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A Complete List of Minimal Diagrams of an Alternating Knot

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ABSTRACT

By the Taite Flyping Conjecture all all possible minimal diagrams of an alternating knot are related by a finite sequence of flypes. We we use this fact to generate all possible minimal diagrams of an alternating knot. Starting with any given minimal diagram our algorithm performs flypes exhaustively until all possible diagrams have been produced. We display our results for all distinct alternating knot diagrams up to 16 crossings. We discuss knots families that have only a small number of distinct minimal diagrams and knot families that have a large number of minimal diagrams.

Keywords: alternating knot; flype; knot diagram, maps and immersions.

Mathematics Subject Classification 2000: 57M25

1. Introduction and basic concepts

In this article we will only consider alternating knots. Let D be a regular minimal diagram of an alternating knot K on the sphere S^2. Then D can be viewed as a 4-regular planar graph G_D, whose vertices are the crossings of D and the under and over information is ignored. The graph G_D defines the knot type up to the mirror image of the knot.

A tangle T in a knot diagram D is a region R bounded by a simple closed curve that intersects D at exactly four points with the restriction that none of these intersections occur at a crossing. A *flype* is an operation on D whereby a tangle is flipped 180 degrees such that a crossing C (connected to T by two arcs) moves to the opposite side of the tangle (see Figure 1).

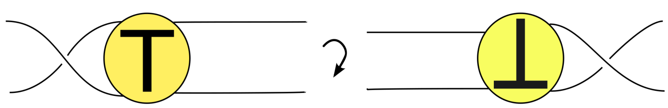


Fig. 1. A flype of tangle T in a knot diagram.

For a flype the *parity* of the tangle T desribes the connection of the strands within the tangle in relation to the position of the crossing C. In Figure 2 we assume that the crossing C is directly connected to either the two arcs on the right or on the left of the tangle T. A tangle can have parity (1), (0), or (∞) as shown in Figure 2. Note that arcs in the tangles in Figure 2 can have an arbitrary number of crossing in between them, the figure just indicates where the strands enter and exist the tangle.

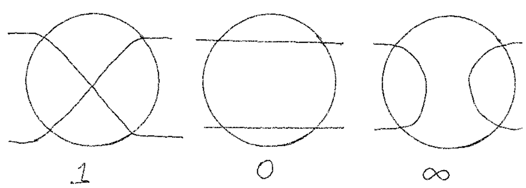


Fig. 2. From left to right the connection of arcs for parity (1), (0), and (∞) tangles where the crossing C is either on the right or left of the tangle.

To record the different knot diagrams we will use the *planar diagram code* (or *PD code*) an encoding of a knot diagram that can be easily processed and manipulated by a computer [5]. To obtain a PD code, a knot diagram must first be *oriented*. Pick any strand and label it with a 1. Follow the orientation that was set and every time a crossing is encountered, increase the label of the new strand by one. At the end of this process there are exactly $2n$ labels where n is the number of crossings in the diagram. Now, at each crossing there are four strands using four labels. Therefore, we can represent each crossing by a quadruple of the four integers that label the four strands. By convention the first index in the quadruple to be the incoming understrand, and the order of the next three indices is determined by moving counter-clockwise around the crossing from the incoming understrand. These n ordered quadruples are the PD code of the diagram. For example, the PD code of the 6-crossing knot in Figure 3 is $\{[1,9,2,8],[5,3,6,2],[3,10,4,11],[9,4,10,5],[11,7,12,6],[7,1,8,12]\}$.

For oriented diagrams we will use the standard crossing sign convention as shown in Figure 4 and we will use the symbol $sign(C)$ to denote the sign of the crossing C.

The Tait-Flyping Conjecture was proved by Menasco and Thistlethwaite [7,8] and we state it as a theorem.

Theorem 1.1. *Given two minimal alternating diagrams D and D' of the same knot type on S^2, then D' can be obtained from D by a finite series of flypes .*

A direct consequence of the conjecture is that given a single minimal diagram

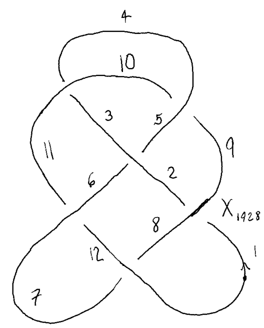


Fig. 3. A 6 crossing knot diagram (strands labeled).

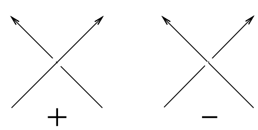


Fig. 4. Left: a positive crossing. Right: a negative crossing.

of an alternating knot, all other minimal diagrams of that knot can be obtained by exhaustively performing all possible flypes on the given diagram and continuing this process recursively until no new diagrams are being generated. The algorithms explained in this paper detail how the generation of all minimal diagrams of a given alternating knot can be accomplished.

REWRITE This project utilizes the Mathematica and C programming languages along with libraries and packages for each. In particular, the KnotTheory package for Mathematica [5] and a library called plcurve written by Jason Cantarella et al. for C [6] were used.

