CSE 15
Discrete Mathematics

Lecture 9 – Functions (2)

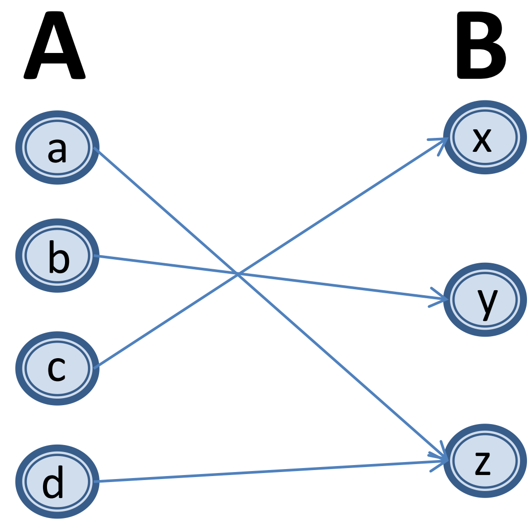
Announcements

- HW #4 out
  - Due **5pm** 10/3 (Wed) with 1 extra day of re-submission.
- Midterm #1 on 10/9 (Tuesday)
- Reading assignment
  - Ch. 2.4-2.6 of textbook

Surjections

Definition: A function *f* from *A* to *B* is called *onto* or *surjective*, if and only if for every element $b \in B$ there is an element $a \in A$ with $f(a) = b$. A function *f* is called a *surjection* if it is onto.

$\forall y \exists x(f(x) = y)$, where x is in the domain and y is the codomain

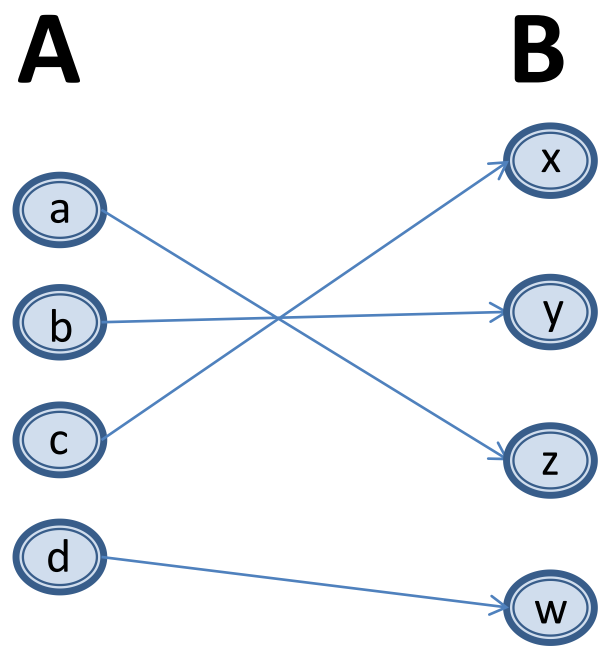


Example

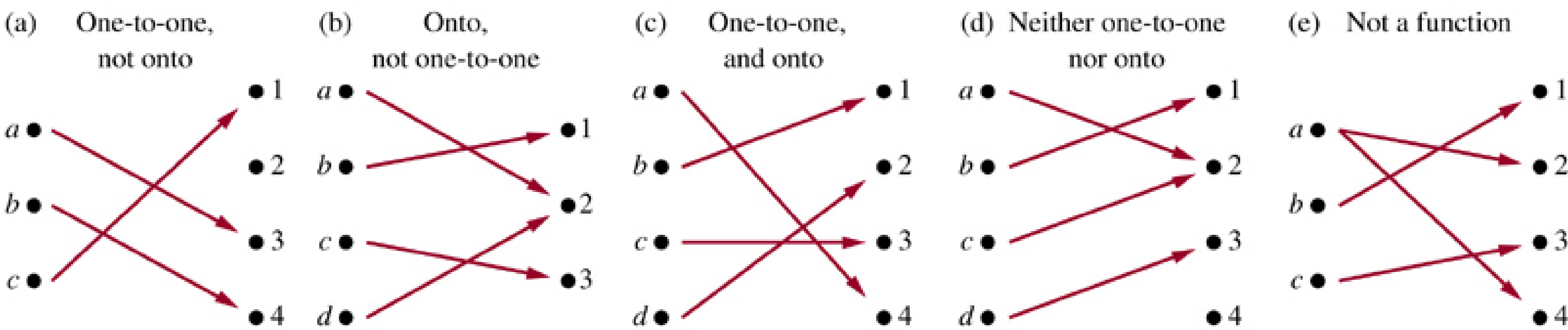
- Is $f(x)=x^2$ from the set of integers to the set of integers onto?
  - No: For what value of x do we have $f(x)=-1$?
- Is $f(x)=x+1$ from the set of integers to the set of integers onto?
  - It is onto, as for each integer y there is an integer x such that $f(x)=y$.
  - To see this, $f(x)=y$ iff $x+1=y$, which holds if and only if $x=y-1$.

