CSE 15
Discrete Mathematics

Lecture 4 – Proposition Logic (4)

Announcement

- HW #2 out on Thursday (9/6)
  - Due at **5pm** 9/14 (Fri) with 1 extra day of submission.
  - Type your answers in a text file and submit it to Catcourses.
  - Or write your answers on papers and scan them into image files and upload to Catcourses.
  - Work on it during and outside lab hours.
- Reading assignment
  - Ch. 1.6 – 1.8 of textbook

Translation from English to Logic

Examples:

1. "Some student in this class has visited Mexico."

   Solution: Let $M(x)$ denote "x has visited Mexico" and $S(x)$ denote "x is a student in this class," and U be all people.

   $$\exists x\ (S(x) \land M(x))$$

2. "Every student in this class has visited Canada or Mexico."

   Solution: Add $C(x)$ denoting "x has visited Canada."

   $$\forall x\ (S(x) \rightarrow (M(x) \lor C(x)))$$

Translation from English to Logic

- U = {fleegles, snurds, thingamabobs}

  F(x): *x* is a fleegle

  S(x): *x* is a snurd

  T(x): x is a thingamabob

  Translate "Everything is a fleegle"

  Solution: $\forall x\, F(x)$

Translation (cont)

- U = {fleegles, snurds, thingamabobs}

  F(x): *x* is a fleegle

  S(x): *x* is a snurd

  T(x): *x* is a thingamabob

  "Nothing is a snurd."

Solution: $\neg\exists x\, S(x)$ What is this equivalent to?

Solution: $\forall x \neg S(x)$

Translation (cont)

- U = {fleegles, snurds, thingamabobs}

  F(x): *x* is a fleegle

  S(x): *x* is a snurd

  T(x): *x* is a thingamabob

“All fleegles are snurds.”

Solution: $\forall x\,(F(x) \rightarrow S(x))$

Translation (cont)

- U = {fleegles, snurds, thingamabobs}

  F(x): *x* is a fleegle

  S(x): *x* is a snurd

  T(x): *x* is a thingamabob

"Some fleegles are thingamabobs."

Solution: $\exists x\ (F(x) \land T(x))$

Translation (cont)

- U = {fleegles, snurds, thingamabobs}

  F(x): *x* is a fleegle

  S(x): *x* is a snurd

  T(x): *x* is a thingamabob

  "No snurd is a thingamabob."

Solution: $\neg\exists x\,(S(x) \wedge T(x))$ What is this equivalent to?

Solution: $\forall x\,(\neg S(x) \vee \neg T(x))$

Translation (cont)

- U = {fleegles, snurds, thingamabobs}

  F(x): x is a fleegle

  S(x): *x* is a snurd

  T(x): *x* is a thingamabob

"If any fleegle is a snurd then it is also a thingamabob."

Solution: $\forall x\ ((F(x) \wedge S(x)) \rightarrow T(x))$

System Specification Example

- Predicate logic is used for specifying properties that systems must satisfy.
- For example, translate into predicate logic:
  - "Every mail message larger than one megabyte will be compressed."
  - "If a user is active, at least one network link will be available."
- Decide on predicates and domains (left implicit here) for the variables:
  - Let $L(m, y)$ be "Mail message m is larger than y megabytes."
  - Let $C(m)$ denote "Mail message m will be compressed."
  - Let $A(u)$ represent "User u is active."
  - Let $S(n, x)$ represent "Network link n is state x.
- Now we have:

$$\forall m(L(m,1) \rightarrow C(m))$$

$$\exists u\, A(u) \rightarrow \exists n\, S(n, available)$$

Lewis Carroll Example

Charles Lutwidge Dodgson
(AKA Lewis Caroll) (1832-1898)

- The first two are called *premises* and the third is called the *conclusion*.
  1. "All lions are fierce."
  2. "Some lions do not drink coffee."
  3. "Some fierce creatures do not drink coffee."
- Here is one way to translate these statements to predicate logic. Let P(x), Q(x), and R(x) be the propositional functions "x is a lion," "x is fierce," and "x drinks coffee," respectively.
  1. $\forall x\, (P(x) \rightarrow Q(x))$
  2. $\exists x\, (P(x) \land \neg R(x))$
  3. $\exists x\, (Q(x) \land \neg R(x))$
- Later we will see how to prove that the conclusion follows from the premises.

