Computational Decision Making for Regular People

04: Modeling Techniques

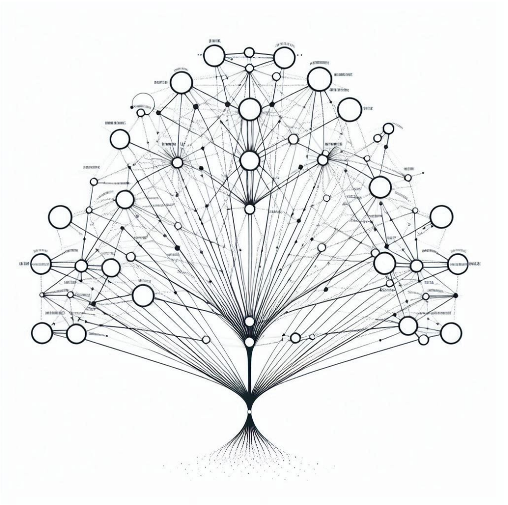
November 5, 2024

Nathan Davis Barrett

Today's Outline

1. Modeling Relationships Between Variables
   - Assignment
   - Knapsack
   - Binary Activation
   - "If"
   - "Max" / "Min"
2. Modeling Real-Life Behaviors
   - Handling Multiple Sets At Once
   - Handling Multiple Time Periods
   - Handling Uncertainty

DISCLAIMER: This is not financial advice. These slides are for educational use only.



This image was created with the assistance of DALL·E 3

A Note on Today's Class

Today we're going to cover a lot of material to set us up for next week when we have to tools to address real-life problems.

We'll cover a variety of individual "tools" that are somewhat disconnected from each other.

With that in mind, I don't expect you to be 100% confident with everything we talk about today. These slides are here to serve as your guide for implementing certain relationships. Go back and reference them often.

My goal today is mostly to **just show you what's out there**. Then, when you need a certain relationship, you can simply go look at the slide to see how to implement it.

Relationships Between Variables

- For the first half of today's lesson, we'll just be focusing on how to model relationships between variables.
- With that in mind, we won't formulate an entire problem, **just individual parts** of a problem.
- We'll have examples throughout for each relationship.
- Use these slides as a reference:

I have a certain relationship I want to model, how can I do it?
Review these slides!

The Assignment (Sub-)Problem

EXAMPLE: I have 5 family members and 10 chores that need to be done. I want to model the relationship that each chore has to be assigned to one person.

- Sets:
  - $p \in \mathbf{P}$: A set of all people.
  - $c \in \mathbf{C}$: A set of all chores.
- Variables:
  - $X_{p,c}$: In indication that chore c as assigned to person p ($X_{p,c} = 1$) or not ($X_{p,c} = 0$). (Binary Variable)
- Constraints:
  - Each chore must be assigned to exactly one person

$$\sum_{p \in \mathbf{P}} X_{p,c} = 1 \quad \forall c \in \mathbf{C}$$

The Assignment (Sub-)Problem

- It doesn't have to be chores that are assigned to people.
- It could be items that are assigned to bins.
- It could be students that are assigned to groups.
- It could be tasks that are assigned to time periods.
- ...

For any **A** *that are assigned to* **B**, *insert the following into your model.*

- Variable:
  - $X_{a,b}$: In indication that element a as assigned to element b ($X_{a,b} = 1$) or not ($X_{a,b} = 0$). (Binary Variable)
- Constraint:
  - Each a must be assigned to exactly one b

$$\sum_{b \in \mathbf{B}} X_{a,b} = 1 \quad \forall a \in \mathbf{A}$$

The Knapsack (Sub-)Problem

EXAMPLE: I have 30 items that could potentially fit in my knapsack. Each item has a fixed size. My knapsack also has a fixed size. I want to model that I can't over-fill my knapsack.

- Sets:
  - $i \in \mathbf{I}$: A set of all items.
- Parameters:
  - κ_i: The size of item i.
  - $\kappa^{KNAPSACK}$: The maximum size of my knapsack
- Variables:
  - X_i: In indication that item i is selected to fit in my knapsack ($X_i = 1$) or not ($X_i = 0$) (Binary Variable)
- Constraints:
  - I can't overfill my knapsack

$$\sum_{i \in \mathbf{I}} \kappa_i X_i \leq \kappa^{KNAPSACK}$$

The Knapsack (Sub-)Problem

- It doesn't have to be items that are fit into a knapsack.
- It could be students that are fit into a group (e.g. with a group size limit).
- It could be tasks that are fit into a time period (e.g. with a time limit).
- It could be major purchases that fit into a bank account (e.g. with a withdrawal limit).
- ...

For any $\underline{\mathbf{A}}$ *(with known size kappa*$_a$*) that has to fit into a grouping of known size* κ^{TOTAL} *insert the following into your model.*

- Variable:
  - X_a: In indication that element a is selected to fit into the grouping.
- Constraint:
  - Each a must be assigned to exactly one b

$$\sum_{a \in \mathbf{A}} \kappa_a X_a \leq \kappa^{TOTAL}$$

Binary Activation

EXAMPLE: I'm considering investing in a variety of investment accounts. However, each one has a minimum deposit amount. How can I model that the amount deposited is greater than this minimum (if I open an account) or zero (if I don't)?

- Parameters:
  - κ_a^{MIN}: The minimum deposit amount for account a
- Variables:
  - X_a: A binary variable indicating whether or not I open account a
  - D_a: The amount I deposit into account a
- Constraints:
  - Minimum Deposit Constraint

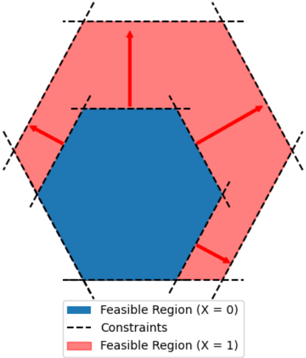
$$D_a \geq \kappa_a^{MIN} - \$1,000,000,000(1 - X_a)$$

- It doesn't have to be $\$1,000,000,000$. It just has to be some large number.
- We normally call this very large number a "Big M".

