1. To prove that set Z is countable, we could find a way to list all of its elements in a unique order.
We list 0 at first, and then list all positive integers from small to large. For each positive integer, put its opposite number behind it.

We can visualize this as:

n	$f(n)$
1	0
2	1
3	-1
4	2
5	-2
6	3
7	-3
...	...

From this list, we can derive the function that counts elements in Z with a unique order:

$$f(n) = \begin{cases} -\frac{n-1}{2} & n \text{ is odd } \& \ n \in N \\ \frac{n}{2} & n \text{ is even } \& \ n \in N \end{cases}$$

Here we can see that the list is one-to-one and onto, so it is a complete list with a unique ordering.

Now, we successfully prove that the set of integers Z is a countably infinite set.

2. We can write all rational numbers in the following table, with the numerators down the columns and the denominators across the rows. The way we list the numerator m and the denominator n is exactly the same as we mentioned in Question 1, that is list positive integers from small to large, and for each positive number, put its opposite number behind it. For the numerator m, we would list an extra 0 at first. Then we would cross out any repeated rational numbers, and follow each diagonal to start ordering as we encounter new rational numbers, as shown below:

$m \backslash n$	1	-1	2	-2	3	...
0	0/1	~~0/-1~~	~~0/2~~	~~0/-2~~	~~0/3~~	...
1	1/1	~~1/-1~~	1/2	~~1/-2~~	1/3	...
-1	-1/1	~~-1/-1~~	-1/2	~~-1/-2~~	-1/3	...
2	2/1	~~2/-1~~	~~2/2~~	~~2/-2~~	2/3	...
-2	-2/1	~~-2/-1~~	~~-2/2~~	~~-2/-2~~	-2/3	...
⋮	⋮	⋮	⋮	⋮	⋮	⋱

So, the order of rational numbers we have devised is:

$$\frac{0}{1},\ \frac{1}{1},\ \frac{-1}{1},\ \frac{2}{1},\ \frac{1}{2},\ \frac{-2}{1},\ \frac{-1}{2},\ \ldots$$

Here, each diagonal is a finite set of rational numbers, and there is a countable number of diagonals, so we could list all rational numbers in a unique order, so the list is one-to-one and onto. (Countable union of finite sets is countable) Now, we successfully prove that the set of all rational numbers is a countably infinite set.

3. First, we would prove that the set of all 2-tuples of $\mathbb{N}$: $\{(i,j) \mid i,j \in \mathbb{N}\}$ is a countably infinite set.
To do this, we can write all 2-tuples in the following table, with i down the columns and j across the rows.
Then we consider the diagonal lines and follow each diagonal to start ordering, as shown below:

i \ j	1	2	3	4	5	$\cdots$
1	(1,1)	(1,2)	(1,3)	(1,4)	(1,5)	$\cdots$
2	(2,1)	(2,2)	(2,3)	(2,4)	(2,5)	$\cdots$
3	(3,1)	(3,2)	(3,3)	(3,4)	(3,5)	$\cdots$
4	(4,1)	(4,2)	(4,3)	(4,4)	(4,5)	$\cdots$
5	(5,1)	(5,2)	(5,3)	(5,4)	(5,5)	$\cdots$
$\vdots$	$\vdots$	$\vdots$	$\vdots$	$\vdots$	$\vdots$	$\cdots$

So, the order of 2-tuples we have devised is:
$(1,1), (2,1), (1,2), (3,1), (2,2), (1,3), \cdots$
Here, each diagonal is a finite set of 2-tuples, and there is a countable number of diagonals, so we could list all 2-tuples in a unique order, so the list is one-to-one and onto. (countable union of finite sets is countable)
Till now, we successfully proved that the set of all 2-tuples of $\mathbb{N}$ is a countably infinite set.

Next, we would introduce a third number $k \in \mathbb{N}$ to all 2-tuples in order to generate the set of all 3-tuples of $\mathbb{N}$: $\{(i,j,k) \mid i,j,k \in \mathbb{N}\}$.

Each third number k would have 3 possible combinations with each 2-tuple (i,j) :
(k,i,j) , (i,k,j) , (i,j,k)
Here we would apply the same method that we used to prove 2-tuples, it also works for 3-tuples.
We can write all 3-tuples in the following table, with k down the columns and (i,j) across the rows. The way we list (i,j) is the same order that we devised for 2-tuples.
Then we consider the diagonal lines and follow each diagonal to start ordering. Since each combination of k and (i,j) would generate 3 3-tuples, we would order up to three 3-tuples at each point, and cross out any repeated 3-tuples as shown below :

k \ (i,j)	(1,1)	(2,1)	(1,2)	(3,1)	⋯
1	(1,1,1)	(1,2,1)	(1,1,2)	(1,3,1)	⋯
	~~(1,1,1)~~	(2,1,1)	~~(1,1,2)~~	(3,1,1)	⋯
	~~(1,1,1)~~	~~(2,1,1)~~	~~(1,2,1)~~	~~(3,1,1)~~	⋯
2	~~(2,1,1)~~	(2,2,1)	(2,1,2)	(2,3,1)	⋯
	~~(1,2,1)~~	~~(2,2,1)~~	(1,2,2)	(3,2,1)	⋯
	~~(1,1,2)~~	~~(2,1,2)~~	~~(1,2,2)~~	(3,1,2)	⋯
⋮	⋮	⋮	⋮	⋮	