Nested Quantifiers (Ch. 1.5)

- Nested Quantifiers
- Order of Quantifiers
- Translating from Nested Quantifiers into English
- Translating Mathematical Statements into Statements involving Nested Quantifiers.
- Translating English Sentences into Logical Expressions.
- Negating Nested Quantifiers.

Nested Quantifiers

- Nested quantifiers are often necessary to express the meaning of sentences in English as well as important concepts in computer science and mathematics.

  Example: "Every real number has an inverse" is

  $$\forall x \exists y (x + y = 0)$$

  where the domains of x and y are the real numbers.

- We can also think of nested propositional functions:

  $\forall x \exists y (x + y = 0)$ can be viewed as $\forall x\, Q(x)$ where $Q(x)$ is $\exists y\, P(x, y)$ where $P(x, y)$ is $(x + y = 0)$

Thinking of Nested Quantification

- Nested Loops
  - To see if $\forall x \forall y P(x,y)$ is true, loop through the values of x:
    - At each step, loop through the values for y.
    - If for some pair of x and y, $P(x,y)$ is false, then $\forall x \forall y P(x,y)$ is false and both the outer and inner loop terminate.

    $\forall x \forall y P(x,y)$ is true if the outer loop ends after stepping through each x.
  - To see if $\forall x \exists y P(x,y)$ is true, loop through the values of x:
    - At each step, loop through the values for y.
    - The inner loop ends when a pair x and y is found such that $P(x, y)$ is true.
    - If no y is found such that $P(x, y)$ is true the outer loop terminates as $\forall x \exists y P(x,y)$ has been shown to be false.

    $\forall x \exists y P(x,y)$ is true if the outer loop ends after stepping through each x.
- If the domains of the variables are infinite, then this process can not actually be carried out.

Order of Quantifiers

Examples:

1. Let $P(x,y)$ be the statement "$x + y = y + x$." Assume that U is the real numbers. Then $\forall x\ \forall y P(x,y)$ and $\forall y\ \forall x P(x,y)$ have the same truth value.

2. Let $Q(x,y)$ be the statement "$x + y = 0$." Assume that U is the real numbers. Then $\forall x \exists y Q(x,y)$ is true, but $\exists y\ \forall x Q(x,y)$ is false.

Questions on Order of Quantifiers

Example 1: Let U be the real numbers,

Define $P(x,y) : x \cdot y = 0$

What is the truth value of the following:

1. $\forall x \forall y P(x,y)$

   Answer: False

2. $\forall x \exists y P(x,y)$

   Answer: True

3. $\exists x \forall y\ P(x,y)$

   Answer: True

4. $\exists x \exists y\ P(x,y)$

   Answer: True

Questions on Order of Quantifiers

Example 2: Let *U* be the real numbers,

Define *P(x,y)* : *x* / *y* = 1

What is the truth value of the following:

1. $\forall x \forall y P(x,y)$

   Answer: False
2. $\forall x \exists y P(x,y)$

   Answer: True
3. $\exists x \forall y\ P(x,y)$

   Answer: False
4. $\exists x \exists\ y\ P(x,y)$

   Answer: True

Quantifications of Two Variables

Statement	When True?	When False
$\forall x \forall y P(x,y)$ $\forall y \forall x P(x,y)$	*P*(*x*,*y*) is true for every pair *x*,*y*.	There is a pair *x, y* for which *P*(*x*,*y*) is false.
$\forall x \exists y P(x,y)$	For every *x* there is a *y* for which *P*(*x*,*y*) is true.	There is an x such that *P*(*x*,*y*) is false for every *y*.
$\exists x \forall y P(x,y)$	There is an *x* for which *P*(*x*,*y*) is true for every *y*.	For every *x* there is a y for which *P*(x,y) is false.
$\exists x \exists y P(x,y)$ $\exists y \exists x P(x,y)$	There is a pair *x, y* for which *P*(*x*,*y*) is true.	*P*(x,y) is false for every pair *x*,*y*

Translating Nested Quantifiers into English

Example 1: Translate the statement

$$\forall x \ (C(x) \lor \exists y \ (C(y) \land F(x, y)))$$

where C(x) is "*x* has a computer," and *F*(*x*,*y*) is "*x* and *y* are friends," and the domain for both *x* and *y* consists of all students in your school.

Solution: Every student in your school has a computer or has a friend who has a computer.

