

# Computational Decision Making for Regular People

## *01: Introduction*

October 15, 2024

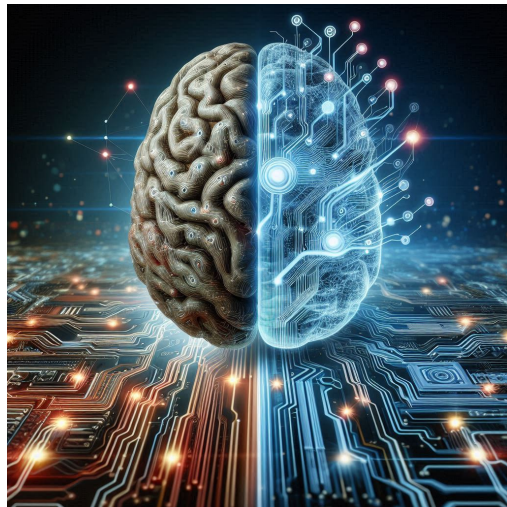
Nathan Davis Barrett



# Today's Outline



1. Mathematical Modeling
2. Optimization
3. What kinds of things can be optimized?
4. General form of an optimization problem
5. A note on optimization theory
6. How to formulate an optimization problem
  - 6.1 The objective function
  - 6.2 Decision variables
  - 6.3 Parameters
  - 6.4 Constraints
7. Basic Examples



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# Mathematical Modeling



## How we think about a problem

- ▶ Consider what a good outcome looks like
  - ▶ Consider different ramifications of different decisions
  - ▶ utilitarian ethics, virtue ethics, deontological ethics, ...
- ▶ Discrete decisions (yes/no, 1,2,3, etc.)
- ▶ Continuous decisions
- ▶ Constraints against undesirable or infeasible decisions

## How a computer thinks about a problem

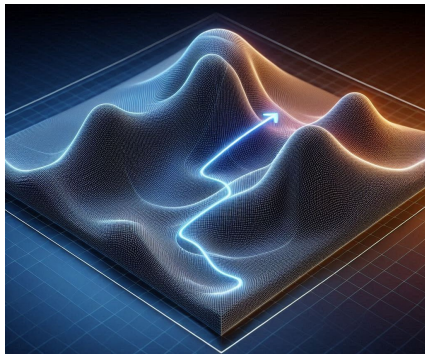
- ▶ Quantify the quality of a solution
  - ▶ Combine all ramifications into one quantity
  - ▶ utilitarian objective only\*
- ▶ Discrete decisions (yes/no, 1,2,3, etc.)
- ▶ Continuous decisions
- ▶ Constraints against undesirable or infeasible decisions

# Optimization



*the action of making the best or most effective use of a situation or resource.*<sup>1</sup>

- ▶ Given a set of possible decisions, determine the best one(s)
- ▶ The way this determination is made depends on the nature of the set of decisions
  - ▶ If a human is executing this determination, the procedure will be unique to that person
  - ▶ If a computer algorithm is executing this determination, the procedure will be unique to that computer algorithm



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<sup>1</sup>Oxford Languages

# What Kinds of Things Can Be Optimized?



- ▶ What kinds of things can be optimized (using a computer)?
- ▶ If you can represent it using mathematical modeling, it (theoretically) can be optimized.
- ▶ Some problems are still too difficult for even modern computer algorithms to solve
  - ▶ Problems with lots and lots of variables
  - ▶ Problems with lots and lots of constraints
  - ▶ Problems that "don't behave well"
    - ▶ Lots of really good outcomes that lie very close to really bad outcomes
    - ▶ Lots of outcomes that are equally good
    - ▶ Good outcomes that are separated by bad outcomes
    - ▶ Really nuanced constraints
    - ▶ etc.

