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A polynomial time circle packing algorithm

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Abstract

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The Andreev–Koebe–Thurston circle packing theorem is generalized and improved in two ways. Simultaneous circle packing representations of the map and its dual map are obtained such that any two edges dual to each other cross at the right angle. The necessary and sufficient condition for a map to have such a primal-dual circle packing representation is that its universal cover graph is 3-connected. A polynomial time algorithm is obtained that given such a map M and a rational number $\varepsilon > 0$ finds an ε-approximation for the primal-dual circle packing representation of M. In particular, there is a polynomial time algorithm that produces simultaneous geodesic line convex drawings of a given map and its dual in a surface with constant curvature, so that only edges dual to each other cross.

1. Introduction

Let Σ be a surface. A *map* on Σ is a pair (G, Σ) where G is a graph which is 2-cell embedded in Σ. A *circle packing* is a set of geodesic circles (disks) with disjoint interiors in a Riemannian surface Σ' with constant curvature. By putting a vertex v_D in the centre of each circle D and joining v_D by geodesics with all points on the boundary of D where the other circles touch D (or where D touches itself), we get a graph on Σ'. If

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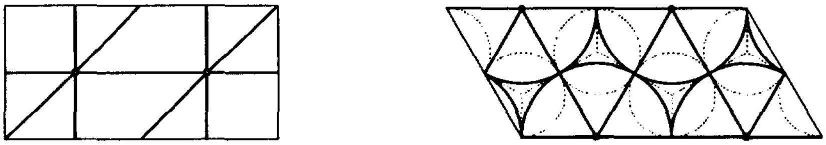

(a) (b)

Fig. 1. A toroidal map and its primal-dual representation.

the obtained graph on the surface is a map, isomorphic to the map $M=(G, \Sigma)$, then we have a *circle packing representation* of M. Simultaneous distinct circle packing representations of a map and its dual map so that in the map and the dual any two edges dual to each other cross at the right angle in the point where the corresponding circles touch is called a *primal-dual circle packing representation.* An example of a toroidal map and its primal-dual circle packing representation is given in Fig. 1.

It was proved by Koebe [6], Andreev [1, 2], and Thurston [9] that if M is a triangulation then it admits a circle packing representation. The proofs of Andreev and Thurston are existential (using a fixed point theorem) but Colin de Verdière [4, 5] found a constructive proof by means of a convergent process. We present an algorithm that for a given reduced map M (see Section 2 for the definition) and a given rational number $\varepsilon > 0$, finds an ε-approximation for the radii and the centres of a circle packing representation of M in a surface of constant curvature (either $+1$, 0, or -1). The time used by our algorithm is polynomial in the size of the input (the number of edges of M plus the size of ε, i.e., $\max\{1, \lceil \log(1/\varepsilon) \rceil\}$).

We generalize the result of Andreev–Koebe–Thurston to the most general maps that admit primal-dual circle packing representation (reduced maps). In particular, every map with a 3-connected graph has a primal-dual circle packing representation. This extends the results of Pulleyblank and Rote [7] and Brightwell and Scheinerman [3, 8] about circle packings of 3-connected planar graphs. With these results we not only characterize maps which admit convex representations but also prove a far reaching generalization of a conjecture of Tutte (proved in [3]) that a 3-connected planar graph and its dual admit a simultaneous straight-line drawing in the plane (with the vertex corresponding to the unbounded face at the infinity) such that each pair of dual edges cross at the right angle.

Our algorithm involves quite detailed analysis on the sizes of the numbers which appear in calculations in the algorithm. The precise computational complexity analysis is necessary to get from it a polynomial time algorithm for the following combinatorial problem.

Problem. Given a reduced map M_0, find simultaneous convex drawings of M_0 and its dual map M_0^* on a surface with constant curvature, such that each edge of M_0 crosses only with its dual edge in M_0^*, and the angle at which they cross is between $\pi/2 - 10^{-1993}$ and $\pi/2 + 10^{-1993}$.

Not only that our results show that there is such a convex representation, but using the circle packing algorithm up to a certain precision one really gets such a representation in time bounded by a polynomial in $|E(M_0)|$. There are other applications of our algorithm in the areas of graph drawing and computer graphics.

