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ON CONVEX POLYHEDRA IN LOBAČEVSKIĬ SPACES

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Abstract. This paper is concerned with the investigation of properties of convex polyhedra in Lobačevskiĭ spaces; it gives a complete description of convex bounded polyhedra with dihedral angles not exceeding 90°, in a three-dimensional Lobačevskiĭ space. This in turn permits a description of the discrete groups generated by reflections acting in the three-dimensional Lobačevskiĭ space with a compact fundamental region.

6 figures; bibliography: 6 items.

The complete classification of the discrete groups generated by reflections in hyperplanes acting on the sphere and in euclidean space is well known. All these groups have been enumerated by Coxeter [5]. The present work is devoted to the finding of all discrete groups generated by reflections in hyperplanes acting in Lobačevskiĭ spaces.

We shall restrict ourselves to discrete groups with compact fundamental region. As in the first two cases, for the solution of the problem one needs only describe all convex bounded polyhedra with dihedral angles π/n, $n \geq 2$.

In the case of the euclidean space (or the sphere) the determination of all these polyhedra is made easier by the fact that all convex bounded polyhedra with angles not exceeding $\pi/2$ are direct products of simplexes (by simplexes) [5]. Therefore a natural generalization of the problem is the following: To describe all convex bounded polyhedra in Lobačevakiĭ spaces whose dihedral angles do not exceed $\pi/2$. The results of the papers [6] and [2] describe the simplexes in Lobačevskiĭ spaces with angles π/n, as well as those with angles not exceeding $\pi/2$; so when we are discussing questions of existence, we shall assume that the polyhedron under consideration is not a simplex.

Let M be a convex bounded n-dimensional polyhedron on the sphere, in euclidean or Lobačevskiĭ space. In the spherical case we shall assume that M is a strictly convex polyhedron, i.e. not containing diametrically opposite points of the sphere. Corresponding to M we consider the $(n-1)$-dimensional abstract nonoriented cell complex $\mathfrak{m}$ which is isomorphic to the boundary of the polyhedron in the category of abstract complexes (§1). Every complex obtained in this manner we shall call an n-dimensional abstract polyhedron.

We shall say that M and M' are polyhedra of one and the same combinatorial type, $M \sim M'$, if the corresponding abstract polyhedra coincide, i.e. if there is a one-to-one correspondence between the boundaries of M and M' preserving dimension and incidence relations. If $\mathfrak{m}$ is an abstract polyhedron, then in each of the three spaces one can find a polyhedron M whose boundary is isomorphic to $\mathfrak{m}$. In this case we shall say that M is a polyhedron of the combinatorial type $\mathfrak{m}$ or that $M \sim \mathfrak{m}$. The set of all polyhedra in each of the spaces splits into classes with respect to the relation, i.e. disjoint classes of polyhedra of the same combinatorial type.

Let $M \sim M'$ be two convex bounded polyhedra in the n-dimensional Lobačevskiĭ space Λ^n, $n \geq 3$. We shall show (§3) that if their corresponding dihedral angles are equal and do not exceed $\pi/2$, then the two polyhedra themselves are equal.

Now let $n = 3$. To every edge of the abstract polyhedron $\mathfrak{m}$ we ascribe a positive number not greater than $\pi/2$. For these numbers we shall determine a simple system of linear inequalities ($\mathfrak{m}0 - \mathfrak{m}4$, §2) equivalent to the existence in the three-dimensional Lobačevskiĭ space of a convex bounded polyhedron $M \sim \mathfrak{m}$ with dihedral angles equal to the given numbers (existence theorem, §§5–6). The assumptions that $n = 3$ and $\mathfrak{m}$ is not a simplex are essential, as will be discussed in detail at the end of §6.

Finally (§8) we shall apply our results to the theory of discrete groups generated by reflections.

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§1. Some properties of polyhedra in Lobačevskiĭ spaces

Let $E^{n,1}$ be an $(n+1)$-dimensional vector space provided with a nondegenerate scalar product with negative index of inertia 1. In $E^{n,1}$ we select a basis such that the scalar product appears in the form $-x_0^2 + x_1^2 + \cdots + x_n^2$. We consider the set $V = \{x \in E^{n,1} \mid (x, x) < 0\}$ and denote by V_+ the one of the two components of V for which $x_0 > 0$.

We identify the points of the Lobačevskiĭ space with the rays in $E^{n,1}$ lying in the cone V_+. To every s-dimensional plane $A \subset \Lambda^n$ corresponds an s-dimensional subspace $\hat{A} \subset E^{n,1}$. The induced scalar product in $\hat{A}$ is nondegenerate and of negative index of inertia 1.

If necessary, we shall use the notation A^s where $s = \dim A^s$, and by X we shall denote the points of the Lobačevskiĭ space.

The group of motions $G(\Lambda^n)$ of the space Λ^n in the given model is realized by the group $\hat{G}(\Lambda^n)$ of all automorphisms of the space $E^{n,1}$ which preserve V_+. To every $g \in G(\Lambda^n)$ corresponds a $\hat{g} \in \hat{G}(\Lambda^n)$ such that $\widehat{(gX)} = \hat{g}\hat{X}$.

We consider now the closure $\overline{V}_+$ of the cone V_+ and denote by V_0 the boundary of the domain $\overline{V}_+$. The set of the rays in $\overline{V}_+$ represents the compactification of the space Λ^n. We shall say that to the rays in V_0 correspond the points X_∞ at infinity of Λ^n. We put $\dim X_\infty = -1$ and $\dim \emptyset = -\infty$.

Let M be a convex polyhedron in Λ^n, i.e.

$$M = \bigcap_{i=1}^{N} \Lambda_i^- (M), \tag{1}$$

where $\overline{\Lambda}_i(M)$ are closed halfspaces in Λ^n; by $H_i(M)$ we denote the hyperplane bounding $\Lambda_i^-(M)$.

To every halfspace $\Lambda_\alpha^- \subset \Lambda^n$ corresponds a halfspace $\hat{\Lambda}_\alpha^- \subset E^{n,1}$, and to every polyhedron a polyhedral angle $\hat{M} = \bigcap_{i=1}^N \hat{\Lambda}_i^-(M)$ with vertex at 0. It is readily seen that this correspondence is one-to-one. It will be useful to point out that $\hat{M} \underset{\neq}{\supset} \{\hat{X} \in M\}$.

We shall assume that M contains a nonempty open set and the system $\Lambda_i^-(M)$ is chosen economically, i.e. for every m

$$M \underset{\neq}{\subset} \bigcap_{i \neq m} \Lambda_i^- (M). \tag{2}$$

Let $n > 1$. The polyhedron is bounded if and only if $\hat{M} \subset V_+$, and it has a finite volume if and only if $\hat{M} \subset \overline{V}_+$, where some of the edges of the angle $\hat{M}$ may lie in V_0.

If $(\bigcap_{i=1}^s H_i^{n-1}) \cap V_+ \subset V_0$ and is not zero, we shall say that the hyperplanes H_i^{n-1}, $i = 1, \cdots, s$, intersect at a point at infinity. If $\hat{M} \cap (\bigcap_{p=1}^s H_{i_p}(M)) \subset V_0$ is also not zero, we say that the polyhedron M has a vertex $\Gamma^{-1}_{i_1 \cdots i_s}(M)$ at infinity.

In general we put

$$\Gamma_{i_1 \ldots i_s}(M) = \left(\bigcap_{p=1}^{s} H_{i_p}(M)\right) \cap M \quad \text{and} \quad H_{i_1 \ldots i_s}(M) = \bigcap_{p=1}^{s} H_{i_p}(M),$$

where $\dim \Gamma_{i_1 \cdots i_s}$ and $\dim H_{i_1 \cdots i_s}$ may assume the following values: $-\infty, -1, 0, \cdots, n$.

If necessary, we shall use the notation $\Gamma^k_{i_1 \cdots i_s}(M)$, where $k = \dim \Gamma^k_{i_1 \cdots i_s}(M)$. We have noted already that $\Gamma^{-1}_{i_1 \cdots i_s}(M)$ is a vertex at infinity; if $k \geq 0$, then $\Gamma^k_{i_1 \cdots i_s}(M)$ is a convex k-dimensional polyhedron appearing as k-dimensional boundary of the polyhedron M. From (2) follows that $\dim \Gamma_i(M) = n - 1$.

An abstract nonorientable cell complex $\mathfrak{K}$ is a finite family of objects ξ, called abstract cells; to every cell belongs a nonnegative integer p called its dimension (we use the notation ξ^p whenever we wish to indicate the dimension explicitly). On the set $\mathfrak{K}$ is given a relation $\prec$, called the incidence relation. It satisfies two axioms.

I. $\prec$ is a partial ordering.

II. From $\xi^p \prec \xi^q$ and $\xi^p \neq \xi^q$ it follows that $p < q$.

The abstract nonoriented complexes generate a category whose morphisms are the mappings $\phi : \mathfrak{K} \longrightarrow \mathfrak{L}$; the complexes as sets of objects satisfy the following axioms:

III. From $\xi_1 \prec \xi_2$ follows $\phi(\xi_1) \prec \phi(\xi_2)$.

IV. From $\eta^p = \phi(\xi^q)$ follows $p \leq q$.

We shall write $\mathfrak{K} \sim \mathfrak{L}$ when in the category the two complexes are isomorphic.

Let $\mathfrak{k}$ be a topological complex. We consider the corresponding abstract complex $\mathfrak{K}$ whose p-dimensional cells are the p-dimensional cells of the complex $\mathfrak{k}$ and $\xi^p \prec \xi^q$ if and only if $\xi^p \subset \xi^q$. We shall say that $\mathfrak{k}$ is a realization of the abstract complex $\mathfrak{K}$. The system of all cells incident with a given cell we shall call the closure of this cell, if their dimension does not exceed its dimension. Evidently by realization the closure becomes the closure in the usual sense.

Now let M be a convex bounded polyhedron. Take it as an n-dimensional topological cell complex and let $M^{(n-1)}$ be the boundary of M, i.e. its $(n-1)$-dimensional frame. By $\mathfrak{m}$ we denote the abstract complex which corresponds to $M^{(n-1)}$.

Then the concept of an abstract polyhedron (cf. the introduction) has been defined in a completely rigorous way.

In the following by a convex bounded polyhedron of the type $\mathfrak{m}$ in Λ^n we shall always mean a pair (M, ϕ), where M is a polyhedron, $M \sim \mathfrak{m}$, and $\phi : \mathfrak{m} \longrightarrow M^{(n-1)}$ is an isomorphism of complexes.

Let $g \in G(\Lambda^n)$ be a motion in the space Λ^n. If M and M_1 are polyhedra and $gM = M_1$, then the motion g determines a morphism g^* of the abstract complexes: $g^* : M^{(n-1)} \longrightarrow M_1^{(n-1)}$. Put $g(M, \phi) = (gM, g^*\phi)$. We shall say that the polyhedra (M, ϕ) and (M_1, ϕ_1) are equal if one can be carried into the other by a motion. Writing $(M, \phi) = (M_1, \phi_1)$ means that the polyhedra are equal and similarly situated.

