CSE 15
Discrete Mathematics

Lecture 7 – Proposition Logic (7)
Sets (1)

Announcement

- HW #3
  - Due **5pm** 9/21 (Fri) with 1 extra day of re-submission.

- Reading assignment
  - Ch. 2.2-2.3 of textbook

Backward Reasoning

Example: Suppose that two people play a game taking turns removing, 1, 2, or 3 stones at a time from a pile that begins with 15 stones. The person who removes the last stone wins the game. Show that the first player can win the game no matter what the second player does.

Backward Reasoning

Proof: Let *n* be the last step of the game.

Step n: Player$_1$ can win if the pile contains 1,2, or 3 stones.

Step n-1: Player$_2$ will have to leave such a pile if the pile that he/she is faced with has 4 stones.

Step n-2: Player$_1$ can leave 4 stones when there are 5,6, or 7 stones left at the beginning of his/her turn.

Step n-3: Player$_2$ must leave such a pile, if there are 8 stones .

Step n-4: Player$_1$ has to have a pile with 9,10, or 11 stones to ensure that there are 8 left.

Step n-5: Player$_2$ needs to be faced with 12 stones to be forced to leave 9,10, or 11.

Step n-6: Player$_1$ can leave 12 stones by removing 3 stones.

Now reasoning forward, the first player can ensure a win by removing 3 stones and leaving 12.

Universally Quantified Assertions

Example: we want to show that x is even if and only if x^2 is even.

Case 1. We show that if x is even then x^2 *is* even using a direct proof (the *only if* part or *necessity*).

If x is even then $x = 2k$ for some integer k.

Hence $x^2 = 4k^2 = 2(2k^2)$ which is even since it is an integer divisible by 2.

This completes the proof of case 1.

Case 2 on next slide →

Universally Quantified Assertions

Case 2. We show that if x^2 is even then x must be even (the *if* part or *sufficiency*). We use a proof by contraposition.

Assume x is not even and then show that x^2 is not even.

If x is not even then it must be odd. So, $x = 2k + 1$ for some k. Then $x^2 = (2k+1)^2 = 4k^2 + 4k + 1 = 2(2k^2 + 2\mathrm{k}) + 1$

which is odd and hence not even. This completes the proof of case 2.

Since x was arbitrary, the result follows by UG.

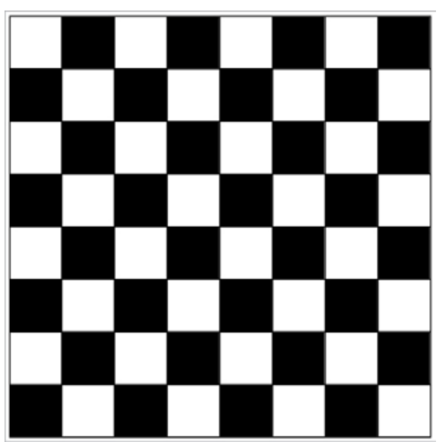
Therefore we have shown that x is even if and only if x^2 is even.

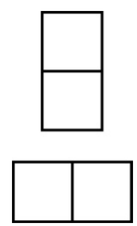
◀

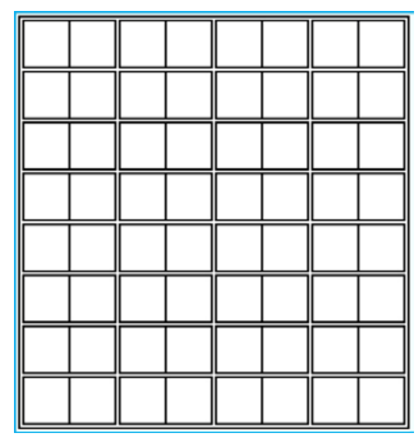
Proof and Disproof: Tilings

Example 1: Can we tile the standard checkerboard using dominos?

Solution: Yes! One example provides a constructive existence proof.


The Standard Checkerboard


Two Dominoes


One Possible Solution

Tilings

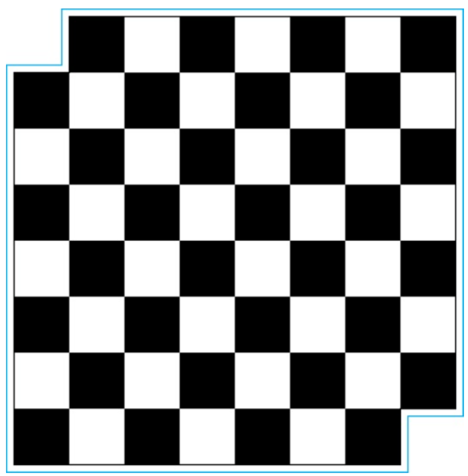
Example 2: Can we tile a checkerboard obtained by removing one of the four corner squares of a standard checkerboard?