"Big M" Constraints

- Use a Big M constraint whenever you have a upper or lower **limit that is contingent on a binary decision**.
- Main Idea: Include an extra term in any constraints that you want to de-activate when a binary variable is 1 (or 0). This extra term should make the constraint so loose that it'll never be binding.
- In general, you should just add a binary variable times some large parameter $\mathcal{M}$
- Example:

$$D_a \geq \kappa_a^{MIN} - \mathcal{M}(1 - X_a)$$



- NOTE OF CAUTION: Choosing a good $\mathcal{M}$ value is the name of the game here:
  - Too small: You'll be chopping off parts of the true feasible region. Your problem is likely to become infeasible.
  - Too big: The solver can get lost and can slow way down.
  - Too too big: Your computer can't handle huge numbers and (comparatively) tiny numbers at the same time. You're likely to encounter weird errors.

“If” Statements

Think of this as a generalization of a Big M constraint.

- Normal "if" statements look like this:

$$A = \begin{cases} B & \textit{if } X \\ C & \textit{else} \end{cases}$$

- We can reformulate this like this:

$$A = A^B + A^C$$

$$A^B = \begin{cases} B & \textit{if } X \\ 0 & \textit{else} \end{cases} = BX$$

$$A^C = \begin{cases} C & \textit{if } (1-X) \\ 0 & \textit{else} \end{cases} = C(1-X)$$

- As an aside, the solver will solve faster if we apply some logical tricks:
- We can reformulate $A = BX$ using a series of Big M Constraints:

$$A \leq \beta^{MAX} X$$

$$A \leq B + \beta^{MIN}(X-1)$$

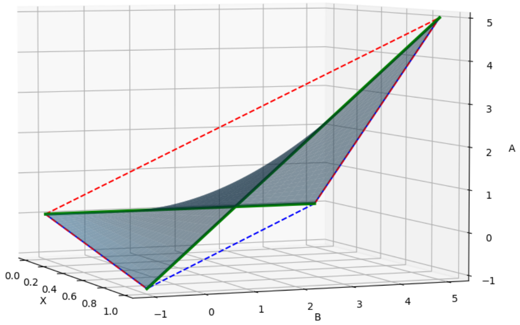
$$A \geq \beta^{MIN} X$$

$$A \geq B + \beta^{MAX}(X-1)$$

- Here, β^{MAX} and β^{MIN} are the maximum and minimum values that B is capable of being. (Big Ms)
- If C is not 0, repeat this process for C

“If” Statement Math (If You’re Interested)

Explanation of how to come up with the linearization of $A = BX$:



$$A \leq \beta^{MAX} X \qquad A \geq \beta^{MIN} X$$
$$A \leq B + \beta^{MIN}(X-1) \qquad A \geq B + \beta^{MAX}(X-1)$$

- Sometimes the objective you define indicates that A will always be maximized or minimized.
- If you’re certain that A will ONLY ever be either maximized or minimized, you can drop two of the constraints.
  - Maximize A, drop ”Greater Than” Constraints
  - Minimize A, drop ”Less Than” Constraints
- Dropping unnecessary constraints means there are less things for the solver to consider: The solver will go faster.

If Statement Example

EXAMPLE: I'm buying a home. The mortgage lender offers me an interest rate of 5%. However, for a certain upfront fee, I can buy down the interest rate to 4.5%. How can I model the interest rate, given that it changes depending on my decision to buy down or not?

- Variables:
  - X: An indication of whether I choose to buy down ($X = 1$) or not ($X = 0$).
  - R: The resulting interest rate.
- Underlying Math:

$$R = \begin{cases} 4.5 & X \\ 5.0 & else \end{cases}$$

- Constraints:

$$R^A = 4.5X$$

$$R^B = 5.0(1 - X)$$

$$R = R^A + R^B$$

"Max" Statements

IMPORTANT CONSIDERATION: Maximizing or Minimizing A

If you will allways be minimizing $A = max(B, C)$,

- Model this as two individual inequalities (no binary variable needed!):

$$A \geq B$$

$$A \geq C$$

If you'll be maximizing $A = max(B, C)$ or you're not sure,

- You'll need to use a binary variable with and the Big M constraints shown on the right.

$$B - C \leq \mathcal{M}Y$$

$$C - B \leq \mathcal{M}(1 - Y)$$

$$A \geq B$$

$$A \geq C$$

$$A \leq B + \mathcal{M}(1 - Y)$$

$$A \leq C + \mathcal{M}Y$$

Here, $\mathcal{M}$ is the maximum possible difference between B and C (e.g. $\max|B - C|$) and Y is a new binary variable.

"Min" Statements

Note that this is the a "Max" statement multiplied by -1.

IMPORTANT CONSIDERATION: Maximizing or Minimizing A

If you will allways be maximizing $A = min(B, C)$,

- Model this as two individual inequalities (no binary variable needed!):

$$A \leq B$$

$$A \leq C$$

If you'll be minimizing $A = max(B, C)$ or you're not sure,

- You'll need to use a binary variable with and the Big M constraints shown on the right.

$$C - B \leq \mathcal{M}Y$$

$$B - C \leq \mathcal{M}(1 - Y)$$

$$A \leq B$$

$$A \leq C$$

$$A \geq B - \mathcal{M}(1 - Y)$$

$$A \geq C - \mathcal{M}Y$$

Here, $\mathcal{M}$ is the maximum possible difference between B and C (e.g. $\max|B - C|$) and Y is a new binary variable.