Bijections

Definition: A function f is a *one-to-one correspondence*, or a *bijection*, if it is both one-to-one and onto (surjective and injective).



Examples


(a) One-to-one, not onto
(b) Onto, not one-to-one
(c) One-to-one, and onto
(d) Neither one-to-one nor onto
(e) Not a function
a
b
c
d
1
2
3
4

Identity function

$$\iota_A : A \to A, \quad \iota_A(x) = x, \forall x \in A$$

- It is one-to-one and onto.

Showing that f is one-to-one or onto

Suppose that $f : A \to B$.

To show that f is injective Show that if $f(x) = f(y)$ for arbitrary $x, y \in A$ with $x \neq y$, then $x = y$.

To show that f is not injective Find particular elements $x, y \in A$ such that $x \neq y$ and $f(x) = f(y)$.

To show that f is surjective Consider an arbitrary element $y \in B$ and find an element $x \in A$ such that $f(x) = y$.

To show that f is not surjective Find a particular $y \in B$ such that $f(x) \neq y$ for all $x \in A$.

Showing that f is one-to-one or onto

Example 1: Let f be the function from $\{a,b,c,d\}$ to $\{1,2,3\}$ defined by $f(a) = 3$, $f(b) = 2$, $f(c) = 1$, and $f(d) = 3$. Is f an onto function?

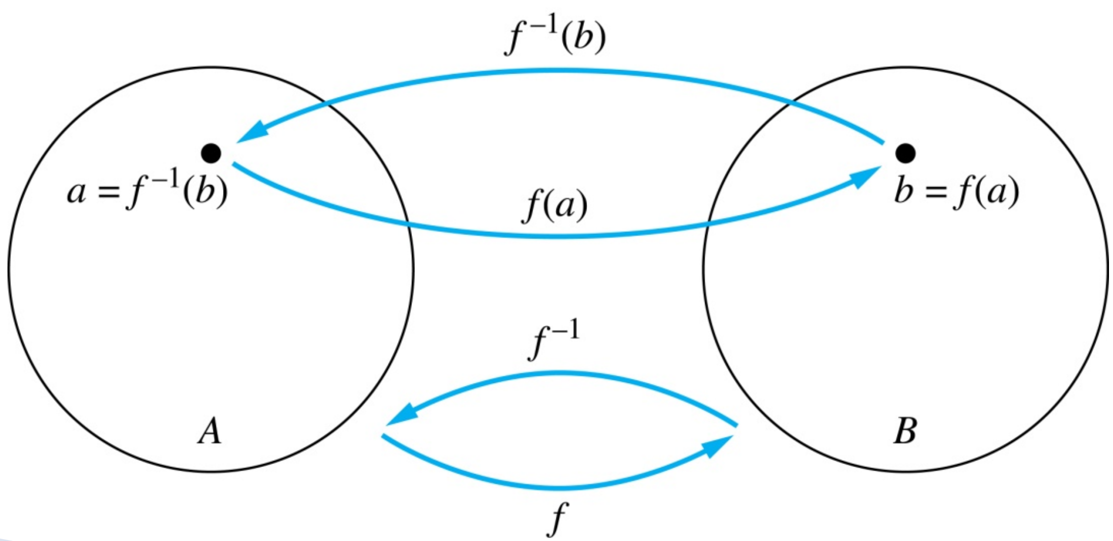
Solution: Yes, f is onto since all three elements of the codomain are images of elements in the domain.