Solution:

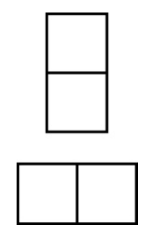
- Our checkerboard has $64 - 1$ = 63 squares.
- Since each domino has two squares, a board with a tiling must have an even number of squares.
- The number 63 is not even.
- We have a contradiction.

Tilings

Example 3: Can we tile a board obtained by removing both the upper left and the lower right squares of a standard checkerboard?



Nonstandard Checkerboard



Dominoes

Continued on next slide →

Tilings

Solution:

- There are 62 squares in this board.
- To tile it we need 31 dominos.
- *Key fact*: Each domino covers one black and one white square.
- Therefore the tiling covers 31 black squares and 31 white squares.
- Our board has either 30 black squares and 32 white squares or 32 black squares and 30 white squares.
- Contradiction!

The Role of Open Problems

- Unsolved problems have motivated much work in mathematics. Fermat's Last Theorem was conjectured more than 300 years ago. It has only recently been finally solved.

Fermat's Last Theorem: The equation $x^n + y^n = z^n$ has no solutions in integers *x*, *y*, and *z*, with $xyz \neq 0$ whenever n is an integer with $n > 2$.

A proof was found by Andrew Wiles in the 1990s.

Additional Proof Methods

- Later we will see many other proof methods:
  - **Mathematical induction**, which is a useful method for proving statements of the form $\forall n\, P(n)$, where the domain consists of all positive integers.
  - **Structural induction**, which can be used to prove such results about recursively defined sets.
  - **Cantor diagonalization** is used to prove results about the size of infinite sets.
  - **Combinatorial proofs** use counting arguments.

Sets (Ch. 2.1)

- Definition of sets
- Describing Sets
  - Roster Method
  - Set-Builder Notation
- Some Important Sets in Mathematics
- Empty Set and Universal Set
- Subsets and Set Equality
- Cardinality of Sets
- Tuples
- Cartesian Product

Introduction

- Sets are one of the basic building blocks for the types of objects considered in discrete mathematics.
  - Important for counting.
  - Programming languages have set operations.
- Set theory is an important branch of mathematics.
  - Many different systems of axioms have been used to develop set theory.
  - Here we are not concerned with a formal set of axioms for set theory. Instead, we will use what is called naïve set theory.

Sets

- A *set* is an **unordered** collection of objects.
  - the students in this class
  - the chairs in this room
- The objects in a set are called the ***elements***, or ***members*** of the set. A set is said to ***contain*** its elements.
- The notation $a \in A$ denotes that a is an element of the set A.
- If a is not a member of A, write $a \notin A$

Describing a Set: Roster Method

- $S = \{a, b, c, d\}$
- Order not important

  $$S = \{a, b, c, d\} = \{b, c, a, d\}$$

- Each distinct object is either a member or not; listing more than once does not change the set.

  $$S = \{a, b, c, d\} = \{a, b, c, b, c, d\}$$

- Ellipses (…) may be used to describe a set without listing all of the members when the pattern is clear.

  $$S = \{a, b, c, d, \ldots\ldots, z\}$$

Roster Method

- Set of all vowels in the English alphabet:

  $V = \{a, e, i, o, u\}$

- Set of all odd positive integers less than 10:

  $O = \{1, 3, 5, 7, 9\}$

- Set of all positive integers less than 100:

  $S = \{1, 2, 3, \ldots\ldots, 99\}$

- Set of all integers less than 0:

  $S = \{\ldots, -3, -2, -1\}$

Some Important Sets

$\mathbf{N}$ = *natural numbers* = $\{0,1,2,3\ldots.\}$

$\mathbf{Z}$ = *integers* = $\{\ldots,-3,-2,-1,0,1,2,3,\ldots\}$

$\mathbf{Z}^{+}$ = *positive integers* = $\{1,2,3,\ldots..\}$

$\mathbf{R}$ = set of *real numbers*

$\mathbf{R}^{+}$ = set of *positive real numbers*

$\mathbf{C}$ = set of *complex numbers*.