Min/Max Statement Example

EXAMPLE: I'm buying a house. As we saw in the previous example, the interest rate for a particular lender can depend on lots of decisions. Let's say that I had two different lenders each with their own buy-down options that I model using "If" statements to prodcue R_1 and R_2. I now want to model the lesser of these two as R^{ACTUAL}. How can I model this?

- Underlying Math:

$$R^{ACUTAL} = \min(R_1, R_2)$$

- In paractice, we want low interest rates. So we'll be minimizing this minimum. So we have to use a binary variable:

- Constraints:

$$R_2 - R_1 \leq \mathcal{M}Y$$
$$R_1 - R_2 \leq \mathcal{M}(1 - Y)$$
$$R^{ACTUAL} \leq R_1$$
$$R^{ACTUAL} \leq R_2$$
$$R^{ACTUAL} \geq R_1 - \mathcal{M}(1 - Y)$$
$$R^{ACTUAL} \geq R_2 - \mathcal{M}Y$$

Modeling Real-Life Behaviors

- That concludes the first half of the lesson.
- Now we'll move on to how to model real-life problems.
- To do that, I'll use an example: Retirement Planning

EXAMPLE: I have 3 retirement accounts: a Roth IRA, a 401k, and a standard brokerage account. For now, we'll assume we know the eact early return for each of these accounts: 7%, 10%, and 12%. The balance on these accounts today are $300,000, $900,00, and $50,000. I'm planning for 25 years of retirement. For now, we'll also assume that we'll withdraw $8,000 per month across all the accounts (increasing 2% per year for inflation). There are certain tax penalties for withdrawing from each account. These penalties can be complex, so for now we'll just assume that for every $1 we withdraw, we only get $1 $\times\alpha_a$ for withdrawing from account a. α_a values are given below. Let's assume that we make withdrawals every month. **How much should I withdraw from each account each month?**

a	α_a
IRA	1
401k	0.8
Brokerage	0.7

Steps of Formulating a Problem

1. Objective Function ($f(X, Y, Z, ...)$)
2. Decision Variables ($X, Y, Z, ...$)
3. Parameters ($\alpha, \beta, \gamma, ...$)
4. Constraints ($X + Y \leq Z...$)

- Throughout: Think of relevant sets that you'd like to repeat variables and/or constraints over.

STEP 1: Identify Objective

EXAMPLE: I have 3 retirement accounts: a Roth IRA, a 401k, and a standard brokerage account. For now, we'll assume we know the eact early return for each of these accounts: 7%, 10%, and 12%. The balance on these accounts today are $300,000, $900,00, and $50,000. I'm planning for 25 years of retirement. For now, we'll also assume that we'll withdraw $8,000 per month across all the accounts (increasing 2% per year for inflation). There are certain tax penalties for withdrawing from each account. These penalties can be complex, so for now we'll just assume that for every $1 we withdraw, we only get $1 $\times \alpha_a$ for withdrawing from account a. α_a values are given below. Let's assume that we make withdrawals every month. **How much should I withdraw from each account each month?**

- What's the objective here?
  - Hint: I didn't explicitly give it to you.

STEP 1: Identify Objective

EXAMPLE: I have 3 retirement accounts: a Roth IRA, a 401k, and a standard brokerage account. For now, we'll assume we know the eact early return for each of these accounts: 7%, 10%, and 12%. The balance on these accounts today are $300,000, $900,00, and $50,000. I'm planning for 25 years of retirement. For now, we'll also assume that we'll withdraw $8,000 per month across all the accounts (increasing 2% per year for inflation). There are certain tax penalties for withdrawing from each account. These penalties can be complex, so for now we'll just assume that for every $1 we withdraw, we only get $1 $\times \alpha_a$ for withdrawing from account a. α_a values are given below. Let's assume that we make withdrawals every month. **How much should I withdraw from each account each month?**

- What's the objective here?
  - Hint: I didn't explicitly give it to you.
- There might be several options (depending on what you actually want).
- One that makes sense to me is to **maximize total ending balance**.

STEP 2: Identify Variables

EXAMPLE: I have 3 retirement accounts: a Roth IRA, a 401k, and a standard brokerage account. For now, we'll assume we know the eact early return for each of these accounts: 7%, 10%, and 12%. The balance on these accounts today are \$300,000, \$900,00, and \$50,000. I'm planning for 25 years of retirement. For now, we'll also assume that we'll withdraw \$8,000 per month across all the accounts (increasing 2% per year for inflation). There are certain tax penalties for withdrawing from each account. These penalties can be complex, so for now we'll just assume that for every \$1 we withdraw, we only get \$1 $\times \alpha_a$ *for withdrawing from account a.* α_a *values are given below. Let's assume that we make withdrawals every month.* **How much should I withdraw from each account each month?**

- What key variables should we consider?
  - What quantities could change based on decisions that I make?

STEP 2: Identify Variables

EXAMPLE: I have 3 retirement accounts: a Roth IRA, a 401k, and a standard brokerage account. For now, we'll assume we know the eact early return for each of these accounts: 7%, 10%, and 12%. The balance on these accounts today are \$300,000, \$900,00, and \$50,000. I'm planning for 25 years of retirement. For now, we'll also assume that we'll withdraw \$8,000 per month across all the accounts (increasing 2% per year for inflation). There are certain tax penalties for withdrawing from each account. These penalties can be complex, so for now we'll just assume that for every \$1 we withdraw, we only get \$1 $\times \alpha_a$ for withdrawing from account a. α_a values are given below. Let's assume that we make withdrawals every month. **How much should I withdraw from each account each month?**

- What key variables should we consider?
  - What quantities could change based on decisions that I make?

