Asymmetric hyperbolic L-spaces, Heegaard genus, and Dehn filling

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Abstract. An L-space is a rational homology 3-sphere with minimal Heegaard Floer homology. We give the first examples of hyperbolic L-spaces with no symmetries. In particular, unlike all previously known L-spaces, these manifolds are not double branched covers of links in S^3. We prove the existence of infinitely many such examples (in several distinct families) using a mix of hyperbolic geometry, Floer theory, and verified computer calculations. Of independent interest is our technique for using interval arithmetic to certify symmetry groups and non-existence of isometries of cusped hyperbolic 3-manifolds. In the process, we give examples of 1-cusped hyperbolic 3-manifolds of Heegaard genus 3 with two distinct lens space fillings. These are the first examples where multiple Dehn fillings drop the Heegaard genus by more than one, which answers a question of Gordon.

1 Introduction

1.1 Asymmetric L-spaces. For a rational homology 3-sphere M, the rank of its Heegaard Floer homology $\widehat{HF}(M)$ is always bounded below by the order of $H_1(M;\mathbb{Z})$, and M is called an *L-space* when this bound is an equality. Lens spaces and other spherical manifolds are all L-spaces, but these are by no means the only examples. In fact, recent work of Boyer, Gordon, and Watson [BGW] shows that each of the eight 3-dimensional geometries has an L-space. Their work is part of broader efforts to characterize L-spaces via properties not obviously connected to Heegaard Floer theory; specifically, they conjecture that a rational homology sphere is an L-space if and only if its fundamental group is not left-orderable. Although the conjecture has been resolved for seven of the geometries, it remains open for the important case of hyperbolic geometry as well as for most manifolds with non-trivial JSJ decompositions. Previous constructions of hyperbolic L-spaces produce them via surgery on strongly invertible manifolds, leading to examples which are double branched covers over links in S^3. One of our main results shows that these are constructions of convenience rather than necessity. Recall that a hyperbolic 3-manifold is *asymmetric* if its only self-isometry is the identity map; by a deep theorem of Gabai, this is equivalent to every self-diffeomorphism

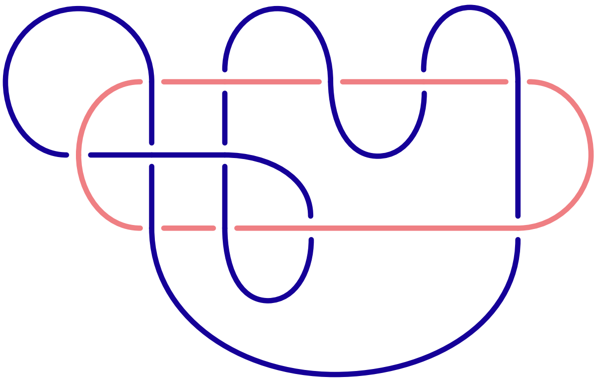

Figure 1.4. The link used in Theorem 3.4 is $L12n1314$ in the Hoste-Thistlewaite census. Our framing conventions for Dehn filling are and are consistent with SnapPy [CDW]. Note there is an orientation-preserving homeomorphism of S^3 which interchanges the two components.

being isotopic to the identity [Gab]. We will show the following:

1.2 Theorem. *There exist infinitely many asymmetric hyperbolic L-spaces. In particular, there are hyperbolic L-spaces which are neither regular covers nor regular branched covers of another 3-manifold.*

Among L-spaces which are *not* double branched covers over links in S^3, hyperbolic examples such as those of Theorem 1.2 are the simplest possible in the sense that any such L-space must have a hyperbolic piece in its prime/JSJ decomposition. This is because any graph manifold which is a rational homology sphere, much less an L-space, is a double branched cover over a link in S^3. This was proved by Montesinos in [Mon, §7.2]; the theorem stated there is paraphrased in the translation below:

1.3 Theorem [Mon, §7.2]. *Let M be a graph manifold whose diagram is a tree with each vertex corresponding to a Seifert fibered space over a (punctured) S^2 or (punctured) $\mathbb{R}\mathrm{P}^2$. Then M is a double branched cover of a link L in S^3.*

Note the rational homology sphere assumption implies that the diagram of the graph manifold is a tree. Also, the cases that arise if the tree is a just single vertex are covered in [Mon, §2-3].

We prove Theorem 1.2 via a combination of hyperbolic geometry, Heegaard Floer theory, and verified computer calculations. The proof of Theorem 1.2 has two parts, the second of which is computer-aided. The first result shows that we need only construct 1-cusped manifolds with certain properties, and the second establishes the existence of such manifolds. Here, the *order* of a lens space is the order of its fundamental group/first homology.

