Computational Decision Making for Regular People

01: Introduction

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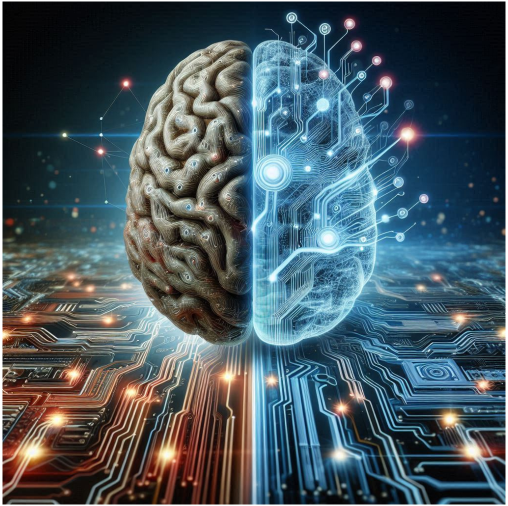
Today's Outline

1. Mathematical Modeling
2. Optimization
3. What kinds of things can be optimized?
4. General form of an optimization problem
5. A note on optimization theory
6. How to formulate an optimization problem
   6.1 The objective function
   6.2 Decision variables
   6.3 Parameters
   6.4 Constraints
7. Basic Examples



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Mathematical Modeling

How we think about a problem	How a computer thinks about a problem

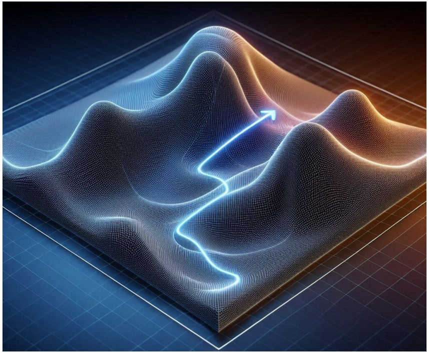
▶ Consider what a good outcome looks like
  ▶ Consider different ramifications of different decisions
  ▶ utilitarian ethics, virtue ethics, deontological ethics, ...
	▶ Quantify the quality of a solution
	  ▶ Combine all ramifications into one quantity
	  ▶ utilitarian objective only*

▶ Discrete decisions (yes/no, 1,2,3, etc.) ▶ Continuous decisions ▶ Constraints against undesirable or infeasible decisions	▶ Discrete decisions (yes/no, 1,2,3, etc.) ▶ Continuous decisions ▶ Constraints against undesirable or infeasible decisions

Optimization

the action of making the best or most effective use of a situation or resource.[1]

- Given a set of possible decisions, determine the best one(s)
- The way this determination is made depends on the nature of the set of decisions
  - If a human is executing this determination, the procedure will be unique to that person
  - If a computer algorithm is executing this determination, the procedure will be unique to that computer algorithm



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[1]Oxford Languages

What Kinds of Things Can Be Optimized?

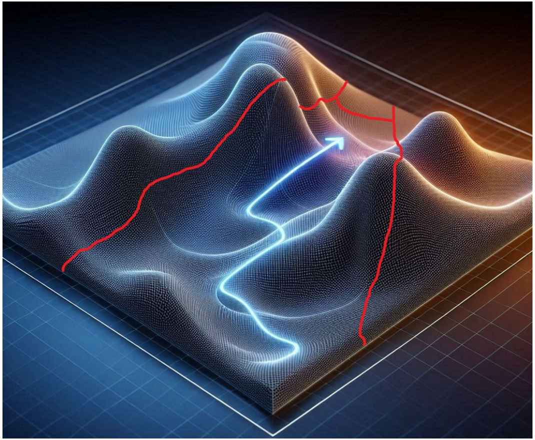
- What kinds of things can be optimized (using a computer)?
- If you can represent it using mathematical modeling, it (theoretically) can be optimized.
- Some problems are still too difficult for even modern computer algorithms to solve
  - Problems with lots and lots of variables
  - Problems with lots and lots of constraints
  - Problems that "don't behave well"
    - Lots of really good outcomes that lie very close to really bad outcomes
    - Lots of outcomes that are equally good
    - Good outcomes that are separated by bad outcomes
    - Really nuanced constraints
    - etc.

General Form of an Optimization Problem

$$\min f(\overline{X})$$

$$---\ \textit{subject to } (s.t.)\ ---$$

$$\overline{X} \in \mathbf{S}$$



- We must define:
  - $\overline{X}$
  - $f(\overline{X})$
  - $\mathbf{S}$

Optimization Theory

- This is a very active field of research
- It gets quite complicated and "mathy" very quickly
- I'll only emphasize two points:

$$\min f(\overline{X})$$

$$s.t.\ \overline{X} \in \mathbf{S}$$

If we can keep $f(\overline{X})$ and S as linear as possible, the computer algorithms are MUCH faster.