# General Form of an Optimization Problem



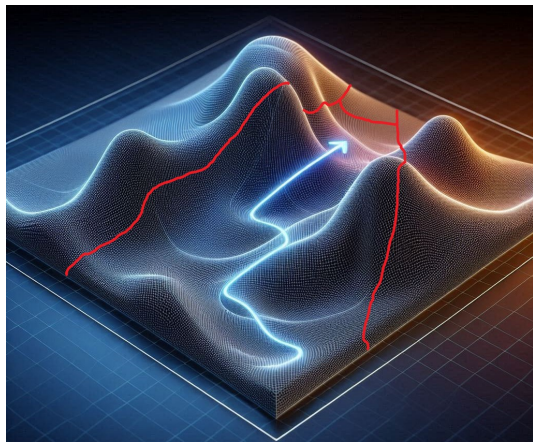
$$\min_{\bar{X}} f(\bar{X})$$

— — — *subject to (s.t.)* — — —

$$\bar{X} \in \mathbf{S}$$

► We must define:

- $\bar{X}$
- $f(\bar{X})$
- $\mathbf{S}$



# Optimization Theory



- ▶ This is a very active field of research
- ▶ It gets quite complicated and "mathy" very quickly
- ▶ I'll only emphasize two points:

$$\min_{\bar{X}} f(\bar{X})$$

$$s.t. \bar{X} \in \mathbf{S}$$

**If we can keep  $f(\bar{X})$  and  $\mathbf{S}$  as linear as possible, the computer algorithms are MUCH faster.**

**If we can't keep things linear, "Bilinear" is the next best thing.**

- ▶ Linear:  $\alpha X + \beta Y + \gamma Z$
- ▶ Bilinear:  $\alpha XY$  or  $\alpha X^2$
- ▶ Non-linear:
  - ▶  $\alpha \sqrt{X}$
  - ▶  $\alpha \log(X)$
  - ▶ etc.

# Formulating Optimization Problems



Key Elements:

In order how I would conceptualize them:

1. Objective Function ( $f(\bar{X})$ )
2. Decision Variables ( $\bar{X}$ )
3. Parameters ( $\bar{\alpha}$ )
4. Constraints (**S**)

In order how I would write / code them:

1. Sets
2. Parameters ( $\bar{\alpha}$ )
3. Decision Variables ( $\bar{X}$ )
4. Constraints (**S**)
5. Objective Function ( $f(\bar{X})$ )



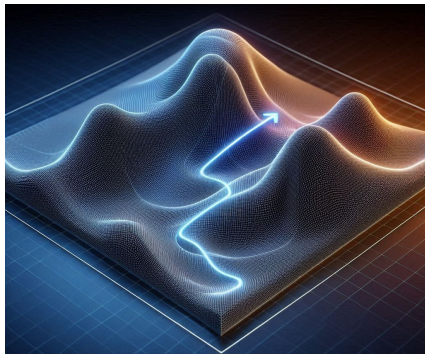
# Objective Function ( $f(\bar{X})$ )



- ▶ What am I trying to accomplish?
- ▶ What am I trying to minimize (or maximize)?
- ▶ How do different decisions change the outcome?

Examples:

- ▶ Minimize cost
- ▶ Minimize risk
- ▶ Maximize probability of reaching a goal
- ▶ Maximize comfort



# Decision Variables ( $\bar{X}$ )



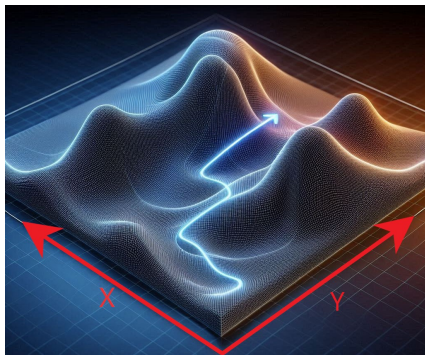
- ▶ What decisions can I make?
- ▶ What values change when I make different decisions?

Important Note:

- ▶ Be open-minded here: often the best way to formulate a problem is to consider each individual part of a problem as it's own variable.
- ▶ Don't just consider the main decisions, consider the smaller decisions that contribute to (or even strictly define) the main decisions

Examples:

- ▶ How much of a certain item to buy
- ▶ How much of a budget category to allot for that item



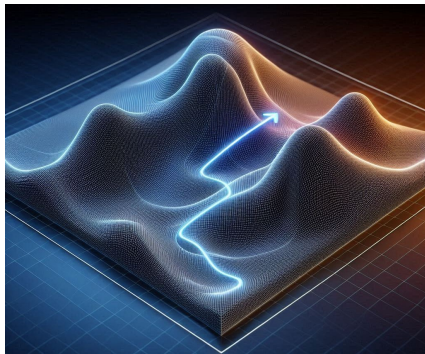
# Parameters ( $\bar{\alpha}$ )



- ▶ What parts of the problem are fixed or immovable?
- ▶ While the Objective function defines the general shape of the mountain, the parameters define the height of the peaks, the steepness of the slopes, etc.