By the set of all polyhedra of combinatorial type $\mathfrak{m}$, to be denoted by $W(\mathfrak{m})$, we shall mean the system of all pairs (M, ϕ), where $\mathfrak{m} \sim M$ and $\phi : \mathfrak{m} \longrightarrow M^{(n-1)}$ is an isomorphism. It is assumed that to equal but differently situated polyhedra correspond different points of the set.

We denote by $\Gamma_1(\mathfrak{m}), \cdots, \Gamma_N(\mathfrak{m})$ all $(n-1)$-dimensional faces of an abstract polyhedron $\mathfrak{m}$. Let F be one of its faces and $I(F) = \{i : F \subset \Gamma_i(\mathfrak{m})\} = \{i_1, \cdots, i_k\}$; put $F = \Gamma_{i_1 \cdots i_k}(\mathfrak{m})$. Let $(M, \phi) \in W(\mathfrak{m})$. By renumbering the halfspaces $\Lambda_i^-(M)$ we can arrange that $\phi(\Gamma_{i_1 \cdots i_k}(\mathfrak{m})) = \Gamma_{i_1 \cdots i_k}(M)$. In what follows, let us suppose that this is done.

Let us denote by Π^1 the set of all closed halfspaces in Λ^n. Let $\hat{\Pi}^1 = \{\hat{\Lambda}^-; \Lambda^- \in \Pi^1\}$. There is a one-to-one correspondence between the vectors $e_\alpha \in E^{n,1}$ with $(e_\alpha, e_\alpha) = 1$ and the halfspaces $\Lambda_\alpha^- \in \Pi^1$, defined by the relation

$$\hat{\Lambda}_\alpha^- = \{x \in E^{n,1}: (x, e_\alpha) < 0\}.$$

This shows that in Π^1 there exists the natural structure of an analytic manifold. Put $\Pi^N = \underbrace{\Pi^1 \times \cdots \times \Pi^1}_{N}$.

To each point $\mathrm{T} \in \Pi^N$, $\mathrm{T} = (\Lambda_1^-, \cdots, \Lambda_N^-)$ corresponds a convex subset $P_\mathrm{T} = \bigcap_{i=1}^N \Lambda_i^- \subset \Lambda^n$. Put $W^N = \{P_\mathrm{T} : \mathrm{T} \in \Pi^N\}$. Having fixed the numbering of the faces of the polyhedron $\mathfrak{m}$, we determine the embedding of the set $W(\mathfrak{m})$ into W^N and the one-to-one mapping $\mathrm{T} : W(\mathfrak{m}) \longrightarrow \Pi^N$, which associates with every $M \in W(\mathfrak{m})$ a collection of halfspaces $\mathrm{T}(M) = (\Lambda_1^-(M), \cdots, \Lambda_N^-(M))$. This association will be very useful in the following, since it enables us to introduce in $W(\mathfrak{m})$ the structure of a manifold.

In the following we shall often write M instead of (M, ϕ), thereby assuming that the isomorphism $\phi : \mathfrak{m} \longrightarrow M^{(n-1)}$ has been chosen and fixed. We shall also write $\Gamma_{i_1 \cdots i_k}$ and $H_{i_1 \cdots i_k}$ instead of $\Gamma_{i_1 \cdots i_k}(M)$ and $H_{i_1 \cdots i_k}(M)$ whenever it is clear from the context which polyhedron is meant.

Two $(n-1)$-dimensional faces Γ_j and Γ_i are adjacent if $\Gamma_i \cap \Gamma_j$ is an $(n-2)$-dimensional face of M. The dihedral angle between adjacent faces Γ_i and Γ_j will be called a dihedral angle of the polyhedron and will be denoted by $\alpha_{ij}(M)$.

We consider now the polyhedron L which is determined by a pair of halfspaces Λ_1^- and Λ_2^-, $H_1 \neq H_2$. Here are the possible cases:

1. $(e_1, e_2) > -1$; then the faces Γ_1 and Γ_2 are adjacent and $(e_1, e_2) = \cos \alpha_{12}(L)$.
2. $(e_1, e_2) = 1$; then $\dim H_{12} = -1$.
3. $(e_1, e_2) < -1$; then $\dim H_{12} = -1$.

Let $\Gamma^0_{i_1 \cdots i_s}$ be a vertex of M. We consider a sphere whose center coincides with this vertex. The hyperplanes H_{i_p}, $p = 1, \cdots, s$, cut out on this sphere a polyhedron which we denote by $S_{i_1 \cdots i_s}$; its dihedral angles are equal to the dihedral angles of M.

Definition 1. A convex polyhedron for every vertex of which the corresponding spherical polyhedron $S_{i_1 \cdots i_s}$ is a simplex will be called a *polyhedron with vertices of simplicial type*, and the corresponding combinatorial type will be called a *combinatorial type with vertices of simplicial type*.

Theorem 1. *Let M be a convex polyhedron in n-dimensional Lobačevskiĭ space, $n \geq 3$, whose dihedral angles do not exceed $\pi/2$. Then*

1) *M is a polyhedron with vertices of simplicial type, and*

2) *every face of M is a polyhedron with angles not greater than $\pi/2$.*

Proof. It is known that every spherical polyhedron with angles not exceeding $\pi/2$ is a simplex; this proves 1). So we proceed to the proof of 2).

In fact, if Γ_{12} and Γ_{13} are adjacent $(n-2)$-dimensional faces of an $(n-1)$-dimensional polyhedron Γ_1, then the dihedral angle α^1_{23} between them can be obtained from the relation

$$\cos \alpha^1_{23} = \frac{\cos \alpha_{23} + \cos \alpha_{13} \cos \alpha_{23}}{\sin \alpha_{13} \sin \alpha_{23}}. \tag{4}$$

This relation follows from formulas of spherical trigonometry if we consider the trihedral angle cut out by Λ_1^-, Λ_2^- and Λ_3^- in the three-dimensional subspace of Λ^n which is orthogonal to H_{123}.

From (4) it follows that $\alpha^1_{23} \le \alpha_{23}$. Moreover, $\alpha^1_{23} = \alpha_{23}$ if and only if two of the angles occurring on the right-hand side are equal to $\pi/2$.

To conclude §1 we introduce the following spherical mapping of Lobačevskiĭ space. It is known that from the topological point of view the set of the rays in the $(n+1)$-dimensional space $E^{n,1}$ is an n-dimensional sphere. Thus to every polyhedral angle in $E^{n,1}$ there corresponds a polyhedron on the sphere, and to the cone V_+, a ball on the sphere.

As a sphere it will be convenient to take the subset S^n of $E^{n,1}$ defined by the condition $x_0^2 + \cdots + x_n^2 = 1$.

By $\widetilde{M}$, $\widetilde{A}$, $\widetilde{\Lambda}^-$, $\widetilde{V}_+$ etc. we denote the intersections of the sets $\hat{M}$, $\hat{A}^-$, $\hat{\Lambda}^-$, V_+ etc. with the sphere S^n.

Let M be a convex N-hedron in Λ^s, $N > 3$. Then $\widetilde{M}$ is a strictly convex spherical set, i.e. $\widetilde{M}$ is convex and does not contain diametrically opposite points of the sphere. In this case there is an abstract polyhedron $\mathfrak{m} \sim \widetilde{M}$.

Definition 2. Let M be a convex N-hedron, $N > 3$, in Λ^3, where every edge of the polyhedron $\widetilde{M}$ has a nonempty intersection with the sphere $\widetilde{V}_+$. Let $\mathfrak{m}$ be an abstract polyhedron and $\phi : \mathfrak{m} \longrightarrow (\widetilde{M})^{(n-1)}$ an isomorphism. Then the pair (M, ϕ) is called an *α-polyhedron of combinatorial type* $\mathfrak{m}$.

The action of the group $G(\Lambda^3)$ on the α-polyhedra is defined in complete analogy with the action on the ordinary polyhedra, with the one difference that $g^*: (\widetilde{M})^{(n-1)} \longrightarrow (\widetilde{M}_1)^{(n-1)}$. In the same way as $W(\mathfrak{m})$, one defines the set $W_\alpha(\mathfrak{m})$ as the set of all α-polyhedra of combinatorial type $\mathfrak{m}$. Finally, the mapping T is continued in the natural way from $W(\mathfrak{m})$ to all the $W_\alpha(\mathfrak{m})$.

§2. Some properties of the convex polygons and the three-dimensional polyhedra

In this section the word polyhedron means three-dimensional polyhedron.

The closures of the cells of a two-dimensional abstract complex of dimension 0, 1 and 2 are called its vertices, edges and faces respectively.

Unlike the general case, in the case of dimension 3 necessary and sufficient conditions are known (Steinitz's theorem [3], §16) under which the complex $\mathfrak{K}$ is an abstract polyhedron. We shall deduce them in a different form, adapted to our purpose.

A1. A complex can be realized as a cell partition of the two-dimensional sphere.

A2. Every edge belongs to exactly two faces.

A3. A nonempty intersection of two faces is either an edge or a vertex.

A4. Every face contains not fewer than three edges.

The polyhedron $\mathfrak{m}$ has vertices of simplicial type if and only if every vertex is incident with exactly three edges.

Along with the complex $\mathfrak{K}$ we consider the dual complex $\mathfrak{K}^*$, i.e. a cell complex such that to each of its s-dimensional cells η^s there corresponds a $(2-s)$-dimensional cell $\xi^{(2-s)}$ of $\mathfrak{K}$. This is described by $\eta^s = \delta(\xi^{(2-s)})$. The function δ has the property that if $\xi_1 \prec \xi_2$, then $\delta(\xi_2) \prec \delta(\xi_1)$.

If $\mathfrak{K}$ is an abstract polyhedron then so is $\mathfrak{K}^*$. Indeed, if $M \sim \mathfrak{m}$, where $\mathfrak{m}$ is an euclidean polyhedron, then the dual polyhedron $M^* \sim \mathfrak{m}^*$.

Let $\mathfrak{m}$ be an abstract polyhedron with vertices of simplicial type. Then the two-dimensional simplicial complex $\mathfrak{m}^*$ is called the scheme of the polyhedron $\mathfrak{m}$.

From Steinitz's theorem quoted above it follows that a two-dimensional simplicial complex is the scheme of an abstract polyhedron if and only if:

C1. Euler's characteristic of $\mathfrak{K}$ equals 2.

C2. Every vertex is incident with at least one edge.

C3. Every edge belongs to exactly two faces.

C4. A nonempty intersection of two faces is either an edge or a vertex.

The scheme $\mathfrak{m}^*$ of the polyhedron $\mathfrak{m}$ can be realized as a simplicial partition of a sphere; we shall assume that $\mathfrak{m}^*$ is realized in just this way.

Lemma 1. *Let M be either* a) *a polygon in the Lobačevskiĭ plane with angles not exceeding $\pi/2$, or* b) *a polyhedron in Λ^3 with angles not greater than $\pi/2$ and without vertices at infinity. Then*

$$\dim \Gamma_{i_1 i_2}(M) = -\dim H_{i_1 i_2}(M). \tag{5}$$

Proof. Let $n=2$. The proof evidently lies in the fact that in Λ^2 there is no triangle (of finite area) where two angles are not smaller than $\pi/2$.