2. Existence of primal-dual circle packings

Let $M_0=(G_0,\Sigma)$ be a map on Σ. Define a new map $M=(G,\Sigma)$ whose vertices are the vertices of G_0 together with the faces of M_0, and whose edges correspond to the vertex-face incidence in M_0. The embedding of G is obtained simply by putting a vertex in each face F of M_0 and joining it to all the vertices on the boundary of F. The map M and the graph G are called the *vertex-face incidence map* and the *vertex-face incidence graph*, respectively. Note that G is bipartite and that every face of M is bounded by precisely four edges of G.

From now on we assume that M_0 is a given map on the closed surface Σ and that M and G are its vertex-face incidence map and vertex-face incidence graph, respectively. We will use the notation $V=V(G)$ throughout the paper. We will denote by n and m the number of vertices and edges of G, respectively. It follows by the Euler's formula that $m=2(n-\chi(\Sigma))$ where $\chi(\Sigma)$ denotes the Euler characteristic of Σ. If $S\subseteq V$ then $E(S)$ denotes the set of edges with both endpoints in S.

Having a primal-dual circle packing representation of M_0 in a surface Σ' with constant curvature we have a circle for each vertex of G. Let r_v be the radius of the circle corresponding to the vertex $v\in V$. If $vuv'u'$ is one of the faces containing v then its diagonals have length $r_v+r_{v'}$ and $r_u+r_{u'}$, respectively, and they cross at the right angle. Assume now that Σ' has constant curvature $+1$ (*spherical* case), 0 (*Euclidean* case), or -1 (*hyperbolic* case). By the elementary geometry we get the following formula for the angle $\alpha=\alpha(r_u,r_v)$ between the edge vu and the diagonal vv' in the face $vuv'u'$:

$$\alpha=\alpha(r_u,r_v)=\begin{cases} \operatorname{arctg}\left(\dfrac{\operatorname{tg} r_u}{\sin r_v}\right), & \text{spherical case;}\\ \operatorname{arctg}\left(\dfrac{r_u}{r_v}\right), & \text{Euclidean case;}\\ \operatorname{arctg}\left(\dfrac{\operatorname{th} r_u}{\operatorname{sh} r_v}\right), & \text{hyperbolic case.}\end{cases} \tag{1}$$

Proposition 2.1. *Let M_0 be a map. A set of numbers $r=(r_v\,|\,v\in V)$ are the radii of a primal-dual circle packing representation of M_0 if and only if for each $v\in V$, $r_v>0$, and the following angle condition is satisfied*:

$$\varphi_v=\sum_{vu\in E(G)}\alpha(r_u,r_v)=\pi, \tag{2}$$

where the sum is taken over all edges vu that are incident to v in G.

The map M_0 is *reduced* if the graph of the universal cover of M_0 is 3-connected. It can be shown that this condition is equivalent to requiring that for every proper subset $S \subset V$, $|S| \geq 2$, of vertices of G we have:

$$2|S| - |E(S)| \geq 2\chi(\Sigma) + 1. \tag{3}$$

If $\chi(\Sigma) > 0$, then (3) should be weakened for $|S| = 2$ to: $|E(S)| \leq 3 - \chi(\Sigma)$. In case when Σ is the 2-sphere, being reduced is equivalent to 3-connectivity, but in general, reduced maps can have loops, parallel edges and nontrivial 2-separations.

Given a set of positive numbers $r = (r_v \mid v \in V)$, $r_v > 0$, one can define the corresponding 'angles' in analogy with (2), $\varphi_v = \Sigma_u \alpha(r_u, r_v)$, where the sum is over all edges uv that are incident to v in G. (In case of multiple edges between u and v, each such edge gives its contribution.) We will write $\Theta(r) = (\vartheta_v \mid v \in V)$ where $\vartheta_v = \varphi_v - \pi$, and use the function

$$\mu(r) = \sum_{v \in V} \vartheta_v^2 \tag{4}$$

to measure how far from the required radii satisfying the angle condition (2) is our choice of r. Call the radii $r = (r_v \mid v \in V)$ *normalized* if $\sum_{v \in V} \vartheta_v = 0$.