- B: The balance in a certain account.
- W: The amount I withdraw from a certain account.

STEP 2: Identify Variables

EXAMPLE: I have 3 retirement accounts: a Roth IRA, a 401k, and a standard brokerage account. For now, we'll assume we know the eact early return for each of these accounts: 7%, 10%, and 12%. The balance on these accounts today are $300,000, $900,00, and $50,000. I'm planning for 25 years of retirement. For now, we'll also assume that we'll withdraw $8,000 per month across all the accounts (increasing 2% per year for inflation). There are certain tax penalties for withdrawing from each account. These penalties can be complex, so for now we'll just assume that for every $1 we withdraw, we only get $1 $\times \alpha_a$ *for withdrawing from account a.* α_a *values are given below. Let's assume that we make withdrawals every month.* **How much should I withdraw from each account each month?**

- What key variables should we consider?
  - What quantities could change based on decisions that I make?
- B: The balance in a certain account.
- W: The amount I withdraw from a certain account.
- But what is the "balance" on a certain account?
  - Is there just one balance for all the accounts for all the months we're considering?
  - How can we fix this?

STEP 2a: Introduce Some Sets

EXAMPLE: I have 3 retirement accounts: a Roth IRA, a 401k, and a standard brokerage account. For now, we'll assume we know the eact early return for each of these accounts: 7%, 10%, and 12%. The balance on these accounts today are $300,000, $900,00, and $50,000. I'm planning for 25 years of retirement. For now, we'll also assume that we'll withdraw $8,000 per month across all the accounts (increasing 2% per year for inflation). There are certain tax penalties for withdrawing from each account. These penalties can be complex, so for now we'll just assume that for every $1 we withdraw, we only get $1 $\times \alpha_a$ *for withdrawing from account a.* α_a *values are given below. Let's assume that we make withdrawals every month.* **How much should I withdraw from each account each month?**

- Sets:
  - **A**: A set of all accounts
  - **T**: A set of all time periods (e.g. months)

$$\downarrow$$

- $B_{a,t}$: The balance on account a at the beggining of month t
- $W_{a,t}$: The amount I withdraw from account a during month t

STEP 3: Idenfity Parameters

EXAMPLE: I have 3 retirement accounts: a Roth IRA, a 401k, and a standard brokerage account. For now, we'll assume we know the eact early return for each of these accounts: 7%, 10%, and 12%. The balance on these accounts today are \$300,000, \$900,00, and \$50,000. I'm planning for 25 years of retirement. For now, we'll also assume that we'll withdraw \$8,000 per month across all the accounts (increasing 2% per year for inflation). There are certain tax penalties for withdrawing from each account. These penalties can be complex, so for now we'll just assume that for every \$1 we withdraw, we only get \$1 $\times \alpha_a$ for withdrawing from account a. α_a values are given below. Let's assume that we make withdrawals every month. **How much should I withdraw from each account each month?**

- What parameters (i.e. constant values) do you see?

STEP 3: Idenfity Parameters

EXAMPLE: I have 3 retirement accounts: a Roth IRA, a 401k, and a standard brokerage account. For now, we'll assume we know the eact early return for each of these accounts: 7%, 10%, and 12%. The balance on these accounts today are \$300,000, \$900,00, and \$50,000. I'm planning for 25 years of retirement. For now, we'll also assume that we'll withdraw \$8,000 per month across all the accounts (increasing 2% per year for inflation). There are certain tax penalties for withdrawing from each account. These penalties can be complex, so for now we'll just assume that for every \$1 we withdraw, we only get \$1 $\times\alpha_a$ for withdrawing from account a. α_a values are given below. Let's assume that we make withdrawals every month. **How much should I withdraw from each account each month?**

- What parameters (i.e. constant values) do you see?
- β_a: The initial balance on account a.
  - I like β since it looks like B which is the variable most related to this parameter.
- δ_a: The (monthly?) interest rate for account a.
- ω_t: The amount of money I'll need to withdraw in month t
  - Remember, this depends on the month (t) since it changes with inflation.
- α_a: The fraction of money withdrawn from account a that I get to keep.

STEP 3a: Compute Parameters

- δ_a: The (monthly?) interest rate for account a.
  - Thinking ahead, we're going to have an equation showing the difference in account balances month-to-month (since $B_{a,t}$ is defined on a monthly basis). This equation will need this value. So it should be given on a monthly basis.
  - I just Google "How to convert yearly interest rate to monthly"

$$\delta_a = \frac{\text{Yearly Interest Rate}}{12}$$

- ω_t: The amount of money I'll need to withdraw in month t
  - I just ask ChatGPT: "I have an initial quantity of money that will increase with inflation. I know the yearly inflation rate is 2%. Please give a formula for that this quantity will be "t" months from now."

$$\omega_t = \$8,000 \times (1 + \frac{0.02}{12})^t$$

STEP 4: Idenfity Constraints

EXAMPLE: I have 3 retirement accounts: a Roth IRA, a 401k, and a standard brokerage account. For now, we'll assume we know the eact early return for each of these accounts: 7%, 10%, and 12%. The balance on these accounts today are \$300,000, \$900,00, and \$50,000. I'm planning for 25 years of retirement. For now, we'll also assume that we'll withdraw \$8,000 per month across all the accounts (increasing 2% per year for inflation). There are certain tax penalties for withdrawing from each account. These penalties can be complex, so for now we'll just assume that for every \$1 we withdraw, we only get \$1 $\times \alpha_a$ *for withdrawing from account a.* α_a *values are given below. Let's assume that we make withdrawals every month.* **How much should I withdraw from each account each month?**

- What constraints do you see?
- If I gave you the balance for account a at the beginning of month $t-1$ (i.e. $B_{a,t-1}$) along with how much you withdrew in that month (i.e. $W_{a,t-1}$), could you tell the balance in that account at month t?