Let $n=3$. If $\Lambda_\alpha^- \subset \Lambda^3$ is a closed halfspace, then we shall denote by Λ_α^+ the closed halfspace in Λ^3 which is uniquely defined by the equation $\Lambda_\alpha^+ \cap \Lambda_\alpha^- = H_\alpha$.

Let $\mathfrak{A}$ be a halfspace, a plane, a polyhedron or a face of a polyhedron in Λ^n. Put $\mathfrak{A}^V = \widetilde{\mathfrak{A}} \cap (\widetilde{V}_+)$.

If the polyhedron M has no vertices at infinity then $\dim \Gamma_{i_1 i_2} = 1$ or $-\infty$ and $\dim H_{i_1 i_2}$ may assume three values: $1, -1, -\infty$. Let us assume that (5) is not satisfied and $\dim \Gamma_{12} = -\infty$, but $\dim H_{12} > -\infty$. Let $\Gamma_3, \cdots, \Gamma_q$ all be adjacent to the face Γ_1 of the polyhedron. From our assumptions follows that

$$\varnothing \neq H_{12}^V \subset H_1^V \setminus \Gamma_1^V, \quad \Gamma_2^V \subset \Big(\bigcap_{i=2}^{q} (\Lambda_i^-)^V\Big) \cap (\Lambda_1^-)^V.$$

Then one can find $3 \le i \le q$ such that $(\Lambda_1^- \cap \Lambda_i^+)^V$ and $(\Lambda_1^- \cap \Lambda_i^-)^V$ contain the points of H_2^V, i.e. $H_2 \cap H_i \cap \Lambda_1^- \neq \emptyset$. Let $\Gamma_3, \cdots, \Gamma_p$, $p \le q$ be all faces for which the last statement is valid. Then two cases are possible:

The special case. There is a $3 \le i \le p$ such that Γ_2 and Γ_i are adjacent.

The general case. $\Gamma_2 \cap \Gamma_i = \emptyset$, $3 \le i \le p$.

These are all the possible cases; for all the angles are not greater than $\pi/2$ and so it follows that at a vertex there intersect exactly three faces, and if $\Gamma_2 \cap \Gamma_i$ contains a vertex, then Γ_{2i} contains an edge.

Let the faces Γ_1 and Γ_2 be adjacent with Γ_3 and let $H_{12}^V \cap (\Lambda_3^+)^V \neq \emptyset$. Then the sum of the angles of the polyhedron $L = \Lambda_1^- \cap \Lambda_2^- \cap \Lambda_3^+$ is greater than or equal to π, and the Gram matrix of the vectors e_1, e_2 and $-e_3$ is nonnegative definite. This means that $\bigcap_{i=1}^3 H_i^V \neq \emptyset$, which contradicts the statement of point a) as applied to the polygon Γ_3. Thus the special case is impossible.

In the general case we apply induction with respect to N and consider the polyhedra $M_0 = \bigcap_{i\geq 2}\Lambda_i^-$ and $M_+ = M_0 \cap \Lambda_1^+$, $M_0 = M_+ \cup M$. Moreover, it is clear that $M_+ = \Lambda_1^+ \cap (\bigcap_{i=2}^q \Lambda_i^-)$.

From the special case examined above we conclude that all the edges of M_0 are extensions of edges of M. Indeed, "new" edges could arise only at the expense of M_+, which is impossible. Consequently the polyhedron M_0 satisfies the conditions of the lemma and by the induction assumption, $\dim \Gamma_{2i}(M_0) = 1$, $i = 3, \cdots, q$. From the construction of M_0 it follows that the same is true for M, and the proof of the lemma is reduced to the special case discussed above.

Definition 3. Let $\Gamma_{i_1}, \cdots, \Gamma_{i_s}$ be a sequence of faces of a three-dimensional polyhedron such that each is adjacent only to the following and the last one only to the first and no three of them intersect at a vertex. We shall say that the faces $\Gamma_{i_1}, \cdots, \Gamma_{i_s}$ are an *s-angled prismatic element*. For "three-angled prismatic element" we shall briefly write "t. p. e.".

Theorem 2. *Let* $\mathfrak{m}$ *be a convex bounded polyhedron in three-dimensional Lobačevskiĭ space* Λ^3 *whose dihedral angles do not exceed* $\pi/2$. *Then the dihedral angles* $\alpha_{ij}(M)$ *satisfy the following system of inequalities depending only on the combinatorial type*:

$\mathfrak{m}0.$
$$0 < \alpha_{ij} \leq \pi/2;$$

$\mathfrak{m}1.$ *If* Γ_{ijk} *is a vertex of* $\mathfrak{m}$ *then*
$$\alpha_{ij} + \alpha_{ik} + \alpha_{jk} > \pi;$$

$\mathfrak{m}2.$ *If* $\Gamma_i, \Gamma_j, \Gamma_k$ *generate a t. p. e. then*
$$\alpha_{ij} + \alpha_{ik} + \alpha_{jk} < \pi;$$

$\mathfrak{m}3.$ *If* $\Gamma_i, \Gamma_j, \Gamma_k, \Gamma_l$ *generate a four-angled prismatic element then*
$$\alpha_{ij} + \alpha_{il} + \alpha_{jk} + \alpha_{kl} < 2\pi;$$

$\mathfrak{m}4.$ *If* Γ_s *is a quadrangle with the sides* $\Gamma_{is}, \Gamma_{js}, \Gamma_{ks}, \Gamma_{ls}$ *enumerated successively, then*
$$\alpha_{is} + \alpha_{ks} + \alpha_{ij} + \alpha_{il} + \alpha_{jk} + \alpha_{kl} < 3\pi,$$
$$\alpha_{js} + \alpha_{ls} + \alpha_{ij} + \alpha_{il} + \alpha_{jk} + \alpha_{kl} < 3\pi.$$

Proof. The sum of the angles of a triangle S_{ijk} equals the sum of the angles at the vertex Γ_{ijk}, and, as we know, is greater than π, which proves $\mathfrak{m}1$.

Let us assume that the inequality $\mathfrak{m}2$ is not satisfied. Then $\dim (H_i \cap H_j \cap H_k) \geq -1$, i.e. $\dim (H_i \cap H_j) \cap (H_k \cap H_j) \geq -1$, but $\dim \Gamma_{ijk} = -\infty$, i.e. $\dim \Gamma_{ik}(\Gamma_j) = -\infty$, and so our assumption is in contradiction to Lemma 1.

We prove the third inequality. If all the angles between the faces $\Gamma_i, \Gamma_j, \Gamma_k, \Gamma_l$ are equal to $\pi/2$, then the Gram matrix of the vectors e_i, e_j, e_k, e_l has the form

$$\begin{pmatrix} 1 & 0 & (e_i,\ e_k) & 0 \\ 0 & 1 & 0 & (e_j,\ e_l) \\ (e_i,\ e_k) & 0 & 1 & 0 \\ 0 & (e_j, e_l) & 0 & 1 \end{pmatrix}.$$

Among the eigenvalues of this matrix there is not more than one negative one; consequently either (e_i, e_k) or $(e_j, e_l) \geq -1$, i.e. $\dim H_{ik} > -\infty$ or $\dim H_{jl} > -\infty$, which is impossible by Lemma 1.

Now let us assume that one of the conditions $\mathfrak{m}4$ is not satisfied; then from (4) it follows that all angles of the quadrangle Γ_s are equal to $\pi/2$. But it is well known that in the Lobačevskiĭ plane there is no such quadrangle.

We recall conditions analogous to $\mathfrak{m}1$ for vertices at infinity of polyhedra of finite volume [2]. If $\Gamma^{-1}_{i_1 \cdots i_s}$ is an infinite vertex, then either $s = 3$ and $\alpha_{i_1 i_2} + \alpha_{i_1 i_3} + \alpha_{i_2 i_3} = \pi$, or $s = 4$ and the Gram matrix of the vectors $e_{i_1}, \cdots, e_{i_4}$ has the form

$$\begin{pmatrix} 1 & 0 & -1 & 0 \\ 0 & 1 & 0 & -1 \\ -1 & 0 & 1 & 0 \\ 0 & -1 & 0 & 1 \end{pmatrix}. \tag{6}$$

Theorem 2 can conveniently be interpreted as follows. Consider the set $W(\mathfrak{m})$ of the polyhedra of a given combinatorial type. Let $\{\Gamma_{ij}\}$ be the system of the edges of the polyhedron $\mathfrak{m}$, where (ij) runs through a certain set of indices $I(\mathfrak{m})$. We may assume that the enumeration of the faces has been fixed in advance. Let R be the number of edges and consider the R-dimensional arithmetical space $\Omega(\mathfrak{m})$ where the coordinates are enumerated by the elements of the set I. Consider the mapping $\alpha: W(\mathfrak{m}) \longrightarrow \Omega(\mathfrak{m})$, which with every $M \in W(\mathfrak{m})$ associates the set of its dihedral angles $\alpha(M) = (\cdots, \alpha_{ij}(M), \cdots)$.

By $W_0(\mathfrak{m})$ we denote the subset of the set $W(\mathfrak{m})$ distinguished by the condition $\alpha_{ij}(M) \leq \pi/2$, and by $\Omega_0(\mathfrak{m})$ the subset of $\Omega(\mathfrak{m})$ satisfying the inequalities $\mathfrak{m}0 - 4$.

Theorem 2 states that

$$\alpha(W_0(\mathfrak{m})) \subset \Omega_0(\mathfrak{m}). \tag{7}$$

The five groups of inequalities $\mathfrak{m}0 - 4$ are not equivalent. The inequalities $\mathfrak{m}1$ and $\mathfrak{m}2$ define in the space $\Omega(\mathfrak{m})$ a polyhedral angle with vertex at the point $(\cdots, \pi/3, \cdots)$, and the other inequalities define some subset of this angle. Clearly the whole system is compatible if and only if the inequalities $\mathfrak{m}1$ and $\mathfrak{m}2$ are compatible. We point out that for all polyhedra except triangular prisms the inequalities $\mathfrak{m}4$ are consequences of the other inequalities.

For $N = 4$ or 5 there is at least one combinatorial type of polyhedra with vertices of simplicial type: the simplex and the triangular prism. It will be convenient not to include these in the subsequent classification of the combinatorial types.

We shall say that the polyhedron is simple if it does not contain a t.p.e. Simplicity is evidently a combinatorial property. Clearly if $\mathfrak{m}$ is simple, then $\mathfrak{m}0 - 4$ are satisfied.

Let us assume that the complex $\mathfrak{m}$ is realized by a cellular partition of the sphere.

Suppose that the polyhedron $\mathfrak{m}$ is not simple and that $\Gamma_1, \Gamma_2, \Gamma_3$ is one of its t.p.e. This means that the set $\mathfrak{m}\setminus(\Gamma_1 \cup \Gamma_2 \cup \Gamma_3)$ consists of two connected subsets $\mathfrak{C}_1$ and $\mathfrak{C}_2$. We consider the complexes $\mathfrak{m}_i$, $i = 1, 2$, obtainable from $\mathfrak{m}$ by fitting in a two-dimensional cell in place of $\mathfrak{C}_i$. The conditions A1–4 for $\mathfrak{m}_i$ are evidently satisfied, i.e. $\mathfrak{m}_i$ is an abstract polyhedron. This is clear also without Steinitz's theorem.

