

Computational Decision Making for Regular People

01: Introduction

October 15, 2024

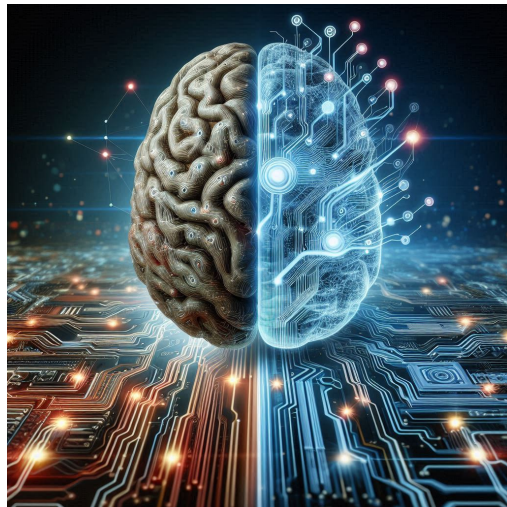
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Today's Outline



1. Mathematical Modeling
2. Optimization
3. What kinds of things can be optimized?
4. General form of an optimization problem
5. A note on optimization theory
6. How to formulate an optimization problem
 - 6.1 The objective function
 - 6.2 Decision variables
 - 6.3 Parameters
 - 6.4 Constraints
7. Basic Examples



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Mathematical Modeling



How we think about a problem

- ▶ Consider what a good outcome looks like
 - ▶ Consider different ramifications of different decisions
 - ▶ utilitarian ethics, virtue ethics, deontological ethics, ...
- ▶ Discrete decisions (yes/no, 1,2,3, etc.)
- ▶ Continuous decisions
- ▶ Constraints against undesirable or infeasible decisions

How a computer thinks about a problem

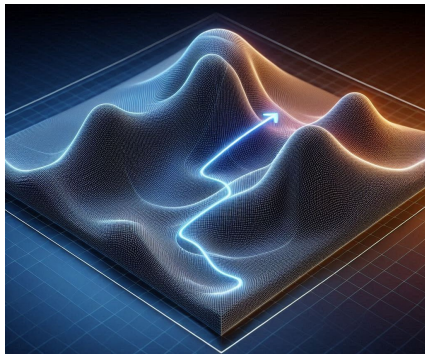
- ▶ Quantify the quality of a solution
 - ▶ Combine all ramifications into one quantity
 - ▶ utilitarian objective only*
- ▶ Discrete decisions (yes/no, 1,2,3, etc.)
- ▶ Continuous decisions
- ▶ Constraints against undesirable or infeasible decisions

Optimization



*the action of making the best or most effective use of a situation or resource.*¹

- ▶ Given a set of possible decisions, determine the best one(s)
- ▶ The way this determination is made depends on the nature of the set of decisions
 - ▶ If a human is executing this determination, the procedure will be unique to that person
 - ▶ If a computer algorithm is executing this determination, the procedure will be unique to that computer algorithm



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¹Oxford Languages

What Kinds of Things Can Be Optimized?



- ▶ What kinds of things can be optimized (using a computer)?
- ▶ If you can represent it using mathematical modeling, it (theoretically) can be optimized.
- ▶ Some problems are still too difficult for even modern computer algorithms to solve
 - ▶ Problems with lots and lots of variables
 - ▶ Problems with lots and lots of constraints
 - ▶ Problems that "don't behave well"
 - ▶ Lots of really good outcomes that lie very close to really bad outcomes
 - ▶ Lots of outcomes that are equally good
 - ▶ Good outcomes that are separated by bad outcomes
 - ▶ Really nuanced constraints
 - ▶ etc.