2. Identifying and Performing Flypes

2.1. *Identifying Flypes*

To perform flypes, we must first be able to identify tangles given a knot diagram D. To do this, we consider the 4-regular planar graph G_D. Recall from the introduction that a tangle is a region in the knot diagram that can be bounded by a simple closed curve that intersects the diagram at exactly four points, not including crossings. When this concept is transferred to the graph G_D, a tangle T is defined by a set of vertices that can be disconnected from the graph G_D through the removal of four edges. In particular, any subset of four edges of the graph G_D that, when removed, disconnects G_D into two components will generate two tangles. Namely, the two sets of vertices that comprise the distinct components correspond to two tangles in the knot diagram D. We can exhaustively check all four edge subsets of the graph and arrive at a complete list of tangles for the knot diagram D. When compiling the list of tangles all four edge subsets connected to a single vertex are ignored because these represent trivial tangles consisting of a single crossing.

To identify all flypes we check which tangles share exactly two strands with another crossing. Note that a tangle cannot be flyped if the four strands coming out of the tangle connect to four separate crossings. Figure 5 shows all the possible flypes for a diagram of the knot 10_{10} where the tangle to be flyped is circled and the crossing it is going to be flyped over is circled as well. Notice that the flypes occur in pairs. That is, if a crossing C can be flyped over a tangle T, then all the crossings the diagram that are not in $T \cup C$ will also form a tangle T'. For example, the flype in the top left of Figure 5 matches the right most graph in row six of the same Figure. Moreover many of the flypes shown are no leading to different diagrams because we are flyping crossings within a group of halftwists. For example, all the diagrams in the top row of Figure 5 except the leftmost are of this type. We will deal with the elimination of duplicate diagrams in Section 3.1.

2.2. *Performing Flypes on Knots*

The methods in Subsection 2.1 can be applied to both knots and links (links are disjoint unions of several knots where the circles are linked). In this section, we assume that we are only dealing with knots.

Given the PD code of an oriented knot diagram, we define *parallel* and $anti-parallel$ flypes depending on the orientation of the two arcs that connect the crossing C to the tangle T, see Figure 6. Notice that the notion of a parallel or anti-parallel flype is independent of the orientation since reversing the orientation of the knot does not change the fact that the flype is parallel or anti-parallel. The difference between a parallel or anti-parallel flype has a substantial effect on the algorithms for executing the flypes. Note that the values in a PD code of a diagram of n crossings are integers between 1 and $2n$, inclusive. Thus, whenever adding a value to a variable creates a number above $2n$, we will reduce the integer by $2n$. Similarly, if subtracting a value creates a value below 1, we will increase the integer by $2n$. The reader should keep this in mind when reading the algorithms of Subsections 2.3 and 2.4.

2.3. *The Parallel Flype Algorithm*

In an oriented alternating knot diagram D a parallel flype is given by a crossing C and a tangle T all encoded in the PD code format. We define the *instrands* of T as the strands that are connected to both C and to T. The other two strands connected to T are called the *outstrands*. In the following, all variables denote the labels given to the arcs of the knot diagram by the PD code. Label the four strands

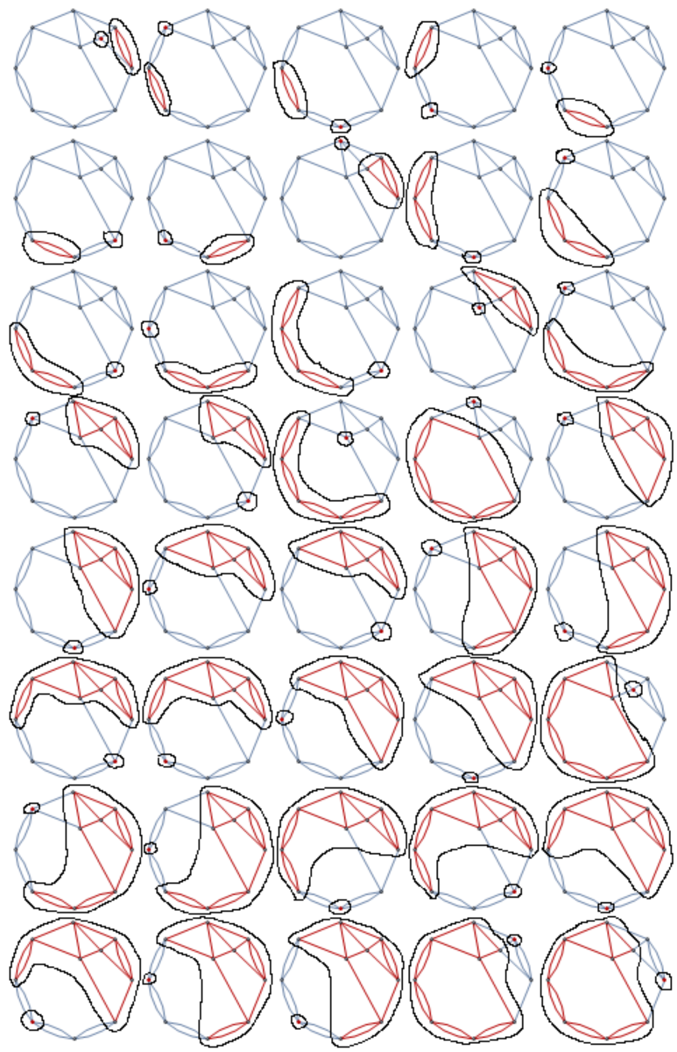


Fig. 5. All possible flypes in Knot 10_{10}.