Indeed, let $M \in W(\mathfrak{m})$; on every edge of the t.p.e. $\Gamma_1(M)$, $\Gamma_2(M)$, $\Gamma_3(M)$ we choose a point and determine the plane H_{N+1} through these. Let $H_{N+1} = \Lambda^+_{N+1} \cap \Lambda^-_{N+1}$ and put $M_1 = \bigcap_{i=1}^{N+1}\Lambda^-_i$ and $M_2 = (\bigcap_{i=1}^{N} \Lambda^-_i) \cap \Lambda^+_{N+1}$. It is easily seen that up to the order of enumeration $M_i \subset W(\mathfrak{m}_i)$.

If now $\mathfrak{m}_1 \sim \mathfrak{m}$, then $\mathfrak{m}_2$ is a triangular prism. In this case we shall say that $\Gamma_1, \Gamma_2, \Gamma_3$ is a truncated t.p.e. *We shall call a simple N-hedron, $N > 5$, truncated if all its t.p.e. are truncated.*

Let $\mathfrak{m}$ be a polyhedron and $\Gamma_{i_p}, \Gamma_{j_p}, \Gamma_{k_p}$, $p = 1, \cdots, s$, all its nontruncated t.p.e.'s. We carry out the truncation of $\mathfrak{m}$ successively on all $\Gamma_{i_p}, \Gamma_{j_p}, \Gamma_{k_p}$, as described above. We shall obtain a system of polyhedra $\mathfrak{m}_1, \cdots, \mathfrak{m}_l$, not all simple, but with all their t.p.e.'s truncated. The polyhedra $\mathfrak{m}_1, \cdots$ $\cdots, \mathfrak{m}_l$ will be called the truncated components of $\mathfrak{m}$.

Let $M \in W(\mathfrak{m})$. Consider the polyhedra M_i, $i = 1, \cdots, l$ obtained from M by dissection at the t.p.e.'s. The polyhedron M is put together using the M_i as follows. For simplicity take $l = 2$ and a single nontruncated t.p.e. in $\mathfrak{m}$: $\Gamma_1, \Gamma_2, \Gamma_3$. Put $\Lambda^-_{N+1} = \Lambda^-_{N+1}(M_1)$ and $\Lambda^+_{N+1} = \Lambda^-_{N+1}(M_2)$; after putting together the faces $\Gamma_{N+1}(M_1)$ and $\Gamma_{N+1}(M_2)$ so that the polyhedra lie on distinct sides of the hyperplane H_{N+1} and the edges $\Gamma_{i,N+1}(M_1)$ and $\Gamma_{i,N+1}(M_2)$, $i = 1, 2, 3$, coincide, we obtain the polyhedron M.

If $\mathfrak{m}$ contains a nontruncated t.p.e. we shall say that $\mathfrak{m}$ is a composite polyhedron.

Now let $\mathfrak{m}$ be a composite polyhedron and $\mathfrak{m}_1, \cdots, \mathfrak{m}_l$ its truncated components. Consider the complex $\mathfrak{p}\mathfrak{m}_i$ obtained from $\mathfrak{m}_i$ by contracting all triangular faces into a point. From the conditions A1–4 for $\mathfrak{m}_i$ it follows that $\mathfrak{p}\mathfrak{m}_i$ is an abstract polyhedron and $\mathfrak{p}\mathfrak{m}_i$ is either simple or a simplex. We shall call the $\mathfrak{p}\mathfrak{m}_i$ the simple components of $\mathfrak{m}$; for a truncated polyhedron there is one simple component.

Let $\mathfrak{m}$ be truncated and $P \in \mathfrak{p}\mathfrak{m}$. We mark those vertices obtained by contracting the triangular faces $\Gamma_i(\mathfrak{m})$ and we draw planes cutting off these vertices. As a result we obtain a polyhedron $P' \in W(\mathfrak{m})$.

§3. The uniqueness theorem

By a familiar freedom of speech, a convex polygon in a Lobačevskiĭ plane $P = \bigcap_{i=1}^{N}\Lambda^-_i$ will be called a parallelogram if for all $1 \le i, j \le N$

$$\dim \Gamma_{ij}(P) = \dim H_{ij}(P). \tag{8}$$

We shall say that P is a parallelogram in the weak sense if P is bounded and the identity (8) may fail when $\dim H_{ij}(P) = -1$. It may be pointed out that parallelograms in the weak sense will occur only in this section, and not in the rest of the paper.

Lemma 2 (on parallelograms). *Let $P \sim P'$ be two unequal parallelograms in the weak sense in the space Λ^2. Let corresponding angles be equal. To the side $\Gamma_i(P)$ let there be associated the sign + or − according to whether it is greater or smaller than the corresponding side $\Gamma_i(P')$, leaving equal sides unmarked. Then on a complete circuit around P the number of sign changes is not smaller than four.*

In the proof of Lemma 2 the word "parallelogram" always means "parallelogram in the weak sense". We shall assume that $H_i \cap H_j = \emptyset$ if $\dim H_{ij} < 0$.

Instead of the sphere S^2 it will be convenient to consider the hyperplane $E^{2,1}$ defined by the condition $x_0 = 1$, in which the structure of an affine space is naturally defined. In the proof of the lemma affine sets will be denoted by the same symbols as the corresponding spherical sets.

Let us consider the m-lateral $M = \bigcap_{i=1}^m \Lambda_i^-$ with the sides $\Gamma_1, \cdots, \Gamma_m$ such that

a) $A_i = \Gamma_i \cap \Gamma_{i+1}$, $i = 1, \cdots, m-1$ are among its vertices,

b) if $H_i \cap H_j \neq \emptyset$, then $H_i \cap H_j \in M$.

(The polygon M is not assumed to be bounded.)

For the proof we need the

Auxiliary Lemma. *Let M and M' be two convex m-laterals satisfying the conditions* a) *and* b), *and let*

$$\angle A_i = \angle A_i' \quad (i = 2, \ldots, m-1),$$
$$|\Gamma_i(M)| \leqslant |\Gamma_i(M')| \qquad (i = 2, \ldots, m-1)$$

with at least one of the inequalities strict. Then $(e_1, e_m) > (e_1', e_m')$.

Proof. Let $\rho(X, Y)$ denote the distance between the points X and Y. Consider the continuous deformation $M_i(t)$ of M defined as follows. Denote by $\pi_i(t)$ the parallel shift along the straight line H_i such that if $t > t' \geq 0$, then

$$\rho(A_{i-1}, \pi_i(t) A_i) > \rho(A_{i-1}, \pi_i(t') A_i). \tag{9}$$

Put

$$M_i(t) = \left(\bigcap_{k=1}^{i} \Lambda_k^-\right) \cap \left(\bigcap_{k=i+1}^{m} \pi_i(t) \Lambda_k^-\right), \quad M_i(0) = M, \tag{10}$$

and then, for $t > t' \geq 0$, we have

$$\angle A_k(t) = \angle A_k(t'), \quad |\Gamma_k(t)| = |\Gamma_k(t')|, \quad k \neq i, \quad |\Gamma_i(t)| > |\Gamma_i(t')|.$$

These inequalities and equations obviously follow from (10).

By $\hat{\pi}_i(t)$ we denote the transformation of the space $E^{2,1}$ corresponding to $\pi_i(t)$. For the proof of the lemma it is sufficient to show that $(e_1, \hat{\pi}_i(t) e_m)$ decreases if t increases. This is equivalent to the following two statements:

Firstly: If M has the properties a) and b), then $d(e_1, \hat{\pi}_i(t) e_m)/dt < 0$;

Secondly: The polyhedron $M_i(t)$ satisfies the conditions of the lemma.

Proof of the first statement. The polygon M can always be decomposed so that $e_i = (0, 0, 1)$ and the midpoint of the side Γ_i coincides with the point O corresponding to the positive ray of the axis Ox_0. Consider the plane $x_0 = 1$ and let $(1, \zeta_k, 0)$, $k \neq i$, be the coordinates of the point $\widetilde{H}_i \cap \widetilde{H}_k$ in

$E^{2,1}$. One of the straight lines $\widetilde{H}_1$, $\widetilde{H}_m$ intersects the line $\widetilde{H}_i$; assume that $\widetilde{H}_1$ does so. Let $\zeta_1 > 0$; then

$$\hat{\pi}_i(t) = \begin{pmatrix} \operatorname{ch} t & \operatorname{sh} t & 0 \\ \operatorname{sh} t & \operatorname{ch} t & 0 \\ 0 & 0 & 1 \end{pmatrix}.$$

If $e_k = (x_{0k}, x_{1k}, x_{2k})$, then the required derivative

$$\frac{d}{dt}(e_1, \pi_i(t)\, e_m) = x_{0m}x_{11} - x_{1m}x_{01}. \tag{12}$$

Consider the two vectors $u_\lambda = (1, \lambda, 0)$ and $v_\lambda = (1, -\lambda, 0)$ and the function $f(\lambda) = (u_\lambda e_1)(v_\lambda e_m) = -\lambda^2(x_{11}x_{1m}) + \lambda(x_{0m}x_{11} - x_{1m}x_{01}) + x_{01}x_{0m}$. We shall show that the function $f(\lambda)$ decreases in a neighborhood of zero. We note that $(u_\lambda e_1)$ is greater than zero when $u_\lambda \notin \hat{\Lambda}_1^-$, equal to zero when $u_\lambda \in \hat{H}_1$ and less than zero when u_λ lies in $\hat{\Lambda}_1^- \setminus \hat{H}_1$. Analogous statements are valid for v_λ. Then it follows, first of all, that $f(0) > 0$. We now find the roots of the equation $f(\lambda) = 0$. The endpoint of the vector u_λ (v_λ) lies on the straight line $\widetilde{H}_i$ and $f(\lambda)$ becomes zero if $\lambda = \zeta_1$ $(-\lambda = \zeta_m)$ respectively. There are three possibilities:

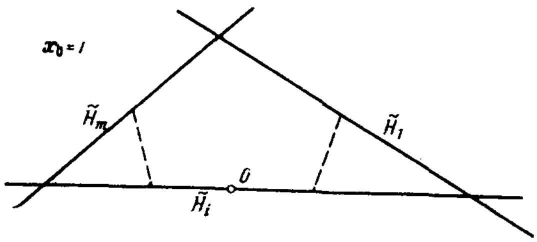


Figure 1

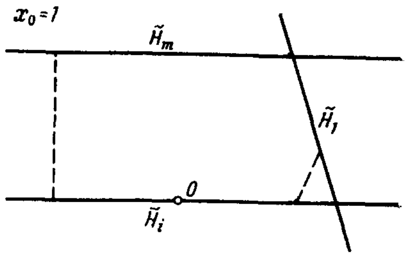


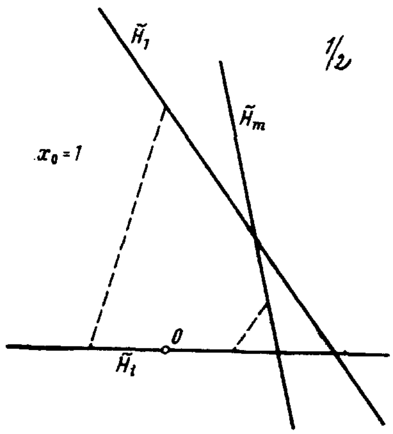
Figure 2

1. $\zeta_m < 0$; then both roots are positive since $\zeta_1 > 0$ (see Figure 1).

