

Due: Wednesday, September 1

## 1. Sets

- If set  $A$  has  $a$  elements and set  $B$  has  $b$  elements, how many elements are in the set  $A \times B$ ? Explain your answer.

## 2. Proofs

- (a) Find the error in the following proof.

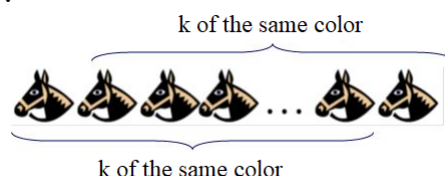
*Claim:* All horses are the same color.

*Proof:* We prove that any collection of horses is monochromatic by induction on the number of horses in the collection.

*Base Case:* Obviously, a set of one horse is a set of horses all with the same color.

*Induction Hypothesis:* Assume that any set of  $k$  horses are all the same color.

*Inductive Step:* Consider a set of  $k + 1$  horses, and stand them all in a line.



The first  $k$  horses in the line form a set of  $k$  horses, and so by the Inductive Hypothesis, are all the same color. The same is true for the last  $k$  horses in the line. Therefore the entire set consists of  $k + 1$  horses of the same color.

(b) Let  $S(n) = 1 + 2 + \cdots + n$  be the sum of the first  $n$  natural numbers and let  $C(n) = 1^3 + 2^3 + \cdots + n^3$  be the sum of the first  $n$  cubes. Prove the following through induction on  $n$ .

(i)  $S(n) = \frac{1}{2}n(n+1)$ .

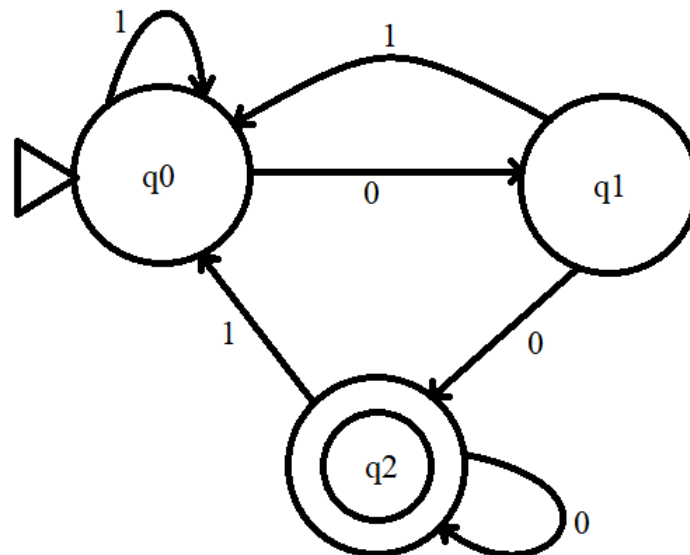
(ii)  $C(n) = \frac{1}{4}(n^4 + 2n^3 + n^2) = \frac{1}{4}n^2(n+1)^2 = S^2(n)$ .

### 3. Describing DFAs

For the following deterministic finite automaton  $M_1$ :

(a) Write out the full mathematical description of  $M_1$ .

(b) Determine what language  $M_1$  recognizes.



### 4. Creating DFAs

Draw DFAs for the following languages (you may assume the alphabet is always  $\{0, 1\}$ ):

(a)  $\{w \mid w \neq \varepsilon\}$

(b)  $\{w \mid w \neq 11 \text{ and } w \neq 111\}$

(c)  $\{w \mid w \text{ begins with a 1 and ends with a 0}\}$

(d)  $\{w \mid w \text{ contains the substring } 0101\}$

In other words,  $w = x0101y$  for some  $x$  and  $y$ . Note that both  $x$  and  $y$  are allowed to be the empty string, so your machine should accept 0101.

(e)  $\{w \mid \text{every odd position of } w \text{ is a } 1\}$

For numbering positions in the string, we will start at position 1. So the string '1' has a 1 in an odd position, while the string '010' has a 1 in an even position and 0s in the odd positions. Your machine should also accept the empty string.