Computational complexity of problems in 3-dimensional topology

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[Novikov 1962] *For $n \geq 5$, there does not exist an algorithm which solves:*

IsSphere: Given a triangulated M^n is it homeomorphic to S^n?

Thm (Geometrization + many results)
There is an algorithm to decide if two compact 3-mflds are homeomorphic.

Today: How hard are these 3-manifold questions? How quickly can we solve them?

Decision Problems: Yes or no answer.

Sorted: Given a list of integers, is it sorted?

SAT: Given $p_1, \ldots p_n \in \mathbb{F}_2[x_1, \ldots, x_k]$ is there $\mathbf{x} \in \mathbb{F}_2^k$ with $p_i(\mathbf{x}) = 0$ for all i?

Unknotted: Given a planar diagram for K in S^3 is K the unknot?

Invertible: Given $A \in M_n(\mathbb{Z})$ does it have an inverse in $M_n(\mathbb{Z})$?

P: Decision problems which can be solved in polynomial time in the input size.

- **Sorted:** $O(\text{length of list})$
- **Invertible:** $O\left(n^{3.5} \log(\text{largest entry})^{1.1}\right)$

NP: Yes answers have proofs that can be checked in polynomial time.

SAT: Given $\mathbf{x} \in \mathbb{F}_2^k$, can check all $p_i(\mathbf{x}) = 0$ in linear time.

Unknotted: A diagram of the unknot with c crossings can be unknotted in $O(c^{11})$ Reidemeister moves. [Lackenby 2013]

coNP: No answers can be checked in polynomial time.

Unknotted: Yes, assuming the GRH [Kuperberg 2011].

Conj: Unknotted is in **P**.