# CSE 15 Homework 9

# Solution (total points = 20)

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**You can also write your answers on papers and scan them into image files for submission.**

**Section 5.1**

8 (2pt)

The proposition to be proved is $P(n)$:

$$2 - 2 \cdot 7 + 2 \cdot 7^2 - \cdots + 2 \cdot (-7)^n = \frac{1 - (-7)^{n+1}}{4}.$$

In order to prove this for all integers $n \geq 0$, we first prove the basis step $P(0)$ and then prove the inductive step, that $P(k)$ implies $P(k+1)$. Now in $P(0)$, the left-hand side has just one term, namely $2$, and the right-hand side is $(1 - (-7)^1)/4 = 8/4 = 2$. Since $2 = 2$, we have verified that $P(0)$ is true. For the inductive step, we *assume* that $P(k)$ is true (i.e., the displayed equation above), and derive from it the truth of $P(k+1)$, which is the equation

$$2 - 2 \cdot 7 + 2 \cdot 7^2 - \cdots + 2 \cdot (-7)^k + 2 \cdot (-7)^{k+1} = \frac{1 - (-7)^{(k+1)+1}}{4}.$$

To prove an equation like this, it is usually best to start with the more complicated side and manipulate it until we arrive at the other side. In this case we start on the left. Note that all but the last term constitute precisely the left-hand side of $P(k)$, and therefore by the inductive hypothesis, we can replace it by the right-hand side of $P(k)$. The rest is algebra:

$$\begin{aligned}
[2 - 2 \cdot 7 + 2 \cdot 7^2 - \cdots + 2 \cdot (-7)^k] + 2 \cdot (-7)^{k+1} &= \frac{1 - (-7)^{k+1}}{4} + 2 \cdot (-7)^{k+1} \\
&= \frac{1 - (-7)^{k+1} + 8 \cdot (-7)^{k+1}}{4} \\
&= \frac{1 + 7 \cdot (-7)^{k+1}}{4} \\
&= \frac{1 - (-7) \cdot (-7)^{k+1}}{4} \\
&= \frac{1 - (-7)^{(k+1)+1}}{4}.
\end{aligned}$$

16 **(2pt)**

The basis step reduces to $6 = 6$. Assuming the inductive hypothesis we have

$$\begin{aligned}
&1 \cdot 2 \cdot 3 + 2 \cdot 3 \cdot 4 + \cdots + k(k+1)(k+2) + (k+1)(k+2)(k+3) \\
&= \frac{k(k+1)(k+2)(k+3)}{4} + (k+1)(k+2)(k+3) \\
&= (k+1)(k+2)(k+3)\left(\frac{k}{4}+1\right) \\
&= \frac{(k+1)(k+2)(k+3)(k+4)}{4}.
\end{aligned}$$

20 **(1pt)**

The basis step is $n = 7$, and indeed $3^7 < 7!$, since $2187 < 5040$. Assume the statement for $k$. Then $3^{k+1} = 3 \cdot 3^k < (k+1) \cdot 3^k < (k+1) \cdot k! = (k+1)!$, the statement for $k+1$.

**Section 5.2**

6 **(1pt each, 3 pt total)**

**a)** We can form the following amounts of postage as indicated: $3 = 3$, $6 = 3+3$, $9 = 3+3+3$, $10 = 10$, $12 = 3+3+3+3$, $13 = 10+3$, $15 = 3+3+3+3+3$, $16 = 10+3+3$, $18 = 3+3+3+3+3+3$, $19 = 10+3+3+3$, $20 = 10+10$. By having considered all the combinations, we know that the gaps in this list cannot be filled. We claim that we can form all amounts of postage greater than or equal to 18 cents using just 3-cent and 10-cent stamps.

**b)** Let $P(n)$ be the statement that we can form $n$ cents of postage using just 3-cent and 10-cent stamps. We want to prove that $P(n)$ is true for all $n \geq 18$. The basis step, $n = 18$, is handled above. Assume that we can form $k$ cents of postage (the inductive hypothesis); we will show how to form $k+1$ cents of postage. If the $k$ cents included two 10-cent stamps, then replace them by seven 3-cent stamps ($7 \cdot 3 = 2 \cdot 10 + 1$). Otherwise, $k$ cents was formed either from just 3-cent stamps, or from one 10-cent stamp and $k-10$ cents in 3-cent stamps. Because $k \geq 18$, there must be at least three 3-cent stamps involved in either case. Replace three 3-cent stamps by one 10-cent stamp, and we have formed $k+1$ cents in postage ($10 = 3 \cdot 3 + 1$).

**c)** $P(n)$ is the same as in part **(b)**. To prove that $P(n)$ is true for all $n \geq 18$, we note for the basis step that from part **(a)**, $P(n)$ is true for $n = 18, 19, 20$. Assume the inductive hypothesis, that $P(j)$ is true for all $j$ with $18 \leq j \leq k$, where $k$ is a fixed integer greater than or equal to 20. We want to show that $P(k+1)$ is true. Because $k-2 \geq 18$, we know that $P(k-2)$ is true, that is, that we can form $k-2$ cents of postage. Put one more 3-cent stamp on the envelope, and we have formed $k+1$ cents of postage, as desired. In this proof our inductive hypothesis included all values between 18 and $k$ inclusive, and that enabled us to jump back three steps to a value for which we knew how to form the desired postage.