If we can't keep things linear, "Bilinear" is the next best thing.

- Linear: $\alpha X + \beta Y + \gamma Z$
- Bilinear: αXY or αX^2
- Non-linear:
  - $\alpha\sqrt{X}$
  - $\alpha \log(X)$
  - etc.

Formulating Optimization Problems

Key Elements:

In order how I would conceptualize them:

1. Objective Function ($f(\overline{X})$)
2. Decision Variables ($\overline{X}$)
3. Parameters ($\overline{\alpha}$)
4. Constraints ($\mathbf{S}$)

In order how I would write / code them:

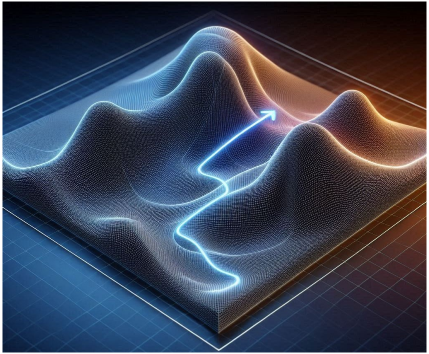
1. Parameters ($\overline{\alpha}$)
2. Decision Variables ($\overline{X}$)
3. Constraints ($\mathbf{S}$)
4. Objective Function ($f(\overline{X})$)

Objective Function ($f(\overline{X})$)

- What am I trying to accomplish?
- What am I trying to minimize (or maximize)?
- How do different decisions change the outcome?

Examples:

- Minimize cost
- Minimize risk
- Maximize probability of reaching a goal
- Maximize comfort



Decision Variables ($\overline{X}$)

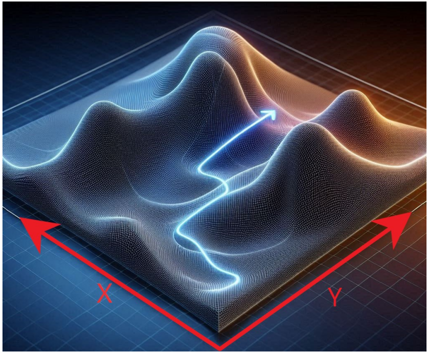
- What decisions can I make?
- What values change when I make different decisions?

Important Note:

- Be open-minded here: often the best way to formulate a problem is to consider each individual part of a problem as it's own variable.
- Don't just consider the main decisions, consider the smaller decisions that contribute to (or even strictly define) the main decisions

Examples:

- How much of a certain item to buy
- How much of a budget category to allot for that item

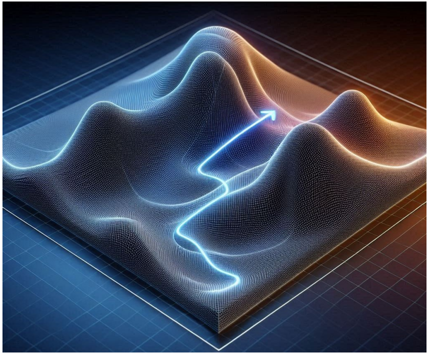


Parameters ($\overline{\alpha}$)

- What parts of the problem are fixed or immovable?
- While the Objective function defines the general shape of the mountain, the parameters define the height of the peaks, the steepness of the slopes, etc.

Examples:

- Cost of a certain item
- Total amount of time available
- The minimum probability of reaching a goal that we are willing to accept

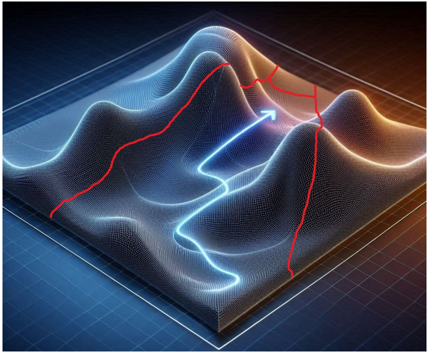


Constraints (S)

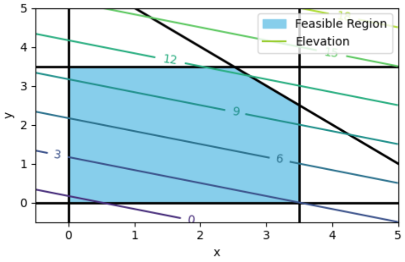
- What sets of decisions are incompatible?
- What is the nature of an individual decision? (Binary, Continuous, etc.)
- How do different decision variables relate to each other?

