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# ON CONVEX POLYHEDRA OF FINITE VOLUME IN LOBAČEVSKIĬ SPACE

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Abstract. The paper contains a complete description of the polyhedra of finite volume with dihedral angles not exceeding 90° in three-dimensional Lobačevskiĭ space.
Bibliography: One item.

In our paper [1] we gave a complete description of the convex bounded polyhedra with dihedral angles not exceeding $\pi/2$ in three-dimensional Lobačevskiĭ space $\Lambda^3$. The methods used there enable us to solve this problem also for the polyhedra of finite volume whose dihedral angles do not exceed $\pi/2$.

We realize the Lobačevskiĭ space by means of Beltrami's model in the open ball $V$ in the three-dimensional euclidean space $E^3$. Let $S$ be the boundary of the ball (sphere). To every $k$-dimensional plane $A \subset \Lambda^3$ there corresponds a euclidean plane $\widetilde{A} \subset E^3$, and to every halfspace $\Lambda^-$ there corresponds a euclidean halfspace $\widetilde{\Lambda}^-$, such that, in the model, $A = \widetilde{A} \cap V$ and $\Lambda^- = \widetilde{\Lambda}^- \cap V$.

Let $M$ be a convex polyhedron in $\Lambda^3$, $M = \bigcap_{i=1}^N \Lambda_i^-(M)$. We shall assume that $M$ contains a nonempty open subset and that $M \neq \bigcap_{i \neq j} \Lambda_i^-(M)$ for every $j$. We let correspond to $M$ the euclidean polyhedron $\widetilde{M} = \bigcap \widetilde{\Lambda}_i^-(M)$. The polyhedron $M$ is bounded if and only if $\widetilde{M} \subset V$, and $M$ has finite volume if and only if $\widetilde{M} \subset \overline{V}$.

Let $M$ be a polyhedron with finite volume, and let $\mathfrak{m}$ be the abstract polyhedron corresponding to $\widetilde{M}$; we shall say then that $M$ has combinatorial type $\mathfrak{m}$. By $\Gamma_i(M)$ we denote a face, by $\Gamma_{ij}(M)$ the edge $\Gamma_i(M) \cap \Gamma_j(M)$ and by $\Gamma_{i_1,\dots,i_k}(M)$ the vertex $\bigcap_{p=1}^k \Gamma_{i_p}(M)$. If $\widetilde{\Gamma}_{i_1,\dots,i_k}(M) \subset S$ is a vertex of the polyhedron $\widetilde{M}$ then we shall say that $\Gamma_{i_1,\dots,i_k}(M)$ is a vertex of $M$ at infinity. Let $H_i(M)$ be the hyperplane of the face $\Gamma_i(M)$ and let $\alpha_{ij}(M)$ represent the dihedral angle of $M$ on the edge $\Gamma_{ij}(M)$.

Theorem 1. *Let $M$ be a polyhedron of finite volume in $\Lambda^3$ with dihedral angles $\alpha_{ij}(M) \le \pi/2$. Then the following statements are true.*

1) *The $\Gamma_i(M)$ are polygons of finite area with angles not exceeding $\pi/2$.*

2) *Every vertex of the polyhedron $M$ is the junction of three faces, and every infinite vertex is the junction of 3 or 4 faces.*

3) $\dim \bigcap_{p=1}^k \Gamma_{i_p} = \dim \bigcap_{p=1}^k H_{i_p}$, *and if $\bigcap_{p=1}^k H_{i_p} \subset S$, then $k = 3$ or 4 and the*

*faces* $\Gamma_{i_1}, \cdots, \Gamma_{i_k}$ *meet in one and the same vertex.*

Proof. 1) The finiteness of the area of the polyhedron $\Gamma_i$ follows from the fact that $\widetilde{\Gamma}_i \subset \overline{V}$. Because of the inequality (5) of the paper [1], its angles do not exceed $\pi/2$. The statement 2) follows from the description of the vertices of a polyhedron of finite volume (cf. [1], §2). Let us prove the statement 3). If $(\bigcap \widetilde{\Gamma}_{i_p}) \cap S = \emptyset$, then this follows immediately from Lemma 1 of [1]; in the opposite case it follows from the description of the infinitely distant vertices of a polyhedron with finite volume.