connecting to T as a, b, c, and d as follows: We set variables a and b to the two instrands where a is the instrand that originates at the underpass in C and b is the instrand that originates at the overpass in C. Set c and d to be the outstrands that, when traversed through the tangle, connect to a and b, respectively. Also note that in Figure 6, either figure in the top can be transformed to the other through a parallel flype. If you traverse the crossing C in the knot diagram D using the given orientation and then immediately enter the tangle T then we call the flype a *non-reversed* parallel flype, otherwise we say it is a *reversed* parallel flype. We decide

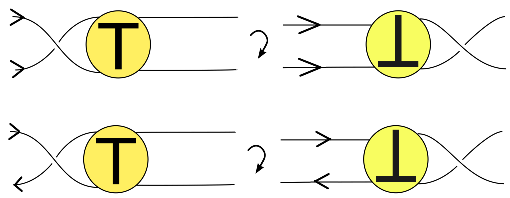


Fig. 6. Top: A parallel flype. Left: non-reversed; Right: reversed. Bottom: An anti-parallel flype.

this by checking where a shows up in the PD code of C. If a is the first number in the quadruple of the PD code corresponding to C, then the flype is reversed. If a is the third number, then the flype is non-reversed. Upon performing a flype, all crossings in T will be flipped. That is, overstrands in T will become understrands and vice versa. The order of the edges around the crossings reverses as well. The transformation is as follows for each crossing in T:

$$[i,j,k,l] = \begin{cases} [l,k,j,i] & \text{if the crossing was positive} \\ [j,i,l,k] & \text{if the crossing was negative} \end{cases} \tag{2.1}$$

Now, all of the values in T need to be changed based on whether the flype was reversed or non-reversed. If the flype was reversed, meaning that the new crossing is now going to show up before entering the tangle, all of the values in the tangle must be incremented by 1. This incrementation is due to the fact that the new crossing in the diagram after the flype will contribute an extra incrementation to the strand labeling before entering the tangle. By using similar logic, if the flype was non-reversed, the values in the tangle must be decremented by 1. The last thing left to do is to determine the new PD code for C which has been moved to the other side of T. To do this, we consider the parity of the tangle along with whether or not the flype was reversed and whether or not C was a positive or negative crossing. The new crossing is decided as follows:

$$C = \begin{cases} \text{if T is parity (1)} & \begin{cases} \text{if reversed} & \begin{cases} [d,c,d+1,c+1] & \text{if } sign(C) < 0 \\ [d,c+1,d+1,c] & \text{if } sign(C) > 0 \end{cases} \\ \text{if non-reversed} & \begin{cases} [d-1,c-1,d,c] & \text{if } sign(C) < 0 \\ [d-1,c,d,c-1] & \text{if } sign(C) > 0 \end{cases} \end{cases} \\ \text{if T is parity (0)} & \begin{cases} \text{if reversed} & \begin{cases} [c,d,c+1,d+1] & \text{if } sign(C) < 0 \\ [c,d+1,c+1,d] & \text{if } sign(C) > 0 \end{cases} \\ \text{if non-reversed} & \begin{cases} [c-1,d-1,c,d] & \text{if } sign(C) < 0 \\ [c-1,d,c,d-1] & \text{if } sign(C) > 0 \end{cases} \end{cases} \end{cases}$$

2.4. *The Anti-Parallel Flype Algorithm*

As in Subsection 2.3 we are given a diagram D, a tangle T, and a crossing C in the PD code format. W.l.o.g. we assume that the crossing C and the tangle T do not contain the crossing where that contains the values 1 and $2n$. (If they do we increment all values in the PD code by a constant.) Let a to the instrand which is entering the tangle and b to the instrand that is exiting the tangle. We split the algorithm in two cases: T does not have parity (∞) and T has parity (∞).

Case 1 T **has parity** (∞)**.** Let c be the outstrand that is entering the tangle and let d be the outstrand that is exiting the tangle. As in the parallel flyping algorithm the crossings in T will be flipped using eqn. (2.1).

We renumber strand labels s that occur in the quadruples of integers representing the crossings according to the following logic. If $a > c$ $(a < c)$ then the crossing where the arc labelled $2n$ changes to the arc labelled 1 does (not) occur when traversing along the knot from the arc labelled b to the arc labelled c (when traversing from d to a). This leads to two cases:

$$s = \begin{cases} s-1 & \text{if } a < c \\ s+1 & \text{if } a > c \end{cases} \text{ for crossings in } T.$$

For crossings not in $T \cup C$ we only change only some of the labels s according to the following criteria

$$s = \begin{cases} s-2 & \text{if } b < s \leq c \text{ is true} \\ s+2 & \text{if } d \leq s < a \text{ is true.} \end{cases}$$

Creation of the new crossing is decided in a similar fashion as with the parallel flyping algorithm. To do this, we consider whether or not $a < c$, whether or not C was a positive or negative crossing, and whether or not c represented a strand that was an overpass in T before the flype occurred. The details to check the various cases that follow will be left to the reader.

