Math 560 Homework 4 Fall 2021

Due: Friday, September 24

Note: You are **not allowed** to simply use the fact that the countable union of *countable* sets is countable, since this is not a trivial proof (you are welcome to prove this before using it). You **are allowed** to use the fact that the countable union of *finite* sets is countable.

1. **Integers** [1 pt]

   Prove that the set of integers, $\mathbb{Z}$, is a countably infinite set.

2. **All Rationals** [1 pt]

   Prove that the set of all rational numbers,

   $$\mathbb{Q} = \left\{ \frac{m}{n} \,\middle|\, m, n \in \mathbb{Z}\,,\; n \neq 0 \right\},$$

   is a countably infinite set.

3. **3-tuples** [2 pts]

   Prove that the set of all 3-tuples of $\mathbb{N}$ is countably infinite, i.e., prove that the set

   $$\{(i, j, k) \mid i, j, k \in \mathbb{N}\}$$

   is a countably infinite set. (Hint: note that positive rational numbers m/n can be rewritten as 2-tuples (m, n) of natural numbers. Can you somehow generalize the idea of that proof?)

4. **Finite Subsets of** $\mathbb{N}$ [3 pts]

   Here you will prove that the set of all **finite** subsets of $\mathbb{N}$ is countably infinite, i.e.,

   $$\{\, \{n_1, n_2, \ldots n_k\} \mid n_1, n_2, \ldots, n_k \in \mathbb{N} \textbf{ and } k \in \mathbb{N} \,\}.$$

   (a) A first attempt at this problem might be to say "list all subsets of size 1, then all subsets of size 2, then all subsets of size 3, etc." Explain why this approach doesn't work.

   (b) Provide the correct proof that the set of all finite subsets of $\mathbb{N}$ is countably infinite. (Hint: the idea from part (a) was a good start, but was not sufficient by itself.)

5. **Computer Programs** [3 pts]

   (a) Given a **finite** alphabet Σ, prove that the set of all **finite** strings that can be generated from the alphabet is a countably infinite set, i.e., $\{w \mid w \in \Sigma^*,\ |w| \in \mathbb{N}\}$. (Hint: very similar to a proof about binary strings we saw in class.)

   (b) Assuming that computers programs are finite in length, use part (a) to prove that there are a countably infinite number of possible computer programs.