STEP 4: Idenfity Constraints

EXAMPLE: I have 3 retirement accounts: a Roth IRA, a 401k, and a standard brokerage account. For now, we'll assume we know the eact early return for each of these accounts: 7%, 10%, and 12%. The balance on these accounts today are $300,000, $900,00, and $50,000. I'm planning for 25 years of retirement. For now, we'll also assume that we'll withdraw $8,000 per month across all the accounts (increasing 2% per year for inflation). There are certain tax penalties for withdrawing from each account. These penalties can be complex, so for now we'll just assume that for every $1 we withdraw, we only get $1 $\times \alpha_a$ *for withdrawing from account a.* α_a *values are given below. Let's assume that we make withdrawals every month.* **How much should I withdraw from each account each month?**

- What constraints do you see?
- If I gave you the balance for account a at the beginning of month $t-1$ (i.e. $B_{a,t-1}$) along with how much you withdrew in that month (i.e. $W_{a,t-1}$), could you tell the balance in that account at month t?

$$B_{a,t} = (B_{a,t-1} - W_{a,t-1}) \times (1 + \delta_a)$$

- But wait!? How should I repeat this equation? If so, how should I repeat it?

STEP 4: Idenfity Constraints

EXAMPLE: I have 3 retirement accounts: a Roth IRA, a 401k, and a standard brokerage account. For now, we'll assume we know the eact early return for each of these accounts: 7%, 10%, and 12%. The balance on these accounts today are \$300,000, \$900,00, and \$50,000. I'm planning for 25 years of retirement. For now, we'll also assume that we'll withdraw \$8,000 per month across all the accounts (increasing 2% per year for inflation). There are certain tax penalties for withdrawing from each account. These penalties can be complex, so for now we'll just assume that for every \$1 we withdraw, we only get \$1 $\times \alpha_a$ *for withdrawing from account a.* α_a *values are given below. Let's assume that we make withdrawals every month.* **How much should I withdraw from each account each month?**

- What constraints do you see?
- If I gave you the balance for account a at the beginning of month $t-1$ (i.e. $B_{a,t-1}$) along with how much you withdrew in that month (i.e. $W_{a,t-1}$), could you tell the balance in that account at month t?

$$B_{a,t} = (B_{a,t-1} - W_{a,t-1}) \times (1 + \delta_a)$$

- But wait!? How should I repeat this equation? If so, how should I repeat it?

$$B_{a,t} = (B_{a,t-1} - W_{a,t-1}) \times (1 + \delta_a) \quad \forall a \in \mathbf{A}, t \in \mathbf{T}$$

- But what about $t = 0$?

STEP 4: Idenfity Constraints

EXAMPLE: I have 3 retirement accounts: a Roth IRA, a 401k, and a standard brokerage account. For now, we'll assume we know the eact early return for each of these accounts: 7%, 10%, and 12%. The balance on these accounts today are \$300,000, \$900,00, and \$50,000. I'm planning for 25 years of retirement. For now, we'll also assume that we'll withdraw \$8,000 per month across all the accounts (increasing 2% per year for inflation). There are certain tax penalties for withdrawing from each account. These penalties can be complex, so for now we'll just assume that for every \$1 we withdraw, we only get \$1 $\times\alpha_a$ for withdrawing from account a. α_a values are given below. Let's assume that we make withdrawals every month. **How much should I withdraw from each account each month?**

- What constraints do you see?
- If I gave you the balance for account a at the beginning of month $t-1$ (i.e. $B_{a,t-1}$) along with how much you withdrew in that month (i.e. $W_{a,t-1}$), could you tell the balance in that account at month t?

$$B_{a,t} = (B_{a,t-1} - W_{a,t-1}) \times (1 + \delta_a)$$

- But wait!? How should I repeat this equation? If so, how should I repeat it?

$$B_{a,t} = (B_{a,t-1} - W_{a,t-1}) \times (1 + \delta_a) \quad \forall a \in \mathbf{A}, t \in \mathbf{T}$$

- But what about $t = 0$?

$$B_{a,t} = (B_{a,t-1} - W_{a,t-1}) \times (1 + \delta_a) \quad \forall a \in \mathbf{A}, t \in \mathbf{T}^{NON-INIT}$$

STEP 4: Idenfity Constraints

EXAMPLE: I have 3 retirement accounts: a Roth IRA, a 401k, and a standard brokerage account. For now, we'll assume we know the eact early return for each of these accounts: 7%, 10%, and 12%. The balance on these accounts today are \$300,000, \$900,00, and \$50,000. I'm planning for 25 years of retirement. For now, we'll also assume that we'll withdraw \$8,000 per month across all the accounts (increasing 2% per year for inflation). There are certain tax penalties for withdrawing from each account. These penalties can be complex, so for now we'll just assume that for every \$1 we withdraw, we only get \$1 $\times \alpha_a$ for withdrawing from account a. α_a values are given below. Let's assume that we make withdrawals every month. **How much should I withdraw from each account each month?**

- What other constraints?
- If my goal is to maximize $B_{a,t}$, having zero withdrawals is a good way to get there. Should we allow this?

STEP 4: Idenfity Constraints

EXAMPLE: I have 3 retirement accounts: a Roth IRA, a 401k, and a standard brokerage account. For now, we'll assume we know the eact early return for each of these accounts: 7%, 10%, and 12%. The balance on these accounts today are $300,000, $900,00, and $50,000. I'm planning for 25 years of retirement. For now, we'll also assume that we'll withdraw $8,000 per month across all the accounts (increasing 2% per year for inflation). There are certain tax penalties for withdrawing from each account. These penalties can be complex, so for now we'll just assume that for every $1 we withdraw, we only get $1 $\times \alpha_a$ *for withdrawing from account a.* α_a *values are given below. Let's assume that we make withdrawals every month.* ***How much should I withdraw from each account each month?***

- What other constraints?
- If my goal is to maximize $B_{a,t}$, having zero withdrawals is a good way to get there. Should we allow this?
- The Gist: Withdrawals has to be equal to ω_t.
  - Hint: We have two sets here **A** and **T**. This constraint will involve both of them, but in different ways. Can you figure out how?

