1. According to the Cartesian Product of sets A and B :

$$A \times B = \{ (a,b) \mid a \in A , b \in B \}$$

each element of set A could be paired with each element of set B in the set $A \times B$.
As mentioned in this question, set A has a elements and set B has b elements. For each element in A, it could be paired with every single element in B, so there are $1 \times b = b$ possible combinations for one element in set A.
Since now we have a elements in set A, then the total number for all possible combinations should be $a \times b$.
Therefore, there are $a \times b$ elements in the set $A \times B$.

2(a). There is an error in Inductive Step.
Assume we have a set of two horses stand in a line, in which the first is black and the second is white.
If we put this case into Inductive Step, here our $k = 1$.
Based on the Inductive Hypothesis, the first $k (=1)$ horses are the same color, and the last $k (=1)$ horses are the same color. Therefore, the entire $k+1$ horses are the same color. However, this conclusion is definitely wrong because our two horses are in different colors, one in black and another in white. Here a contradiction occurs, the proof could not deal with our special case.
In conclusion, the proof fails at a set of two horses. When the two horses are in different colors, this proof would come to a completely opposite answer.

2(b). (i) $S(n) = \frac{1}{2}n(n+1)$

Base Case : $S(1) = 1 = \frac{1}{2} \times 1 \times (1+1)$

which clearly satisfies the formula

Inductive Hypothesis : consider the integer, k.

Assume that $S(k) = 1+2+\cdots+k = \frac{1}{2}k(k+1)$

Inductive Step : consider the integer $k+1$.

$$
\begin{aligned}
S(k+1) &= 1+2+\cdots+k+(k+1) \\
&= (1+2+\cdots+k)+(k+1) \\
&= S(k)+(k+1)
\end{aligned}
$$

From the Inductive Hypothesis, we have assumed that

$$S(k) = \frac{1}{2}k(k+1)$$

Hence

$$
\begin{aligned}
S(k+1) &= \frac{1}{2}k(k+1)+(k+1) \\
&= \frac{1}{2}(k+1) \times k + \frac{1}{2}(k+1) \times 2 \\
&= \frac{1}{2}(k+1)(k+2) \\
&= \frac{1}{2}(k+1)[(k+1)+1]
\end{aligned}
$$

which satisfies the formula

Till here, $S(n) = \frac{1}{2}n(n+1)$ get proved.

2(b). (ii) $C(n) = \frac{1}{4}(n^4 + 2n^3 + n^2) = \frac{1}{4}n^2(n+1)^2 = S^2(n)$

Base Case : $C(1) = 1^3 = 1$

$= \frac{1}{4} \times (1^4 + 2 \times 1^3 + 1^2)$

$= \frac{1}{4} \times 1^2 \times (1+1)^2$

$= S^2(1)$

which clearly satisfies the formula

Inductive Hypothesis : consider the integer k.

Assume that $C(k) = 1^3 + 2^3 + \cdots + k^3$

$= \frac{1}{4}(k^4 + 2k^3 + k^2)$

$= \frac{1}{4}k^2(k+1)^2$

$= S^2(k)$

Inductive Step : consider the integer $k+1$.

$C(k+1) = 1^3 + 2^3 + \cdots + k^3 + (k+1)^3$

$= (1^3 + 2^3 + \cdots + k^3) + (k+1)^3$

$= C(k) + (k+1)^3$

From the Inductive Hypothesis, we have assumed that

$C(k) = \frac{1}{4}(k^4 + 2k^3 + k^2) = \frac{1}{4}k^2(k+1)^2 = S^2(k)$

Hence $C(k+1) = \frac{1}{4}(k^4 + 2k^3 + k^2) + (k+1)^3$

$= \frac{1}{4}k^2(k^2+2k+1) + \frac{1}{4}[4(k+1)^3]$

$= \frac{1}{4}[k^2(k+1)^2 + 4(k+1)^3]$

$= \frac{1}{4}(k+1)^2[k^2 + 4(k+1)]$

$= \frac{1}{4}(k+1)^2(k^2+4k+4)$

$= \frac{1}{4}(k+1)^2[(k^2+2k+1) + (2k+2) + 1]$

$= \frac{1}{4}(k+1)^2[(k+1)^2 + 2(k+1) + 1]$

$= \frac{1}{4}[(k+1)^4 + 2(k+1)^3 + (k+1)^2]$ ①

$= \frac{1}{4}(k+1)^2[(k+1)^2 + 2(k+1) + 1]$

$= \frac{1}{4}(k+1)^2(k^2+4k+4)$

$= \frac{1}{4}(k+1)^2(k+2)^2 = \frac{1}{4}(k+1)^2[(k+1)+1]^2$ ②

$= S^2(k+1)$ ③

combine line ①, ② and ③ together:

