Homework 3 Solution

Type your answers in a text file and submit it in CatCourses.
You can also write your answers on papers and scan them into image files for submission.

1 pt for each problem and sub-problem (total 20 pts)

Section 1.6

4 (a) (c) (e)

a) We have taken the conjunction of two propositions and asserted one of them. This is, according to Table 1, simplification.

c) modus ponens

e) hypothetical syllogism

6

Let r be the proposition "It rains," let f be the proposition "It is foggy," let s be the proposition "The sailing race will be held," let l be the proposition "The life saving demonstration will go on," and let t be the proposition "The trophy will be awarded." We are given premises $(\neg r \vee \neg f) \to (s \wedge l)$, $s \to t$, and $\neg t$. We want to conclude r. We set up the proof in two columns, with reasons, as in Example 6. Note that it is valid to replace subexpressions by other expressions logically equivalent to them.

Step	**Reason**
1. $\neg t$	Hypothesis
2. $s \to t$	Hypothesis
3. $\neg s$	Modus tollens using (1) and (2)
4. $(\neg r \vee \neg f) \to (s \wedge l)$	Hypothesis
5. $(\neg(s \wedge l)) \to \neg(\neg r \vee \neg f)$	Contrapositive of (4)
6. $(\neg s \vee \neg l) \to (r \wedge f)$	De Morgan's law and double negative
7. $\neg s \vee \neg l$	Addition, using (3)
8. $r \wedge f$	Modus ponens using (6) and (7)
9. r	Simplification using (8)

8

First we use universal instantiation to conclude from "For all x, if x is a man, then x is not an island" the special case of interest, "If Manhattan is a man, then Manhattan is not an island." Then we form the contrapositive (using also double negative): "If Manhattan is an island, then Manhattan is not a man." Finally we use modus ponens to conclude that Manhattan is not a man. Alternatively, we could apply modus tollens.

10 (a) (b)

a) If we use modus tollens starting from the back, then we conclude that I am not sore. Another application of modus tollens then tells us that I did not play hockey.

b) We really can't conclude anything specific here.

14 (a) (b)

In each case we set up the proof in two columns, with reasons, as in Example 6.

a) Let $c(x)$ be "x is in this class," let $r(x)$ be "x owns a red convertible," and let $t(x)$ be "x has gotten a speeding ticket." We are given premises $c(\text{Linda})$, $r(\text{Linda})$, $\forall x(r(x) \to t(x))$, and we want to conclude $\exists x(c(x) \wedge t(x))$.

Step	**Reason**
1. $\forall x(r(x) \to t(x))$	Hypothesis
2. $r(\text{Linda}) \to t(\text{Linda})$	Universal instantiation using (1)
3. $r(\text{Linda})$	Hypothesis
4. $t(\text{Linda})$	Modus ponens using (2) and (3)
5. $c(\text{Linda})$	Hypothesis
6. $c(\text{Linda}) \wedge t(\text{Linda})$	Conjunction using (4) and (5)
7. $\exists x(c(x) \wedge t(x))$	Existential generalization using (6)

b) Let $r(x)$ be "r is one of the five roommates listed," let $d(x)$ be "x has taken a course in discrete mathematics," and let $a(x)$ be "x can take a course in algorithms." We are given premises $\forall x(r(x) \to d(x))$ and $\forall x(d(x) \to a(x))$, and we want to conclude $\forall x(r(x) \to a(x))$. In what follows y represents an arbitrary person.

Step	**Reason**
1. $\forall x(r(x) \to d(x))$	Hypothesis
2. $r(y) \to d(y)$	Universal instantiation using (1)
3. $\forall x(d(x) \to a(x))$	Hypothesis
4. $d(y) \to a(y)$	Universal instantiation using (3)
5. $r(y) \to a(y)$	Hypothetical syllogism using (2) and (4)
6. $\forall x(r(x) \to a(x))$	Universal generalization using (5)

24

Steps 3 and 5 are incorrect; simplification applies to conjunctions, not disjunctions.

30

Let a be "Allen is a good boy"; let h be "Hillary is a good girl"; let d be "David is happy." Then our assumptions are $\neg a \vee h$ and $a \vee d$. Using resolution gives us $h \vee d$, as desired.

Section 1.7

6

An odd number is one of the form $2n+1$, where n is an integer. We are given two odd numbers, say $2a+1$ and $2b+1$. Their product is $(2a+1)(2b+1) = 4ab+2a+2b+1 = 2(2ab+a+b)+1$. This last expression shows that the product is odd, since it is of the form $2n+1$, with $n = 2ab+a+b$.