STEP 4: Idenfity Constraints

EXAMPLE: I have 3 retirement accounts: a Roth IRA, a 401k, and a standard brokerage account. For now, we'll assume we know the eact early return for each of these accounts: 7%, 10%, and 12%. The balance on these accounts today are $300,000, $900,00, and $50,000. I'm planning for 25 years of retirement. For now, we'll also assume that we'll withdraw $8,000 per month across all the accounts (increasing 2% per year for inflation). There are certain tax penalties for withdrawing from each account. These penalties can be complex, so for now we'll just assume that for every $1 we withdraw, we only get $1 $\times\alpha_a$ for withdrawing from account a. α_a values are given below. Let's assume that we make withdrawals every month. **How much should I withdraw from each account each month?**

- What other constraints?
- If my goal is to maximize $B_{a,t}$, having zero withdrawals is a good way to get there. Should we allow this?
- The Gist: Withdrawals has to be equal to ω_t.
  - Hint: We have two sets here **A** and **T**. This constraint will involve both of them, but in different ways. Can you figure out how?

$$\sum_{a \in \mathbf{A}} \alpha_a W_{a,t} = \omega_t \quad \forall t \in \mathbf{T}$$

- Any consideration of **T** versus $\mathbf{T}^{NON-INIT}$?

STEP 4: Idenfity Constraints

EXAMPLE: I have 3 retirement accounts: a Roth IRA, a 401k, and a standard brokerage account. For now, we'll assume we know the eact early return for each of these accounts: 7%, 10%, and 12%. The balance on these accounts today are \$300,000, \$900,00, and \$50,000. I'm planning for 25 years of retirement. For now, we'll also assume that we'll withdraw \$8,000 per month across all the accounts (increasing 2% per year for inflation). There are certain tax penalties for withdrawing from each account. These penalties can be complex, so for now we'll just assume that for every \$1 we withdraw, we only get \$1 $\times\alpha_a$ for withdrawing from account a. α_a values are given below. Let's assume that we make withdrawals every month. **How much should I withdraw from each account each month?**

- What other constraints?
- If my goal is to maximize $B_{a,t}$, having zero withdrawals is a good way to get there. Should we allow this?
- The Gist: Withdrawals has to be equal to ω_t.
  - Hint: We have two sets here **A** and **T**. This constraint will involve both of them, but in different ways. Can you figure out how?

$$\sum_{a \in \mathbf{A}} \alpha_a W_{a,t} = \omega_t \quad \forall t \in \mathbf{T}$$

- Any consideration of **T** versus $\mathbf{T}^{NON-INIT}$?
  - No. Special considerations for the initial (or final time period) usually only happen when our constraint involves both t AND $t-1$.

STEP 4: Idenfity Constraints

EXAMPLE: I have 3 retirement accounts: a Roth IRA, a 401k, and a standard brokerage account. For now, we'll assume we know the eact early return for each of these accounts: 7%, 10%, and 12%. The balance on these accounts today are $300,000, $900,00, and $50,000. I'm planning for 25 years of retirement. For now, we'll also assume that we'll withdraw $8,000 per month across all the accounts (increasing 2% per year for inflation). There are certain tax penalties for withdrawing from each account. These penalties can be complex, so for now we'll just assume that for every $1 we withdraw, we only get $1 $\times \alpha_a$ for withdrawing from account a. α_a values are given below. Let's assume that we make withdrawals every month. **How much should I withdraw from each account each month?**

- What other constraints?
- Another way to get really large ending balances would be to start with infinitely large initial balances. Is this allowed?

STEP 4: Idenfity Constraints

EXAMPLE: I have 3 retirement accounts: a Roth IRA, a 401k, and a standard brokerage account. For now, we'll assume we know the eact early return for each of these accounts: 7%, 10%, and 12%. The balance on these accounts today are $300,000, $900,00, and $50,000. I'm planning for 25 years of retirement. For now, we'll also assume that we'll withdraw $8,000 per month across all the accounts (increasing 2% per year for inflation). There are certain tax penalties for withdrawing from each account. These penalties can be complex, so for now we'll just assume that for every $1 we withdraw, we only get $1 $\times \alpha_a$ for withdrawing from account a. α_a values are given below. Let's assume that we make withdrawals every month. **How much should I withdraw from each account each month?**

- What other constraints?
- Another way to get really large ending balances would be to start with infinitely large initial balances. Is this allowed?
- The Gist: The initial balance should be equal to β_a.
- Should we repeat over **T**?

STEP 4: Idenfity Constraints

EXAMPLE: I have 3 retirement accounts: a Roth IRA, a 401k, and a standard brokerage account. For now, we'll assume we know the eact early return for each of these accounts: 7%, 10%, and 12%. The balance on these accounts today are $300,000, $900,00, and $50,000. I'm planning for 25 years of retirement. For now, we'll also assume that we'll withdraw $8,000 per month across all the accounts (increasing 2% per year for inflation). There are certain tax penalties for withdrawing from each account. These penalties can be complex, so for now we'll just assume that for every $1 we withdraw, we only get $1 $\times \alpha_a$ for withdrawing from account a. α_a values are given below. Let's assume that we make withdrawals every month. **How much should I withdraw from each account each month?**

- What other constraints?
- Another way to get really large ending balances would be to start with infinitely large initial balances. Is this allowed?
- The Gist: The initial balance should be equal to β_a.
- Should we repeat over **T**?

$$B_{a,0} = \beta a \quad \forall a \in \mathbf{A}$$

- Finally, another way to get really big ending balances would be to have negative withdrawals. Is this allowed?