2.1 Theorem. *Suppose M is a 1-cusped hyperbolic 3-manifold. If M is asymmetric and has two lens space Dehn fillings of coprime order, then there are infinitely many Dehn fillings of M which are asymmetric hyperbolic L-spaces. Moreover, M is the complement of a knot in an integral homology 3-sphere and fibers over the circle with fiber a once-punctured surface.*

3.4 Theorem. *There exist infinitely many 1-cusped hyperbolic 3-manifolds which are asymmetric and have two lens space fillings of coprime order. Specifically, if N is the exterior of the link in Figure 1.4, then for all large $k \in \mathbb{Z}$, the $(6k \pm 1, k)$ Dehn filling on either component of N yields such a manifold.*

In addition to Theorem 3.4, Theorem 3.1 offers a finite number of explicit examples for which the proof is slightly easier. A Heegaard diagram of the simplest of these examples is given in Figure 3.3.

1.5 Heegaard genus, Dehn filling, and the Berge conjecture. Our second main result answers a question of Gordon [Gor] regarding the existence of manifolds where multiple fillings drop the Heegaard genus by more than one:

1.6 Corollary. *There exist infinitely many 1-cusped hyperbolic 3-manifolds of Heegaard genus three which admit two distinct lens space fillings.*

This corollary follows immediately from Theorem 3.4, as manifolds with genus two Heegaard splittings always have symmetries; the examples of Theorem 3.4 must have Heegaard genus exactly three since the link in Figure 1.4 is 3-bridge.

The interest in L-spaces stems in part from open questions about lens space surgery, with the Berge Conjecture as the chief example. Another interesting feature of Corollary 1.6 is that it provides counterexamples to the following generalization of the Berge Conjecture, since the exterior of any $(1,1)$–knot has Heegaard genus two:

1.7 Conjecture [BDH, Conjecture 9]. *If knots $K_1 \subset L(p_1, q_1)$ and $K_2 \subset L(p_2, q_2)$ are longitudinal surgery duals, then up to reindexing, K_2 is a $(1,1)$–knot and $p_2 \geq p_1$.*

We note that these examples do not contradict the Berge Conjecture itself because they are not knot complements in S^3; see the proof of Theorem 3.4 for details.

2 Asymmetric L-spaces from cusped manifolds

This section is devoted to the proof of the following result:

2.1 Theorem. *Suppose M is a 1-cusped hyperbolic 3-manifold. If M is asymmetric and has two lens space Dehn fillings of coprime order, then there are infinitely many Dehn fillings of M which are asymmetric hyperbolic L-spaces. Moreover, M is the complement of a knot in an integral homology 3-sphere and fibers over the circle with fiber a once-punctured surface.*

This theorem follows immediately from the next two lemmas, where in the second one we set $N \setminus \partial N \cong M$.

2.2 Lemma. *Suppose M is an asymmetric 1-cusped hyperbolic 3-manifold. Then all but finitely many Dehn fillings of M are hyperbolic and asymmetric.*

2.3 Lemma. *Suppose N is a compact 3-manifold with ∂N a torus. If N has two lens space Dehn fillings of coprime order, then N has infinitely many Dehn fillings which are L-spaces. Moreover, N is the exterior of a knot in an integral homology sphere and fibers over the circle with fiber a surface with one boundary component.*

The proofs of these two lemmas are completely independent and will be familiar to experts in the areas of 3-dimensional hyperbolic geometry and Heegaard Floer theory, respectively.

3 Asymmetric manifolds with lens space fillings

Before proving Theorem 3.4, we warm up with the following easier and more concrete result, which, when combined with Theorem 2.1, also suffices to prove Theorem 1.2.

3.1 Theorem. *Table 3.2 lists 22 distinct 1-cusped hyperbolic 3-manifolds which are asymmetric and have two lens space fillings of coprime order.*

We provide a rigorous computer-assisted proof of Theorem 3.1 using SnapPy [CDW], the verification scheme of [HIKMOT]. These examples were found in the census of 1-cusped hyperbolic 3-manifolds with at most 9 tetrahedra [Bur, CHW] by a brute-force search through these 59,107 manifolds.

We next extend the phenomena exhibited in Theorem 3.1 to an infinite family of examples; note that our conventions for Dehn filling are specified in Figure 1.4.

3.4 Theorem. *There exist infinitely many 1-cusped hyperbolic 3-manifolds which are asymmetric and have two lens space fillings of coprime order. Specifically, if N is the exterior of the link in Figure 1.4, then for all large $k \in \mathbb{Z}$, the $(6k \pm 1, k)$ Dehn filling on either component of N yields such a manifold.*

References

[BDH] K. L. Baker, B. G. Doleshal, and N. Hoffman. On manifolds with multiple lens space filings. To appear in Boletín de la Sociedad Matemática Mexicana, 2013. `arXiv:1308.5002`.