Consider a sequence $M_m$, $m = 1, 2, \cdots$, of polyhedra in $\Lambda^3$. The notion of generalized limit introduced in [1] is equivalent to the following. We shall say that the convex set $M$ is the generalized limit of the sequence $M_m$ if $M \neq \emptyset$ and $\widetilde{M} = \lim \widetilde{M}_m$, where the limit is to be taken in the sense of convergence of convex sets in $E^3$. We denote by $\gamma_i$ the generalized limit of the faces $\Gamma_i(M_m)$; either $\gamma_i$ is an infinitely distant point or $\gamma_i \subset \Lambda^3$. Let $h_i$ be the limit of the hyperplanes $H_i(M_m)$; then also either $h_i \subset \Lambda^3$ or $h_i$ is an infinitely distant point.

Let $K = \{i, h_i \subset \Lambda^3\}$. We suppose that $i, j \in K$ and $h_i \cap h_j \neq \emptyset$. We denote by $\omega_{ij}$ the dihedral angle between the planes containing $M$.

Lemma. *Let* $M_m$, $m = 1, 2, \cdots$, *be a sequence of polyhedra in* $\Lambda^3$ *of finite volume and of combinatorial type* $\mathfrak{m}$ *with angles not exceeding* $\pi/2$. *Let* $X$ *be a point in* $\Lambda^3$ *and let* $X \in M_m$ *for all* $m > m_0$. *Then the sequence* $M_m$ *contains a generalized convergent subsequence; its limit* $M$ *is a convex set, and the following statements are true.*

1) *If* $M$ *is a polyhedron (polygon) then its volume (area) is finite.*

2) *If a face or an edge is degenerate, but the limit is contained in* $\Lambda^3$, *then the volume of* $M$ *is zero.*

3) *If the edge* $\Gamma_{ij}$ *with the endpoints* $\Gamma_{ijk}$, $\Gamma_{ijl}$ *degenerates into a point, then* $M = \gamma_k = \gamma_l$ *and* $h_i, h_j \perp h_k$.

4) *If* $M$ *is a polygon with nonzero area, then* $M = \gamma_i, \gamma_j$ *and there exist* $h_{k_1}, h_{k_2}, h_{k_3}$, *such that* $\Gamma_{k_p}$ *and* $\Gamma_i, \Gamma_j$ *are adjacent and* $h_j \perp h_{k_p}$, $p = 1, 2, 3$.

5) $0 \leq \omega_{ij} \leq \pi/2$, *and if* $\Gamma_{ij}$ *is an edge of the polyhedron and* $\lim \alpha_{ij}(m) = \alpha_{ij}$, *then* $\omega_{ij} = \alpha_{ij}$. *If, however,* $\dim(\Gamma_i \cap \Gamma_j) \leq 0$, *then* $\omega_{ij} = 0$.

The proof of this lemma coincides exactly with the proof of the corresponding statements for sequences of bounded polyhedra which is contained in §4 of the paper [1].

Theorem 2. *Let* $\mathfrak{m}$ *be an abstract three-dimensional polyhedron not a simplex such that three or four faces meet at every vertex. The following conditions are necessary and sufficient for the existence in* $\Lambda^3$ *of a convex polyhedron of finite volume of the combinatorial type* $\mathfrak{m}$ *with the angles* $\alpha_{ij} \leq \pi/2$:

$\mathfrak{m}0$. $0 < \alpha_{ij} \leq \pi/2$.