Examples:

- ▶ Cost of a certain item
- ▶ Total amount of time available
- ▶ The minimum probability of reaching a goal that we are willing to accept



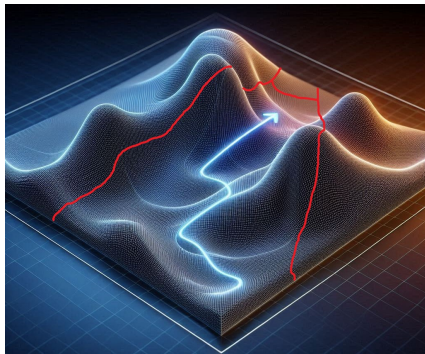
# Constraints (S)



- ▶ What sets of decisions are incompatible?
- ▶ What is the nature of an individual decision? (Binary, Continuous, etc.)
- ▶ How do different decision variables relate to each other?

Examples:

- ▶ We cannot spend more in a budget category than the total allotment for that category.
- ▶ The amount of money spent in a budget category is strictly equal to the sum of transactions that lies in that category.
- ▶ The calculated probability of reaching a goal must be greater than the minimum value we specified.





- ▶ Often there are several variables, parameters, constraints, etc. that are repeated for a given set of values.
- ▶ We can write the general idea behind that variable, parameter, constraint, etc. by simply writing it once and indicating that it should be repeated for every element in a set.
- ▶ In math, it looks like this:

$$\forall e \in \mathbf{E}$$

"for all elements  $e$  in the set  $\mathbf{E}$ "

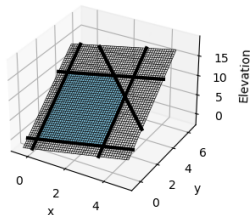
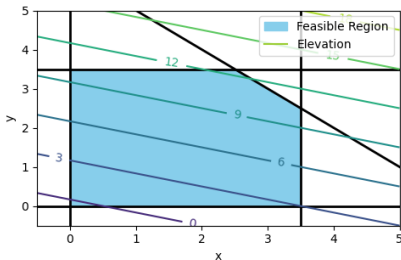
- ▶ Some variables, parameters, constraints, etc. can be defined over multiple sets:

$$\forall c \in \mathbf{C}, t \in \mathbf{T}$$

Examples:

- ▶ A set of all time periods
  - ▶ An individual variable must be defined for each time period
- ▶ A set of all budget categories
  - ▶ A spending limit constraint must be specified for each budget category
- ▶ A set of all tasks that need to be done
  - ▶ A parameter specifying the time it takes to perform each task must be specified for each task
- ▶ Variables belonging to multiple sets:
  - ▶ An budget allotment variable must be defined for each budget category  $c \in \mathbf{C}$  for each time period  $t \in \mathbf{T}$ .
  - ▶ We'll call this variable " $A_{c,t}$ "

# A Visual Example (Linear)



$$\max_{X,Y} X + 3Y - 0.5$$

$$s.t. \quad X \geq 0$$

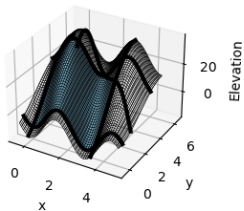
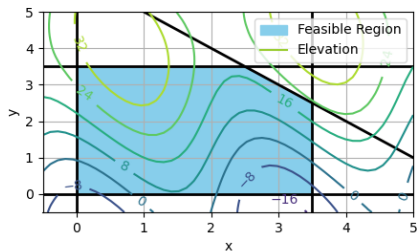
$$Y \geq 0$$

$$X \leq 3.5$$

$$Y \leq 3.5$$

$$X + Y \leq 6$$

# A Visual Example (Nonlinear)



$$\max_{X,Y} f(X, Y)$$

$$s.t. \quad X \geq 0$$

$$Y \geq 0$$

$$X \leq 3.5$$

$$Y \leq 3.5$$

$$X + Y \leq 6$$

# Example: Classic Economics Problem



*Jack and Jill find themselves on an abandoned island and need food and shelter. Jack can either build 5 shingles per hour or can catch 2 fish per hour. Jill can either build 9 shingles per hour or can catch 3 fish per hour. They only need 25 shingles but would welcome as much fish as they can get. They each have 8 hours to work in a day. What's the best way they can spend their time to maximize the amount of food and shelter (equally prioritized)?*