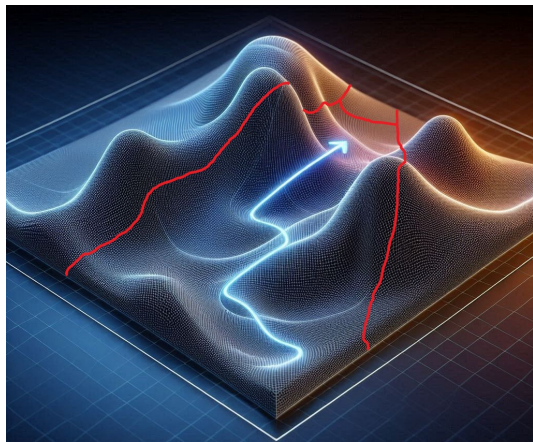
General Form of an Optimization Problem



$$\begin{aligned} & \min f(\bar{X}) \\ & \text{---} \text{---} \text{---} \textit{subject to (s.t.)} \text{---} \text{---} \text{---} \\ & \bar{X} \in \mathbf{S} \end{aligned}$$

► We must define:

- \bar{X}
- $f(\bar{X})$
- \mathbf{S}



Optimization Theory



- ▶ This is a very active field of research
- ▶ It gets quite complicated and "mathy" very quickly
- ▶ I'll only emphasize two points:

$$\min f(\bar{X})$$

$$s.t. \bar{X} \in \mathbf{S}$$

If we can keep $f(\bar{X})$ and \mathbf{S} as linear as possible, the computer algorithms are MUCH faster.

If we can't keep things linear, "Bilinear" is the next best thing.

- ▶ Linear: $\alpha X + \beta Y + \gamma Z$
- ▶ Bilinear: αXY or αX^2
- ▶ Non-linear:
 - ▶ $\alpha\sqrt{X}$
 - ▶ $\alpha \log(X)$
 - ▶ etc.

Formulating Optimization Problems



Key Elements:

In order how I would conceptualize them:

1. Objective Function ($f(\bar{X})$)
2. Decision Variables (\bar{X})
3. Parameters ($\bar{\alpha}$)
4. Constraints (**S**)

In order how I would write / code them:

1. Parameters ($\bar{\alpha}$)
2. Decision Variables (\bar{X})
3. Constraints (**S**)
4. Objective Function ($f(\bar{X})$)

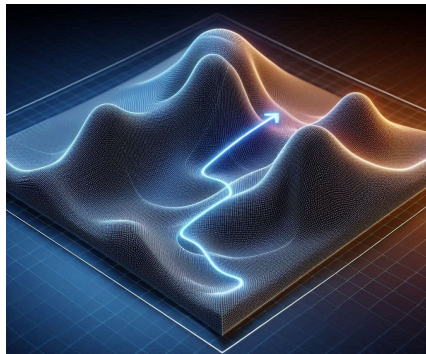
Objective Function ($f(\bar{X})$)



- ▶ What am I trying to accomplish?
- ▶ What am I trying to minimize (or maximize)?
- ▶ How do different decisions change the outcome?

Examples:

- ▶ Minimize cost
- ▶ Minimize risk
- ▶ Maximize probability of reaching a goal
- ▶ Maximize comfort



Decision Variables (\bar{X})



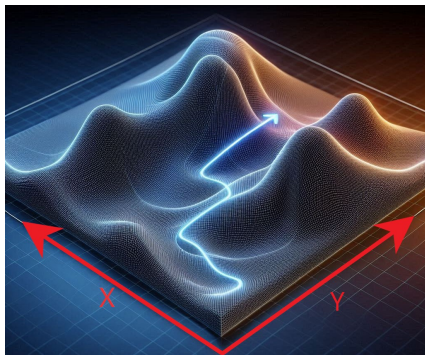
- ▶ What decisions can I make?
- ▶ What values change when I make different decisions?