2. $\widetilde{H}_i \| \widetilde{H}_m$, i.e. $x_{1m} = 0$. The function $f(\lambda)$ is linear and its only root is positive (see Figure 2).

3. $\zeta_m > 0$; then because of the convexity of M we have $\zeta_m > \zeta_1$, i.e. the equation has roots of opposite signs and the negative root has its absolute value greater than the positive root (see Figure 3).

The function $f(\lambda)$ is a quadratic trinomial. Our information about it determines its behavior in a neighborhood of zero and shows that there $f(\lambda)$ is decreasing. Consequently $0 > f_\lambda'(0) = x_{0m}x_{11} - x_{1m}x_{01}$, and by (12) the first statement is established.



1/2

Figure 3

The second statement is proved by induction on m. For $m = 3$, there is nothing to prove. If we assume that the statement has been established for all $m' < m$ we thereby assume that the auxiliary lemma is satisfied. Verification of the fact that $M_i(t)$ satisfies the condition of the lemma is reduced, in view of the nature of the deformation, to showing that $H_p(t)$ and $H_q(t)$ never intersect for any t, or any $p < i < q$, except perhaps when $p = 1$ and $q = m$. However $H_p(t) \cap H_q(t) = \emptyset$ is equivalent to $(e_p(t), e_q(t)) \le -1$ because $(e_p(0), e_q(0)) \le -1$, and for pairs of polygons $M_{pq}(0) = \bigcap_{k=p}^{q} \Lambda_i^-(0)$ and $M_{pq}(t) = \bigcap_{k=p}^{q} \Lambda_i^-(t)$ the lemma is true.

Using the auxiliary lemma, we can now prove the lemma on parallelograms. Assume the opposite to be true. Since the number of sign changes is even, it must be either zero or two. So we have two cases. First case: The broken line bounding P can be decomposed into two nonempty parts by deleting two of its sides (denote them by Γ_p and Γ_q) in such a way that there are distinguished sides in at least one part and in each part all the distinguished sides have the same sign.

This construction is impossible if and only if all the sides are marked by the same sign, which constitutes the second case.

The sides of P may be enumerated consecutively, and $p < q$.

At least one of the two pairs of polygons $M = (\bigcap_{k=1}^{p} \Lambda_k^-) \cap (\bigcap_{l=q}^{N} \Lambda_l^-)$ and M' (where M' is defined analogously on the basis of P') or $M_1 = \bigcap_{k=p}^{q} \Lambda_k^-$ and $M'_1 = \bigcap_{k=p}^{q} \Lambda_k'^-$ satisfies the auxiliary lemma, according to which

$$(e_p, e_q) \leqslant (e_p', e_q') \leqslant (e_p, e_q),$$

where at least one of the inequalities is strict, and thus the first case is excluded. In the second case P and P' themselves satisfy the conditions of the auxiliary lemma, according to which $(e_1 e_N) \neq (e'_1, e'_N)$. This proves the lemma.

Lemma 3. *Let $M \sim M'$ be two convex bounded polyhedra in Λ^3, all of whose two-dimensional faces are parallelograms in the weak sense (without assuming that the vertices are simplicial) and whose corresponding plane angles are equal. Then M and M' are two equal polyhedra.*

Proof. From the equality of the plane angles it follows that the dihedral angles are equal (cf. [1], Chapter III, §2); hence the theorem will be proved when we demonstrate equality of the edges. To each edge $\Gamma_{ij}(M)$ we associate the sign + or − according to whether it is greater or smaller than the edge $\Gamma_{ij}(M')$, equal edges remaining unmarked. By the parallelogram lemma, if only one edge in a face is marked, then the number of sign changes for a complete circuit is not smaller than four. This is in contradiction to the following lemma of Cauchy.

Lemma (Cauchy). *Let $\mathfrak{m}$ be an abstract three-dimensional polyhedron with signs + or − prescribed for some edges. Then there is a face of the polyhedron which contains a marked edge and for which the number of sign changes for a circuit is smaller than four.*

A proof can be found in Chapter 2, §1 of the book [1].

Theorem 3. *Let $M, M' \in \mathfrak{M}_a(\mathfrak{m})$ be two a-polyhedra with vertices of simplicial type all of whose faces are parallelograms and all of whose corresponding dihedral angles are equal to each other. Then M and M' are equal polyhedra.*

Proof. Equality of the angles implies that for a pair of corresponding vertices of M and M' both of them are in $\widetilde{V}_+$ or else neither. Suppose that $\widetilde{\Gamma}_{i_p j_p k_p}$ and $\widetilde{\Gamma}'_{i_p j_p k_p}$, $p = 1, \cdots, l$, are all the vertices of $\widetilde{M}$ and $\widetilde{M}'$ respectively which do not lie in $\widetilde{V}_+$. For every p there is a motion of the space

Λ^3 superposing $\widetilde{H}_i$ on $\widetilde{H}'_i$, $\widetilde{H}_j$ on $\widetilde{H}'_j$, and $\widetilde{H}_k$ on $\widetilde{H}'_k$. Therefore we can select half-spaces Λ^-_{N+p} and Λ'^-_{N+p} which are also superposed by this motion, where the plane H_{N+p} intersects the edges Γ_{ij}, Γ_{ik}, Γ_{jk} and no other sides of the polygons Γ_i, Γ_j, Γ_k. The plane H'_{N+p} has analogous properties.

Consider the polyhedra $Q = \bigcap_{m=1}^{N+l} \Lambda^-_m$ and $Q' = \bigcap_{m=1}^{N+l} \Lambda'^-_m$. All their faces are parallelograms in the weak sense. Moreover, all their vertices are of simplicial type, which, together with the equality of the dihedral angles, in view of (4) implies the equality of the plane angles. By Lemma 2 $Q = gQ'$, $g \in G(\Lambda^3)$, and so M and M' are equal.

Corollary. *Let $M, M' \in W_\alpha(\mathfrak{m})$ be two α-polyhedra with vertices of simplicial type and angles not exceeding $\pi/2$, where corresponding dihedral angles are equal. Then M and M' are equal polyhedra.*

By $W_{\alpha p}(\mathfrak{m})$ and $W_p(\mathfrak{m})$ we shall denote subsets of $W_\alpha(\mathfrak{m})$ and $W(\mathfrak{m})$ distinguished by the condition "all the faces are parallelograms". Evidently $W_{\alpha 0} \subset W_{\alpha p}$ (Lemma 1). This proves the corollary.

Uniqueness Theorem. *Let M and M' denote two convex bounded polyhedra in n-dimensional Lobačevskiĭ space, $n \geq 3$, and let $M \sim M'$, with corresponding dihedral angles equal and not exceeding $\pi/2$. Then M and M' are equal polyhedra.*

Proof. For three-dimensional space the theorem has already been proved, since a convex bounded polyhedron with angles not exceeding $\pi/2$ is an α-polyhedron. Theorem 1 of §1 provides for induction, which comples the proof.

§4. Manifold of polyhedra

All the subsequent results refer to three-dimensional Lobačevskiĭ space, so that in this and in the two following sections we shall assume that $n = 3$ and that the word "polyhedron" means "a convex three-dimensional polyhedron with vertices of simplicial type" if nothing else is indicated.

Lemma 4. *Suppose that $M \in W_\alpha(\mathfrak{m})$ is an α-polyhedron (recall: with vertices of simplicial type) in Λ^3. There is a neighborhood $U \subset \Pi^N$ of the point $\mathrm{T}(M)$ such that $P_{\mathrm{T}} = M'$ for every $\mathrm{T} \in U$ is an α-polyhedron of the combinatorial type $\mathfrak{m}$. Moreover, by reducing the neighborhood U we may assume that M and M' both have, or else both do not have the following properties:*

1) $\hat{\Gamma}_{ijk} \in V_+$;
2) $\hat{\Gamma}_{ijk} \notin \bar{V}_+$;
3) *all the faces are parallelograms;*
4) *all dihedral angles are acute.*

The lemma becomes evident if one remembers that at every vertex there intersect exactly three planes from $H_j(M)$.

The lemma means that $\mathrm{T}(W_\alpha(\mathfrak{m}))$ and $\mathrm{T}(W(\mathfrak{m}))$ are open subsets of an analytical manifold Π^N. Moreover, T is a one-to-one mapping, so that in $W_\alpha(\mathfrak{m})$ we can introduce the structure of an analytical manifold, assuming that T is a diffeomorphism.

Lemma 4 states that $W(\mathfrak{m})$ and $W_{\alpha p}(\mathfrak{m})$ are submanifolds of the variety $W_\alpha(\mathfrak{m})$.

The action of the group $G(\Lambda^3)$ was defined on the set $W_\alpha(\mathfrak{m})$; in particular, if $(M, \phi) \in W_\alpha(\mathfrak{m})$ and $g \in G(\Lambda^3)$ then $g(M, \phi) = (gM, g\phi)$. The group $G(\Lambda^3)$ also acts on the variety Π^N; and it is easily seen that $\mathrm{T}g = g\mathrm{T}$.

The definition of an α-polyhedron implies that $N > 4$ and that the rank of the vector system $e_1(M), \cdots, e_N(M)$ equals 4; this proves that $G(\Lambda^3)$ acts on $\mathrm{T}(W_\alpha(\mathfrak{m}))$ and on $W_\alpha(\mathfrak{m})$ without fixed points and effectively.

By $V_\alpha(\mathfrak{m})$ we denote the factor space $W_\alpha(\mathfrak{m})/G(\Lambda^3)$. We shall assume that the suffixes on V and W have the same meaning. Let $\chi: W_\alpha(\mathfrak{m}) \longrightarrow V_\alpha(\mathfrak{m})$ be the canonical mapping.

We shall show that on $V_\alpha(\mathfrak{m})$ one can introduce the structure of a smooth manifold where the mapping χ is analytic. It suffices to show that for two distinct orbits $G(\Lambda^3)M_1$ and $G(\Lambda^3)M_2$ there are $G(\Lambda^3)$-invariant regions U_1 and U_2 such that $G(\Lambda^3)M_i \subset U_i$, $i = 1, 2$, and $U_1 \cap U_2 = \emptyset$. If $M_2 \notin G(\Lambda^3)M_1$, then i and j can be found so that $(e_i(M_1), e_j(M_1)) \neq (e_i(M_2), e_j(M_2))$. In the opposite case the Gram matrices of the vector systems would coincide, and, as pointed out above, their rank would be 4, i.e. there is an element $\hat{g} \in \hat{G}(\Lambda^3)$ which carries $e_i(M_1)$ into $e_i(M_2)$ for all $i = 1, \cdots, N$. This means that $gM_1 = M_2$. The function $f(M) = (e_i(M), e_j(M))$ is smooth and a $G(\Lambda^3)$-invariant. From it the regions U, and U_2 can be chosen in the standard manner.