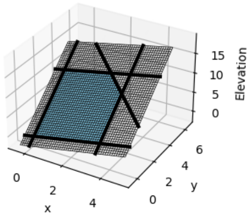
Examples:

- We cannot spend more in a budget category than the total allotment for that category.
- The amount of money spent in a budget category is strictly equal to the sum of transactions that lies in that category.
- The calculated probability of reaching a goal must be greater than the minimum value we specified.



A Visual Example (Linear)





$$\max X + 3Y - 0.5$$

$$s.t. \quad X \geq 0$$

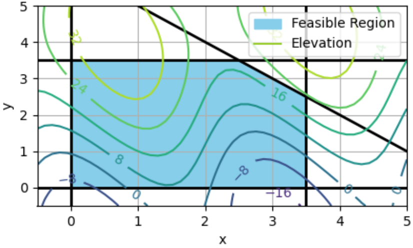
$$Y \geq 0$$

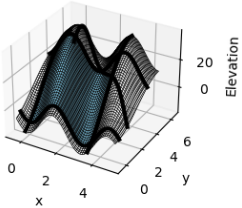
$$X \leq 3.5$$

$$Y \leq 3.5$$

$$X + Y \leq 6$$

A Visual Example (Nonlinear)





$$\max f(X, Y)$$

$$s.t. \quad X \geq 0$$

$$Y \geq 0$$

$$X \leq 3.5$$

$$Y \leq 3.5$$

$$X + Y \leq 6$$

Example: Etsy Shop Problem

Jack and Jill own an Etsy shop that sells t-shirts and wooden sculptures. Jack can either make 5 t-shirts per hour or he can make 2 sculptures per hour. Jill can either make 9 t-shirts per hour or she can make 3 sculptures per hour. Their most recent order indicates that they only need 25 t-shirts but would welcome as many sculptures as they can create. They each how 8 hours to work today. How should they spend their time today so that they can maximize the total number of t-shirts and sculptures they make?

Take a minute to think about the problem on your own:

1. Objective Function ($f(\overline{X})$)
2. Decision Variables ($\overline{X}$)
3. Parameters ($\overline{\alpha}$)
4. Constraints (**S**)

Example: Etsy Shop Problem

First conceptualize the problem:

- Objective: Maximize the total number of products produced
- Variables:
  - How many hours each person spends on producing each product
  - The amount of each product that gets made
- Parameters:
  - How much product can be made by each person per hour
  - How must time there is in a day
  - The number of t-shirts they need
  - The priority between each product
- Constraints:
  - Each person can only work between 0 and 8 hours in a day
  - The amount of each product produced should be directly determined by how many hours each person spends making that product
  - The number of t-shirts should be limited to 25

Example: Etsy Shop Problem

Let's write the problem using mathematical modeling:

1. Define Parameters:
   - How much product can be made by each person per hour
     - $\alpha_{Jack,Sculptures} = 2$
     - $\alpha_{Jack,Shirts} = 5$
     - $\alpha_{Jill,Sculptures} = 3$
     - $\alpha_{Jill,Shirts} = 9$
   - $\tau^{DAY} = 8$: How much time there is to work in a day
   - $\kappa = 25$: The number of t-shirts they need

2. Define Decision Variables:
   - The number of hours each person should spend making each product in a day
     - $H_{Jack,Sculptures}$
     - $H_{Jack,Shirts}$
     - $H_{Jill,Sculptures}$
     - $H_{Jill,Shirts}$
   - The amount of each product that gets made
     - $N_{Sculptures}$
     - N_{Shirts}

Example: Etsy Shop Problem

3. Constraints:
   - Each person can only work between 0 and 8 hours in a day:

$$0 \leq H_{Jack,Sculptures} + H_{Jack,Shirts} \leq \tau^{DAY}$$

$$0 \leq H_{Jill,Sculptures} + H_{Jill,Shirts} \leq \tau^{DAY}$$

   - The amount of each product produced should be directly determined by how many hours each person spends making that product:

$$N_{Sculptures} = \alpha_{Jack,Sculptures} H_{Jack,Sculptures} + \alpha_{Jill,Sculptures} H_{Jack,Sculptures}$$

$$N_{Shirts} = \alpha_{Jack,Shirts} H_{Jack,Shirts} + \alpha_{Jill,Shirts} H_{Jack,Shirts}$$