If the codomain were changed to $\{1,2,3,4\}$, f would not be onto.

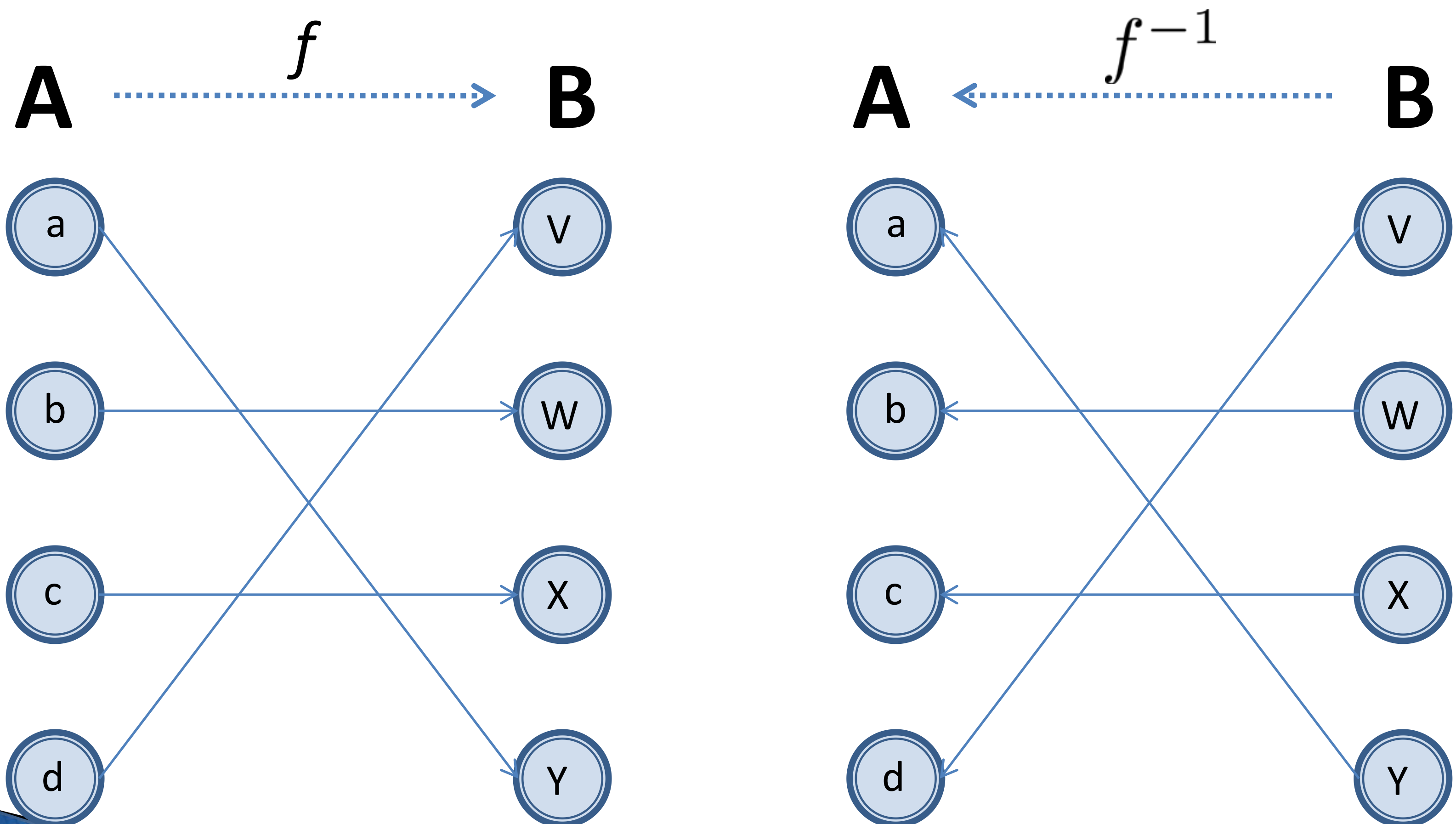
Inverse Functions

Definition: Let *f* be a bijection from *A* to *B*. Then the *inverse* of *f*, denoted f^{-1}, is the function from *B* to *A* defined as $f^{-1}(y) = x \text{ iff } f(x) = y$

No inverse exists unless *f* is a bijection. Why?



Inverse Functions



Questions

Example 1: Let f be the function from $\{a,b,c\}$ to $\{1,2,3\}$ such that $f(a) = 2$, $f(b) = 3$, and $f(c) = 1$. Is f invertible and if so what is its inverse?

Solution: The function f is invertible because it is a one-to-one correspondence. The inverse function f^{-1} reverses the correspondence given by f, so $f^{-1}(1) = c$, $f^{-1}(2) = a$, and $f^{-1}(3) = b$.

Questions

Example 2: Let f: **Z** $\rightarrow$ **Z** be such that $f(x) = x + 1$. Is f invertible, and if so, what is its inverse?

Solution: The function f is invertible because it is a one-to-one correspondence. The inverse function f^{-1} reverses the correspondence so $f^{-1}(y) = y - 1$.

Questions

Example 3: Let *f:* **R** $\rightarrow$ **R** be such that $f(x) = x^2$. Is *f* invertible, and if so, what is its inverse?

Solution: The function *f* is not invertible because it is not one-to-one .