Note that the mapping $\alpha: W_\alpha(\mathfrak{m}) \longrightarrow \Omega(\mathfrak{m})$ is carried over to $V_\alpha(\mathfrak{m})$ in the natural way. The mapping $\alpha: V_\alpha(\mathfrak{m}) \longrightarrow \Omega(\mathfrak{m})$ is analytic. The restriction of α to $V_{\alpha p}(\mathfrak{m})$ is a monomorphism (Theorem 3).

We shall not introduce a special notation for the points of the set $V_\alpha(\mathfrak{m})$; we shall denote them just as the points of the set $W_\alpha(\mathfrak{m})$.

Lemma 5. *Let $M \in V_{\alpha p}(\mathfrak{m})$; then there are neighborhoods U and Φ, $M \in U \subset V_{\alpha p}(\mathfrak{m})$ and $\alpha(M) \in \Phi \subset \Omega(\mathfrak{m})$, of the points of M and $\alpha(M)$ respectively such that the restriction of α to U is a homeomorphism of U onto Φ.*

Proof. Choose a neighborhood U of a point of M such that $\overline{U} \subset V_{\alpha p}$ is compact. The restriction of α to $\overline{U}$ is a one-to-one and continuous mapping of the compact $\overline{U}$ onto the compact $\alpha(\overline{U})$. Consequently $\alpha: \overline{U} \longrightarrow \alpha(\overline{U})$ is a homeomorphism.

The dimension of the manifold $V_{\alpha p}(\mathfrak{m})$ equals the dimension of the manifold $V_\alpha(\mathfrak{m})$ and is obtained from the relation

$$\dim V_\alpha(\mathfrak{m}) = \dim W_\alpha(\mathfrak{m}) - \dim G(\Lambda^3),$$

so that the group $G(\Lambda^3)$ acts effectively and without fixed points. Thus $\dim V_{\alpha p}(\mathfrak{m}) = 3N - 6$. The dimension of the space $\Omega(\mathfrak{m})$ equals the number of the edges of the polyhedron $\mathfrak{m}$; we have denoted it by R. Let v be the number of vertices of the polyhedron. Their simpliciality means that $2R = 3v$. Putting this in Euler's formula we find that $R = 3N - 6$, i.e. $\dim V_{\alpha p}(\mathfrak{m}) = \dim \Omega(\mathfrak{m})$. To finish the proof we apply the theorem that under a homeomorphism regions are mapped into regions. From Lemma 5 follows

Corollary. *If in Lobačevskiĭ space there is an α-polyhedron $M \in W_{\alpha p}(\mathfrak{m})$ with the angles $\alpha_{ij}(M)$, then there is an $\epsilon > 0$ such that for every selection of α_{ij}, $|\alpha_{ij}(M) - \alpha_{ij}| < \epsilon$ there is an $M' \in W_\alpha(\mathfrak{m})$ whose angles equal the numbers α_{ij}. In particular, if there is an α-polyhedron with a finite volume and angles smaller than $\pi/2$, then there is also a bounded polyhedron of the same combinatorial type and with acute angles.*

The remaining part of this section is devoted to the study of the convergence problem in the manifold $W(\mathfrak{m})$. We begin with the required definitions. Let Λ_m^-, $m = 1, 2, \cdots$, be a sequence of closed halfspaces in n-dimensional Lobačevskiĭ space, $n \geq 2$. A topology introduced into the set Π^1 means that the sequence Λ_m^- tends to the closed halfspace Λ_0^-, $\lim \Lambda_m^- = \Lambda_0^-$ if and only if the sequence of the vectors e_m, $m = 1, 2, \cdots$, determined from (3) tends to a vector e_0, $\lim e_m = e_0$ in the sense of the ordinary coordinate convergence. Remember that $(e_0, e_0) = 1$.

Consider the sequence $\hat{\Lambda}_m^-$, $m = 1, 2, \cdots$. The given definition means that the sequence $\hat{\Lambda}_m^-$ tends to a closed halfspace E_0^- in the sense of ordinary convergence. The set of all halfspaces in E^{n+1} is compact, i.e. in every sequence of halfspaces $\hat{\Lambda}_m^-$ a convergent subsequence can be selected, but its limit (call it E_0^-) possibly does not lie in $\hat{\Pi}^1$. If $E_0^- \notin \hat{\Pi}^1$, then there are the two possibilities: $E_0^- \cap V_+ = \emptyset$ or $\overline{V}_+ \subset E_0^-$.

Let us say that the sequence Λ_m^-, $m = 1, 2, \cdots$ is *generalized convergent* if the sequence $\hat{\Lambda}_m^-$, $m = 1, 2, \cdots$, is convergent, $\lim \hat{\Lambda}_m^- = E_0^-$ and $E_0^- \cap V_+ \neq \emptyset$. We introduce the notation $\lim \Lambda_m^- = \lambda_0^-$ where either $\lambda_0^- \in \Pi^1$ and $\lim \Lambda_m^- = \lim \Lambda_m^-$, or λ_0^- is an improper halfspace whose interior consists of all Λ^n and the boundary is the infinite point to which corresponds the ray $V_0 \cap E_0^-$.

Let M_m, $m = 1, 2, \cdots$, be a sequence of convex bounded polyhedra of combinatorial type $\mathfrak{m}$ in Λ^n, $n \geq 2$.

We shall say that the sequence is generalized convergent to the polyhedron M, $\lim M_m = M$, if for every $i \leq N$ the sequence $\Lambda_i^-(M_m)$ is generalized convergent and $M = \bigcap_{i=1}^N \lambda_i^-$. If $M \in W(\mathfrak{m})$, then the convergence concept coincides with the topology in $W(\mathfrak{m})$ introduced earlier, and in this case we shall say that $M = \lim M_m$ but the sequence converges in the ordinary sense.

It is natural to define the generalized limit of hyperplanes $H_i(M_m)$ as the boundary of the halfspace λ_i^-; we shall say that $\lim H_i(M_m) = h_i$ where h_i is either a hyperplane or a point at infinity. We denote by J the set of all those i, $1 \leq i \leq N$, for which h_i is an ordinary hyperplane. Clearly $M = \bigcap_{i \in J} \lambda_i^-$.

The generalized limit of a sequence of t-dimensional, $t > -\infty$, planes $H_{i_1 \cdots i_k}(M_m)$ is the set $h_{i_1 \cdots i_k} = \bigcap_{s=1}^k h_{i_s}$. If $i_s \in J$, $1 \leq s \leq k$ then $h_{i_1 \cdots i_k}$ is an ordinary plane. In the opposite case $h_{i_s} = h_{i_p}$ for all i_s, $i_p \notin J$, $1 \leq p, s \leq k$ and $h_{i_1 \cdots i_k} = h_{i_s}$, $i_s \notin J$. The generalized limit of the boundaries $\Gamma_{i_1 \cdots i_k}(M_m)$ is the boundary $\Gamma_{i_1 \cdots i_m}(M) = h_{i_1 \cdots i_k} \cap M$.

Let $i, j \in J$ and $\dim h_i \cap h_j > n - 2$; by ω_{ij} we denote the dihedral angle between the planes h_i and h_j situated in $\lambda_i^- \cap \lambda_j^-$.

Lemma 6. *Let $M_m \in W(\mathfrak{m})$, $m = 1, 2, \cdots$, be a sequence of convex polyhedra in the space Λ^n, $n \geq 2$, X a point in Λ^n and $X \in M_m$ for $m > m_0$. Then there is a generalized convergent subsequence M_{i_m}. The limit of every generalized convergent sequence of bounded polyhedra is a polyhedron of finite volume.*

Proof. The first statement follows evidently from the analogous property of sequences of halfspaces. For the second statement we remember that the volume of a polyhedron M is finite if and only if $\hat{M} \subset \overline{V}_+$, and that $M_m \subset V_+$.

Lemma 7. *Let $M_m \in W(\mathfrak{m})$ be a generalized convergent sequence of convex bounded polyhedra in Λ^n. Then $(e_i(M), e_j(M)) = \lim (e_i(M_m), e_j(M_m))$ for all $i, j \in J$.*

The proof is obvious from the definition of convergence and the continuity of (,).

From Lemmas 1 and 7 follows

Corollary. *Suppose that the conditions of Lemma 7 are satisfied and $n = 2, 3$, and $\alpha_{ij}(M_m) \leq \pi/2$. Then $(e_i(M), e_j(M)) \leq 0$, $1 \leq i, j \leq N$, and $(e_i(M), e_j(M)) \leq -1$, $(ij) \notin J$. This means that $0 \leq \omega_{ij} \leq \pi/2$ and $\omega_{ij} = 0$ if $(ij) \notin I$ but the angle ω_{ij} is well defined. (In fact, this is true for arbitrary n.)*

Lemma 8. *Let M_m be a generalized convergent sequence of convex bounded polyhedra of combinatorial type m in the space Λ^n, $n = 2$ or 3, with angles not greater than $\pi/2$. Then*

1. $\lambda_i^- \neq \lambda_j^-$, $i, j \in J$, $i \neq j$.

2. *No two of the hyperplanes* h_{i_s}, $i_s \in J$, $s = 1, 2, 3$ *coincide.*

3. *If* $i \in J$, *then* $\dim \Gamma_i(m) \geq 0$. *In the case* $n = 3$ *there is a sequence of polyhedra* P_m *in* Λ^2 *such that* P_m *equals* $\Gamma_i(M_m)$ *and* $\lim P_m$ *exists and equals* $\Gamma_i(M)$.

4. *If* $h_i = h_j$, $i, j \in J$, *then* $\omega_{ij} = 0$, $M \in h_j$ *and equals* $\Gamma_j(M)$.

5. *If* $n = 3$ *but* $\lim M_m = M$ *is a polygon, then* M *is a polygon of finite area.*

Proof. 1. If $\lambda_i = \lambda_j$ then $\omega_{ij} = \pi$, which contradicts Lemma 6.

2. Let $h_{i_1} = h_{i_2} = h_{i_3}$. Then two of the halfspaces $\lambda_{i_s}^-$, $s = 1, 2, 3$ coincide, which in view of assertion 1 is impossible.

3. In a polyhedron whose angles are not greater than $\pi/2$ the foot of a perpendicular from a point of the polyhedron onto the plane of a face is actually a point of the face. We select a point $X \in M$; there is a sequence $X_m \in M_m$, $m = 1, 2, \cdots$, whose limit is X. The proof of this statement is obvious. The basis of the perpendicular from X_m onto the plane $H_i(M_m)$ lies in $\Gamma_i(M_m)$. This property is preserved in the limit case, i.e. $\dim \Gamma_i(M) \geq 0$.