   - The number of shirts should be limited to 25:

$$N_{Shirts} = \kappa$$

4. Define Objective:
   - Maximize the total number of products produced

$$\max N_{Shirts} + N_{Sculptures}$$

Example: Etsy Shop Problem

FULL FORMULATION

$$\max N_{Shirts} + N_{Sculptures}$$

$$- - - \ \textit{subject to } (s.t.) \ - - -$$

$$0 \leq H_{Jack,Sculptures} + H_{Jack,Shirts} \leq \tau^{DAY}$$

$$0 \leq H_{Jill,Sculptures} + H_{Jill,Shirts} \leq \tau^{DAY}$$

$$N_{Shirts} = \alpha_{Jack,Shirts} H_{Jack,Shirts} + \alpha_{Jill,Shirts} H_{Jack,Shirts}$$

$$\begin{aligned} N_{Sculptures} = \ & \alpha_{Jack,Sculptures} H_{Jack,Sculptures} \\ & + \alpha_{Jill,Sculptures} H_{Jack,Sculptures} \end{aligned}$$

$$N_{Shirts} = \kappa$$

- Notice how much repetition there is.
- How could I change this problem if I had 1000 people and 75 products?

Sets

- Often there are several variables, parameters, constraints, etc. that are repeated for a given set of values.
- We can write the general idea behind that variable, parameter, constraint, etc. by simply writing it once and indicating that it should be repeated for every element in a set.
- In math, it looks like this:

$$\forall e \in \mathbf{E}$$

"for all elements e in the set $\mathbf{E}$"

$$0 \leq H_{Jack,Sculptures} + H_{Jack,Shirts} \leq \tau^{DAY}$$

$$0 \leq H_{Jill,Sculptures} + H_{Jill,Shirts} \leq \tau^{DAY}$$

$$\downarrow$$

$$0 \leq H_{p,Sculptures} + H_{p,Shirts} \leq \tau^{DAY} \quad \forall p \in \mathbf{P}$$

$$\downarrow$$

$$0 \leq \sum_{r \in \mathbf{R}} H_{p,r} \leq \tau^{DAY} \quad \forall p \in \mathbf{P}$$

$p \in \mathbf{P}$: A set of all People

$r \in \mathbf{R}$: A set of all Products

Sets

$$\alpha_{Jack,Sculptures} = 2$$

$$\alpha_{Jack,Shirts} = 5$$

$$\alpha_{Jill,Sculptures} = 3$$

$$\alpha_{Jill,Shirts} = 9$$

$$\downarrow$$

$$\alpha_{p,r} \quad \forall p \in \mathbf{P}, r \in \mathbf{R}$$

$\alpha_{p,r}$	*Jack*	*Jill*
Shirts	5	9
Sculptures	2	3

$$N_{Shirts} = \alpha_{Jack,Shirts} H_{Jack,Shirts} + \alpha_{Jill,Shirts} H_{Jack,Shirts}$$

$$\begin{aligned} N_{Sculptures} = {} & \alpha_{Jack,Sculptures} H_{Jack,Sculptures} \\ & + \alpha_{Jill,Sculptures} H_{Jack,Sculptures} \end{aligned}$$

$$\downarrow$$

$$N_r = \alpha_{Jack,r} H_{Jack,r} + \alpha_{Jill,r} H_{Jack,r} \quad \forall r \in \mathbf{R}$$

$$\downarrow$$

$$N_r = \sum_{p \in \mathbf{P}} \alpha_{p,r} H_{p,r} \quad \forall r \in \mathbf{R}$$

Example: Etsy Shop Problem with Sets

$$\max N_{Shirts} + N_{Sculptures}$$
$$---\ subject\ to\ (s.t.)\ ---$$
$$0 \leq H_{Jack,Sculptures} + H_{Jack,Shirts} \leq \tau^{DAY}$$
$$0 \leq H_{Jill,Sculptures} + H_{Jill,Shirts} \leq \tau^{DAY}$$
$$N_{Shirts} = \alpha_{Jack,Shirts} H_{Jack,Shirts} + \alpha_{Jill,Shirts} H_{Jack,Shirts}$$
$$N_{Sculptures} = \alpha_{Jack,Sculptures} H_{Jack,Sculptures} + \alpha_{Jill,Sculptures} H_{Jack,Sculptures}$$
$$N_{Shirts} = \kappa$$

$$\max \sum_{r \in \mathbf{R}} N_r$$
$$---\ subject\ to\ (s.t.)\ ---$$
$$0 \leq \sum_{r \in \mathbf{R}} H_{p,r} \leq \tau^{DAY} \quad \forall p \in \mathbf{P}$$
$$N_r = \sum_{p \in \mathbf{P}} \alpha_{p,r} H_{p,r} \quad \forall r \in \mathbf{R}$$
$$N_{Shirts} = \kappa$$

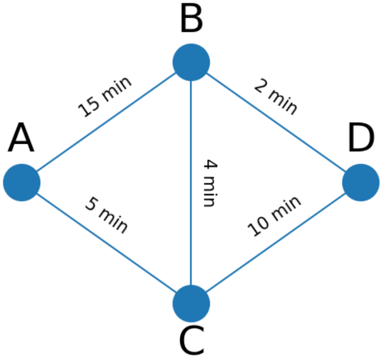
This one can handle any number of people or products.