The angles and the corresponding radii will be computed by means of an iteration process. Given $r = (\mathrm{r}_v \mid v \in V)$, order the vertices $u_1, u_2, \ldots, u_n$ of G such that $\vartheta_{u_1} \geq \vartheta_{u_2} \geq \cdots \geq \vartheta_{u_q} \geq 0 > \vartheta_{u_{q+1}} \geq \cdots \geq \vartheta_{u_n}$. Let

$$\sigma(r) = \max_{q \leq i < n} (\vartheta_{u_i}^- - \vartheta_{u_{i+1}}), \tag{5}$$

where $\vartheta_{u_q}^- = 0$ and $\vartheta_{u_i}^- = \vartheta_{u_i}$ if $i > q$. Let t be the smallest index i where the maximum in (5) is attained. (We define $\sigma(r) = 0$ and $t = n$ if $q = n$.) Set $S = S(r) = \{u_1, \ldots, u_t\}$, and let r' be defined by

$$r'_v = \begin{cases} \beta r_v, & \text{if } v \in S; \\ \gamma r_v, & \text{otherwise} \end{cases} \tag{6}$$

where $\beta \geq 1$ and $\gamma > 0$ are constants such that r' is normalized. Let $(\vartheta'_v;\, v \in V) = \Theta(r')$, and let

$$f(\beta, \gamma) = \sum_{v \in S} (\vartheta_v - \vartheta'_v).$$

Call the pair (β, γ) *suitable* if $\vartheta'_v \geq \vartheta'_u$ for all $v \in S$ and $u \notin S$, and $f(\beta, \gamma) \geq \sigma(r)/6$. We are ready to present our algorithm. We will give details only for the hyperbolic case which we assume henceforth. The spherical and the Euclidean case are not much different.

Instance: A reduced map M_0 on a surface with negative Euler characteristic and a rational number $\delta > 0$.

Task: For the vertex-face incidence map M of M_0 find radii $r=(r_v\,|\,v\in V)$ and points P_v, $v\in V$, in the fundamental polygon of the surface in the hyperbolic plane such that there is a primal-dual circle packing of M_0 with radii $r^0=(r_v^0\,|\,v\in V)$ and centres $P_v^0, v\in V$, and for each vertex v of M we have $|r_v^0-r_v|\leqslant\delta$ and $\operatorname{dist}(P_v^0,P_v)\leqslant\delta$.

Algorithm A

1. Construct $M, n:=|V|$, $m:=|E(M)|$.
2. Set $\delta_1=2^{-2n-4}m^{-2n-5}\delta$ and $\varepsilon=2^{-4n}m^{-4n-18}\delta_1^2$.
3. Let $p=20n\lceil\log_2 m\rceil+\lceil\log_2(1/\varepsilon)\rceil$ be the number of binary digits used in all the computations in the following steps.
4. Set $r_v:=\operatorname{Ar\,ch}(\operatorname{ctg}(n\pi/2m))$, $v\in V$.
5. **while** $\mu(r)>\varepsilon/2$ **do**
   - 5.1 Determine $\sigma=\sigma(r)$ and the set $S=S(r)\subset V$.
   - 5.2 Find a suitable pair (β,γ) as follows. If $|E(S)|+2\chi(\Sigma)\geqslant 0$ then apply bisection on the interval (0, 1) to find appropriate γ, and then determine β (by bisection) so that r' given by (6) will be normalized. If $|E(S)|+2\chi(\Sigma)<0$ then find β by using bisection on the interval $(1, 2m^n\lceil\log m\rceil)$, and determine γ so that r' is normalized.
   - 5.3 $r:=r'$, where r' is defined by (6).
6. Compute $P_v, v\in V$.
7. For $v\in V$, output r_v and P_v.

It can be shown that the choice of ε implies that for every $v\in V$ we have $|r_v-r_v^0|\leqslant\delta_1$. Let us describe how to obtain the centres P_v in Step 6. Choose an arbitrary vertex $v_0\in V$ and put it in the origin of the hyperbolic plane. By using the elementary hyperbolic geometry we can calculate the coordinates P_v for all vertices v that are adjacent to v_0 in G. Using the obtained points P_v we similarly compute the coordinates of their neighbours, etc. We can prove that for every $v\in V$, the error in coordinates is bounded by δ, i.e., $\operatorname{dist}(P_v,P_v^0)\leqslant\delta$.