12 **(2pt)**

The basis step is to note that $1 = 2^0$. Notice for subsequent steps that $2 = 2^1$, $3 = 2^1 + 2^0$, $4 = 2^2$, $5 = 2^2 + 2^0$, and so on. Indeed this is simply the representation of a number in binary form (base two). Assume the inductive hypothesis, that every positive integer up to $k$ can be written as a sum of distinct powers of $2$. We must show that $k+1$ can be written as a sum of distinct powers of $2$. If $k+1$ is odd, then $k$ is even, so $2^0$ was not part of the sum for $k$. Therefore the sum for $k+1$ is the same as the sum for $k$ with the extra term $2^0$ added. If $k+1$ is even, then $(k+1)/2$ is a positive integer, so by the inductive hypothesis $(k+1)/2$ can be written as a sum of distinct powers of $2$. Increasing each exponent by $1$ doubles the value and gives us the desired sum for $k+1$.

32 **(1pt)**

The proof is invalid for $k = 4$. We cannot increase the postage from $4$ cents to $5$ cents by either of the replacements indicated, because there is no 3-cent stamp present and there is only one 4-cent stamp present. There is also a minor flaw in the inductive step, because the condition that $j \geq 3$ is not mentioned.

## Section 5.3

4 **(1pt each, 2 pt total)**

**c)** $f(2) = f(1)^2 + f(0)^3 = 1^2 + 1^3 = 2$, $f(3) = f(2)^2 + f(1)^3 = 2^2 + 1^3 = 5$, $f(4) = f(3)^2 + f(2)^3 = 5^2 + 2^3 = 33$, $f(5) = f(4)^2 + f(3)^3 = 33^2 + 5^3 = 1214$

**d)** Clearly $f(n) = 1$ for all $n$, since $1/1 = 1$.

8 **(1pt each, 2 pt total)**

**c)** The sequence starts out $2$, $6$, $12$, $20$, $30$, and so on. The differences between successive terms are $4$, $6$, $8$, $10$, and so on. Thus the $n^{\text{th}}$ term is $2n$ greater than the term preceding it; in symbols: $a_n = a_{n-1} + 2n$. Together with the initial condition $a_1 = 2$, this defines the sequence recursively.

**d)** The sequence starts out $1$, $4$, $9$, $16$, $25$, and so on. The differences between successive terms are $3$, $5$, $7$, $9$, and so on—the odd numbers. Thus the $n^{\text{th}}$ term is $2n-1$ greater than the term preceding it; in symbols: $a_n = a_{n-1} + 2n - 1$. Together with the initial condition $a_1 = 1$, this defines the sequence recursively.

14 (Use inductive proof) **(2pt)**

The basis step $(n = 1)$ is clear, since $f_2 f_0 - f_1^2 = 1 \cdot 0 - 1^2 = -1 = (-1)^1$. Assume the inductive hypothesis. Then we have

$$\begin{aligned} f_{n+2}f_n - f_{n+1}^2 &= (f_{n+1} + f_n)f_n - f_{n+1}^2 \\ &= f_{n+1}f_n + f_n^2 - f_{n+1}^2 \\ &= -f_{n+1}(f_{n+1} - f_n) + f_n^2 \\ &= -f_{n+1}f_{n-1} + f_n^2 \\ &= -(f_{n+1}f_{n-1} - f_n^2) \\ &= -(-1)^n = (-1)^{n+1}. \end{aligned}$$

26 **(1pt each, 3 pt total)**

**a)** If we apply each of the recursive step rules to the only element given in the basis step, we see that $(2,3)$ and $(3,2)$ are in $S$. If we apply the recursive step to these we add $(4,6)$, $(5,5)$, and $(6,4)$. The next round gives us $(6,9)$, $(7,8)$, $(8,7)$, and $(9,6)$. A fourth set of applications adds $(8,12)$, $(9,11)$, $(10,10)$, $(11,9)$, and $(12,8)$; and a fifth set of applications adds $(10,15)$, $(11,14)$, $(12,13)$, $(13,12)$, $(14,11)$, and $(15,10)$.

**b)** Let $P(n)$ be the statement that $5 \mid a+b$ whenever $(a,b) \in S$ is obtained by $n$ applications of the recursive step. For the basis step, $P(0)$ is true, since the only element of $S$ obtained with no applications of the recursive step is $(0,0)$, and indeed $5 \mid 0+0$. Assume the strong inductive hypothesis that $5 \mid a+b$ whenever $(a,b) \in S$ is obtained by $k$ or fewer applications of the recursive step, and consider an element obtained with $k+1$ applications of the recursive step. Since the final application of the recursive step to an element $(a,b)$ must be applied to an element obtained with fewer applications of the recursive step, we know that $5 \mid a+b$. So we just need to check that this inequality implies $5 \mid a+2+b+3$ and $5 \mid a+3+b+2$. But this is clear, since each is equivalent to $5 \mid a+b+5$, and $5$ divides both $a+b$ and $5$.

**c)** This holds for the basis step, since $5 \mid 0+0$. If this holds for $(a,b)$, then it also holds for the elements obtained from $(a,b)$ in the recursive step by the same argument as in part **(b)**.