CSE 15
Discrete Mathematics

Lecture 3 – Proposition Logic (3)

Announcement

- HW #1 (reminder)
  - Due **5pm** 9/5 (Wed) with 1 extra day of re-submission.
  - Type your answers in a text file and submit it in Catcourses.
  - Or write your answers on papers and scan them into image files for submission.
  - Work on it during and outside lab hours.

- Reading assignment
  - Ch. 1.6 – 1.8 of textbook

Propositional Equivalences (Ch. 1.3)

- Tautologies, Contradictions, and Contingencies.
- Logical Equivalence
  - Showing Logical Equivalence

Constructing New Logical Equivalences

- We can show that two expressions are logically equivalent by developing a series of logically equivalent statements.
- To prove that $A \equiv B$ we produce a series of equivalences beginning with A and ending with B.

$$\begin{aligned} A &\equiv A_1 \\ &\vdots \\ A_n &\equiv B \end{aligned}$$

- Keep in mind that whenever a proposition (represented by a propositional variable) occurs in the equivalences listed earlier, it may be replaced by an arbitrarily complex compound proposition.

Equivalence Proofs

Example: Show that $\neg(p \vee (\neg p \wedge q))$

is logically equivalent to $\neg p \wedge \neg q$

Solution:

$$
\begin{array}{lcll}
\neg(p \vee (\neg p \wedge q)) & \equiv & \neg p \wedge \neg(\neg p \wedge q) & \text{by the second De Morgan law} \\
 & \equiv & \neg p \wedge [\neg(\neg p) \vee \neg q] & \text{by the first De Morgan law} \\
 & \equiv & \neg p \wedge (p \vee \neg q) & \text{by the double negation law} \\
 & \equiv & (\neg p \wedge p) \vee (\neg p \wedge \neg q) & \text{by the second distributive law} \\
 & \equiv & F \vee (\neg p \wedge \neg q) & \text{because } \neg p \wedge p \equiv F \\
 & \equiv & (\neg p \wedge \neg q) \vee F & \text{by the commutative law} \\
 & & & \text{for disjunction} \\
 & \equiv & (\neg p \wedge \neg q) & \text{by the identity law for } \mathbf{F}
\end{array}
$$

Equivalence Proofs

Example: Show that

is a tautology. $(p \wedge q) \rightarrow (p \vee q)$

Solution:

$$
\begin{aligned}
(p \wedge q) \rightarrow (p \vee q) \quad &\equiv \quad \neg(p \wedge q) \vee (p \vee q) && \text{by truth table for } \rightarrow \\
&\equiv \quad (\neg p \vee \neg q) \vee (p \vee q) && \text{by the first De Morgan law} \\
&\equiv \quad (\neg p \vee p) \vee (\neg q \vee q) && \text{by associative and} \\
& && \text{commutative laws} \\
& && \text{laws for disjunction} \\
&\equiv \quad T \vee T && \text{by truth tables} \\
&\equiv \quad T && \text{by the domination law}
\end{aligned}
$$

Predicates & Quantifiers (Ch. 1.4)

- Predicates
- Variables
- Quantifiers
  - Universal Quantifier
  - Existential Quantifier
- Negating Quantifiers
  - De Morgan’s Laws for Quantifiers
- Translating English to Logic

Propositional Logic Not Enough

- If we have:

  “All men are mortal.”

  “Socrates is a man.”
- Does it follow that “Socrates is mortal?”
- Can’t be represented in propositional logic. Need a language that talks about objects, their properties, and their relations.
- Later we’ll see how to draw inferences.

Limitations of proposition logic

- Proposition logic cannot adequately express the meaning of statements
- Suppose we know

  "*<u>Every</u>* computer connected to the university network is functioning properly"
- No rules of propositional logic allow us to conclude

  "SE100 is functioning properly"

  where SE100 is one of the computers connected to the university network

Introducing Predicate Logic

- Predicate logic uses the following new features:
  - Variables: x, y, z
  - Predicates
  - Quantifiers (*to be covered in a few slides*)
- *Propositional functions* are a generalization of propositions.
  - They contain variables and a predicate, e.g., $P(x)$
  - Variables can be replaced by elements from their *domain*.

Propositional Functions

- Propositional functions become propositions (and have truth values) when their variables are each replaced by a value from the *domain* (or *bound* by a quantifier, as we will see later).
- The statement *P(x)* is said to be the value of the propositional function *P* at *x*.
- For example, let *P(x)* denote "$x > 0$" and the domain be the integers. Then:

  P(-3) is false.

  P(0) is false.

  P(3) is true.
- Often the domain is denoted by *U*. So in this example *U* is the integers.

