$	Premise
2. $Man(Socrates) \rightarrow Mortal(Socrates)$	UI from (4)
3. $Man(Socrates)$	Premise
4. $Mortal(Socrates)$	MP from (2) and (3)

Universal Modus Ponens

Universal Modus Ponens combines universal instantiation and modus ponens into one rule.

$$
\begin{array}{l}
\forall x (P(x) \to Q(x)) \\
P(a), \text{where } a \text{ is a particular} \\
\quad\quad \text{element in the domain} \\
\hline
\therefore Q(a)
\end{array}
$$

This rule could be used in the Socrates example.

Introduction to Proofs (Ch. 1.7)

- Mathematical Proofs
- Forms of Theorems
- Direct Proofs
- Indirect Proofs
  - Proof of the Contrapositive
  - Proof by Contradiction

Proofs of Mathematical Statements

- A ***proof*** is a valid argument that establishes the truth of a statement.
- ***Informal proofs:*** generally shorter and easier:
  - More than one rule of inference are often used in a step.
  - Steps may be skipped.
  - The rules of inference used are not explicitly stated.
  - Easier to understand and to explain to people.
  - But it is also easier to introduce errors.
- Proofs have many practical applications:
  - verification that computer programs are correct
  - establishing that operating systems are secure
  - enabling programs to make inferences in artificial intelligence
  - showing that system specifications are consistent

Definitions

- A ***theorem*** is a statement that can be shown to be true using:
  - definitions
  - other theorems
  - *axioms* (statements which are given as true)
  - rules of inference
- A ***lemma*** is a 'helping theorem' or a result which is needed to prove a theorem.
- A ***corollary*** is a result which follows directly from a theorem.
- Less important theorems are sometimes called ***propositions***.
- A ***conjecture*** is a statement that is being proposed to be true. Once a proof of a conjecture is found, it becomes a theorem. It may turn out to be false.

Forms of Theorems

- Many theorems assert that a property holds for all elements in a domain, but often the universal quantifier (needed for a precise statement of a theorem) is omitted by standard mathematical convention.

  For example, the statement:

  "If $x > y$, where x and y are positive real numbers, then $x^2 > y^2$ "

  really means

  "For all positive real numbers x and y, if $x > y$, then $x^2 > y^2$."

Proving Theorems

- Many theorems have the form:

$$\forall x(P(x) \rightarrow Q(x))$$

- To prove them, we show that where *c* is an arbitrary element of the domain, $P(c) \rightarrow Q(c)$
- By universal generalization, the truth of the original formula follows.
- So, we must prove something of the form: $p \rightarrow q$

Proving Conditional Statements: $p \rightarrow q$

- *Trivial Proof*: If we know q is true, then
  $p \rightarrow q$ is true as well.

"If it is raining then $1=1$."

- *Vacuous Proof*: If we know p is false then
  $p \rightarrow q$ is true as well.

"If I am both rich and poor then $2 + 2 = 5$."

[Even though these examples seem silly, both trivial and vacuous proofs are often used in mathematical induction, as we will see in Chapter 5)]

Proving Conditional Statements: $p \rightarrow q$

- *Direct Proof*: Assume that p is true. Use rules of inference, axioms, and logical equivalences to show that q must also be true.

Example: Give a direct proof of the theorem "If n is an odd integer, then n^2 is odd."

Solution: Assume that n is odd. Then $n = 2k + 1$ for an integer k. Squaring both sides of the equation, we get:

$n^2 = (2k + 1)^2 = 4k^2 + 4k + 1 = 2(2k^2 + 2k) + 1 = 2r + 1$,

where $r = 2k^2 + 2k$, an integer.

We have proved that if n is an odd integer, then n^2 is an odd integer. ◀

(◀ marks the end of the proof. Sometimes **QED** "*quod erat demonstrandum*" or "*which had to be demonstrated*" is used instead.)

Proving Conditional Statements: *p* →*q*

Definition: The real number *r* is *rational* if there exist integers *p* and *q* where $q \neq 0$ such that *r* = *p*/*q*

Example: Prove that the sum of two rational numbers is rational.

Solution*:* Assume *r* and *s* are two rational numbers. Then there must be integers *p, q* and also *t, u* such that

$$r = p/q, \quad s = t/u, \quad u \neq 0, \ q \neq 0$$

$$r + s = \frac{p}{q} + \frac{t}{u} = \frac{pu+qt}{qu} = \frac{v}{w}$$

where $v = pu + qt$

$w = \mathrm{qu} \neq 0$

Thus the sum is rational. ◀

Proving Conditional Statements: $p \rightarrow q$

- ***Proof by Contraposition***: Assume $\neg q$. Show $\neg p$ is true also. This is sometimes called an ***indirect proof*** method. If we give a direct proof of $\neg q \rightarrow \neg p$ then we have a proof of $p \rightarrow q$.

  Why does this work?

Example: Prove that if n is an integer and $3n + 2$ is odd, then n is odd.

Solution: Assume n is even. So, $n = 2k$ for some integer k. Thus

$$3n + 2 = 3(2k) + 2 = 6k + 2 = 2(3k + 1) = 2j \text{ for } j = 3k + 1$$

Therefore $3n + 2$ is even. Since we have shown $\neg q \rightarrow \neg p$, $p \rightarrow q$ must hold as well. If n is an integer and $3n + 2$ is odd (not even), then n is odd (not even). ◀

Proving Conditional Statements: $p \rightarrow q$

Example: Prove that for an integer *n*, if n^2 is odd, then *n* is odd.

Solution: Use proof by contraposition. Assume *n* is even (i.e., not odd). Therefore, there exists an integer *k* such that $n = 2k$. Hence,

$$n^2 = 4k^2 = 2\,(2k^2)$$

and n^2 is even(i.e., not odd).

We have shown that if *n* is an even integer, then n^2 is even. Therefore by contraposition, for an integer *n*, if n^2 is odd, then *n* is odd. ◀