$$C = \begin{cases} \text{if } a < c & \begin{cases} \text{sign}(C) > 0 & \begin{cases} d-1, c-1, d, c-2 & c \text{ an overpass in } T \\ c-2, d, c-1, d-1 & c \text{ an underpass in } T \end{cases} \\ \text{sign}(C) < 0 & \begin{cases} d-1, c-2, d, c-1 & c \text{ an overpass in } T \\ c-2, d-1, c-1, d & c \text{ an underpass in } T \end{cases} \end{cases} \\ \text{if } a > c & \begin{cases} \text{sign}(C) > 0 & \begin{cases} d+1, c+1, d+2, c & c \text{ an overpass in } T \\ c, d+2, c+1, d+1 & c \text{ an underpass in } T \end{cases} \\ \text{sign}(C) < 0 & \begin{cases} d+1, c, d+2, c+1 & c \text{ an overpass in } T \\ c, d+1, c+1, d+2 & c \text{ an underpass in } T \end{cases} \end{cases} \end{cases}$$

Case 2 T **does not have parity** (∞)**.**

Let c and d be the outstrands that connect to a and b respectively. As in the parallel flyping algorithm the crossings in T will be flipped using eqn. (2.1).

We renumber strand labels s according to the following logic as follows: For each crossing in T:

$$s = \begin{cases} s-1 & \text{if } a \leq s \leq c \\ s+1 & \text{if } d \leq s \leq b \end{cases}$$

No changes of need to be made for the crossings outside of T: Creation of the new crossing is decided in a similar fashion as in the previous cases and we leave the details to the reader. To do this, we consider whether or not C was a positive or negative crossing, and whether or not c represented a strand that was an overpass in T before the flype occurred.

$$C = \begin{cases} \text{sign}(C) > 0 & \begin{cases} d, c, d+1, c-1 & c \text{ an overpass in } T \\ c-1, d+1, c, d & c \text{ an underpass in } T \end{cases} \\ \text{sign}(C) < 0 & \begin{cases} d, c-1, d+1, c & c \text{ an overpass in } T \\ c-1, d, c, d+1 & c \text{ an underpass in } T \end{cases} \end{cases}$$

Example 2.1. Example of the flyping algorithm.

In Figure 7 four different diagrams of the knot 8_{13} are shown. The top-left diagram has PD-code $\{[1,8,2,9],[9,2,10,3],[3,13,4,12],[15,5,16,4],[5,15,6,14],[13,7,14,6],[11,1,12,16],[7,10,8,11]\}$. Flyping across the tangle encircled by the dashed curve using the crossing encircled with the dashed curve produces the diagram at the bottom left in Figure 7 (this is a parallel flype). Flyping across the tangle encircled by the solid curve using the crossing encircled with the solid curve produces the diagram at the bottom right in Figure 7 (this is an anti-parallel flype). Using the diagram at the bottom left to flype across the encircled tangle using the encircled crossing produces the diagram at the top right in Figure 7 (this is also an anti-parallel flype). The four diagrams in Figure 7 are the complete list of diagrams of the knot 8_{13}, see section 3.3.

We first discuss the parallel flype with the tangle $T = \{[13,7,14,6],[5,15,6,14],[15,5,16,4]\}$ and $C = [3,13,4,12]$. Here $a = 4$ and $b = 13$ are the instrands, $c = 7$ and $d = 16$ are the outstrands and the flype is non-reversed. Flipping the crossings in T (all are positive) and decrementing by one (since the flype is non-reversed) yields the new tangle $T = \{[5,13,6,12],[13,5,14,4],[3,15,4,14]\}$. The parity of the tangle T is (1) and $sign(C) > 0$ so the new crossing will be $[d-1,c,d,c-1] = [15,7,16,6]$. The four crossings not in $T \cup C$ remain unchanged. Therefore we obtain the PD-code for the diagram at the bottom left in Figure 7 as $\{[1,8,2,9],[9,2,10,3],[11,1,12,16],[7,10,8,11],[5,13,6,12],[13,5,14,4],[3,15,4,14],[15,7,16,6]\}$. Next we discuss the anti-parallel flype with the tangle $T = \{[1,8,2,9],[9,2,10,3]\}$ and $C = [7,10,8,11]$. Then $a = 8$ and $b =$

10 are the instrands, $c = 1$ and $d = 3$ are the outstrands the flype T has parity (∞). Flipping the crossings in T (all are negative) and incrementing by one (since $a > c$) yields the new tangle $T = \{[9, 2, 10, 3], [3, 10, 4, 11]\}$. $sign(C) < 0$, $a > c$ and c connects to an underpass in T so the new crossing will be $[c, d+1, c+1, d+2] = [1, 4, 2, 5]$. The five crossings not in $T \cup C$ are $\{[3, 13, 4, 12], [15, 5, 16, 4], [5, 15, 6, 14], [13, 7, 14, 6], [11, 1, 12, 16]\}$. We have $d < a$ so we add two all values $d \leq s < a$ and obtain $\{[5, 13, 6, 12], [15, 7, 16, 6], [7, 15, 8, 14], [13, 9, 14, 8], [11, 1, 12, 16]\}$. This results in the PD-code for the diagram at the bottom right in Figure 7 as $\{[5, 13, 6, 12], [15, 7, 16, 6], [7, 15, 8, 14], [13, 9, 14, 8], [11, 1, 12, 16], [1, 4, 2, 5], [9, 2, 10, 3], [3, 10, 4, 11]\}$.