If $i \in J$, then there is a sequence $g_m \in G(\Lambda^3)$ such that $\lim g_m = 1 \in G(\Lambda^3)$ and $g_m H_i(M_m) = h_i$. It is readily seen that $\lim g_m M_m = \lim g_m \cdot \lim M_m = M$. Consider the sequence $g_m M_m$; we have $g_m \Gamma_i(M_m) = \Gamma_i(g_m M_m)$. Moreover, $\Gamma_i(g_m M_m)$ is a sequence of polygons in Λ^2, and according to the definition the sequence $\Gamma_i(g_m M_m)$ is generalized convergent to $\Gamma_i(M)$.

4. If $h_i = h_j$, then in view of assertion 1 we have $\lambda_i^- \cap \lambda_j^- = h_j$. The rest of the proof is obvious.

5. If M is a polygon then we have the situation of assertion 4 and $M = \Gamma_i(M)$. The polygon $\Gamma_i(M)$ is the limit of a sequence of polygons (assertion 3) and consequently (Lemma 5) has finite area.

Let $M_m \in W(\mathfrak{m})$, $m = 1, 2, \cdots$, $M = \lim M_m$ and $\Gamma_{i_1 \cdots i_k}^{t'}(M) = \lim \Gamma_{i_1 \cdots i_k}^{t}(M_m)$. We shall say that the face $\Gamma_{i_1 \cdots i_k}$ degenerates (in the sequence M_m) if $t \neq t'$. Turning to our case $n = 3$, we shall use the word "polyhedron" again in the sense fixed in the beginning of this section. Moreover, we shall assume it to be bounded and in $W_0(\mathfrak{m})$. Let $\mathfrak{m}$ be not a simplex. Evidently if $0 \leq t \leq 3$, then $-1 \leq t' \leq 3$, but not always $t > t'$. From our definitions it follows that the limit of an edge may be a plane region when one of its angles tends to 0 or to π. (In the case of a vertex everything is analogous and we shall not consider this case in particular.) The second case is impossible when the angles are not greater than $\pi/2$. In the first case the polyhedron itself degenerates into a plane. Thus the degeneration of an edge is unessential, since it follows from the stronger degeneration where $t > t'$. Degeneration of a polyhedron will not be the object of our study. Thus we shall assume that degeneration means $t > t'$. We shall say that an edge (side) degenerates boundedly if its limit is a point and if none of the faces (itself a polygon) containing this edge degenerates into a point.

Lemma 9. *Let* P_m, $m = 1, 2, \cdots$, *be a sequence of* k*-gons with angles not exceeding* $\pi/2$, *and let* $P = \lim P_m$. *Then the following assertions are valid.*

1. *If* P *is a bounded polygon, then its area equals the limit of the areas of the polygons* P_m. *The sequence degenerates if and only if the limit of the sum of the angles of* P_m *equals* $\pi(k-2)$.

2. *If the side* Γ_i *degenerates boundedly, then its adjacent angles tend to* $\pi/2$.

Proof. 1. If P is a bounded polygon then $P = \bigcap_{i=1}^{k} \lim \Lambda_i^-(P_m)$.

The function $s(\mathrm{T}) =$ area of P_{T}, $\mathrm{T} \in \Pi^k$, is continuous everywhere in its domain of definition.

Moreover $s(\mathrm{T}) = \pi(k-2)$ − (the sum of the angles of the polygon P_{T}) if only P_{T} is a k-gon. This completes the proof of statement 1.

2. The statement is trivial when P_m is a sequence of triangles. Let $k > 3$. Extend the sides of P_m adjacent to Γ_i into the halfspace Λ_i^-. By Lemma 1 they do not intersect, i.e. $|\Gamma_i|$ is greater than the length of the base of the triangle, where the base angles are equal to the angles at Γ_i, but the third vertex is at infinity. If the length of this base tends to zero then so does the area of the triangle, which means that the sum of the base angles tends to π.

Let Γ_i be the face of a polyhedron $\mathfrak{m}$ with the sides $\Gamma_{ii_1}, \cdots, \Gamma_{ii_p}$. The system of inequalities consisting of all those inequalities $\mathfrak{m}0-4$ which depend only on the indices $i, i_1, \cdots, i_p$ will be called the restriction of the system $\mathfrak{m}0-4$ to Γ_i.

Lemma 10. *Let the sequence of polyhedra in Λ^3 $M_m \in W_0(\mathfrak{m})$, $m = 1, 2, \cdots$, be generalized convergent to M. Then:*

1. *If $i \in J$ and $\Gamma_i(M)$ is bounded and the numbers $\alpha_{ij} = \lim \alpha_{ij}(M_m)$ satisfy the system $\mathfrak{m}0-4$ restricted to Γ_i, then Γ_i does not degenerate.*

2. *If the edge Γ_{ij} with the endpoints Γ_{ijk} and Γ_{ijl} degenerates boundedly, then $i, j, k, l \in J$, $h_{ij} \perp h_k$ and $h_{ij} \perp h_l$, $\Gamma_k(M) = \Gamma_l(M)$ and the faces Γ_i and Γ_j degenerate.*

3. *If $i \in J$ and the face Γ_i degenerates into a ray, a straight line or a segment, then it has two adjacent faces Γ_j and Γ_k, $j, k \in J$, such that $h_j = h_k$ and h_k is orthogonal to h_i.*

Proof. 1. If a sequence of k-gons with angles not greater than $\pi/2$ degenerates, but its limit is a bounded set, then (Lemma 9) the sum of the angles tends to $\pi(k-2)$, i.e. $k = 3$ or $k = 4$.

Suppose that $i \in J$, that the face Γ_i degenerates and that its limit is a bounded set. From (4) it follows that the sum of the angles of the polygon $\Gamma_i(M_m)$ is less than or equal to $\alpha_{i_1i_k}(M_m) + \alpha_{i_1i_2}(M_m) + \cdots + \alpha_{i_{k-1}i_k}(M_m)$ and equality is reached if $\Gamma_i(M_m) \perp \Gamma_{i_p}(M_m)$, $p = 1, \cdots, k$. The same is true in the limit situtation. This not only proves the statement as formulated, but also shows that under the described degeneration $i_p \in J$, $p = 1, \cdots, k$ and $h_i \perp h_p$.

2. This follows immediately from (4) and statement 2 of Lemma 9.

3. The ray (or straight line) $\Gamma_i(M) = h_i \cap (\bigcap_{i \in J} \lambda_l^-)$. Thus there exist $j, k \in J$ such that $\lambda_j^- \cap \lambda_k^- \cap h_i$ is a straight line. Then $\omega_{ij} + \omega_{ik} + \omega_{jk} = \pi$ and all the angles are smaller than $\pi/2$. This completes the proof.

Corollary. *If under the conditions of Lemma 10 we have $i \in J$ and a face degenerates or some edge degenerates into a point, then M is a polygon.*

The proof follows immediately from Lemma 10.

Let Γ_{123} be a vertex of the polyhedron M and X a point in Λ^3. Consider a sequence of polyhedra $M_m \in W_0(\mathfrak{m})$ where $\Gamma_{123}(M_m) = X$.

Theorem 4. *Let M_m, $m = 1, 2, \cdots$, be a sequence of convex bounded polyhedra in Λ^3 of combinatorial type $\mathfrak{m}$ where $\mathfrak{m}$ is not a simplex, with angles not greater than $\pi/2$, where $\Gamma_{123}(M_m) = X$, $m = 1, 2, \cdots$. Suppose that for all $(ij) \subset I(\mathfrak{m})$ there is a limit $\lim \alpha_{ij}(M_m)$ equal to α_{ij}.*

Then from the sequence one can select a generalized convergent subsequence M_{i_m}, $M = \lim M_{i_m}$, such that:

1. *If the numbers α_{ij} satisfy the inequalities $\mathfrak{m}0-4$, then $M \in W(\mathfrak{m})$ and $\alpha_{ij}(M) = \alpha_{ij}$.*

2. *If* Γ_{pqr} *and* Γ_{pqs} *are two vertices of* $\mathfrak{m}$ *different from* Γ_{123} *and* $\alpha_{pq} = 0$, $\alpha_{pr} = \alpha_{ps} = \alpha_{qr} = \alpha_{qs} = \pi/2$, *but the other* $\alpha_{ij} < \pi/2$, *and if all the inequalities* $\mathfrak{m}0 - 4$ [1] *which do not depend on* p, q, r, s *are valid, then* M *is a convex* N*-hedron of finite volume with a unique vertex* Γ^{-1}_{pqrs} *at infinity, no face except* Γ_{pq} *degenerate and* $\alpha_{ij}(M) = \alpha_{ij}$ *if* $(ij) \neq (pq)$.

Proof. Using Lemma 6, we assume that the sequence M_m itself is generalized convergent.

The theorem will be proved if we can show that M is a polyhedron of nonzero volume without vertex at infinity under the conditions of assertion 1, and with a unique infinite vertex $\Gamma^{-1}_{pqrs}(M)$ under those of assertion 2. Then in view of Lemma 10, M will have all the combinatorial properties, and in view of Lemma 7 the angles will be equal to the numbers α_{ij}.

Because of $\omega_{12} + \omega_{13} + \omega_{23} > \pi$ we have h_{12}, h_{13}, h_{23} all distinct. Under the conditions of assertion 2 we have $\omega_{12}, \omega_{13}, \omega_{23} < \pi/2$, and according to assertion 2 of Lemma 10 the edges Γ_{12}, Γ_{13}, and Γ_{23} do not degenerate into a point provided that their faces do not degenerate into a point, which is impossible by assertion 1 of the same lemma.

Under the conditions of assertion 1 we need a refined argument. Let us begin with the assumption that of the three edges $\Gamma_{12}, \Gamma_{13}, \Gamma_{23}$ only one may degenerate into a point, since the degeneration of two of them is impossible if the faces do not degenerate. Indeed, since the angles are not greater than $\pi/2$ the contraction into a single point of two edges implies the contraction into a point of the whole polygon.

Let the edge Γ_{12} degenerate into a point and let Γ_{124} be its second endpoint. By assertion 2 of Lemma 10 $\alpha_{13} = \alpha_{14} = \alpha_{23} = \alpha_{24} = \pi/2$ and $\Gamma_3(M) = \Gamma_4(M) = M$ is a polygon of finite (cf. parts 4 and 5 of Lemma 8) and nonzero area (the edges Γ_{13} and Γ_{23} do not degenerate). There is a face Γ_j not degenerating in the "third" side $\Gamma_j(M)$. Thus $\Gamma_j(M)$ is a ray or a straight line because (assertion 1 of Lemma 10) nothing else is possible. But in this case $\omega_{3j} = \omega_{4j} = \pi/2$ are also equal to the corresponding α_{ij} (Lemma 7, Corollary). Three cases are possible:

1. Γ_1 and Γ_j are not adjacent; then the angles α_{ij} do not satisfy the inequality $\mathfrak{m}3$ for $\Gamma_1, \Gamma_3, \Gamma_j, \Gamma_4$ which form a four-angled prismatic element because Γ_1 and Γ_3 are not adjacent ($\omega_{13} = 0$, and $\alpha_{ij} > 0$, $(ij) \in J$).

2. Γ_1 and Γ_j are adjacent and either $\Gamma_1, \Gamma_3, \Gamma_j$ or $\Gamma_1, \Gamma_4, \Gamma_j$ form a t.p.e.; then $\mathfrak{m}2$ fails.