$\mathfrak{m}1^*$. *If* $\Gamma_{ijk}$ *is a vertex of* $\mathfrak{m}$ *then* $\alpha_{ij} + \alpha_{jk} + \alpha_{ki} \geq \pi$, *and if* $\Gamma_{ijkl}$ *is a vertex then* $\alpha_{ij} + \alpha_{jk} + \alpha_{kl} + \alpha_{li} = 2\pi$.

𝔪 2. *If* $\Gamma_i, \Gamma_j, \Gamma_k$ *is a triangular prismatic element (as defined in* [1], §2), *then* $\alpha_{ij} + \alpha_{jk} + \alpha_{ki} < \pi$.

𝔪 3. *If* $\Gamma_i, \Gamma_j, \Gamma_k, \Gamma_l$ *is a quadrangular prismatic element, then* $\alpha_{ij} + \alpha_{jk} + \alpha_{kl} + \alpha_{li} < 2\pi$.

𝔪 4. *If* 𝔪 *is a triangular prism with bases* $\Gamma_1$ *and* $\Gamma_2$, *then* $\alpha_{13} + \alpha_{14} + \alpha_{15} + \alpha_{23} + \alpha_{24} + \alpha_{25} < 3\pi$.

𝔪 5. *If among the faces* $\Gamma_i, \Gamma_j, \Gamma_k$ *we have* $\Gamma_i$ *and* $\Gamma_j$, $\Gamma_j$ *and* $\Gamma_k$ *adjacent, but* $\Gamma_i$ *and* $\Gamma_k$ *not adjacent, but concurrent in one vertex (we may say: they touch each other) and all three do not meet in one vertex, then* $\alpha_{ij} + \alpha_{jk} < \pi$.

**Proof.** Necessity. The condition 𝔪1 follows from the description of the vertices (cf. [1], §2). Conditions 𝔪2 and 𝔪3 derive from Theorem 1, assertion 3), and can be proved exactly as the corresponding statements in [1], Theorem 2. The condition 𝔪5 is satisfied because from a point it is impossible to draw two perpendiculars onto a plane.

We point out that if $F$ is a simplicial vertex and the sum of the angles at $F$ is much greater than $\pi$, then $F$ is an ordinary vertex. Therefore, assuming that the condition 𝔪4 is not satisfied in $M$, we see immediately that $M$ is a bounded polyhedron. But it was shown in [1] (Theorem 2) that the condition 𝔪4 is satisfied in the bounded case.

Sufficiency. Let $P \subset E^3$ be a polyhedron of the combinatorial type 𝔪. We cut off all its nonsimplicial vertices by means of planes which cut quadrangular pyramids out of $P$. We denote by $B$ the resulting polyhedron and by 𝔟 the corresponding abstract polyhedron. Suppose that $P$ has $N$ faces and $l$ nonsimplicial vertices. We shall denote the new faces 𝔟 by $\Gamma_{N+1}, \cdots, \Gamma_{N+l}$, while retaining the numbers of the old ones. Contracting in 𝔟 the quadrangles $\Gamma_{N+1}, \cdots, \Gamma_{N+l}$ into points, we obtain 𝔪.

We proceed immediately to the construction of the required polyhedron. We distinguish two cases: 1. For some vertex of 𝔪, say $\Gamma_{123}$, one of the inequalities in 𝔪1* is strict. 2. Equality holds in 𝔪1* for all vertices.

If to all vertices of the polyhedron there correspond strict inequalities 𝔪1, then we are in the case explained in the paper [1] and the theorem follows.

Let 𝔪 be an abstract polyhedron. By $\Omega(\mathfrak{m})$ we denote the arithmetic space where the coordinates are given by pairs $(ij)$ corresponding to the edges of the polyhedron 𝔪.

We begin with the first case. We consider the point $\alpha(m)$ of the space $\Omega(\mathfrak{b})$ with the coordinates

$$\alpha_{ij}(m) = \begin{cases} \alpha_{ij} + \dfrac{1}{m}, & \text{if} \quad \alpha_{ij} \neq \pi/2, \\ \\ \alpha_{ij} - \dfrac{1}{m}, & \text{if} \quad \alpha_{ij} = \pi/2, \end{cases}$$

when $i, j \leq N$ and $\alpha_{i,N+j} = \pi/2 - 1/m$.