We leave the details of how to obtain the diagram on the top right in Figure 7 to the reader.

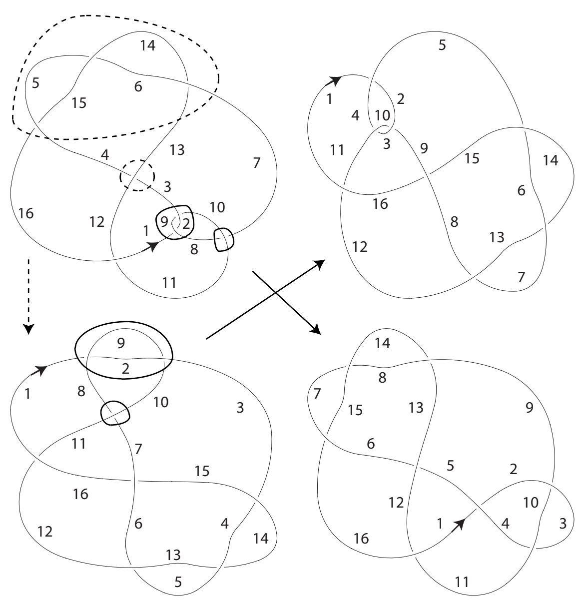


Fig. 7. The knot 8_{13} has four non-equivalent diagrams.

3. Results

3.1. *The Algorithm to Generate All Minimal Diagrams*

The algorithm described in Section 2 can be used to generate all diagrams that can be obtained with a single flype from a given alternating diagram. However we now need to ask the question when are different diagrams actually distinct.

A program in the plcurve library [6] is used to answer this question. The program checks for diagram isotopy assuming that the knot is oriented and that the diagram is embedded on an unoriented sphere. We will call this the the OU^* (oriented knot, unoriented S^2) category similar to the language used in [2]. Define the set $allMinimalDiagrams$ as the set of all minimal diagrams (given by their PD-codes) obtainable from the given diagram D by flypes. Note that D is oriented by its PD-code. Initially $allMinimalDiagrams$ contains only the PD code of D. We then detect and execute all possible flypes in D. This results in a set $newMinimalDiagrams$ of all minimal diagrams that are one flype away from D. For each diagram in $newMinimalDiagrams$, we check to see if it represents a diagram that is already in $allMinimalDiagrams$ using the plcurve library. If it does, the algorithm does not consider that specific diagram anymore. If it does not, the PD code of the diagram is appended to $allMinimalDiagrams$ and the process of detecting and performing all possible flypes is continued on that diagram. This approach is recursive so there is not a set number of iterations for this process. However the algorithm will generate all minimal diagrams obtainable from the diagram D by flypes. By Theorem 1.1 there are only finally many such diagrams the process must after a finite number of iterations. This obtains the set of all minimal diagrams - distinct under the OU^* category - of the knot type that related to the oriented diagram D by a finite sequence of flypes. We note that this may not be the complete set of minimal diagrams of the knot type, additional diagrams in the OU^* category may result from considering the mirror image of the knot type for chiral knots and by reversing the orientation of the diagram D.

3.2. *Counting diagrams in the OU category*

3.3. *Comparing the count in the OU^* category with other results*

In [2] entitled "maps, immersions and permutations" the authors study the problem of counting and of listing topologically inequivalent planar 4-valent maps with a single component and a given number n of vertices. In particular, they produce a count of irreducible indecomposable spherical immersions of an oriented circle in an unoriented 2-sphere and denote this as OU category. Here a diagram is irreducible and indecomposable, if it cannot be disconnected by the removal of a vertex or by cutting two distinct edges. An alternating diagram D of a prime knot can be thought of as an indecomposable spherical immersion if one forgets the over and under information in the diagram and just considers the diagram as the 4-regular graph G_D on the sphere. Thus the numbers from our computation in the OU^* category should very closely related to the numbers given in [2] for the OU category. In the table below we compare the two counts.

crossings	3	4	5	6	7	8	9	10	11
OU-diagrams[2]	1	1	2	3	11	38	156	638	2973
OU^*-diagrams	1	1	2	4	11	41	152	639	2646

DJ - I DO NO REMEMBER WERE DID THE 11 OU NUMBER COME FROM? READ THE STUFF BELOW CAREFULLY

We can get agreement between the two categories as follows: First, we doubled the number of PD-codes using the PD-codes in the set of OU^*-diagrams by adding the diagrams with the orientation reversed. We removed duplicates using the plcurve library [6] and this resulted in the OU-diagram numbers.

However we can also explain the differences without additional computation. We first deal with the odd crossing numbers: For 9 crossings there are two non reversible knots (9_{32} and 9_{33}) with four diagrams and for 11 crossings there are 123 non reversible knots with 327 diagrams. Thus we have for a fixed odd crossing number that the number of OU^*-diagrams plus number of diagrams of non reversible knots among the OU^*-diagrams equals the number of OU-diagrams.

