CSE 15
Discrete Mathematics

Lecture 22– Counting

Advanced Counting Techniques

Announcement

- HW #9 this week
  - Due **5pm** 12/5 (Wed) with 1 extra day of re-submission.
- Reading assignment
  - Ch.9.1 and 9.5 of textbook

Permutations and Combinations (Ch. 6.3)

- Permutations
- Combinations
- Combinatorial Proofs

Permutations

Definition: A *permutation* of a set of distinct objects is an **ordered** arrangement of these objects.

An ordered arrangement of r elements of a set is called an *r-permutation*.

Example: Let $S = \{1,2,3\}$.

- The ordered arrangement $3,1,2$ is a permutation of S.
- The ordered arrangement $3,2$ is a 2-permutation of S.

- The number of r-permutations of a set with n elements is denoted by $P(n,r)$.
  - The 2-permutations of $S = \{1,2,3\}$ are $1,2$; $1,3$; $2,1$; $2,3$; $3,1$; and $3,2$. Hence, $P(3,2) = 6$.

A Formula for the Number of Permutations

Theorem 1: If n is a positive integer and r is an integer with $1 \leq r \leq n$, then there are $P(n, r) = n(n-1)(n-2)\cdots(n-r+1)$ r-permutations of a set with n distinct elements.

Proof: Use the product rule.

The first element can be chosen in n ways.

The second in $n-1$ ways, and so on until there are $(n-(r-1))$ ways to choose the last element.

- Note that $P(n,0) = 1$, since there is only one way to order zero elements.

Corollary 1: If n and r are integers with $1 \leq r \leq n$ then

$$P(n, r) = \frac{n!}{(n-r)!}$$

Solving Counting Problems by Counting Permutations

Example: How many ways are there to select a first-prize winner, a second-prize winner, and a third-prize winner from 100 different people who have entered a contest?

Solution:

$$P(100,3) = 100 \cdot 99 \cdot 98 = 970{,}200$$

Solving Counting Problems by Counting Permutations

Example: Suppose that a saleswoman has to visit eight different cities. She must begin her trip in a specified city, but she can visit the other seven cities in any order she wishes. How many possible orders can the saleswoman use when visiting these cities?

Solution: The first city is chosen, and the rest are ordered arbitrarily.

Hence the orders are:

$$7! = 7 \cdot 6 \cdot 5 \cdot 4 \cdot 3 \cdot 2 \cdot 1 = 5040$$

If she wants to find the tour with the shortest path that visits all the cities, she must consider 5040 paths!

Solving Counting Problems by Counting Permutations

Example: How many permutations of the letters *ABCDEFGH* contain the string *ABC* ?

Solution: We solve this problem by counting the permutations of six objects, *ABC*, *D*, *E*, *F*, *G*, and *H*.

$$6! = 6 \cdot 5 \cdot 4 \cdot 3 \cdot 2 \cdot 1 = 720$$

Combinations

Definition: An *r-combination* of elements of a set is an **unordered** selection of *r* elements from the set.

Thus, an *r*-combination is simply a subset of the set with *r* elements.

- The number of *r*-combinations of a set with *n* distinct elements is denoted by $C(n, r)$.
- The notation $\binom{n}{r}$ is also used and is called a *binomial coefficient*.

Example: Let *S* be the set $\{a, b, c, d\}$.

Then $\{a, c, d\}$ is a 3-combination from S. It is the same as $\{d, c, a\}$ since the order listed does not matter.

- $C(4,2) = 6$ because the 2-combinations of $\{a, b, c, d\}$ are the six subsets $\{a, b\}$, $\{a, c\}$, $\{a, d\}$, $\{b, c\}$, $\{b, d\}$, and $\{c, d\}$.

Combinations

Theorem 2: The number of *r*-combinations of a set with *n* elements, where $n \geq r \geq 0$, equals

$$C(n,r) = \frac{n!}{(n-r)!r!}.$$

Proof: By the product rule $P(n, r) = C(n,r) \cdot P(r,r)$. Therefore,

$$C(n,r) = \frac{P(n,r)}{P(r,r)} = \frac{n!/(n-r)!}{r!/(r-r)!} = \frac{n!}{(n-r)!r!}.$$

Combinations

Example: How many poker hands of five cards can be dealt from a standard deck of 52 cards? Also, how many ways are there to select 47 cards from a deck of 52 cards?

Solution: Since the order in which the cards are dealt does not matter, the number of five card hands is:

$$C(52,5) = \frac{52!}{5!47!}$$

$$= \frac{52 \cdot 51 \cdot 50 \cdot 49 \cdot 48}{5 \cdot 4 \cdot 3 \cdot 2 \cdot 1} = 26 \cdot 17 \cdot 10 \cdot 49 \cdot 12 = 2,598,960$$

- The different ways to select 47 cards from 52 is

$$C(52,47) = \frac{52!}{47!5!} = C(52,5) = 2,598,960.$$

This is a special case of a general result. →

Combinations

Corollary 2: Let n and r be nonnegative integers with $r \leq n$. Then $C(n, r) = C(n, n - r)$.