Important Note:

- ▶ Be open-minded here: often the best way to formulate a problem is to consider each individual part of a problem as it's own variable.
- ▶ Don't just consider the main decisions, consider the smaller decisions that contribute to (or even strictly define) the main decisions

Examples:

- ▶ How much of a certain item to buy
- ▶ How much of a budget category to allot for that item



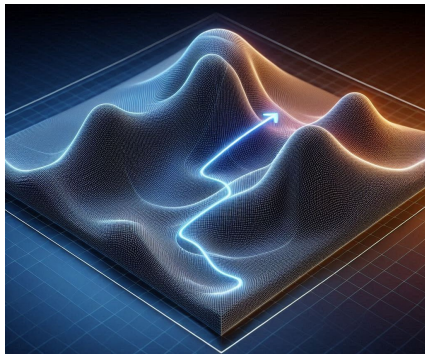
Parameters ($\bar{\alpha}$)



- ▶ What parts of the problem are fixed or immovable?
- ▶ While the Objective function defines the general shape of the mountain, the parameters define the height of the peaks, the steepness of the slopes, etc.

Examples:

- ▶ Cost of a certain item
- ▶ Total amount of time available
- ▶ The minimum probability of reaching a goal that we are willing to accept



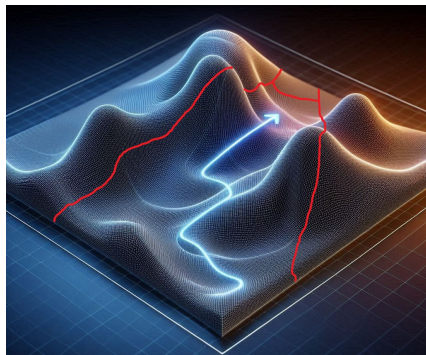
Constraints (S)



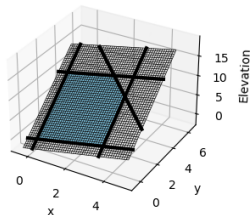
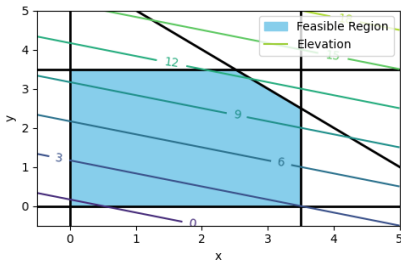
- ▶ What sets of decisions are incompatible?
- ▶ What is the nature of an individual decision? (Binary, Continuous, etc.)
- ▶ How do different decision variables relate to each other?

Examples:

- ▶ We cannot spend more in a budget category than the total allotment for that category.
- ▶ The amount of money spent in a budget category is strictly equal to the sum of transactions that lies in that category.
- ▶ The calculated probability of reaching a goal must be greater than the minimum value we specified.



A Visual Example (Linear)



$$\max X + 3Y - 0.5$$

$$s.t. \quad X \geq 0$$

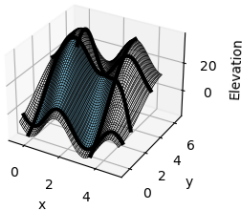
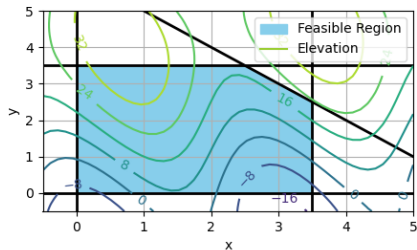
$$Y \geq 0$$

$$X \leq 3.5$$

$$Y \leq 3.5$$

$$X + Y \leq 6$$

A Visual Example (Nonlinear)



$$\max f(X, Y)$$

$$s.t. \quad X \geq 0$$

$$Y \geq 0$$

$$X \leq 3.5$$

$$Y \leq 3.5$$

$$X + Y \leq 6$$

Example: Etsy Shop Problem



Jack and Jill own an Etsy shop that sells t-shirts and wooden sculptures. Jack can either make 5 t-shirts per hour or he can make 2 sculptures per hour. Jill can either make 9 t-shirts per hour or she can make 3 sculptures per hour. Their most recent order indicates that they only need 25 t-shirts but would welcome as many sculptures as they can create. They each have 8 hours to work today. How should they spend their time today so that they can maximize the total number of t-shirts and sculptures they make?