Q = set of rational numbers

Set-Builder Notation

- Specify the property or properties that all members must satisfy:

  $S = \{x \mid x \text{ is a positive integer less than } 100\}$

  $O = \{x \mid x \text{ is an odd positive integer less than } 10\}$

  $O = \{x \in \mathbf{Z}^+ \mid x \text{ is odd and } x < 10\}$
- A predicate may be used:

  $$S = \{x \mid \mathrm{P}(x)\}$$
- Example: $S = \{x \mid \mathrm{Prime}(x)\}$
- Positive rational numbers:

  $$\mathbf{Q}^+ = \{x \in \mathbf{R} \mid x = p/q, \text{ for some positive integers } p, q\}$$

Interval Notation

$$[a,b] = \{x \mid a \le x \le b\}$$
$$[a,b) = \{x \mid a \le x < b\}$$
$$(a,b] = \{x \mid a < x \le b\}$$
$$(a,b) = \{x \mid a < x < b\}$$

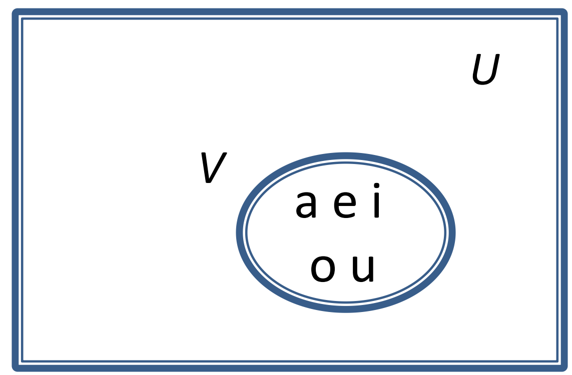
closed interval [a,b]

open interval (a,b)

Universal Set and Empty Set

- The *universal set U* is the set containing everything currently under consideration.
  - Sometimes implicit
  - Sometimes explicitly stated.
  - Contents depend on the context.
- The empty set is the set with no elements. Symbolized ∅, but {} also used.

Venn Diagram



Some things to remember

- Sets can be elements of sets.

  $\{\{1,2,3\}, a, \{b,c\}\}$

  $\{\mathbf{N}, \mathbf{Z}, \mathbf{Q}, \mathbf{R}\}$

- The empty set is different from a set containing the empty set.

  $\emptyset \neq \{\emptyset\}$

Set Equality

Definition: Two sets are *equal* if and only if they have the same elements.

- Therefore if A and B are sets, then A and B are equal if and only if $\forall x(x \in A \leftrightarrow x \in B)$.

- We write *A* = *B* if *A* and *B* are equal sets.

$$\{1,3,5\} = \{3, 5, 1\}$$

$$\{1,5,5,5,3,3,1\} = \{1,3,5\}$$

Subsets

Definition: The set *A* is a *subset* of *B*, if and only if every element of *A* is also an element of *B*.

- The notation $A \subseteq B$ is used to indicate that A is a subset of the set B.
- $A \subseteq B$ holds if and only if $\forall x(x \in A \rightarrow x \in B)$ is true.
  1. Because $a \in \emptyset$ is always false, $\emptyset \subseteq S$, for every set *S*.
  2. Because $a \in S \rightarrow a \in S$, $S \subseteq S$, for every set *S*.

Showing a Set is or is not a Subset of Another Set

- **Showing that A is a Subset of B**: To show that $A \subseteq B$, show that if x belongs to A, then x also belongs to B.
- **Showing that A is not a Subset of B**: To show that A is not a subset of B, $A \nsubseteq B$, find an element $x \in A$ with $x \notin B$. (Such an x is a counterexample to the claim that $x \in A$ implies $x \in B$.)

Examples:

1. The set of all computer science majors at your school is a subset of all students at your school.
2. The set of integers with squares less than 100 is not a subset of the set of nonnegative integers.

Another look at Equality of Sets

- Recall that two sets A and B are *equal*, denoted by $A = B$, iff

$$\forall x(x \in A \leftrightarrow x \in B)$$

- Using logical equivalences we have that $A = B$ iff

$$\forall x[(x \in A \rightarrow x \in B) \land (x \in B \rightarrow x \in A)]$$

- This is equivalent to

$$A \subseteq B \qquad \text{and} \qquad B \subseteq A$$

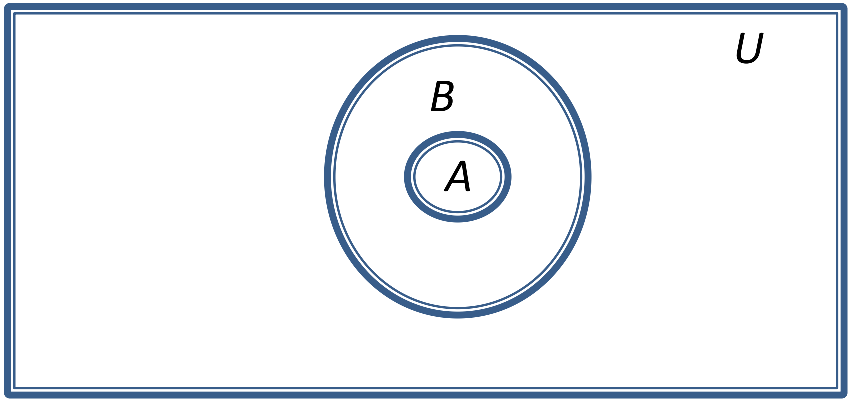
Proper Subsets

Definition: If $A \subseteq B$, but $A \neq B$, then we say A is a *proper subset* of B, denoted by $A \subset B$. If $A \subset B$, then

$$\forall x(x \in A \rightarrow x \in B) \wedge \exists x(x \in B \wedge x \notin A)$$

is true.