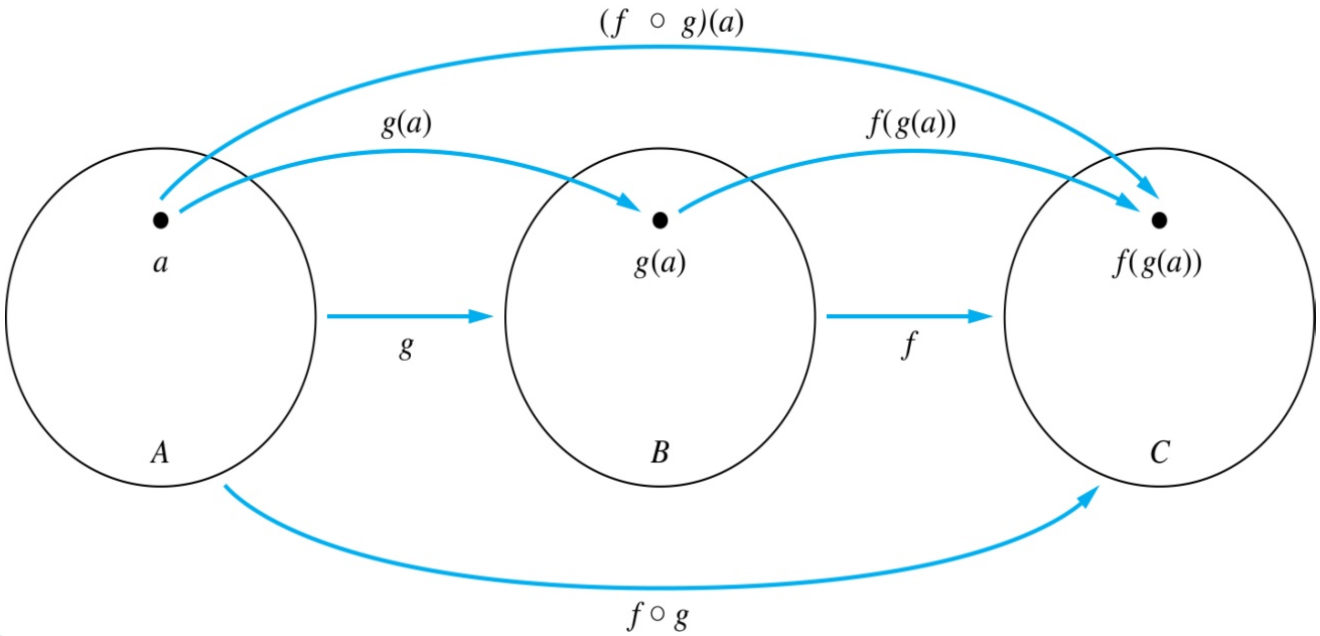
Example

- Sometimes we restrict the domain or the codomain of a function or both, to have an invertible function.
- The function $f(x)=x^2$, from R^+ to R^+ is
  - one-to-one : If $f(x)=f(y)$, then $x^2=y^2$, then $x+y=0$ or $x-y=0$, so $x=-y$ or $x=y$.
  - onto: $y= x^2$, every non-negative real number has a square root.
  - inverse function:

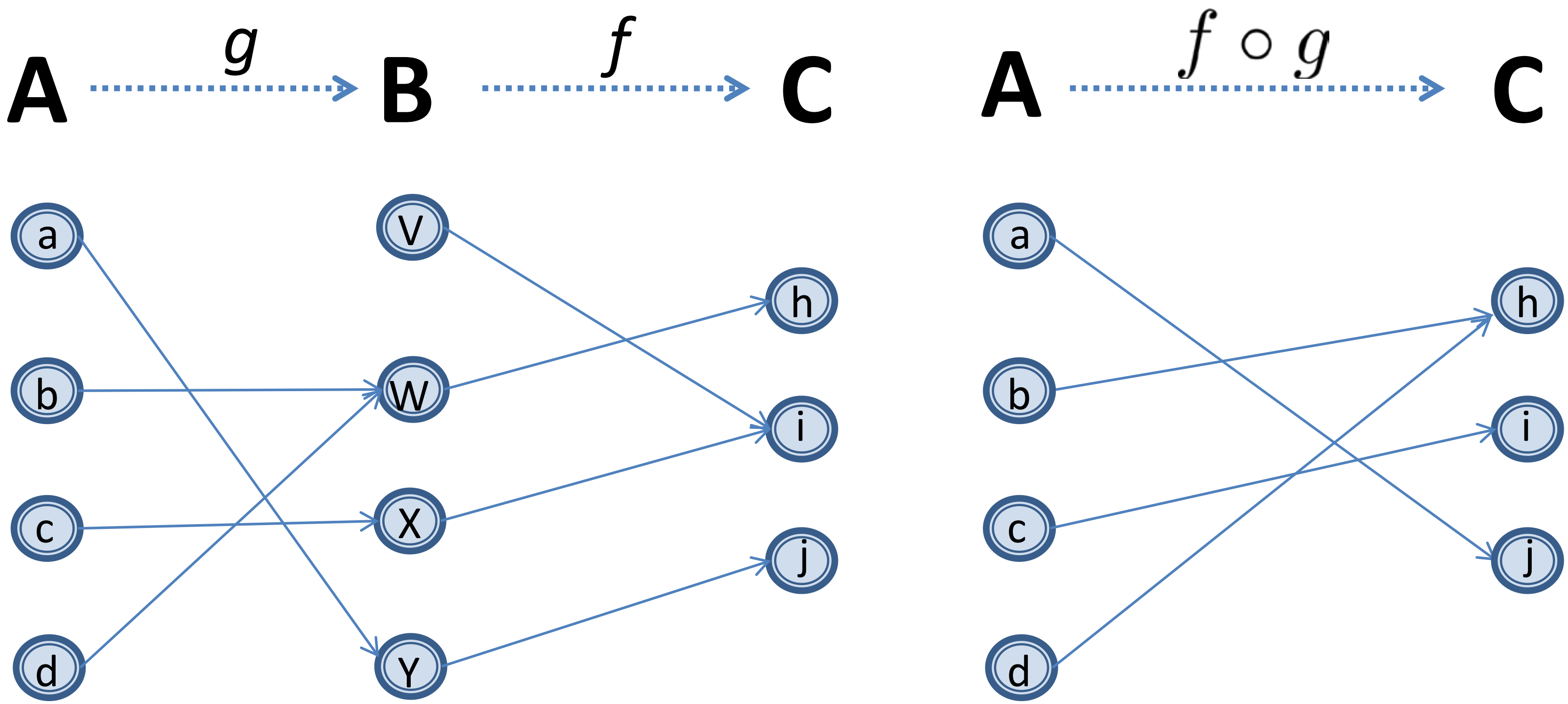
$$f^{-1}(y) = \sqrt{y}$$

Composition

- **Definition**: Let *f*: *B* → *C*, *g*: *A* → *B*. The *composition of f with g*, denoted $f \circ g$ is the function from *A* to *C* defined by $f \circ g(x) = f(g(x))$



Composition



Composition

Example 1: If $f(x) = x^2$ and $g(x) = 2x + 1$,
then $f(g(x)) = (2x + 1)^2$

and $g(f(x)) = 2x^2 + 1$

Composition Questions

Example 2: Let g be the function from the set $\{a,b,c\}$ to itself such that $g(a) = b$, $g(b) = c$, and $g(c) = a$. Let f be the function from the set $\{a,b,c\}$ to the set $\{1,2,3\}$ such that $f(a) = 3$, $f(b) = 2$, and $f(c) = 1$.

What is the composition of f and g, and what is the composition of g and f.

Solution: The composition $f \circ g$ is defined by

$$f \circ g\ (a) = f(g(a)) = f(b) = 2.$$
$$f \circ g\ (b) = f(g(b)) = f(c) = 1.$$
$$f \circ g\ (c) = f(g(c)) = f(a) = 3.$$

Note that $g \circ f$ is not defined, because the range of f is not a subset of the domain of g.