Theorem 2.2. *Given a reduced map M_0 and a rational number $\delta>0$ one can find in polynomial time δ-approximations for the centres and the radii of a primal-dual circle packing representation of M_0.*

Sketch of the proof. First we show that there is a suitable pair (β,γ) asked for in Step 5.2 of Algorithm A. To prove this we first verify that $f(\beta,\gamma)\geqslant\sigma(r)/2$ if β is large enough, or γ is sufficiently close to 0. Next it is easy to see that if $f(\beta,\gamma)\leqslant\sigma(r)/2$ then for arbitrary vertices $v\in S$, $u\notin S$ we have $\vartheta_v'\geqslant\vartheta_u'$. Moreover, if $f(\beta_1,\gamma_1)\leqslant\sigma(r)/6$, and $\sigma(r)/3\leqslant f(\beta_2,\gamma_2)\leqslant\sigma(r)/2$, then $\beta_2-\beta_1$ and $\gamma_1-\gamma_2$ cannot be too small. This implies that the bisection with our precision p will find a suitable pair. The only trouble in the search of a suitable pair is that given β there does not necessarily exist a γ such that r' is normalized. It turns out that this trouble can be overcome using the strategy applied in 5.2. If $|E(S)|+2\chi(\Sigma)\geqslant 0$ then for every γ between 0 and 1 there is a β such that r' is normalized. Therefore we may search for γ on the interval $(0, 1)$. Similarly, if $|E(S)|+2\chi(\Sigma)<0$ then we search for β on $(1, \infty)$. Here and later on we use the following important fact. During the execution of Algorithm A we have

$$\max_{v\in V} r_v \leqslant 2\log m \quad \text{and} \quad \min_{v\in V} r_v \geqslant m^{-n}. \tag{7}$$

We know now that Step 5.2 of Algorithm A in polynomial time discovers a suitable pair (β, γ). It is important that $\mu(r')$ is smaller than $\mu(r)$, where r' is the new value for the function r obtained in Step 5.2. More precisely, $\mu(r')\leqslant(1-n^{-4}/3))\mu(r)$.

This proves that in Steps 1 to 5, Algorithm A finds radii r such that $\mu(r)\leqslant\varepsilon/2$. Next, it is easy to see that in our Algorithm A, viewed as an infinite process and with exact arithmetic, the radii converge to a limit. Once we see this, we have a guarantee on the existence of a primal-dual circle packing representation. It remains to show that our polynomial time convergence process always converges to the same solution. We need an estimate analogous to (7) that holds in general, and not only for solutions obtained by the algorithm. Finally, we derive a uniqueness result: Suppose that $r^0=(r_v^0\,|\,v\in V)$ is an exact solution for the primal-dual circle packing radii. If $\mu(r)<2^{-4n}m^{-4n-16}\varepsilon$, where $\varepsilon<1/4$, then for every $v\in V$ we have $1-\sqrt{\varepsilon}<r_v/r_v^0<1+\sqrt{\varepsilon}$.

We have already explained how to obtain from the known radii approximations, the approximations for the centres P_v. Since our ε was chosen small enough, it can be shown that the distance of P_v from the exact points P_v^0 is at most δ. $\square$

The following corollary is an immediate consequence of the preceding proof.

Corollary 2.3. *A map M_0 on a surface Σ with negative Euler characteristic admits a primal-dual circle packing representation if and only if it is reduced. The primal-dual circle packing radii of M_0 are uniquely determined.*

The submaps of reduced maps are more general than submaps of triangulations in the sense that they may contain loops or parallel edges. Therefore, our results in particular prove the existence of circle packings of more general maps than implied by the Andreev–Koebe–Thurston's Theorem.

Corollary 2.4. *A map M_0 on a surface Σ with nonpositive Euler characteristic admits a circle packing representation if and only if M_0 does not contain contractible loops or pairs of edges (possibly loops) with the same endpoint(s) that are homotopic relative their endpoint(s).*

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