Venn Diagram



Set Cardinality

Definition: If there are exactly n distinct elements in *S* where *n* is a nonnegative integer, we say that *S* is ***finite***. Otherwise it is ***infinite***.

Definition: The ***cardinality*** of a finite set *A*, denoted by $|A|$, is the number of (distinct) elements of *A*.

Examples:

1. $|\emptyset| = 0$
2. Let S be the letters of the English alphabet. Then $|S| = 26$
3. $|\{1,2,3\}| = 3$
4. $|\{\emptyset\}| = 1$
5. The set of integers is infinite.

Power Sets

Definition: The set of all subsets of a set *A*, denoted P(*A*), is called the *power set* of *A*.

Example: If $A = \{a,b\}$ then

$$\mathcal{P}(A) = \{\emptyset, \{a\},\{b\},\{a,b\}\}$$

- If a set has *n* elements, then the cardinality of the power set is 2^n.

Example

- What is the power set of the empty set?
  - $P(\emptyset)=\{\emptyset\}$
- The set $\{\emptyset\}$ has exactly two subsets, i.e., $\emptyset$, and the set $\{\emptyset\}$. Thus $P(\{\emptyset\})=\{\emptyset,\{\emptyset\}\}$.

Tuples

- The ***ordered n-tuple*** $(a_1,a_2,\ldots..,a_n)$ is the **ordered collection** that has a_1 as its first element and a_2 as its second element and so on until a_n as its last element.
- Two n-tuples are equal if and only if their corresponding elements are equal.
- 2-tuples are called *ordered pairs*.
- The ordered pairs (a,b) and (c,d) are equal if and only if $a = c$ and $b = d$.

Cartesian Product

Definition: The ***Cartesian Product*** of two sets A and B, denoted by $A \times B$ is the set of ordered pairs (a,b) where $a \in A$ and $b \in B$.

Example:

$$A \times B = \{(a, b) | a \in A \wedge b \in B\}$$

$A = \{a,b\}$ $B = \{1,2,3\}$

$A \times B = \{(a,1),(a,2),(a,3),(b,1),(b,2),(b,3)\}$

- **Definition**: A subset R of the Cartesian product $A \times B$ is called a ***relation*** from the set A to the set B. (Relations will be covered in depth in Chapter 9.)

Cartesian Product

Definition: The cartesian products of the sets $A_1, A_2, \ldots\ldots, A_n$, denoted by $A_1 \times A_2 \times \ldots\ldots \times A_n$, is the set of ordered n-tuples $(a_1, a_2, \ldots\ldots, a_n)$ where a_i belongs to A_i for $i = 1, \ldots n$.

$$A_1 \times A_2 \times \cdots \times A_n = \{(a_1, a_2, \ldots, a_n) | a_i \in A_i \text{ for } i = 1, 2, \ldots n\}$$

Example: What is $A \times B \times C$ where $A = \{0,1\}$, $B = \{1,2\}$ and $C = \{0,1,2\}$

Solution: $A \times B \times C$ **=** $\{(0,1,0), (0,1,1), (0,1,2), (0,2,0), (0,2,1), (0,2,2), (1,1,0), (1,1,1), (1,1,2), (1,2,0), (1,2,1), (1,1,2)\}$

Using set notation with quantifiers

- $\forall x \in S(P(x))$ denotes the universal quantification P(x) over all elements in the set S.
- Shorthand for $\forall x(x \in S \rightarrow (P(x)))$.
- $\exists x \in S(P(x))$ is shorthand for $\exists x(x \in S \wedge P(x))$.
- What do these two statements mean?

$$\forall x \in R(x^2 \geq 0), \exists x \in Z(x^2 = 1)$$

  - The square of every real number is non-negative.
  - There is an integer whose square is 1.

Truth Sets of Quantifiers

- Given a predicate *P* and a domain *D*, we define the ***truth set*** of *P* to be the set of elements in *D* for which *P*(*x*) is true. The truth set of *P*(x) is denoted by

$$\{x \in D | P(x)\}$$

- **Example**: The truth set of *P*(*x*) where the domain is the integers and *P*(*x*) is "$|x| = 1$" is the set {-1,1}