So the order of 3-tuples we have devised is :
(1,1,1), (1,2,1), (2,1,1), (1,1,2), (1,3,1), ⋯
Here, each diagonal is a finite set of 3-tuples, and there is a countable number of diagonals, so we could list all 3-tuples in a unique order, so the list is one-to-one and onto. (countable union of finite ~~sets~~ sets is countable)
Now, we successfully prove that the set of all 3-tuples of N is countably infinite.

4(a). Here, all subsets of size 1 is infinite, all subsets of size 2 is infinite, all subsets of size 3 is infinite, etc.
This attempt is trying to union countable infinite sets together by a countable way. However, we could not say that the countable union of these countable infinite sets is countable because this attempt gives a bad order. Since the set of all subsets of size 1 is infinite, where should we start to list all subsets of size 2?
To sum up, this approach doesn't work because it gives a bad order and we could not get a complete list with a unique ordering.

4(b). We can list the set of all finite subsets of $\mathbb{N}$ in a different ~~number~~ way:

First, the empty set ϕ

Then, all subsets not exceeding size 1 and the maximum allowed number is 1 (cross out repeated subsets):

$\cancel{\phi}$, $\{1\}$

Then, all subsets not exceeding size 2 and the maximum allowed number is 2 (cross out repeated subsets):

$\cancel{\phi}$, $\cancel{\{1\}}$, $\{2\}$, $\{1,2\}$

Then, all subsets not exceeding size 3 and the maximum allowed number is 3 (cross out repeated subsets):

$\cancel{\phi}$, $\cancel{\{1\}}$, $\cancel{\{2\}}$, $\{3\}$, $\cancel{\{1,2\}}$, $\{1,3\}$, $\{2,3\}$, $\{1,2,3\}$

Then, continue ......

So the order of finite subsets of N we have devised is :

$$\phi, \{1\}, \{2\}, \{1,2\}, \{3\}, \{1,3\}, \{2,3\}, \{1,2,3\} \cdots$$

Here, "all subsets not exceeding size k and the maximum allowed number is k ($k \in N$)" is used to describe our sets for unique ordering, and each of this kind of set is a countable finite set of finite subsets of N. Also, there is a countable number of our designed sets $(1, 2, 3, \cdots)$.

Since the countable union of finite sets is countable, now we have a countable number of designed sets and each designed set is finite, we can list all finite subsets of N in a unique order, and the list is one-to-one and onto.

Therefore, we have successfully proved that the set of all finite subsets of N is countably infinite.

5(a). The alphabet Σ is finite, assume its size is N (N different symbols)

We would list all finite strings that can be generated from Σ in this way :

First, the empty set ϕ

Then, finite strings with length 1 :
since there are N different symbols in Σ, there would be N finite strings with length 1, the size of this set is N.
since this set is finite, we can give a complete list of its elements with a unique ordering.

Then, finite strings with length 2 :
since there are N different symbols in Σ, and each symbol could be concatenated with N different symbols, there would be N^2 finite strings with length 2, the size of this set is N^2.
since this set is finite, we can give a complete list of its elements with a unique ordering.

Then, finite strings with length 3 :
......

......

......

So the order of finite strings generated from Σ we have devised is :

ϕ

N finite strings with length 1
N^2 finite strings with length 2
N^3 finite strings with length 3
....

Here, each set with a specific length of strings is a finite set of finite strings, and there is a countable number for these sets (with the growth in length —— 1, 2, 3, ⋯).

Since the countable union of finite sets is countable, now we have a countable number of sets with strings in fixed length, and each set is finite, we can list all finite strings generated from Σ in a unique number, and the list is one-to-one and onto.

Therefore, we have successfully proved that the set of all finite strings that can be generated from Σ is countably infinite.

5(b). From 5(a), we already know that each finite string generated from Σ has a unique number.
Here, we could create a list of all possible computer programs in this way:

Check each finite string that can be generated from Σ:
~~#~~ if the string is not valid for a computer progam, skip it and keep going; if the string is valid for a computer program, write it into my list and keep going.

Since, each finite string generated from Σ has a unique number, now we just selected part of those strings into our list.
As a result, in my list of all possible computer programs, this is still a complete list with unique ordering, because it is one-to-one (each number represents a unique string) and onto (every string has a unique number).
Therefore, we have successfully proved that there are a countably infinite number of possible computer programs!

if number is different, the representing string is different.