[BGW] S. Boyer, C. M. Gordon, and L. Watson. On L-spaces and left-orderable fundamental groups. *Math. Ann.* **356** (2013), 1213–1245. `arXiv:1107.5016`, MR3072799.

[Bur] B. A. Burton. The cusped hyperbolic census is complete. Preprint 2014, 32 pages. `arXiv:1405.2695`.

[BBP+] B. A. Burton, R. Budney, W. Pettersson, et al. Regina 4.95: Software for 3-manifold topology and normal surface theory. Available from `http://regina.sourceforge.net/`, 1999–2014.

[CHW] P. J. Callahan, M. V. Hildebrand, and J. R. Weeks. A census of cusped hyperbolic 3-manifolds. *Math. Comp.* **68** (1999), 321–332. With microfiche supplement. MR1620219.

M	#tets	$M_{(1,0)}$	$M_{(0,1)}$	g	vol(M)	systole
$v3372^*$	7	$L(7,1)$	$L(19,7)$	10	6.541194	0.952884
$t10397$	8	$L(11,2)$	$L(14,3)$	12	6.880362	0.911798
$t10448$	8	$L(17,5)$	$L(29,8)$	15	6.891314	0.716411
$t11289^*$	8	$L(11,2)$	$L(26,7)$	15	7.084874	0.576033
$t11581$	8	$L(7,1)$	$L(31,12)$	16	7.180413	0.767839
$t11780$	8	$L(23,7)$	$L(6,1)$	12	7.232671	0.643558
$t11824$	8	$L(34,13)$	$L(19,4)$	19	7.246332	0.480409
$t12685$	8	$L(14,3)$	$L(29,8)$	18	7.674889	0.693829
$o9^*_{34328}$	10	$L(13,2)$	$L(34,13)$	19	7.529794	0.312418
$o9_{35609}$	10	$L(50,19)$	$L(29,8)$	27	7.631975	0.237482
$o9^*_{35746}$	10	$L(17,3)$	$L(41,12)$	24	7.642118	0.238001
$o9_{36591}$	9	$L(55,21)$	$L(31,7)$	31	7.707673	0.188586
$o9_{37290}$	9	$L(31,12)$	$L(19,4)$	22	7.762770	0.442218
$o9_{37552}$	9	$L(35,8)$	$L(13,3)$	18	7.781895	0.408545
$o9_{38147}$	9	$L(29,12)$	$L(41,11)$	27	7.831770	0.392648
$o9_{38375}$	9	$L(17,3)$	$L(29,8)$	24	7.851404	0.349858
$o9_{38845}$	9	$L(13,2)$	$L(18,5)$	15	7.896384	0.770335
$o9_{39220}$	10	$L(13,2)$	$L(46,17)$	28	7.930877	0.304931
$o9_{41039}$	10	$L(13,2)$	$L(21,8)$	16	8.122543	0.916284
$o9_{41063}$	9	$L(26,7)$	$L(41,11)$	30	8.126169	0.386869
$o9_{41329}$	9	$L(34,9)$	$L(49,18)$	34	8.159350	0.364220
$o9_{43248}$	10	$L(37,8)$	$L(18,5)$	23	8.444914	0.689245

Table 3.2. The 22 manifolds of Theorem 3.1. Here, "#tets" refers to the canonical triangulation supplied in [DHL] and g is the genus of the fibration of M over the circle (whose existence follows from Theorem 2.1) computed via the Alexander polynomial. The lens spaces were identified using Regina [BBP$^+$]. The manifolds marked with a $*$ also appear in Theorem 3.4. The data is all rigorous with the exception of the volume and systole columns, which were approximated numerically, as the methods of [HIKMOT] have not yet been extended to those quantities. Note that none of these manifolds are knot complements in S^3, since the pair of lens space surgeries have fundamental groups whose orders differ by more than one.