Take a minute to think about the problem on your own:

1. Objective Function ($f(\bar{X})$)
2. Decision Variables (\bar{X})
3. Parameters ($\bar{\alpha}$)
4. Constraints (**S**)

Example: Etsy Shop Problem



First conceptualize the problem:

- ▶ Objective: Maximize the total number of products produced
- ▶ Variables:
 - ▶ How many hours each person spends on producing each product
 - ▶ The amount of each product that gets made
- ▶ Parameters:
 - ▶ How much product can be made by each person per hour
 - ▶ How much time there is in a day
 - ▶ The number of t-shirts they need
 - ▶ The priority between each product
- ▶ Constraints:
 - ▶ Each person can only work between 0 and 8 hours in a day
 - ▶ The amount of each product produced should be directly determined by how many hours each person spends making that product
 - ▶ The number of t-shirts should be limited to 25

Example: Etsy Shop Problem



Let's write the problem using mathematical modeling:

1. Define Parameters:

- ▶ How much product can be made by each person per hour
 - ▶ $\alpha_{Jack, Sculptures} = 2$
 - ▶ $\alpha_{Jack, Shirts} = 5$
 - ▶ $\alpha_{Jill, Sculptures} = 3$
 - ▶ $\alpha_{Jill, Shirts} = 9$
- ▶ $\tau^{DAY} = 8$: How much time there is to work in a day
- ▶ $\kappa = 25$: The number of t-shirts they need

2. Define Decision Variables:

- ▶ The number of hours each person should spend making each product in a day
 - ▶ $H_{Jack, Sculptures}$
 - ▶ $H_{Jack, Shirts}$
 - ▶ $H_{Jill, Sculptures}$
 - ▶ $H_{Jill, Shirts}$
- ▶ The amount of each product that gets made
 - ▶ $N_{Sculptures}$
 - ▶ N_{Shirts}

Example: Etsy Shop Problem



3. Constraints:

- ▶ Each person can only work between 0 and 8 hours in a day:

$$0 \leq H_{Jack, Sculptures} + H_{Jack, Shirts} \leq \tau^{DAY}$$

$$0 \leq H_{Jill, Sculptures} + H_{Jill, Shirts} \leq \tau^{DAY}$$

- ▶ The amount of each product produced should be directly determined by how many hours each person spends making that product:

$$N_{Sculptures} = \alpha_{Jack, Sculptures} H_{Jack, Sculptures} + \alpha_{Jill, Sculptures} H_{Jill, Sculptures}$$

$$N_{Shirts} = \alpha_{Jack, Shirts} H_{Jack, Shirts} + \alpha_{Jill, Shirts} H_{Jill, Shirts}$$

- ▶ The number of shirts should be limited to 25:

$$N_{Shirts} = \kappa$$

4. Define Objective:

- ▶ Maximize the total number of products produced

$$\max N_{Shirts} + N_{Sculptures}$$

Example: Etsy Shop Problem



FULL FORMULATION

$$\max N_{Shirts} + N_{Sculptures}$$

— — — *subject to (s.t.)* — — —

$$0 \leq H_{Jack, Sculptures} + H_{Jack, Shirts} \leq \tau^{DAY}$$

$$0 \leq H_{Jill, Sculptures} + H_{Jill, Shirts} \leq \tau^{DAY}$$

$$N_{Shirts} = \alpha_{Jack, Shirts} H_{Jack, Shirts} + \alpha_{Jill, Shirts} H_{Jack, Shirts}$$

$$N_{Sculptures} = \alpha_{Jack, Sculptures} H_{Jack, Sculptures} \\ + \alpha_{Jill, Sculptures} H_{Jack, Sculptures}$$

$$N_{Shirts} = \kappa$$

- ▶ Notice how much repetition there is.
- ▶ How could I change this problem if I had 1000 people and 75 products?