Composition Questions

Example 2: Let f and g be functions from the set of integers to the set of integers defined by $f(x) = 2x + 3$ and $g(x) = 3x + 2$.

What is the composition of *f* and *g*, and also the composition of *g* and *f* ?

Solution:

$$f \circ g\ (x) = f(g(x)) = f(3x + 2) = 2(3x + 2) + 3 = 6x + 7$$

$$g \circ f\ (x) = g(f(x)) = g(2x + 3) = 3(2x + 3) + 2 = 6x + 11$$

- Note that f ◦ g and g ◦ f are defined in this example, but they are not equal.
- The commutative law does not hold for composition of functions.

f and f^{-1}

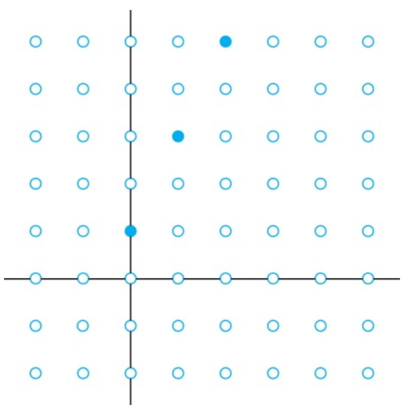
- f and f^{-1} form an identity function in any order.
- Let f: A →B with f(a)=b.
- Suppose f is one-to-one correspondence from A to B.
- Then f^{-1} is one-to-one correspondence from B to A.
- The inverse function reverses the correspondence of f, so $f^{-1}(b)=a$ when f(a)=b, and f(a)=b when $f^{-1}(b)=a$.
- $(f^{-1} \circ f)(a)=f^{-1}(f(a))=f^{-1}(b)=a$, and
- $(f \circ f^{-1})(b)=f(f^{-1})(b))=f(a)=b$.