Examples of Propositional Functions

- Let "$x + y = z$" be denoted by $R(x, y, z)$ and U (for all three variables) be the integers. Find these truth values:

  R(2,-1,5)

  Solution: F

  R(3,4,7)

  Solution: T

  R(x, 3, z)

  Solution: Not a Proposition

- Now let "$x - y = z$" be denoted by $Q(x, y, z)$, with U as the integers. Find these truth values:

  Q(2,-1,3)

  Solution: T

  Q(3,4,7)

  Solution: F

  Q(x, 3, z)

  Solution: Not a Proposition

Compound Expressions

- Connectives from propositional logic carry over to predicate logic.
- If *P(x)* denotes "$x > 0$," find these truth values:

  $P(3) \vee P(-1)$ **Solution**: T

  $P(3) \wedge P(-1)$ **Solution**: F

  $P(3) \rightarrow P(-1)$ **Solution**: F

  $P(-1) \rightarrow P(3)$ **Solution**: T
- Expressions with variables are not propositions and therefore do not have truth values. For example,

  $P(3) \wedge P(y)$

  $P(x) \rightarrow P(y)$
- When used with quantifiers (to be introduced next), these expressions (propositional functions) become propositions.

Quantifiers

Charles Peirce (1839-1914)

- We need *quantifiers* to express the meaning of English words including *all* and *some*:
  - "All men are Mortal."
  - "Some cats do not have fur."
- The two most important quantifiers are:
  - *Universal Quantifier,* **"**For all," **symbol:** $\forall$
  - *Existential Quantifier*, "There exists," **symbol:** $\exists$
- We write as in $\forall x\, P(x)$ and $\exists x\, P(x)$.
- $\forall x\, P(x)$ asserts $P(x)$ is true for <u>every</u> x in the *domain*.
- $\exists x\, P(x)$ asserts $P(x)$ is true for <u>some</u> x in the *domain*.
- The quantifiers are said to *bind* the variable x in these expressions.

Universal Quantifier

- $\forall x\, P(x)$ is read as "For all x, P(x)" or "For every x, P(x)"

Examples:

1) If $P(x)$ denotes "$x > 0$" and U is the integers, then $\forall x\, P(x)$ is false.
2) If $P(x)$ denotes "$x > 0$" and U is the positive integers, then $\forall x\, P(x)$ is true.
3) If $P(x)$ denotes "x is even" and U is the integers, then $\forall\, x\, P(x)$ is false.

Existential Quantifier

- $\exists x\, P(x)$ is read as "For some x, $P(x)$", or as "There is an x such that $P(x)$," or "For at least one x, $P(x)$."

Examples:

1. If $P(x)$ denotes "$x > 0$" and U is the integers, then $\exists x\, P(x)$ is true. It is also true if U is the positive integers.
2. If $P(x)$ denotes "$x < 0$" and U is the positive integers, then $\exists x\, P(x)$ is false.
3. If $P(x)$ denotes "x is even" and U is the integers, then $\exists x\, P(x)$ is true.

Thinking about Quantifiers

- When the domain of discourse is finite, we can think of quantification as looping through the elements of the domain.
- To evaluate $\forall x\, P(x)$, loop through all x in the domain.
  - If at every step $P(x)$ is true, then $\forall x\, P(x)$ is true.
  - If at a step $P(x)$ is false, then $\forall x\, P(x)$ is false and the loop terminates.
- To evaluate $\exists x\, P(x)$, loop through all x in the domain.
  - If at some step, $P(x)$ is true, then $\exists x\, P(x)$ is true and the loop terminates.
  - If the loop ends without finding an x for which $P(x)$ is true, then $\exists x\, P(x)$ is false.
- Even if the domains are infinite, we can still think of the quantifiers this fashion, but the loops will not terminate in some cases.

Thinking about Quantifiers as Conjunctions and Disjunctions

- If the domain is finite, a universally quantified proposition is equivalent to a **conjunction** of propositions without quantifiers and an existentially quantified proposition is equivalent to a **disjunction** of propositions without quantifiers.
- If *U* consists of the integers 1,2, and 3:

$$\forall x P(x) \equiv P(1) \wedge P(2) \wedge P(3)$$

$$\exists x P(x) \equiv P(1) \vee P(2) \vee P(3)$$

- Even if the domains are infinite, you can still think of the quantifiers in this fashion, but the equivalent expressions without quantifiers will be infinitely long.

Properties of Quantifiers

- The truth value of $\exists x\, P(x)$ and $\forall x\, P(x)$ depend on both the propositional function ***P(x)*** and on the domain ***U***.
- **Examples**:
  1. If U is the positive integers and $P(x)$ is the statement "$x < 2$", then $\exists x\, P(x)$ is true, but $\forall x\, P(x)$ is false.
  2. If U is the negative integers and $P(x)$ is the statement "$x < 2$", then both $\exists x\, P(x)$ and $\forall x\, P(x)$ are true.
  3. If U consists of 3, 4, and 5, and $P(x)$ is the statement "$x > 2$", then both $\exists x\, P(x)$ and $\forall x\, P(x)$ are true. But if $P(x)$ is the statement "$x < 2$", then both $\exists x\, P(x)$ and $\forall x$ $P(x)$ are false.