- ▶ Often there are several variables, parameters, constraints, etc. that are repeated for a given set of values.
- ▶ We can write the general idea behind that variable, parameter, constraint, etc. by simply writing it once and indicating that it should be repeated for every element in a set.
- ▶ In math, it looks like this:

$$\forall e \in \mathbf{E}$$

"for all elements e in the set \mathbf{E} "

$$0 \leq H_{Jack, Sculptures} + H_{Jack, Shirts} \leq \tau^{DAY}$$

$$0 \leq H_{Jill, Sculptures} + H_{Jill, Shirts} \leq \tau^{DAY}$$

↓

$$0 \leq H_{p, Sculptures} + H_{p, Shirts} \leq \tau^{DAY} \quad \forall p \in \mathbf{P}$$

↓

$$0 \leq \sum_{r \in \mathbf{R}} H_{p,r} \leq \tau^{DAY} \quad \forall p \in \mathbf{P}$$

$p \in \mathbf{P}$: A set of all People

$r \in \mathbf{R}$: A set of all Products

Sets



$$\alpha_{Jack, Sculptures} = 2$$

$$\alpha_{Jack, Shirts} = 5$$

$$\alpha_{Jill, Sculptures} = 3$$

$$\alpha_{Jill, Shirts} = 9$$



$$\alpha_{p,r} \quad \forall p \in \mathbf{P}, r \in \mathbf{R}$$

$\alpha_{p,r}$	Jack	Jill
Shirts	5	9
Sculptures	2	3

$$N_{Shirts} = \alpha_{Jack, Shirts} H_{Jack, Shirts} + \alpha_{Jill, Shirts} H_{Jack, Shirts}$$

$$N_{Sculptures} = \alpha_{Jack, Sculptures} H_{Jack, Sculptures} + \alpha_{Jill, Sculptures} H_{Jack, Sculptures}$$



$$N_r = \alpha_{Jack,r} H_{Jack,r} + \alpha_{Jill,r} H_{Jack,r} \quad \forall r \in \mathbf{R}$$



$$N_r = \sum_{p \in \mathbf{P}} \alpha_{p,r} H_{p,r} \quad \forall r \in \mathbf{R}$$

Example: Etsy Shop Problem with Sets



$$\max N_{Shirts} + N_{Sculptures}$$

— — — *subject to (s.t.)* — — —

$$0 \leq H_{Jack, Sculptures} + H_{Jack, Shirts} \leq \tau^{DAY}$$

$$0 \leq H_{Jill, Sculptures} + H_{Jill, Shirts} \leq \tau^{DAY}$$

$$N_{Shirts} = \alpha_{Jack, Shirts} H_{Jack, Shirts} + \alpha_{Jill, Shirts} H_{Jill, Shirts}$$

$$N_{Sculptures} = \alpha_{Jack, Sculptures} H_{Jack, Sculptures} + \alpha_{Jill, Sculptures} H_{Jill, Sculptures}$$

$$N_{Shirts} = \kappa$$

$$\max \sum_{r \in \mathbf{R}} N_r$$

— — — *subject to (s.t.)* — — —

$$0 \leq \sum_{r \in \mathbf{R}} H_{p,r} \leq \tau^{DAY} \quad \forall p \in \mathbf{P}$$

$$N_r = \sum_{p \in \mathbf{P}} \alpha_{p,r} H_{p,r} \quad \forall r \in \mathbf{R}$$

$$N_{Shirts} = \kappa$$

This one can handle any number of people or products.

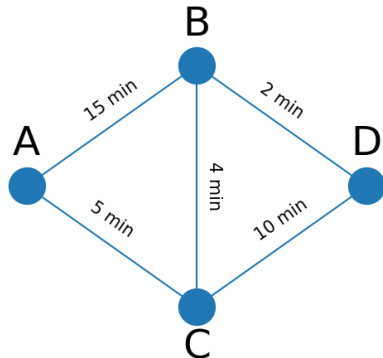
Example: Route Planning Problem



Given the roads indicated in the graph below, what's the fastest way to get from point A to point D?