Example 2: Translate the statement

$$\exists x \forall y \ \forall z \ ((F(x, y) \land F(x,z) \land (y \neq z)) \rightarrow \neg F(y,z))$$

Solution: There is a student none of whose friends are also friends with each other.

Translating Mathematical Statements into Predicate Logic

Example : Translate "The sum of two positive integers is always positive" into a logical expression.

Solution:

1. Rewrite the statement to make the implied quantifiers and domains explicit:

   "For every two integers, if these integers are both positive, then the sum of these integers is positive."

2. Introduce the variables *x* and *y*, and specify the domain, to obtain:

   "For all positive integers *x* and *y*, *x* + *y* is positive."

3. The result is:

$$\forall x \forall y((x > 0) \land (y > 0) \rightarrow (x + y > 0))$$

where the domain of both variables consists of all integers

Translating English into Logical Expressions Example

Example: Use quantifiers to express the statement "There is a woman who has taken a flight on every airline in the world."

Solution:

1. Let $P(w,f)$ be "w has taken f" and $Q(f,a)$ be "f is a flight on a."
2. The domain of w is all women, the domain of f is all flights, and the domain of a is all airlines.
3. Then the statement can be expressed as:

$$\exists w\, \forall a\, \exists f\, (P(w,f) \wedge Q(f,a))$$

Negating Nested Quantifiers

Part 1: Use quantifiers to express the statement that "There does not exist a woman who has taken a flight on every airline in the world."

Solution: $\neg\exists w \forall a \exists f\, (P(w,f) \wedge Q(f,a))$

Part 2: Now use De Morgan's Laws to move the negation as far inwards as possible.

Solution:

1. $\neg\exists w \forall a \exists f\, (P(w,f) \wedge Q(f,a))$
2. $\forall w \neg \forall a \exists f\, (P(w,f) \wedge Q(f,a))$ by De Morgan's for $\exists$
3. $\forall w \exists a \neg \exists f\, (P(w,f) \wedge Q(f,a))$ by De Morgan's for $\forall$
4. $\forall w \exists a \forall f \neg (P(w,f) \wedge Q(f,a))$ by De Morgan's for $\exists$
5. $\forall w \exists a \forall f (\neg P(w,f) \vee \neg Q(f,a))$ by De Morgan's for $\wedge$.

Negating Nested Quantifiers

Part 3: Can you translate the result back into English?

$$\forall w \exists a \forall f (\neg P(w,f) \vee \neg Q(f,a))$$

Solution:

"For every woman there is an airline such that for all flights, this woman has not taken that flight or that flight is not on this airline"

Proofs: Rules of Inference (Ch. 1.6)

- Valid Arguments
- Inference Rules for Propositional Logic
- Using Rules of Inference to Build Arguments
- Rules of Inference for Quantified Statements
- Building Arguments for Quantified Statements

Terminologies

- **Proof**: valid arguments that establish the truth of a mathematical statement
- **Argument**: a sequence of statements that end with a conclusion
- **Valid**: the conclusion or final statement of the argument must follow the truth of proceeding statements or **premise** of the argument

Revisiting the Socrates Example

- We have the two premises:
  - "All men are mortal."
  - "Socrates is a man."
- And the conclusion:
  - "Socrates is mortal."
- How do we get the conclusion from the premises?

The Argument

- We can express the premises (above the line) and the conclusion (below the line) in predicate logic as an argument:

$$\begin{array}{c} \forall x(Man(x) \rightarrow Mortal(x)) \\ Man(Socrates) \\ \hline \therefore \quad Mortal(Socrates) \end{array}$$

- We will see shortly that this is a valid argument.

Valid Arguments

- We will show how to construct valid arguments in two stages; first for propositional logic and then for predicate logic. The rules of inference are the essential building block in the construction of valid arguments.

1. Propositional Logic

   Inference Rules

2. Predicate Logic

   Inference rules for propositional logic plus additional inference rules to handle variables and quantifiers.

Arguments in Propositional Logic

- A *argument* in propositional logic is a sequence of propositions.
  - All but the final proposition are called *premises*.
  - The last statement is the *conclusion*.
- The argument is valid if the premises imply the conclusion.
- If the premises are $p_1, p_2, \ldots, p_n$ and the conclusion is q then

  $(p_1 \wedge p_2 \wedge \ldots \wedge p_n) \rightarrow q$ is a tautology.