Precedence of Quantifiers

- The quantifiers $\forall$ and $\exists$ have higher precedence than all the logical operators.
- For example, $\forall x\, P(x) \vee Q(x)$ means $(\forall x\, P(x)) \vee Q(x)$
- $\forall x\, (P(x) \vee Q(x))$ means something different.
- Unfortunately, often people write $\forall x\, P(x) \vee Q(x)$ when they mean $\forall x\, (P(x) \vee Q(x))$.

Returning to the Socrates Example

- Introduce the propositional functions *Man(x)* denoting "*x* is a man" and *Mortal(x)* denoting "*x* is mortal." Specify the domain as all people.
- The two premises are:

$$\forall x Man(x) \rightarrow Mortal(x)$$

$$Man(Socrates)$$

- The conclusion is:

$$Mortal(Socrates)$$

- Later we will show how to prove that the conclusion follows from the premises.

Equivalences in Predicate Logic

- Statements involving predicates and quantifiers are *logically equivalent* if and only if they have the same truth value
  - for every predicate substituted into these statements and
  - for every domain of discourse used for the variables in the expressions.
- The notation $S \equiv T$ indicates that S and T are logically equivalent.
- **Example**: $\forall x \neg\neg S(x) \equiv \forall x\, S(x)$

Negating Quantified Expressions

- Consider $\forall x\, J(x)$

  "Every student in your class has taken a course in Java."

  Here $J(x)$ is "x has taken a course in java" and

  the domain is students in your class.

- Negating the original statement gives "It is not the case that every student in your class has taken Java." This implies that "There is a student in your class who has not taken java."

  Symbolically $\neg \forall x\, J(x)$ and $\exists x\, \neg J(x)$ are equivalent

…Negating Quantified Expressions

- Now Consider $\exists x\, J(x)$

  "There is a student in this class who has taken a course in Java."

  Where $J(x)$ is "x has taken a course in Java."
- Negating the original statement gives "It is not the case that there is a student in this class who has taken Java." This implies that "Every student in this class has not taken Java"

  Symbolically $\neg\exists x\, J(x)$ and $\forall x\, \neg J(x)$ are equivalent

De Morgan's Laws for Quantifiers

- The rules for negating quantifiers are:

TABLE 2 **De Morgan's Laws for Quantifiers.**

Negation	*Equivalent Statement*	*When Is Negation True?*	*When False?*
$\neg\exists x\, P(x)$	$\forall x \neg P(x)$	For every x, $P(x)$ is false.	There is an x for which $P(x)$ is true.
$\neg\forall x\, P(x)$	$\exists x \neg P(x)$	There is an x for which $P(x)$ is false.	$P(x)$ is true for every x.

- The reasoning in the table shows that:

$$\neg\forall x P(x) \equiv \exists x \neg P(x)$$

$$\neg\exists x P(x) \equiv \forall x \neg P(x)$$

- These are important. You will be using these a lot!

Translating from English to Logic

Example 1: Translate the following sentence into predicate logic: "Every student in this class has taken a course in Java."

Solution:

First decide on the domain *U*.

Solution 1: If *U* is all students in this class, define a propositional function J(x) denoting "x has taken a course in Java" and translate as $\forall x\, J(x)$.

Solution 2: But if *U* is all people, also define a propositional function S(x) denoting "x is a student in this class" and translate as $\forall x\, (S(x) \rightarrow J(x))$.

...Translating from English to Logic

Example 2: Translate the following sentence into predicate logic: “Some student in this class has taken a course in Java.”

Solution:

First decide on the domain *U*.

Solution 1: If *U* is all students in this class, translate as

$$\exists x\, J(x)$$

Solution 1: But if *U* is all people, then translate as

$$\exists x\, (S(x) \land J(x))$$

$\exists x\, (S(x) \rightarrow J(x))$ is not correct. What does it mean?

Translation from English to Logic

More examples:

1. "Some student in this class has visited Mexico."

   Solution: Let $M(x)$ denote "x has visited Mexico" and $S(x)$ denote "x is a student in this class," and U be all people.

   $$\exists x \ (S(x) \land M(x))$$

2. "Every student in this class has visited Canada or Mexico."

   Solution: Add $C(x)$ denoting "x has visited Canada."

   $$\forall x \ (S(x) \rightarrow (M(x) \lor C(x)))$$