Example: Route Planning Problem

Given the roads indicated in the graph below, what's the fastest way to get from point A to point D?

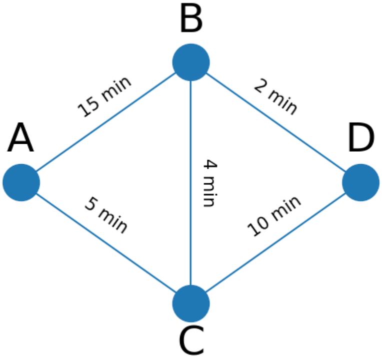
Take a minute to think about the problem on your own:

1. Objective Function ($f(\overline{X})$)
2. Decision Variables ($\overline{X}$)
3. Parameters ($\overline{\alpha}$)
4. Constraints (**S**)



Example: Route Planning Problem

First conceptualize the problem:



- Objective: Minimize the time spent traveling
- Variables:
  - Whether or not to travel down each stretch of road
  - Whether or not to visit each point
- Parameters:
  - How long each stretch of road is
  - Which roads connect to which points
- Constraints:
  - If I travel into a point, I must travel out of it.
  - I have to visit point A and point D

Example: Route Planning Problem

Then write the problem using mathematical modeling:

1. Define Sets:
   - $p \in \mathbf{P} = \{A, B, C, D\}$: A set of all points
   - $p \in \mathbf{P}^{NON-TERM} = \{B, C\}$: A set of all points other than the terminal (starting and ending) points
   - $p \in \mathbf{P}^{TERM} = \{A, D\}$: A set of all the terminal points
   - $r \in \mathbf{R} = \{AB, AC, BC, BD, CD\}$: A set of all roads
   - $r \in \mathbf{R}_p$: A set of all roads that touch point p

p	$\mathbf{R}_p$
A	$\{AB, AC\}$
B	$\{AB, BC, BD\}$
C	$\{AC, BC, CD\}$
D	$\{BD, CD\}$

2. Define Parameters:
   - δ_r: The time it takes to travel road r

r	$\delta_r(minutes)$
AB	15
AC	5
BC	4
BD	2
CD	10

3. Define Decision Variables:
   - $X_r \quad \forall r \in \mathbf{R}$: Whether or not to travel down road r (**Binary Variable**)
   - $Y_p \quad \forall p \in \mathbf{P}$: Whether or not to visit point p (**Binary Variable**)

Example: Route Planning Problem

4. Define Constraints:
   - If I travel into a point, I must travel out of it

$$\sum_{r \in \mathbf{R}_p} X_r = 2Y_p \quad \forall p \in \mathbf{P}^{NON-TERM}$$

$$\sum_{r \in \mathbf{R}_p} X_r = Y_p \quad \forall p \in \mathbf{P}^{TERM}$$

   - I have to visit point A and point D

$$Y_p = 1 \quad \forall p \in \mathbf{P}^{TERM}$$

5. Define Objective:
   - Minimize the amount of time spent traveling

$$\min_{X_r, Y_p} \sum_{r \in \mathbf{R}} \delta_r X_r$$

– – – FULL FORMULATION – – –

$$\min_{X_r, Y_p} \sum_{r \in \mathbf{R}} \delta_r X_r$$

$$s.t. \quad \sum_{r \in \mathbf{R}_p} X_r = 2Y_p \quad \forall p \in \mathbf{P}^{NON-TERM}$$

$$\sum_{r \in \mathbf{R}_p} X_r = Y_p \quad \forall p \in \mathbf{P}^{TERM}$$

$$Y_p = 1 \quad \forall p \in \mathbf{P}^{TERM}$$

$$X_r, Y_p \in \{0, 1\}$$

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* Notice how the objective and all constraints are linear

Next Class

- Visit the course website:

QR Code to Course Website

- These slides are posted in the Lecture 01 folder.

- Python / Optimization coding crash course
- Go the the Lecture 02 folder on the course website
  - Follow the instructions on "slides.pdf" in the Lecture 02 folder.
  - The more you can do on your own before class the more time we'll have to answer questions and do example next class.
- Please bring your laptops to class!