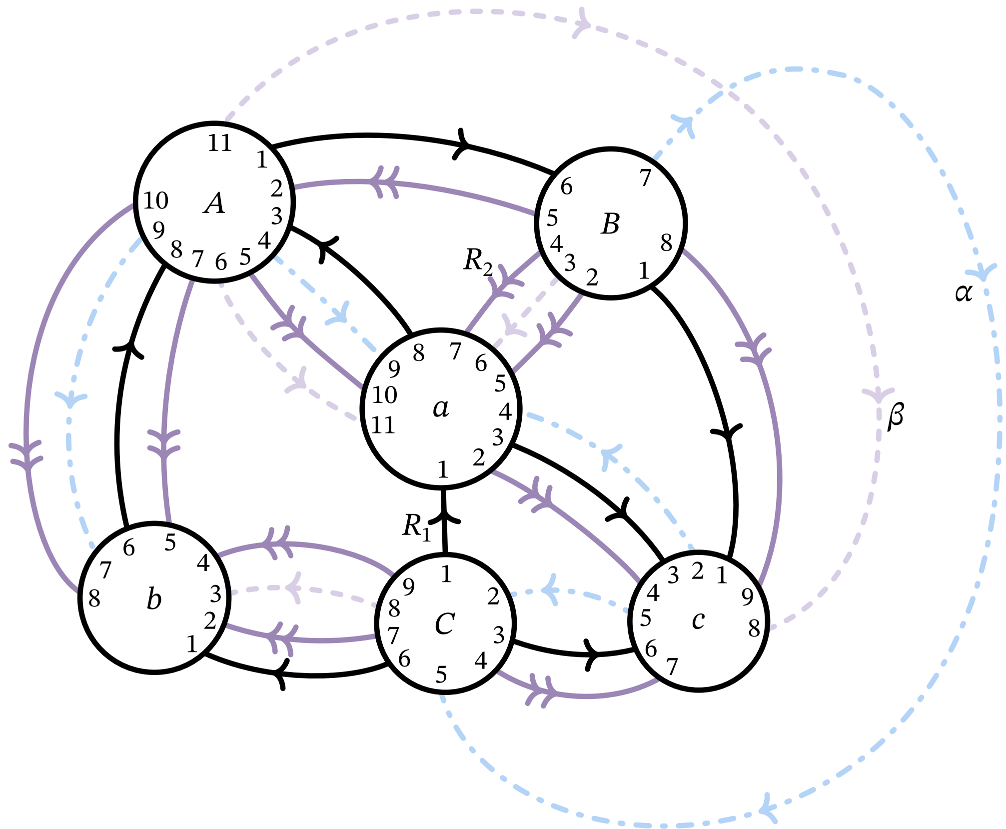

Figure 3.3. A Heegaard diagram for the first manifold $v3372$ in Table 3.2, corresponding to $\langle a, b, c \mid R_1 := ab^{-1}a^{-2}c^2bc = 1,\ R_2 := aba^{-1}c^2ba^2bcb = 1 \rangle$. Also shown are the slopes $\alpha = c^{-2}a^2b$ and $\beta = cba^2$ which give lens spaces $L(7,1)$ and $L(19,7)$, oriented so any positive combination of them gives an L-space.

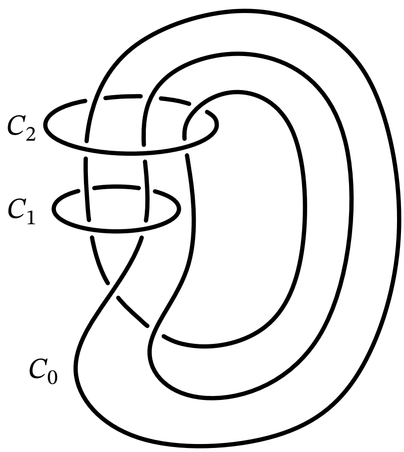

Figure 3.5. The link L.

[CDW] M. Culler, N. M. Dunfield, and J. R. Weeks. SnapPy v2.2, a computer program for studying the geometry and topology of 3-manifolds. Available from http://snappy.computop.org, (2014/7/8).

[DHL] N. M. Dunfield, N. R. Hoffman, and J. E. Licata. Ancillary files with the arXiv version of this paper. arXiv:TBA.

[Gab] D. Gabai. The Smale conjecture for hyperbolic 3-manifolds: $\mathrm{Isom}(M^3) \simeq \mathrm{Diff}(M^3)$. *J. Differential Geom.* **58** (2001), 113–149. MR1895350.

[Gor] C. M. Gordon. Lecture at AIM Workshop on Triangulations, Heegaard Splittings, and Hyperbolic Geometry. December, 2007.

[HIKMOT] N. Hoffman, K. Ichihara, M. Kashiwagi, H. Masai, S. Oishi, and A. Takayasu. Verified computations for hyperbolic 3-manifolds. code available from: http://www.oishi.info.waseda.ac.jp/~takayasu/hikmot/, 2013. arXiv:1310.3410.

[Mon] J. M. Montesinos. Variedades de seifert que son recubricadores ciclicos rami cados de dos hojas, Boletino Soc. *Mat. Mexicana* **18** (1973), 1–32.