Homework 4 Solution

1 point each problem or sub-problem, total 20 points

Section 1.8

16

We know from algebra that the following equations are equivalent: $ax + b = c$, $ax = c - b$. $x = (c - b)/a$. This shows, constructively, what the unique solution of the given equation is.

44

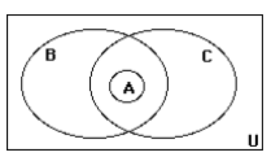
Without loss of generality, we number the squares from 1 to 25, starting in the top row and proceeding left to right in each row; and we assume that squares 5 (upper right corner), 21 (lower left corner), and 25 (lower right corner) are the missing ones. We argue that there is no way to cover the remaining squares with dominoes.

By symmetry we can assume that there is a domino placed in 1-2 (using the obvious notation). If square 3 is covered by 3-8, then the following dominoes are forced in turn: 4-9, 10-15, 19-20, 23-24, 17-22, and 13-18, and now no domino can cover square 14. Therefore we must use 3-4 along with 1-2. If we use all of 17-22, 18-23, and 19-24, then we are again quickly forced into a sequence of placements that lead to a contradiction. Therefore without loss of generality, we can assume that we use 22-23, which then forces 19-24, 15-20, 9-10, 13-14, 7-8, 6-11, and 12-17, and we are stuck once again. This completes the proof by contradiction that no placement is possible.

Section 2.1

16

We allow B and C to overlap, because we are told nothing about their relationship. The set A must be a subset of each of them, and that forces it to be positioned as shown. We cannot actually show the properness of the subset relationships in the diagram, because we don't know where the elements in B and C that are not in A are located—there might be only one (which is in both B and C), or they might be located in portions of B and/or C outside the other. Thus the answer is as shown, but with the added condition that there must be at least one element of B not in A and one element of C not in A.



20 (a) (c)

The cardinality of a set is the number of elements it has.

a) The empty set has no elements, so its cardinality is 0.

c) This set has two elements, so its cardinality is 2.

24 (b) (c)

b) This is the power set of $\{a\}$.

c) This set has three elements. Since 3 is not a power of 2, this set cannot be the power set of any set.

26

We need to show that every element of $A \times B$ is also an element of $C \times D$. By definition, a typical element of $A \times B$ is a pair (a, b) where $a \in A$ and $b \in B$. Because $A \subseteq C$, we know that $a \in C$; similarly, $b \in D$. Therefore $(a, b) \in C \times D$.

32 (a) (d)

In each case the answer is a set of 3-tuples.

a) $\{(a,x,0),(a,x,1),(a,y,0),(a,y,1),(b,x,0),(b,x,1),(b,y,0),(b,y,1),(c,x,0),(c,x,1),(c,y,0),(c,y,1)\}$

d) $\{(x,x,x),(x,x,y),(x,y,x),(x,y,y),(y,x,x),(y,x,y),(y,y,x),(y,y,y)\}$

38

Suppose $A \neq B$ and neither A nor B is empty. We must prove that $A \times B \neq B \times A$. Since $A \neq B$, either we can find an element x that is in A but not B, or vice versa. The two cases are similar, so without loss of generality, let us assume that x is in A but not B. Also, since B is not empty, there is some element $y \in B$. Then (x, y) is in $A \times B$ by definition, but it is not in $B \times A$ since $x \notin B$. Therefore $A \times B \neq B \times A$.

44 (b) (c)

In each case we want the set of all values of x in the domain (the set of integers) that satisfy the given equation or inequality.

b) The square roots of 2 are not integers, so the truth set is the empty set, $\emptyset$.

c) Negative integers certainly satisfy this inequality, as do all positive integers greater than 1. However, $0 \not< 0^2$ and $1 \not< 1^2$. Thus the truth set is $\{x \in \mathbf{Z} \mid x < x^2\} = \{x \in \mathbf{Z} \mid x \neq 0 \wedge x \neq 1\} = \{\ldots, -3, -2, -1, 2, 3, \ldots\}$.

Section 2.2

4 (c) (d)

Note that $A \subseteq B$.

c) There are no elements in A that are not in B, so the answer is $\emptyset$.

d) $\{f, g, h\}$

18 (c) (d)

c) Suppose that $x \in (A - B) - C$. Then x is in $A - B$ but not in C. Since $x \in A - B$, we know that $x \in A$ (we also know that $x \notin B$, but that won't be used here). Since we have established that $x \in A$ but $x \notin C$, we have proved that $x \in A - C$.

d) To show that the set given on the left-hand side is empty, it suffices to assume that x is some element in that set and derive a contradiction, thereby showing that no such x exists. So suppose that $x \in (A - C) \cap (C - B)$. Then $x \in A - C$ and $x \in C - B$. The first of these statements implies by definition that $x \notin C$, while the second implies that $x \in C$. This is impossible, so our proof by contradiction is complete.

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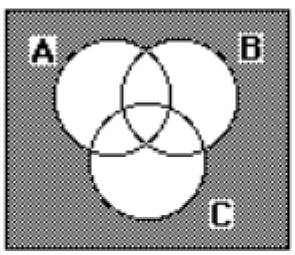
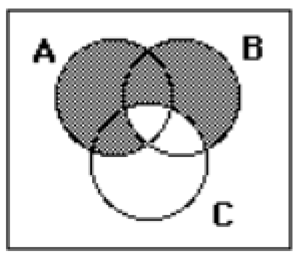
First suppose x is in the left-hand side. Then x must be in A but in neither B nor C. Thus $x \in A - C$, but $x \notin B - C$, so x is in the right-hand side. Next suppose that x is in the right-hand side. Thus x must be in $A - C$ and not in $B - C$. The first of these implies that $x \in A$ and $x \notin C$. But now it must also be the case that $x \notin B$, since otherwise we would have $x \in B - C$. Thus we have shown that x is in A but in neither B nor C, which implies that x is in the left-hand side.

26 (b) (c)

The set is shaded in each case.



(b)



(c)