STEP 4: Idenfity Constraints

EXAMPLE: I have 3 retirement accounts: a Roth IRA, a 401k, and a standard brokerage account. For now, we'll assume we know the eact early return for each of these accounts: 7%, 10%, and 12%. The balance on these accounts today are $300,000, $900,00, and $50,000. I'm planning for 25 years of retirement. For now, we'll also assume that we'll withdraw $8,000 per month across all the accounts (increasing 2% per year for inflation). There are certain tax penalties for withdrawing from each account. These penalties can be complex, so for now we'll just assume that for every $1 we withdraw, we only get $1 $\times \alpha_a$ for withdrawing from account a. α_a values are given below. Let's assume that we make withdrawals every month. **How much should I withdraw from each account each month?**

- What other constraints?
- Another way to get really large ending balances would be to start with infinitely large initial balances. Is this allowed?
- The Gist: The initial balance should be equal to β_a.
- Should we repeat over **T**?

$$B_{a,0} = \beta a \quad \forall a \in \mathbf{A}$$

- Finally, another way to get really big ending balances would be to have negative withdrawals. Is this allowed?

$$W_{a,t} \geq 0 \quad \forall a \in \mathbf{A}, t \in \mathbf{T}$$

- We can define this in Pyomo buy simply saying that $W_{a,t}$ is in the "NonNegativeReals" domain.

Bringing It All Together

- Let's revisit the objective since we defined it only using words earlier: // "Maximize total ending balance".
- Let's define it using math:

Bringing It All Together

- Let's revisit the objective since we defined it only using words earlier: // "Maximize total ending balance".
- Let's define it using math:

$$max \sum_{a \in \mathbf{A}} B_{a,t^{END}}$$

- Now let's go gather everything from the previous slides.

Bringing It All Together

- Let's revisit the objective since we defined it only using words earlier: // "Maximize total ending balance".
- Let's define it using math:

$$max \sum_{a \in \mathbf{A}} B_{a,t^{END}}$$

- Now let's go gather everything from the previous slides.

$$max \sum_{a \in \mathbf{A}} B_{a,t^{END}}$$

$$---- \text{ subject to } ----$$

$$B_{a,t} = (B_{a,t-1} - W_{a,t-1}) \times (1 + \delta_a)$$
$$\forall a \in \mathbf{A}, t \in \mathbf{T}^{NON-INIT}$$

$$\sum_{a \in \mathbf{A}} \alpha_a W_{a,t} = \omega_t \quad \forall t \in \mathbf{T}$$

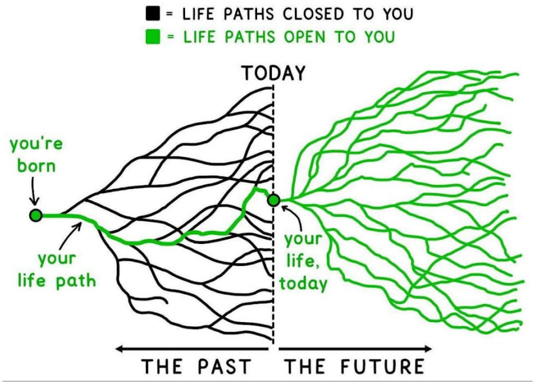
$$B_{a,0} = \beta a \quad \forall a \in \mathbf{A}$$

$$W_{a,t} \geq 0 \quad \forall a \in \mathbf{A}, t \in \mathbf{T}$$

This is your sticky note to have by when you code this model in Pyomo.

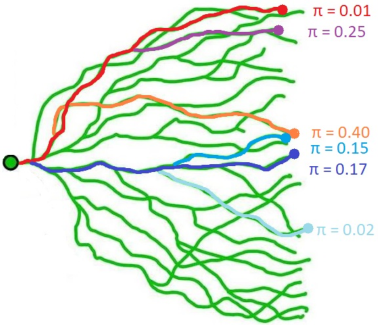
Handling Uncertainty

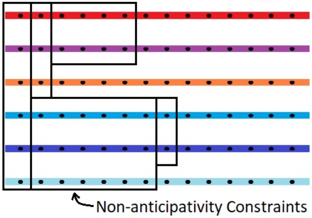
- This is an active field of research. If you're interested in learning more, the words to look up are "Stochastic Programming".
- Scenarios ($s \in \mathbf{S}$)
  - An enumeration of all possible outcomes
  - Each path should have an associated probability π_s
  - All probabilities should sum to 1 (100%)
- Coming up with these is one of the hardest parts
  - Virtually impossible to enumerate **all** outcomes.
  - Very hard to estimate the probability of each outcome.



Handling Uncertainty (Generic)

- Some variables are shared by all scenarios
  - Decisions that need to be made here and now that impact all scenarios moving forward.
- Some variables can be made in the future (and are thus specific to each scenario)
  - Each scenario get's a unique copy of that variable (e.g. iterate over **S**)
  - But at certain points in time, certain scenarios are not yet distinguishable. In order to prevent the solution from "anticipating" future changes that are not yet clear, we need "non-anticipativity constraints"
  - For each box, select on representative scenario (s_1), all other scenarios (s_2) should equal that scenario
  - $X_{s_1,t} = X_{s_2,t} \quad \forall s_2 \in \mathbf{S}^{BOX}, t \in \mathbf{T}^{BOX}$

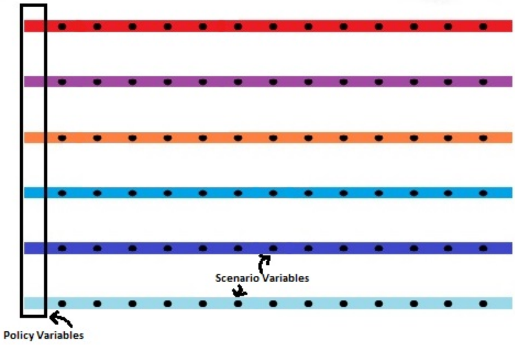




Handling Uncertainty (Typical)

- A tree-like structure can be unreasonably difficult for even a computer to solve if we have a LOT of scenarios that are all interconnected.
- A simplification that works well is called a "two-stage" approach.