Proof: From Theorem 2, it follows that

$$C(n,r) = \frac{n!}{(n-r)!r!}$$

and

$$C(n,n-r) = \frac{n!}{(n-r)![n-(n-r)]!} = \frac{n!}{(n-r)!r!} .$$

Hence, $\boldsymbol{C(n, r) = C(n, n - r)}$. ◀

This result can be proved without using algebraic manipulation. →

Combinations

Example: How many ways are there to select five players from a 10-member tennis team.

Solution: By Theorem 2, the number of combinations is

$$C(10, 5) = \frac{10!}{5!5!} = 252.$$

Example: A group of 30 people have been trained as astronauts to go on the first mission to Mars. How many ways are there to select a crew of six people to go on this mission?

Solution: By Theorem 2, the number of possible crews is

$$C(30, 6) = \frac{30!}{6!24!} = \frac{30\cdot 29\cdot 28\cdot 27\cdot 26\cdot 25}{6\cdot 5\cdot 4\cdot 3\cdot 2\cdot 1} = 593,775\ .$$

Applications of Recurrence Relations (Ch. 8.1)

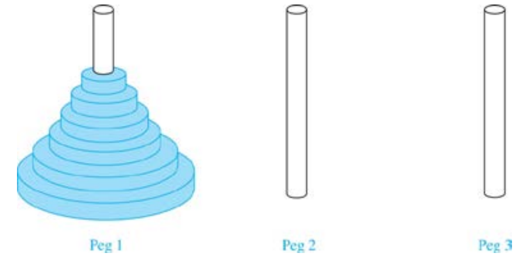
- Applications of Recurrence Relations
  - The Tower of Hanoi

Recurrence Relations
(recalling definitions from Chapter 2)

Definition: A *recurrence relation* for the sequence $\{a_n\}$ is an equation that expresses a_n in terms of one or more of the previous terms of the sequence, namely, $a_0, a_1, \ldots, a_{n-1}$, for all integers n with $n \geq n_0$, where n_0 is a nonnegative integer.

- A sequence is called a *solution* of a recurrence relation if its terms satisfy the recurrence relation.

The Tower of Hanoi



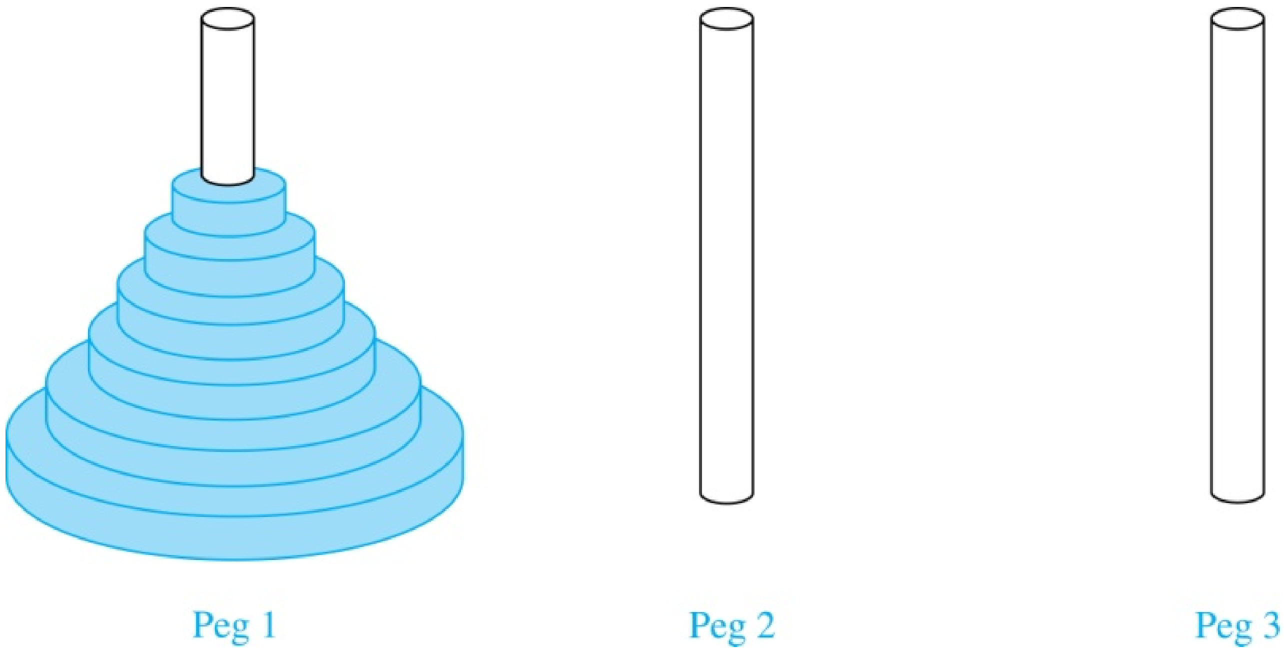
In the late nineteenth century, the French mathematician Édouard Lucas invented a puzzle consisting of three pegs on a board with disks of different sizes.

Initially all of the disks are on the first peg in order of size, with the largest on the bottom.