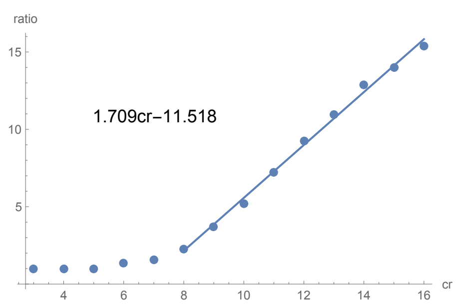
For even crossing numbers the story is more complicated because we also need to consider achiral knots. (Since alternating achiral knot have writhe zero there are none for the odd crossing numbers.) The knot 6_3 is fully amphichiral and we obtain this knot twice representing diagrams that are mirror images. So we need to subtract one from the OU^* number to obtain the OU number. For 8 crossings there are four knots (8_3, 8_9, 8_{12} and 8_{18}) that are fully amphichiral with 9 diagrams (1,2,5 and 1 diagrams respectively). These diagrams contain 3 pairs or mirror images. In addition the knot 8_{17} is negatively amphicheiral that is, it is equal to the inverse of its mirror, with one diagram. Thus if we subtract three (the three pairs of mirror images) from the OU^* number to obtain the OU number. For 10 crossings there are seven knots (10_{17}, 10_{33}, 10_{37}, 10_{43}, 10_{45}, 10_{99}, and 10_{123}) that are fully amphichiral with 70 diagrams (2,8,8,18,32,1, and 1 diagrams respectively). These diagrams contain 34 pairs or mirror images. There are 21 chiral knots that are non-reversible with 33 diagrams. In addition, there knot six negatively amphicheiral knots (10_{79}, 10_{81}, 10_{88}, 10_{109}, 10_{115}, and 10_{118}) with 12 diagrams that are also non-reversible. However since they are also amphicheiral they do not need to be considered. Thus if we subtract 34 (the 34 pairs or mirror images) and add 33 (the 33 diagrams of chiral knots) to the OU^* number one obtains the OU number.

3.4. *Results for higher crossing numbers and the distribution of the number of flype equivalent diagrams with a fix crossing number.*

The table shows our results about the number of distinct OU^*-diagrams for all alternating knot types in the knot table up to 16 crossings. The last row shows the ratios of the number of diagrams divided by the number of alternating knot types.

crossings	10	11	12	13	14	15	16
alt knot types	123	367	1,288	4,878	19,536	85,263	379,799
OU^*-diagrams	639	2,646	11,928	53,471	251,324	1,195,160	5,851,499
ratio	5.20	7.21	9.26	10.96	12.86	14.02	15.41

Figure 3.4 on the left shows the ratios of the number of diagrams divided by the number of alternating knot types for crossing numbers from three to sixteen. There is best fit line using as data the crossing numbers from eight to sixteen that illustrated that the ratio seems to increase roughly linearly with the crossing number. Figure 3.4 on the right shows the distribution of diagrams with sixteen crossings. The number of diagrams (diagrams-axis) for a given number of diagrams (flypes-axis) for 16 crossings. In the plot the points are connected by a line segments and it is clear that most knot types have only a very small number of diagrams. We will discuss in more detail in the next paragraph.



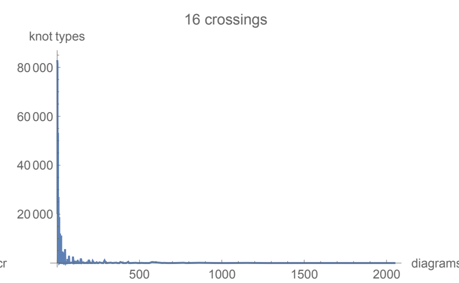


Fig. 8. Left:The ratios of the number of diagrams divided by the number of alternating knot types for a fixed crossing number. Right: The number of knots types (knot types-axis) for a given number of diagrams (digrams-axis) for 16 crossings.

For 16 crossing there are 148 different values of the number of diagrams, ranging from one to 2048. Out of these values only fourteen are prims given by $\{2, 3, 5, 7, 11, 13, 17, 19, 23, 31, 41, 43, 113, 163\}$; all powers of two are present (up to 2^{12}) and all products of three times a power of two are present (up to 3×2^9).

Om order to compare the distributions of the number of diagrams for different crossing numbers, we report a relative frequency. That is for a fixed crossing number we divide the number of knot types with a given number of diagrams by the total number of alternating knot types for that given crossing number. Figure 3.4 shows the results for the first seventeen data points for crossing numbers 11 to 16. (We do not report on the smaller crossing numbers since there are not even seventeen data points.) It is striking to see how similar the distribution looks.

The table below summarizes the similarity of the distributions in a different way for crossing numbers 14, 15 and 16. The first column gives the percentage of

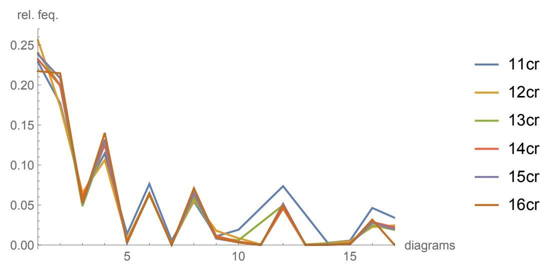


Fig. 9. Left:The ratios of the number of diagrams divided by the number of alternating knot types for a fixed crossing number. Right: The number of knots types (knot types-axis) for a given number of diagrams (digrams-axis) for 16 crossings.