10

A rational number is a number that can be written in the form x/y where x and y are integers and $y \neq 0$. Suppose that we have two rational numbers, say a/b and c/d. Then their product is, by the usual rules for multiplication of fractions, $(ac)/(bd)$. Note that both the numerator and the denominator are integers, and that $bd \neq 0$ since b and d were both nonzero. Therefore the product is, by definition, a rational number.

20

We need to prove the proposition "If 1 is a positive integer, then $1^2 \geq 1$." The conclusion is the true statement $1 \geq 1$. Therefore the conditional statement is true. This is an example of a trivial proof, since we merely showed that the conclusion was true.

22

We give a proof by contradiction. Suppose that we don't get a pair of blue socks or a pair of black socks. Then we drew at most one of each color. This accounts for only two socks. But we are drawing three socks. Therefore our supposition that we did not get a pair of blue socks or a pair of black socks is incorrect, and our proof is complete.

26

We need to prove two things, since this is an "if and only if" statement. First let us prove directly that if n is even then $7n+4$ is even. Since n is even, it can be written as $2k$ for some integer k. Then $7n+4 = 14k+4 = 2(7k+2)$. This is 2 times an integer, so it is even, as desired. Next we give a proof by contraposition that if $7n+4$ is even then n is even. So suppose that n is not even, i.e., that n is odd. Then n can be written as $2k+1$ for some integer k. Thus $7n+4 = 14k+11 = 2(7k+5)+1$. This is 1 more than 2 times an integer, so it is odd. That completes the proof by contraposition.

28

There are two things to prove. For the "if" part, there are two cases. If $m = n$, then of course $m^2 = n^2$; if $m = -n$, then $m^2 = (-n)^2 = (-1)^2 n^2 = n^2$. For the "only if" part, we suppose that $m^2 = n^2$. Putting everything on the left and factoring, we have $(m+n)(m-n) = 0$. Now the only way that a product of two numbers can be zero is if one of them is zero. Therefore we conclude that either $m+n = 0$ (in which case $m = -n$), or else $m - n = 0$ (in which case $m = n$), and our proof is complete.

30

We write these in symbols: $a < b$, $(a+b)/2 > a$, and $(a+b)/2 < b$. The latter two are equivalent to $a+b > 2a$ and $a+b < 2b$, respectively, and these are in turn equivalent to $b > a$ and $a < b$, respectively. It is now clear that all three statements are equivalent.

32

We give direct proofs that (i) implies (ii), that (ii) implies (iii), and that (iii) implies (i). That will suffice. For the first, suppose that $x = p/q$ where p and q are integers with $q \neq 0$. Then $x/2 = p/(2q)$, and this is rational, since p and $2q$ are integers with $2q \neq 0$. For the second, suppose that $x/2 = p/q$ where p and q are integers with $q \neq 0$. Then $x = (2p)/q$, so $3x - 1 = (6p)/q - 1 = (6p - q)/q$ and this is rational, since $6p - q$ and q are integers with $q \neq 0$. For the last, suppose that $3x - 1 = p/q$ where p and q are integers with $q \neq 0$. Then $x = (p/q + 1)/3 = (p+q)/(3q)$, and this is rational, since $p + q$ and $3q$ are integers with $3q \neq 0$.

42

We show that each of these is equivalent to the statement (v) n is odd, say $n = 2k+1$. Example 1 showed that (v) implies (i), and Example 8 showed that (i) implies (v). For $(v) \rightarrow (ii)$ we see that $1 - n = 1 - (2k+1) = 2(-k)$ is even. Conversely, if n were even, say $n = 2m$, then we would have $1 - n = 1 - 2m = 2(-m) + 1$, so $1 - n$ would be odd, and this completes the proof by contraposition that $(ii) \rightarrow (v)$. For $(v) \rightarrow (iii)$, we see that $n^3 = (2k+1)^3 = 8k^3 + 12k^2 + 6k + 1 = 2(4k^3 + 6k^2 + 3k) + 1$ is odd. Conversely, if n were even, say $n = 2m$, then we would have $n^3 = 2(4m^3)$, so n^3 would be even, and this completes the proof by contraposition that $(iii) \rightarrow (v)$. Finally, for $(v) \rightarrow (iv)$, we see that $n^2 + 1 = (2k+1)^2 + 1 = 4k^2 + 4k + 2 = 2(2k^2 + 2k + 1)$ is even. Conversely, if n were even, say $n = 2m$, then we would have $n^2 + 1 = 2(2m^2) + 1$, so $n^2 + 1$ would be odd, and this completes the proof by contraposition that $(iv) \rightarrow (v)$.