1. Assume that you make one set of choices today that will be constant regardless of which path you take. I'll call these "policy variables" (a.k.a. "1^{st} stage variables").
2. Then assume that all scenarios are totally independent.
3. Apply the policy decisions to each scenario.
    - Each scenario will have it's own set of unique "scenario variables" that will be determined as a result of the policy variables. (a.k.a. "2^{nd} stage variables")



Two-Stage Example

- Imagine I have the same retirement planning problem we had earlier.
- But now, instead of knowing the exact rate of return for each account at each month, we know some possible scenarios (s)

s	π_s	IRA	401k	Brokerage
1	0.05	2%	3%	-2%
2	0.10	7%	6%	20%
3	0.60	7%	10%	12%
4	0.25	8%	5%	0%

- The first thing we need to do is establish some "policy variables".
  - My policy is that I'll withdraw a certain porportion from each account each month.
  - For example, perhaps I'll draw out 30% of my monthly expenses from my IRA, 60% from my 401k, and 10% from my brokerage.
  - But I don't know the best percentage values: 30%, 60%, 10%. We'll use the solver to find the optimal policy.
- You can think of these policy variables repeated for all the different scenarios. But since the Non-anticipativity constraint would make them all equal, we'll only define **one set of policy variables that we'll apply to all the scenarios**.

Two-Stage Example

- Policy Variables:
  - F_a: The fraction of my monthly expenses that I'll withdraw from account a
- Policy Constraints:
  - Fractions must add up to 1 (i.e. 100%)

$$\sum_{a \in \mathbf{A}} F_a = 1$$

- Now we need to modify our original problem in three ways:
  1. We need to repeat the original problem for each scenario.
  2. We need to relate each W value to our policy variables.
  3. We need to modify the objective to account for the scenarios.

1. Repeat the original problem for each scenario.

$$max \sum_{a \in \mathbf{A}} B_{a,t^{END},s}$$

$$- - - - \text{ subject to } - - - -$$

$$B_{a,t,s} = (B_{a,t-1,s} - W_{a,t-1,s}) \times (1 + \delta_{a,s})$$
$$\forall a \in \mathbf{A}, t \in \mathbf{T}^{NON-INIT}, s \in \mathbf{S}$$

$$\sum_{a \in \mathbf{A}} \alpha_a W_{a,t,s} = \omega_t \quad \forall t \in \mathbf{T}, s \in \mathbf{S}$$

$$B_{a,0,s} = \beta a \quad \forall a \in \mathbf{A}, s \in \mathbf{S}$$

$$W_{a,t,s} \leq 0 \quad \forall a \in \mathbf{A}, t \in \mathbf{T}, s \in \mathbf{S}$$

$$\sum_{a \in \mathbf{A}} F_a = 1$$

Two-Stage Example

2. Relate each W value to our policy variables.

$$max \sum_{a \in \mathbf{A}} B_{a,t^{END},s}$$

$$----\text{ subject to }----$$

$$B_{a,t,s} = (B_{a,t-1,s} - W_{a,t-1,s}) \times (1 + \delta_{a,s})$$
$$\forall a \in \mathbf{A}, t \in \mathbf{T}^{NON-INIT}, s \in \mathbf{S}$$

$$\alpha_a W_{a,t,s} = \omega_t F_a \quad \forall a \in \mathbf{A}, t \in \mathbf{T}, s \in \mathbf{S},$$

$$B_{a,0,s} = \beta a \quad \forall a \in \mathbf{A}, s \in \mathbf{S}$$

$$W_{a,t,s} \leq 0 \quad \forall a \in \mathbf{A}, t \in \mathbf{T}, s \in \mathbf{S}$$

$$\sum_{a \in \mathbf{A}} F_a = 1$$

3. Modify the objective to account for the scenarios.

$$max \sum_{s \in \mathbf{S}} \pi_s \sum_{a \in \mathbf{A}} B_{a,t^{END},s}$$

$$----\text{ subject to }----$$

$$B_{a,t,s} = (B_{a,t-1,s} - W_{a,t-1,s}) \times (1 + \delta_{a,s})$$
$$\forall a \in \mathbf{A}, t \in \mathbf{T}^{NON-INIT}, s \in \mathbf{S}$$

$$\alpha_a W_{a,t,s} = \omega_t F_a \quad \forall a \in \mathbf{A}, t \in \mathbf{T}, s \in \mathbf{S},$$

$$B_{a,0,s} = \beta a \quad \forall a \in \mathbf{A}, s \in \mathbf{S}$$

$$W_{a,t,s} \leq 0 \quad \forall a \in \mathbf{A}, t \in \mathbf{T}, s \in \mathbf{S}$$

$$\sum_{a \in \mathbf{A}} F_a = 1$$

Two-Stage Example Comments

- I can have as many scenarios as I want (so long as my computer can handle it)
- The policy structure can get very nuanced. I've just proposed a very simple one here.
- The danger here is that the scenarios I pick have to represent reality well. (That's why Wallstreet exists)