knot types with one diagram, the columns labelled 2, 4, 8, 16 and 32 gives the percentage of knot types where the number of diagrams is divisible by 2^k but not divisible by 2^{k+1}. The second to last column gives the percentage of knot types with three diagrams and the last column gives the percentage of knot types where the number of diagrams is a prime greater than three. We note that there are very few knot types that have a prime number (greater than three) of diagrams.

diagrams	1	2	4	8	16	32	3	prime
14 cross %	23.2	29.2	19.8	11.0	5.4	2.4	6.0	.5
15 cross %	23.8	29.5	20.5	11.0	5.2	2.1	5.2	.3
16 cross %	21.7	30.0	20.9	11.4	5.5	2.3	5.3	.3

3.5. *Results on the number of flype equivalent diagrams*

Theorem 3.1. *For any $n \in \mathbb{N}$ there exists alternating knot types with exactly n distinct minimal diagrams.*

Proof. All we need to do is the construct knot types with only a single non trivial flyping circuit that admits exactly n distinct flyping positions. This can be done in many ways, for example we can use two distinct Conway basic polyhedra together with $n-1$ halftwists to construct a single flyping circuit. An example is shown in Figure 10. Here we use the Conway basic polyhedras 8^* and 10^* together with $n = 3$ halftwists giving rise to four distinct diagrams. □

Next we address the question: Given an alternating knot type with crossing number n, how many distinct minimal diagrams are possible? What knot families admit the largest number of flypes? The next theorem will give a partial answer to this question.

Theorem 3.2. *For any $n \in \mathbb{N}$ with $n \geq 6$ there exists knot types with exactly 2^{n-5} distinct minimal diagrams for n even and $3 * 2^{n-7}$ distinct minimal diagrams for n odd.*

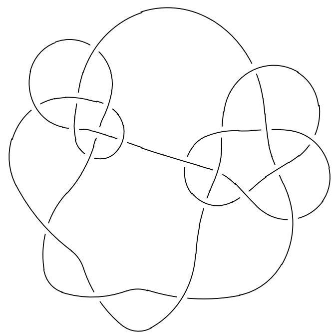


Fig. 10. A 21 crossing knot with a single non trivial flyping circuit that gives rise to four minimal non-equivalent diagrams.

We conjecture that for a given crossing number n 2^{n-5} for n even and $3{*}2^{n-7}$ for n odd are actually the largest numbers of distinct minimal diagrams of a single knot type that are possible. We have confirmed this for crossing numbers $6 \leq n \leq 16$ through our computations.

Proof. We consider the family of 4-plats given by the Conway symbol $211\ldots12$. Let k be the number of ones then the crossing number is $cr = 4+k$. $211\ldots12$ is a link if $cr \equiv 2(3)$ and a knot otherwise. The continued fraction of $211\ldots12$ is of the form $fib(cr-1)/fib(cr+1)$ where $fib(k)$ is the kth Fibbonnaci number. Moreover if cr is even then the knot or link is amphichiral. In Figure 11 two equivalent square diagrams of the 4-plat 211112 are shown. We immediately see that each of the four single crossings can be flyped to the other side (either right and left or the bottom and top). If we assume that all diagrams of $211\ldots12$ have the two crossings in the center as horizontal half twists then this gives rise to 2^{cr-4} different locations for the $cr-4$ crossing represented by the ones. However if we rotate such a diagram by 180 degrees and then move the two vertical half twists on the outside back to the other side. This last step looks like it can be done by two flypes - however this can also be accomplished by a planar isotopy on the sphere. This operation gives a pairing of the location of the crossings represented by the ones, and thus we have 2^{cr-5} different pairings. Next we observe that the two outside crossings are horizontal if cr is odd and are vertical if cr is even. In the case of an odd crossing number this allows for an additional symmetry, which essentially essentially moves the outside two crossing to the inside and changes the single crossings as follows: It keeps the position of the single crossings at the top and bottom, and exchanges the crossings from right to left and conversely. Figure 11 on the right shows the result when one starts with the diagram on the left. If we use B fo bottom, T for top, R for right and L for left, then moving from the center outwards the location of the single crossings can be described as $BLBR$ for the diagram on the left of Figure 11.

For the diagram on the right of Figure 11 this changes to $RTLT$ and we see that the order is reversed and T has replaced the B symbol. Note that this operation is an isotopy on the sphere S^2 (there are no flypes or Reidemeister moves necessary). Thus we have only 2^{cr-6} different pairings when cr is odd. Using the same type of isotopy in a case where cr is even (as in Figure 11), produces a square diagram where the two halftwists at the center are now vertical instead of horizontal. After a rotation by ninety degrees we observe that we now have a configuration whose mirror image can by obtained by flypes from the the configuration we started with. Moreover the location of the single crossings can be described by $BRBL$, that is we keep B and T unchanged and swap R and L.