It is easy to see that there is an $m_0 > 0$ such that for all $m > m_0$ the numbers $\alpha_{ij}(m)$ satisfy the conditions $\mathfrak{b}0 - \mathfrak{b}4$ from [1], and this means that there is a polyhedron of combinatorial type $\mathfrak{b}$ and $\alpha(B_m) = \alpha(m)$.

We decompose the polyhedra $B_m$ so that the vertex $\Gamma_{123}(B_m)$ lies at a certain fixed point of the space, and we select from the sequence $B_m$, $m > m_0$, a generalized convergent subsequence. Let $M$ be its limit.

We shall prove that $M$ is a nondegenerate polyhedron. Consider the edges $\Gamma_{12}$, $\Gamma_{13}$, $\Gamma_{23}$ of the polyhedron $B_m$. If two of them meet in a point, then this point is a vertex of a complete face, say $\Gamma_1$. Let $k$ be the number of sides (edges) of the polygon $\Gamma_1$. Then the sum of its angles tends to $\pi(k-2)$, which is impossible in view of the equality (5) in [1] and the inequalities $\mathfrak{m}2 - \mathfrak{m}4$. If one edge, say $\Gamma_{12}$, degenerates into a point, then $M = \gamma_3$; and if $\Gamma_{124}$ is the second endpoint of the edge $\Gamma_{12}$, then $\gamma_3 = \gamma_4$ and $h_1, h_2 \perp h_3$. Moreover there is a face $\Gamma_j$ such that $h_j \perp h_3$. Because $\alpha_{ij}$ satisfies the condition $\mathfrak{m}5$, we have $j \leq N$.

Thus 1, 2, 3, 4, $j \leq N$ and $\alpha_{13} = \alpha_{14} = \alpha_{23} = \alpha_{24} = \alpha_{3j} = \alpha_{4j} = \pi/2$. One easily sees that the existence of the six right angles violates one of the conditions $\mathfrak{m}2 - \mathfrak{m}5$. We have shown that the edges $\Gamma_{12}$, $\Gamma_{13}$, $\Gamma_{23}$ do not degenerate; from $\alpha_{12} + \alpha_{13} + \alpha_{23} > \pi$ it follows that the straight lines forming these edges are pairwise distinct and do not lie in one and the same plane. Thus we have proved that $M$ is a nondegenerate polyhedron.

The lemma implies that no face of $M$ ever degenerates, except that it may contain an infinitely distant point, and that its angles do not exceed $\pi/2$ and its faces $\gamma_i$ and $\gamma_j$ are adjacent if and only if the corresponding faces of $\mathfrak{b}$ are adjacent. The conditions satisfied by the dihedral angles between the faces adjacent at an infinitely distant vertex (see [1], §2) show that no edge can degenerate by itself into an infinitely distant point, i.e. that for this to happen it is necessary that the angle at this edge tends to zero and that at an infinitely distant point only the faces $\Gamma_{n+j}$ may degenerate. Finally, they necessarily degenerate if their area tends to zero. Hence $\tilde{M} \sim \mathfrak{m}$ and Theorem 2 is proved in the first case.

Proceeding to the second case, we consider again the polyhedron $\mathfrak{b}$ and in it we contract to a point those and only those quadrangles $\Gamma_{N+j}$ for which $\Gamma_{N+i}(\mathfrak{b}) \cap \Gamma_{12}(\mathfrak{b}) = \emptyset$, where $\Gamma_{12}(\mathfrak{m})$ is some edge. So we obtain an abstract polyhedron which we denote by $\mathfrak{c}$.