Take a minute to think about the problem on your own:

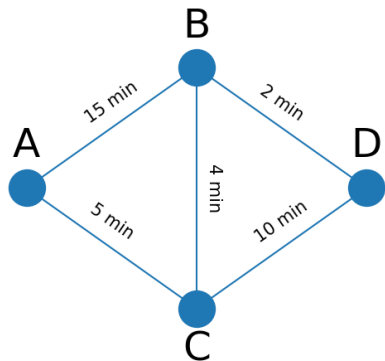
1. Objective Function ($f(\bar{X})$)
2. Decision Variables (\bar{X})
3. Parameters ($\bar{\alpha}$)
4. Constraints (**S**)



Example: Route Planning Problem



First conceptualize the problem:



- ▶ Objective: Minimize the time spent traveling
- ▶ Variables:
 - ▶ Whether or not to travel down each stretch of road
 - ▶ Whether or not to visit each point
- ▶ Parameters:
 - ▶ How long each stretch of road is
 - ▶ Which roads connect to which points
- ▶ Constraints:
 - ▶ If I travel into a point, I must travel out of it.
 - ▶ I have to visit point A and point D

Example: Route Planning Problem



Then write the problem using mathematical modeling:

1. Define Sets:

- ▶ $p \in \mathbf{P} = \{A, B, C, D\}$: A set of all points
- ▶ $p \in \mathbf{P}^{NON-TERM} = \{B, C\}$: A set of all points other than the terminal (starting and ending) points
- ▶ $p \in \mathbf{P}^{TERM} = \{A, D\}$: A set of all the terminal points
- ▶ $r \in \mathbf{R} = \{AB, AC, BC, BD, CD\}$: A set of all roads
- ▶ $r \in \mathbf{R}_p$: A set of all roads that touch point p

p	\mathbf{R}_p
A	$\{AB, AC\}$
B	$\{AB, BC, BD\}$
C	$\{AC, BC, CD\}$
D	$\{BD, CD\}$

2. Define Parameters:

- ▶ δ_r : The time it takes to travel road r

r	$\delta_r(\text{minutes})$
AB	15
AC	5
BC	4
BD	2
CD	10

3. Define Decision Variables:

- ▶ $X_r \ \forall r \in \mathbf{R}$: Whether or not to travel down road r (**Binary Variable**)
- ▶ $Y_p \ \forall p \in \mathbf{P}$: Whether or not to visit point p (**Binary Variable**)

Example: Route Planning Problem



4. Define Constraints:

- ▶ If I travel into a point, I must travel out of it

$$\sum_{r \in \mathbf{R}_p} X_r = 2Y_p \quad \forall p \in \mathbf{P}^{NON-TERM}$$

$$\sum_{r \in \mathbf{R}_p} X_r = Y_p \quad \forall p \in \mathbf{P}^{TERM}$$

- ▶ I have to visit point A and point D

$$Y_p = 1 \quad \forall p \in \mathbf{P}^{TERM}$$

5. Define Objective:

- ▶ Minimize the amount of time spent traveling

$$\min_{X_r, Y_p} \sum_{r \in \mathbf{R}} \delta_r X_r$$

— — — FULL FORMULATION — — —

$$\min_{X_r, Y_p} \sum_{r \in \mathbf{R}} \delta_r X_r$$

$$\text{s.t.} \quad \sum_{r \in \mathbf{R}_p} X_r = 2Y_p \quad \forall p \in \mathbf{P}^{NON-TERM}$$

$$\sum_{r \in \mathbf{R}_p} X_r = Y_p \quad \forall p \in \mathbf{P}^{TERM}$$

$$Y_p = 1 \quad \forall p \in \mathbf{P}^{TERM}$$

$$X_r, Y_p \in \{0, 1\}$$

* Notice how the objective and all constraints are linear

Next Class



- ▶ Visit the course website:



QR Code to Course Website

- ▶ These slides are posted in the Lecture 01 folder.

- ▶ Python / Optimization coding crash course
- ▶ Go to the Lecture 02 folder on the course website
 - ▶ Follow the instructions on "slides.pdf" in the Lecture 02 folder.
 - ▶ The more you can do on your own before class the more time we'll have to answer questions and do example next class.
- ▶ Please bring your laptops to class!