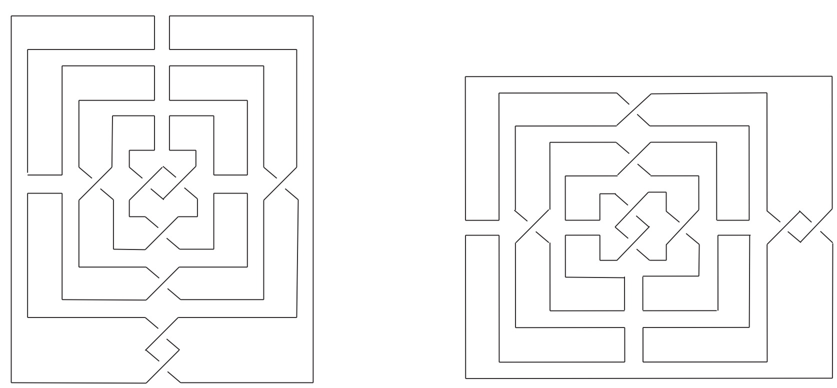


Fig. 11. On the left: A square diagram of the 4-plat 211112. On the right: An equivalent diagram of the same 4-plat.

Next we need to show that all of these diagrams are actually distinct in the OU^*-category. We will only consider the case when cr is even. The crossings represented by the ones can ordered, first there is a single crossing connected by strands to one of two crossings at the end of the Conway symbol $211\ldots12$, next is the single crossing connected by one strand to the two crossings and by a second strand to the first single crossing, next is the single crossing connected by one strand to the first single crossing crossing and by a second strand to the second single crossing or the central two crossings and so on. This gives a linear order of the single crossings that us unique up to completely reversing this order. We note that flypes in the diagram do not effect this order, Now we observe that there is a unique path (in the direction from the inside two crossings to the outside two crossings) consisting of $k-1$ edges in G_D connecting the single crossings in exactly this order. At each crossing such a path either moves to the opposite edge (that is follows the knot) of makes a turn (that is jumps from the overpass to the underpass of from the underpass to the overpass). If we label each move to an opposite edge by an O and each turn by a T, then such a pass can be described by a word of length $k-2$ in the two letters O and T. Clearly there are 2^{k-2} such paths. Since each path must be preserved in the OU^*-category, this leads to 2^{cr-6} inequivalent diagrams. In the case when cr is even then through the planar isotopy we described (see Figure 11) we can obtain the mirror image giving rise to a second set of 2^{cr-6} inequivalent diagrams.

Next we consider the family of 4-plats given by the Conway symbol $2211\ldots12$. Let k be the number of ones then the crossing number is $cr = 6+k$. $2211\ldots12$ is a link if $cr \equiv 0(3)$ and a knot otherwise. The continued fraction of $2211\ldots12$ is of the form $fib(cr-1)/fib(cr-1,2,5)$ where $fib(k)$ is the kth Fibbonnaci number and $fib(k,2,5)$ is the kth Fibbonnaci number of a Fibonnaci squence starting with 2 and 5. If we assume that all diagrams of $2211\ldots12$ have the two full twists in the center starting with horizontal half twists then this gives rise to 2^{cr-6} different positions for the $cr-6$ crossing represented by the ones. In addition we need to consider the second two in the Conway symbol. The two crossings have three possible positions, they are either below or above the horizontal two half-twists in the center, or they are split up with one below and one above the horizontal two half-twists in the center. Assume that the these two half-twists are not split up then as before if we rotate such a diagram by 180 degrees and then move the two vertical half twists on the outside back to the other side we can assume that the crossings from the second two in the Conway symbol are below the two half-twists in the center. With the different positions the ones can take this gives rise to 2^{cr-6} different diagrams. In addition we have the diagrams where the two crossings are split. Here we now have a rotational symmetry as before giving rise to 2^{cr-7} different diagrams for total of $3*2^{n-7}$ distinct minimal diagrams. To show that all of these diagrams are actually distinct in the OU^*-category is similar to the first case and will only provide a brief summary. As before we have a linear order of the single crossings that is unique if start with the single crossings attached to the two twos in the center of the diagram. In the case when the second two crossings are not split we assume that the center crossings are horizontal and the the second two is represented by a set of vertical crossings below the horizontal crossings. Then we have a unique path of length $cr-5$ starting at the bottom of the second two crossings and moving through the single crossings in their linear order. These paths give rise to 2^{cr-7} different symbols of the two letters O and T. Moreover there are 2^{cr-7} different symbols of the two letters O and T for each set of path where the initial segment moves right or moves left. This gives 2^{cr-6} different diagrams in the $OU*$ category in the case when the second two crossings are not split.

In the case when the second two crossings are split then we have a unique path of length $cr-6$ starting at the first single crossing and moving through the single crossings in their linear order giving rise to 2^{cr-5} different symbols of the two letters O and T. Considering if the path starts on the right or left of the center two crossings gives rise to 2^{cr-7} different diagrams in the $OU*$ category and thus there are a total of $3 * 2^{cr-7}$ different diagrams in the $OU*$ category. □

3.6. *The distribution of the number of flype equivalent diagrams with a fix crossing number.*

diagrams with large number for flypes

diagrams with small number for flypes

4. Technical Details

NOT SURE HOW MUCH OF THIS TO PUT IN – NEEDS TO BE MUCH SHORTER

To implement the algorithm described in Section 4, several technical problems must be overcome. The plcurve library [6] is written in C and all flyping code is written in Mathematica. Thus, one problem is the need to switch between the two languages. Another problem is that with increasing crossing number, the number of distinct diagrams increases rapidly requiring a parallel computing approach. These issues are addressed in the following subsections.

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