Rules of Inference for Propositional Logic: Modus Ponens

$$\begin{array}{l} p \to q \\ p \\ \hline \therefore q \end{array}$$

Corresponding Tautology:

$(p \wedge (p \to q)) \to q$

Example:

Let p be “It is snowing.”
Let q be “I will study discrete math.”

“If it is snowing, then I will study discrete math.”
“It is snowing.”

“Therefore , I will study discrete math.”

Modus Tollens

$$\begin{array}{l} p \to q \\ \neg q \\ \hline \therefore \neg p \end{array}$$

Corresponding Tautology:

$(\neg q \wedge (p \to q)) \to \neg p$

Example:

Let *p* be “it is snowing.”
Let *q* be “I will study discrete math.”

“If it is snowing, then I will study discrete math.”
“I will not study discrete math.”

“Therefore , it is not snowing.”

Hypothetical Syllogism

$$\begin{array}{l} p \rightarrow q \\ q \rightarrow r \\ \hline \therefore p \rightarrow r \end{array}$$

Corresponding Tautology:
$((p \rightarrow q) \wedge (q \rightarrow r)) \rightarrow (p \rightarrow r)$

Example:
Let *p* be "it snows."
Let *q* be "I will study discrete math."
Let *r* be "I will get an A."

"If it snows, then I will study discrete math."
"If I study discrete math, I will get an A."

"Therefore , If it snows, I will get an A."

Disjunctive Syllogism

$$\begin{array}{l} p \vee q \\ \neg p \\ \hline \therefore q \end{array}$$

Corresponding Tautology:
$(\neg p \wedge (p \vee q)) \rightarrow q$

Example:
Let p be "I will study discrete math."
Let q be "I will study English literature."

"I will study discrete math or I will study English literature."
"I will not study discrete math."

"Therefore , I will study English literature."

Addition

$$\frac{p}{\therefore p \vee q}$$

Corresponding Tautology:

$p \rightarrow (p \vee q)$

Example:

Let *p* be “I will study discrete math.”

Let *q* be “I will visit Las Vegas.”

“I will study discrete math.”

“Therefore, I will study discrete math or I will visit Las Vegas.”

Simplification

Corresponding Tautology:

$(p \wedge q) \rightarrow p$

$$\frac{p \wedge q}{\therefore p} \quad \textbf{\textit{OR}} \quad \frac{p \wedge q}{\therefore q}$$

Example:

Let *p* be "I will study discrete math."
Let *q* be "I will study English literature."

"I will study discrete math and English literature"

"Therefore, I will study discrete math."

Conjunction

$$\begin{array}{l} p \\ q \\ \hline \therefore p \wedge q \end{array}$$

Corresponding Tautology:
$((p) \wedge (q)) \rightarrow (p \wedge q)$

Example:
Let *p* be "I will study discrete math."
Let *q* be "I will study English literature."

"I will study discrete math."
"I will study English literature."

"Therefore, I will study discrete math and I will study English literature."

Resolution

Resolution plays an important role in AI and is used in Prolog.

$$\begin{array}{l} \neg p \vee r \\ p \vee q \\ \hline \therefore q \vee r \end{array}$$

Corresponding Tautology:
$((\neg p \vee r) \wedge (p \vee q)) \rightarrow (q \vee r)$

Example:
Let *p* be "I will study discrete math."
Let *r* be "I will study English literature."
Let q be "I will study databases."

"I will not study discrete math or I will study English literature."
"I will study discrete math or I will study databases."

"Therefore, I will study databases or I will study English literature."

Using the Rules of Inference to Build Valid Arguments

- A *valid argument* is a sequence of statements. Each statement is either a premise or follows from previous statements by rules of inference. The last statement is called conclusion.
- A valid argument takes the following form:

$$
\begin{array}{l}
\quad S_1 \\
\quad S_2 \\
\quad . \\
\quad . \\
\quad . \\
\quad S_n \\
\\
\therefore C
\end{array}
$$

Valid Arguments

Example 1: From the single proposition

$$p \wedge (p \to q)$$

Show that *q* is a conclusion.

Solution:

Step	**Reason**
1. $p \wedge (p \to q)$	Premise
2. p	Conjunction using (1)
3. $p \to q$	Conjunction using (1)
4. q	Modus Ponens using (2) and (3)