$$C(k+1) = \frac{1}{4}[(k+1)^4 + 2(k+1)^3 + (k+1)^2]$$
$$= \frac{1}{4}(k+1)^2(k+2)^2 = \frac{1}{4}(k+1)^2[(k+1)+1]^2$$
$$= S^2(k+1)$$

which satisfies the formula

Till here, $C(n) = \frac{1}{4}(n^4+2n^3+n^2) = \frac{1}{4}n^2(n+1)^2 = S^2(n)$ get proved.

3(a). $Q = \{q_0, q_1, q_2\}$

$\Sigma = \{0, 1\}$

δ is defined by:

$\delta(q_0, 0) = q_1$, $\delta(q_0, 1) = q_0$,

$\delta(q_1, 0) = q_2$, $\delta(q_1, 1) = q_0$,

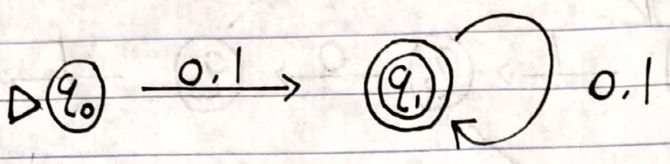
$\delta(q_2, 0) = q_2$, $\delta(q_2, 1) = q_0$.

q_0 is given as the start state

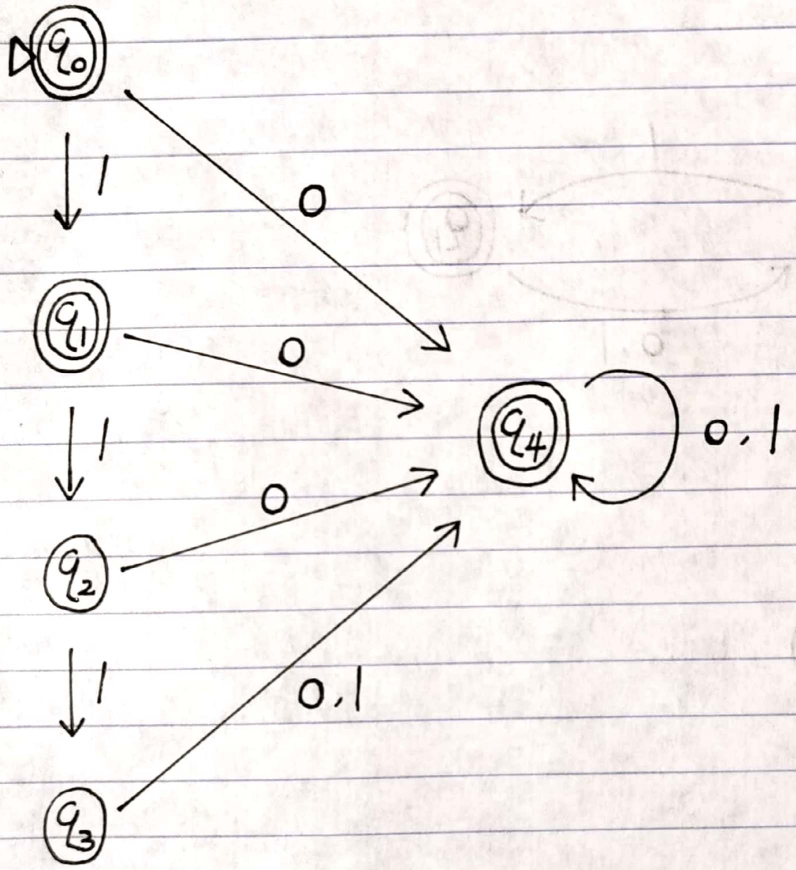
$F = \{q_2\}$

3(b). $\{w \mid w$ ends with at least two 0 $\}$

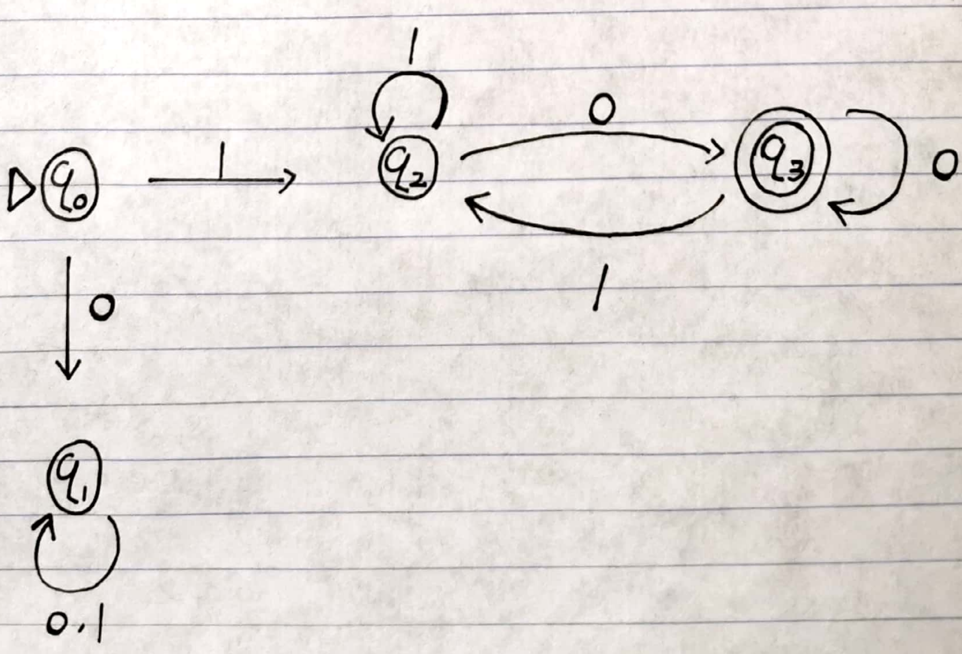
4(a).



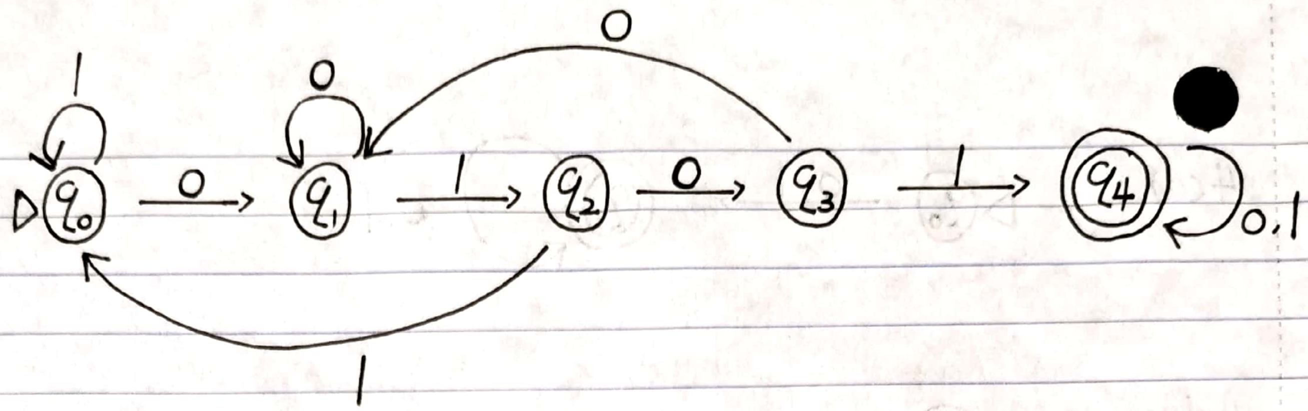
4(b).



4(c).



4(d).



4(e).