If $\alpha_{ij} = \pi/2$ then all the vertices of $\mathfrak{m}$ are nonsimplicial. We assume that $N$, $R$, $V$ are the numbers of the faces, edges and vertices respectively of the polyhedron $\mathfrak{m}$. In the present case $R = 2V$ and $N + V - R = 2$; and if $\mathfrak{m}$ is not a simplex, then $V > 4$ and $N > 6$. Let $\Gamma_{12}(\mathfrak{m})$ be an edge; there are only 6 faces of $\mathfrak{m}$ for which $\Gamma_i \cap \Gamma_{12} = \emptyset$, and this means that there is a certain face, say $\Gamma_4$, whose intersection with $\Gamma_{12}$ is empty.

If all the $\alpha_{ij} = \pi/2$, then there is an edge, say $\Gamma_{12}$, such that $\alpha_{12} < \pi/2$. Both endpoints of $\Gamma_{12}$ are simplicial vertices. The argument is similar to the one carried through above and shows that there is a face, say $\Gamma_4$, such that $\Gamma_{12}(\mathrm{m}) \cap \Gamma_4(\mathrm{m}) = \emptyset$.

In the space $\Omega(\mathrm{c})$ we consider the point with the coordinates $\alpha_{ij}(m)$:

$$\alpha_{12}(m) = \begin{cases} \alpha_{12} + \dfrac{1}{m}, & \text{if} \quad \alpha_{12} \neq \pi/2, \\ \alpha_{12} - \dfrac{1}{m}, & \text{if} \quad \alpha_{12} = \pi/2. \end{cases}$$

If $(i, j) \neq (1, 2)$, but $i, j \leq N$, then $\alpha_{ij}(m) = \alpha_{ij}$ and $\alpha_{i, N+j}(m) = \pi/2 - 1/m$.

It is readily seen that there is an $m_0 > 0$ such that for $m > m_0$ the numbers $\alpha_{ij}(m)$ satisfy the conditions $\mathrm{c}1$ and $\mathrm{c}2$ – $\mathrm{c}5$, and that they occur in the same situation as in the Case 1 considered above. (As $\Gamma_{123}$ one has to choose one of the endpoints of the edge $\Gamma_{12}$.) With regard to the fact proven above, there is a sequence of polyhedra $C_m$, $m > m_0$, such that $\alpha(C_m) = \alpha(m)$. Since $\Gamma_{12}(\mathrm{m}) \cap \Gamma_4(\mathrm{m}) = \emptyset$, with regard to the conditions characterizing the second case it follows that all the vertices of the polyhedron $C_m$ belonging to the face $\Gamma_4$ are at infinity. Let $X_1$, $X_2$, $X_3$ be three distinct points at infinity and $\Gamma_{4p}$ and $\Gamma_{4q}$ two neighboring edges of the polyhedron $C_m$. We decompose $C_m$ so that the endpoints of the separating edges lie at the points $X_1$, $X_2$ and $X_3$ for $m > m_0$. Then the sequence $C_m$ contains a generalized convergent sequence; let $M$ be its limit.

The points $X_i \subset \gamma_4$, where $\gamma_4$ is a nondegenerate polygon. If $M = \gamma_4$, then $h_p$, $h_q \perp h_4$, and one can find $r$ and $s$ such that $\gamma_4 = \gamma_r$, $h_s \perp h_4$. Clearly in this case $\Gamma_{4qpl}$, which is the common endpoint of the edges $\Gamma_{4p}$ and $\Gamma_{4q}$, is a nonsimplical vertex. Moreover, $r \leq N$, so that the area of $\gamma_r$ is not zero; and $l = s$ in virtue of condition $\mathrm{m}5$. But then condition $\mathrm{m}5$ is again violated for the triplet $4, l, r$. This means that $M$ is a polyhedron.

That $\widetilde{M} \sim \mathrm{m}$ and $\alpha_{ij}(M) = \alpha_{ij}$ can be verified as in the first case. This completes the proof of the theorem.

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