$f^{-1} \circ f = \iota_A,\ f \circ f^{-1} = \iota_B;\ \iota_A, \iota_B$ are identity functions for A and B

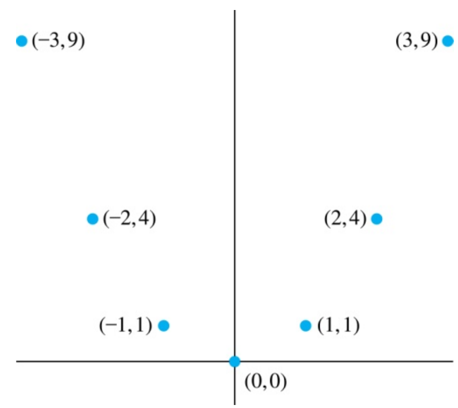
$(f^{-1})^{-1} = f$

Graphs of Functions

- Let f be a function from the set A to the set B. The *graph* of the function f is the set of ordered pairs $\{(a,b) \mid a \in A \text{ and } f(a) = b\}$.



Graph of $f(n) = 2n + 1$ from Z to Z



Graph of $f(x) = x^2$ from Z to Z

Some Important Functions

- The *floor* function, denoted
  $$f(x) = \lfloor x \rfloor$$
  is the largest integer less than or equal to *x*.
- The *ceiling* function, denoted
  $$f(x) = \lceil x \rceil$$
  is the smallest integer greater than or equal to *x*.

Examples: $\lceil 3.5 \rceil = 4$ $\lfloor 3.5 \rfloor = 3$

$\lceil -1.5 \rceil = -1$ $\lfloor -1.5 \rfloor = -2$

Floor and Ceiling Functions

TABLE 1 Useful Properties of the Floor and Ceiling Functions.
(n is an integer, x is a real number)

(1a)	$\lfloor x \rfloor = n$ if and only if $n \le x < n + 1$
(1b)	$\lceil x \rceil = n$ if and only if $n - 1 < x \le n$
(1c)	$\lfloor x \rfloor = n$ if and only if $x - 1 < n \le x$
(1d)	$\lceil x \rceil = n$ if and only if $x \le n < x + 1$
(2)	$x - 1 < \lfloor x \rfloor \le x \le \lceil x \rceil < x + 1$
(3a)	$\lfloor -x \rfloor = -\lceil x \rceil$
(3b)	$\lceil -x \rceil = -\lfloor x \rfloor$
(4a)	$\lfloor x + n \rfloor = \lfloor x \rfloor + n$
(4b)	$\lceil x + n \rceil = \lceil x \rceil + n$

Proving Properties of Functions

Example: Prove that if x is a real number, then

$$\lfloor 2x \rfloor = \lfloor x \rfloor + \lfloor x + 1/2 \rfloor$$

Solution: Let $x = n + \varepsilon$, where n is an integer and $0 \leq \varepsilon < 1$.

Case 1: $\varepsilon < ½$

- $2x = 2n + 2\varepsilon$ and $\lfloor 2x \rfloor = 2n$, since $0 \leq 2\varepsilon < 1$.
- $\lfloor x + 1/2 \rfloor = n$, since $x + ½ = n + (1/2 + \varepsilon)$ and $0 \leq ½ + \varepsilon < 1$.
- Hence, $\lfloor 2x \rfloor = 2n$ and $\lfloor x \rfloor + \lfloor x + 1/2 \rfloor = n + n = 2n$.

Case 2: $\varepsilon \geq ½$

- $2x = 2n + 2\varepsilon = (2n + 1) + (2\varepsilon - 1)$ and $\lfloor 2x \rfloor = 2n + 1$, since $0 \leq 2\varepsilon - 1 < 1$.
- $\lfloor x + 1/2 \rfloor = \lfloor n + (1/2 + \varepsilon) \rfloor = \lfloor n + 1 + (\varepsilon - 1/2) \rfloor = n + 1$ since $0 \leq \varepsilon - 1/2 < 1$.
- Hence, $\lfloor 2x \rfloor = 2n + 1$ and $\lfloor x \rfloor + \lfloor x + 1/2 \rfloor = n + (n + 1) = 2n + 1$.