Take a minute to think about the problem on your own:

1. Objective Function ( $f(\bar{X})$ )
2. Decision Variables ( $\bar{X}$ )
3. Parameters ( $\bar{\alpha}$ )
4. Constraints (**S**)



# Example: Classic Economics Problem



## First conceptualize the problem:

- ▶ Objective: Maximize the number of products produced according to some priority scheme
- ▶ Variables:
  - ▶ How many hours each person spends on producing each product
  - ▶ The amount of each product that gets made
- ▶ Parameters:
  - ▶ How much product can be made by each person per hour
  - ▶ How much time there is in a day
  - ▶ The number of shingles they need
  - ▶ The priority between each product
- ▶ Constraints:
  - ▶ Each person can only work between 0 and 8 hours in a day
  - ▶ The amount of each product produced should be directly determined by how many hours each person spends making that product
  - ▶ The number of shingles should be limited to 25
- ▶ Sets:
  - ▶ A set of all people involved
  - ▶ A set of all products involved

# Example: Classic Economics Problem



Then write the problem using mathematical modeling:

## 1. Define Sets:

- ▶  $p \in \mathbf{P} = \{Jack, Jill\}$ : A set of all people involved
- ▶  $r \in \mathbf{R} = \{Shingles, Fish\}$ : A set of all products involved

## 2. Define Parameters:

- ▶  $\alpha_{p,r}$ : How much product can be made by each person per hour

$\alpha_{p,r}$	Jack	Jill
Shingles	5	9
Fish	2	3

- ▶  $\tau^{DAY} = 8$ : How much time there is to work in a day
- ▶  $\kappa = 25$ : The number of shingles they need

## 2. Parameters (cont.):

- ▶  $\rho_r = 1$ : The priority between each product

## 3. Define Decision Variables:

- ▶  $H_{p,r} \forall p \in \mathbf{P}, r \in \mathbf{R}$ : The number of hours each person should spend making each product in a day
- ▶  $N_r \forall r \in \mathbf{R}$ : The amount of each product that gets made

## 4. Define Constraints:

- ▶ Each person can only work between 0 and 8 hours in a day

$$0 \leq \sum_{r \in \mathbf{R}} H_{p,r} \leq \tau^{DAY} \quad \forall p \in \mathbf{P}$$

# Example: Classic Economics Problem



## 4. Constraints (cont.):

- ▶ The amount of each product produced should be directly determined by how many hours each person spends making that product

$$N_r = \sum_{p \in \mathbf{P}} \alpha_{p,r} H_{p,r} \quad \forall r \in \mathbf{R}$$

- ▶ The number of shingles should be limited to 25

$$N_{Shingles} = \kappa$$

## 5. Define Objective:

- ▶ Maximize the number of products produced according to some priority scheme

$$\max \sum_{r \in \mathbf{R}} \rho_r N_r$$

— — — FULL FORMULATION — — —

$$\max_{H_{p,r}, N_r} \sum_{r \in \mathbf{R}} \rho_r N_r$$

$$\text{s.t.} \quad 0 \leq \sum_{r \in \mathbf{R}} H_{p,r} \leq \tau^{DAY} \quad \forall p \in \mathbf{P}$$

$$N_r = \sum_{p \in \mathbf{P}} \alpha_{p,r} H_{p,r} \quad \forall r \in \mathbf{R}$$

$$N_{Shingles} = \kappa$$

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\* Notice how the objective and all constraints are linear

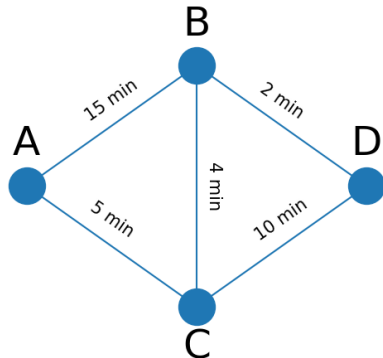
# Example: Route Planning Problem



*Given the roads indicated in the graph below, what's the fastest way to get from point A to point D?*

Take a minute to think about the problem on your own:

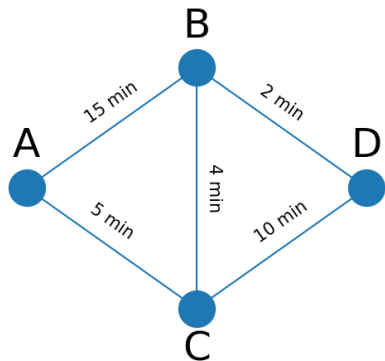
1. Objective Function ( $f(\bar{X})$ )
2. Decision Variables ( $\bar{X}$ )
3. Parameters ( $\bar{\alpha}$ )
4. Constraints (**S**)



# Example: Route Planning Problem



First conceptualize the problem:



- ▶ Objective: Minimize the time spent traveling
- ▶ Variables:
  - ▶ Whether or not to travel down each stretch of road
  - ▶ Whether or not to visit each point
- ▶ Parameters:
  - ▶ How long each stretch of road is
  - ▶ Which roads connect to which points
- ▶ Constraints:
  - ▶ If I travel into a point, I must travel out of it.
  - ▶ I have to visit point A and point D

# Example: Route Planning Problem



Then write the problem using mathematical modeling:

## 1. Define Sets:

- ▶  $p \in \mathbf{P} = \{A, B, C, D\}$ : A set of all points
- ▶  $p \in \mathbf{P}^{NON-TERM} = \{B, C\}$ : A set of all points other than the terminal (starting and ending) points
- ▶  $p \in \mathbf{P}^{TERM} = \{A, D\}$ : A set of all the terminal points
- ▶  $r \in \mathbf{R} = \{AB, AC, BC, BD, CD\}$ : A set of all roads
- ▶  $r \in \mathbf{R}_p$ : A set of all roads that touch point  $p$

$p$	$\mathbf{R}_p$
$A$	$\{AB, AC\}$
$B$	$\{AB, BC, BD\}$
$C$	$\{AC, BC, CD\}$
$D$	$\{BD, CD\}$

## 2. Define Parameters:

- ▶  $\delta_r$ : The time it takes to travel road  $r$

$r$	$\delta_r(\text{minutes})$
$AB$	15
$AC$	5
$BC$	4
$BD$	2
$CD$	10

## 3. Define Decision Variables:

- ▶  $X_r \quad \forall r \in \mathbf{R}$ : Whether or not to travel down road  $r$  (**Binary Variable**)
- ▶  $Y_p \quad \forall p \in \mathbf{P}$ : Whether or not to visit point  $p$  (**Binary Variable**)

# Example: Route Planning Problem



## 4. Define Constraints:

- ▶ If I travel into a point, I must travel out of it

$$\sum_{r \in \mathbf{R}_p} X_r = 2Y_p \quad \forall p \in \mathbf{P}^{NON-TERM}$$

$$\sum_{r \in \mathbf{R}_p} X_r = Y_p \quad \forall p \in \mathbf{P}^{TERM}$$

- ▶ I have to visit point A and point D

$$Y_p = 1 \quad \forall p \in \mathbf{P}^{TERM}$$

## 5. Define Objective:

- ▶ Minimize the amount of time spent traveling

$$\min_{X_r, Y_p} \sum_{r \in \mathbf{R}} \delta_r X_r$$

— — — FULL FORMULATION — — —

$$\min_{X_r, Y_p} \sum_{r \in \mathbf{R}} \delta_r X_r$$

$$\text{s.t.} \quad \sum_{r \in \mathbf{R}_p} X_r = 2Y_p \quad \forall p \in \mathbf{P}^{NON-TERM}$$

$$\sum_{r \in \mathbf{R}_p} X_r = Y_p \quad \forall p \in \mathbf{P}^{TERM}$$

$$Y_p = 1 \quad \forall p \in \mathbf{P}^{TERM}$$

$$X_r, Y_p \in \{0, 1\}$$

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\* Notice how the objective and all constraints are linear

# Next Class



- ▶ Visit the course website:



QR Code to Course Website

- ▶ These slides are posted in the Lecture 01 folder.

- ▶ Python / Optimization coding crash course
- ▶ Go to the Lecture 02 folder on the course website
  - ▶ Follow the instructions on "slides.pdf" in the Lecture 02 folder.
  - ▶ The more you can do on your own before class the more time we'll have to answer questions and do example next class.
- ▶ Please bring your laptops to class!