Rules: You are allowed to move the disks one at a time from one peg to another as long as a larger disk is never placed on a smaller.

Goal: Using allowable moves, end up with all the disks on the second peg in order of size with largest on the bottom.

The Tower of Hanoi



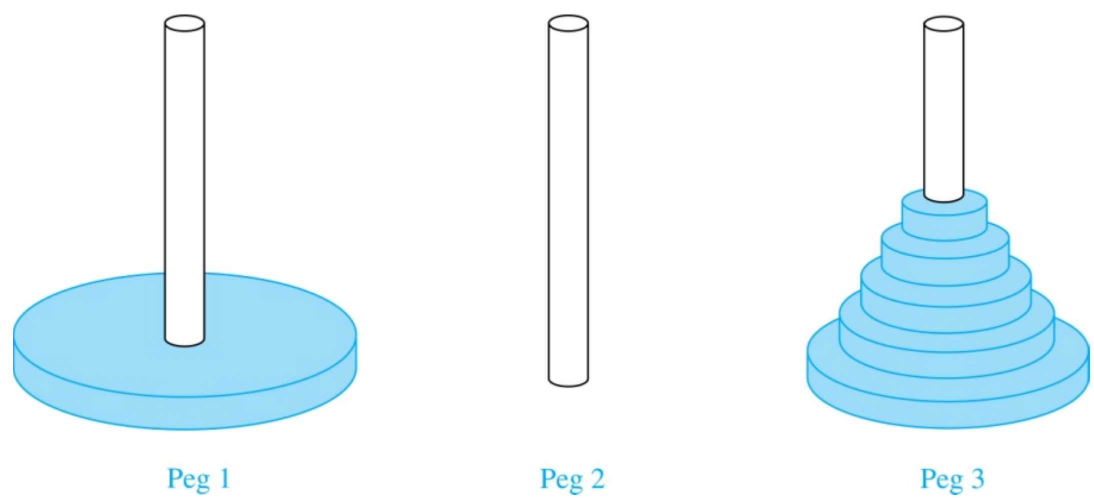
The Initial Position in the Tower of Hanoi Puzzle

The Tower of Hanoi

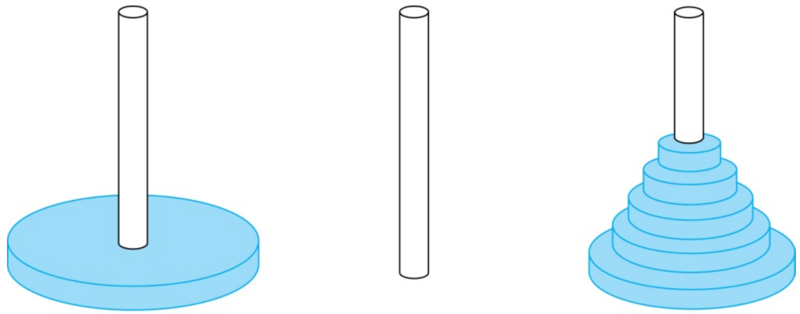
Solution: Let $\{H_n\}$ denote the number of moves needed to solve the Tower of Hanoi Puzzle with n disks.

Set up a recurrence relation for the sequence $\{H_n\}$.

Begin with n disks on peg 1. We can transfer the top $n - 1$ disks, following the rules of the puzzle, to peg 3 using H_{n-1} moves.



The Tower of Hanoi



First, we use 1 move to transfer the largest disk to the second peg.

Then we transfer the $n-1$ disks from peg 3 to peg 2 using H_{n-1} additional moves. This can not be done in fewer steps.

Hence, $H_n = 2H_{n-1} + 1$.

The initial condition is $H_1 = 1$ since a single disk can be transferred from peg 1 to peg 2 in one move.

The Tower of Hanoi

We can use an iterative approach to solve this recurrence relation by repeatedly expressing H_n in terms of the previous terms of the sequence.

$$
\begin{aligned}
H_n &= 2H_{n-1} + 1 \\
&= 2(2H_{n-2} + 1) + 1 = 2^2 H_{n-2} + 2 + 1 \\
&= 2^2(2H_{n-3} + 1) + 2 + 1 = 2^3 H_{n-3} + 2^2 + 2 + 1 \\
&\vdots \\
&= 2^{n-1}H_1 + 2^{n-2} + 2^{n-3} + \dots + 2 + 1 \\
&= 2^{n-1} + 2^{n-2} + 2^{n-3} + \dots + 2 + 1 \qquad \textit{because } H_1 = 1 \\
&= 2^n - 1 \qquad \textit{using the formula for the sum of the terms of a geometric series}
\end{aligned}
$$

The Tower of Hanoi

- There was a myth created with the puzzle. Monks in a tower in Hanoi are transferring 64 gold disks from one peg to another following the rules of the puzzle. They move one disk each day. When the puzzle is finished, the world will end.
- Using this formula for the 64 gold disks of the myth,

$$2^{64} - 1 = 18{,}446{,}744{,}073{,}709{,}551{,}615$$

days are needed to solve the puzzle, which is more than 500 billion years.