3. Γ_1 and Γ_j are adjacent and Γ_{13j} and Γ_{14j} are vertices; then Γ_1 does not satisfy the inequality $\mathfrak{m}4$.

Assuming that the edge Γ_{12} degenerates we arrive at a contradiction, i.e. M in reality is a polyhedron.

Let $\Gamma^{-1}_{i_1 \cdots i_s}(M)$ be the vertex at infinity. Then $s = 3$ or $s = 4$.

$s = 3$. Then the faces $\Gamma_{i_1}, \Gamma_{i_2}, \Gamma_{i_s}$ are adjacent in pairs and the angles between them are greater than 0, i.e. the corresponding faces of M are adjacent; but in view of the conditions $\mathfrak{m}1 - \mathfrak{m}2$ we have $\alpha_{i_1 i_2} + \alpha_{i_1 i_3} + \alpha_{i_2 i_4} \neq \pi$, which makes the existence of this vertex impossible.

$s = 4$. Then the Gram matrix of the vectors $e_{i_1}, e_{i_2}, e_{i_3}, e_{i_4}$ has the form (6), i.e. the faces $\Gamma_{i_1}, \Gamma_{i_2}, \Gamma_{i_3}, \Gamma_{i_4}$ are in this order adjacent. If they form a quadrangular prismatic element, then from the form of the matrix it follows that the condition $\mathfrak{m}3$ fails to hold.

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1) We point out that this implies the validity of all inequalities $\mathfrak{m}2$.

Let $\Gamma_{i_1 i_3}(\mathfrak{m})$ be an edge; then $\alpha_{i_1 i_3} = 0$, which is possible only under the condition of assertion 2 if $(i_1 i_3) = (pq)$. Finally, in the case of assertion 2 the edge Γ_{pq} is necessarily degenerate, as otherwise the volume of M would equal zero.

Supplement to Theorem 4. *Let $M_m \in W(\mathfrak{m})$ be simple polyhedra, and let $\alpha_{ij}(M_m) = \pi/3 + 1/m$, $(ij) \in I(\mathfrak{m})$; then there is a polyhedron $M \in W_\alpha(\mathfrak{m})$ of finite volume all of whose angles equal $\pi/3$ and all of whose vertices are at infinity.*

Proof. Suppose that M_m, $m = 1, 2, \cdots$, is a polyhedron with the middle of the edge Γ_{12} lying at the point O. We have shown that the faces Γ_1 and Γ_2 do not degenerate into a straight line and the edge Γ_{12} does not degenerate into a point (the angles are smaller than $\pi/2$); hence $\lim M_m = M$ is a nondegenerate polyhedron. In the further argument one repeats exactly the proof of the theorem.

§5. The existence theorem. Beginning of the proof

We recall our agreement concerning the use of the word polyhedron in the beginning of §4. Moreover, we now assume that all polyhedra to be considered are bounded.

Existence theorem. *Let $\mathfrak{m}$ be a convex abstract three-dimensional polyhedron with vertices of simplicial type, but not a simplex. The conditions $\mathfrak{m}0 - \mathfrak{m}4$ are necessary and sufficient for the existence of a convex bounded polyhedron M in three-dimensional Lobačevskiĭ space Λ^3 with dihedral angles not greater than $\pi/2$ such that $\alpha_{ij}(M) = \alpha_{ij}$.*

The necessity of the conditions follows from Theorem 2, represented by the formula (7). Now we have to show that

$$\alpha(W_0(\mathfrak{m})) = \Omega_0(\mathfrak{m}). \tag{13}$$

The proof is carried through in three steps:

First step: $W_0(\mathfrak{m}) \neq \emptyset \Longrightarrow \alpha(W_0(\mathfrak{m})) = \Omega_0(\mathfrak{m})$.

Second step: If $\mathfrak{m}$ is simple, then $W_0(\mathfrak{m}) \neq \emptyset$.

Third step: $\Omega_0(\mathfrak{m}) \neq \emptyset \Longrightarrow W_0(\mathfrak{m}) \neq \emptyset$.

Theorem 5. *If $\mathfrak{m}$ is an abstract three-dimensional polyhedron but not a simplex and $W_0(\mathfrak{m}) \neq \emptyset$, then $\alpha(W_0(\mathfrak{m})) = \Omega_0(\mathfrak{m})$.*

Proof. By Lemma 4 we know that $\alpha(W_0(\mathfrak{m}))$ is an open subset of $\Omega_0(\mathfrak{m})$. Indeed, for every point $\alpha(M)$, $M \in W_0(\mathfrak{m})$ there is a neighborhood Φ in the space $\Omega(\mathfrak{m})$ such that $\Phi \subset \alpha(W(\mathfrak{m}))$ and $\Phi \cap \Omega_0(\mathfrak{m}) \subset \alpha(W_0(\mathfrak{m}))$.

On the other hand, if $\alpha' \in \overline{\alpha(W_0(\mathfrak{m}))} \cap \Omega_0(\mathfrak{m})$ then by Theorem 4 there is an $M' \in \Omega_0(\mathfrak{m})$ such that $\alpha' = \alpha(M')$. Altogether we conclude that $\alpha(W_0(\mathfrak{m}))$ is a simultaneously open and closed subset of the convex set $\Omega_0(\mathfrak{m})$, and by the condition of the theorem we have $\alpha(W_0(\mathfrak{m})) \neq \emptyset$, i.e. (13) is valid.

Let $\mathfrak{m}$ be a simple polyhedron and $W_0(\mathfrak{m}) \neq \emptyset$. Let Γ_{pq} be an edge with the endpoints Γ_{pqr} and Γ_{pqs}. Then the points $\alpha(m) \in \Omega(\mathfrak{m})$ with the coordinates $\alpha_{pq}(m) = 1/m$, $\alpha_{pr}(m) = \alpha_{ps}(m) = \alpha_{qr}(m) = \alpha_{qs}(m) = \pi/2$ whereas the others $\alpha_{ij}(m) = 2\pi/5$, $m = 1, 2, \cdots$, lie in $\Omega_0(\mathfrak{m})$. By Theorem 5 a polyhedron $M_m \in W_0(\mathfrak{m})$ exists such that $\alpha(M_m) = \alpha(m)$. This sequence satisfies assertion 2 of Theorem 4, which means that there is a polyhedron M with a finite volume and angles not exceeding $\pi/2$ and a single vertex $\Gamma^{-1}_{pqrs}(M)$ at infinity.

By a motion in Λ^3 the polyhedron M can be displaced so that $e_p = (1, 0, 1, 1)$, $e_q = (t, 0, -1, t)$, $e_r = (1, 1, 0, 1)$, $e_s = (1, -1, 0, 1)$, $t > 0$.

Let us consider the polyhedron $M(u)$ for which $\Lambda_i^-(M(u)) = \Lambda_i^-(M)$, $i \neq p, q, r, s$ and

$$e_p(M(u)) = \frac{1}{\sqrt{1-4u-4u^2}}(1-2u, 0, 1, 1),$$

$$e_q(M(u)) = \frac{1}{\sqrt{1+4tu-4t^2u^2}}(t-2tu, 0, -1, t), \tag{14}$$

$$e_r(M(u)) = \frac{1}{\sqrt{1-2u-u^2}}(1+u, 1, 0, 1), \quad e_s(M(u)) = \frac{1}{\sqrt{1-2u-u^2}}(1+u, -1, 0, 1).$$

Then $M(0) = M$ and $\mathrm{T}(M(u))$ is a continuous curve in Π^N. We can select u so small that adjacent faces remain adjacent and acute angles remain acute. After having formed the Gram matrix of the vectors $e_p(M(u)), \cdots, e_s(M(u))$, on the basis of the formulas (14) it is not hard to see that of the faces $\Gamma_p, \Gamma_q, \Gamma_r, \Gamma_s$ each is adjacent to the following one and the last is adjacent to the first. If u is sufficiently small, then $H_p(u)$ and $H_q(u)$ do not intersect, the angles Λ^3, α_{pr}, α_{qr}, α_{ps}, α_{qs}, α_{qs} are smaller than $\pi/2$ and $M(u)$ is a convex bounded polyhedron with angles not exceeding $\pi/2$. Let $M(u) \sim \mathfrak{m}'$. The abstract polyhedron $\mathfrak{m}'$ is obtained from $\mathfrak{m}$ by a change of the edge Γ_{pq} onto the edge Γ_{rs}. We point out that in the preceding discussion the simplicity of $\mathfrak{m}$ was used only for the construction of the sequence M_m.

For the following it will be very useful to remember the schemes introduced in §2. Thus let $\mathfrak{m}$ be an abstract polyhedron and $\mathfrak{m}^*$ its scheme. Let us denote by $\langle i \rangle$ the vertex of the complex $\mathfrak{m}^*$ which corresponds to the face Γ_i, by $\langle ij \rangle$ the edge corresponding to the edge Γ_{ij} and by Δ_{ijk} the face corresponding to the vertex Γ_{ijk} of the abstract polyhedron $\mathfrak{m}$.

A subcomplex $\mathfrak{K}$ of the complex $\mathfrak{m}^*$ will be called a subscheme *if a cell of $\mathfrak{m}$ enters into $\mathfrak{K}$ if* and only if its boundary is in $\mathfrak{K}$. Thus a subscheme is uniquely defined by its vertices and it consists of the given vertices, of edges joining these vertices and faces whose vertices are among the chosen vertices.

Let us assume that the subscheme corresponding to the vertices $\langle 1 \rangle, \cdots, \langle k \rangle$ is a cyclic graph. Excluding it from $\mathfrak{m}^*$ we decompose the scheme into two homeomorphic plane regions $\mathfrak{A}_1$ and $\mathfrak{A}_2$. By the definition of a subscheme, $\mathfrak{A}_1$ and $\mathfrak{A}_2$ contain certain vertices of the scheme. The faces $\Gamma_1, \cdots$ $\cdots, \Gamma_k$ of the polyhedron $\mathfrak{m}$ corresponding to vertices of the cyclic graph form a k-angled prismatic element. If $k = 3$, then the faces $\Gamma_1, \Gamma_2, \Gamma_3$ form a t. p. e. (trihedral prismatic element). The subscheme corresponding to the t. p. e. $\Gamma_i, \Gamma_j, \Gamma_k$ we denote by ∇_{ijk}. The t. p. e. is truncated if either in $\mathfrak{A}_1$ or in $\mathfrak{A}_2$ there is only one vertex of the scheme.

If $\mathfrak{m}$ is a simple polyhedron, then the subscheme corresponding to three arbitrary vertices is not a cyclic graph.

Now we turn to the polyhedra M_m and $M(u)$. Let $M(u) \in W(\mathfrak{m}')$. The transformation of the abstract polyhedron implies the following: In the scheme $\mathfrak{m}^*$ we separate the two triangles Δ_{pqr} and Δ_{pqs} which have the edge $\langle pq \rangle$ in common. We consider the subscheme with the vertices

$\langle p\rangle, \langle q\rangle, \langle r\rangle, \langle s\rangle$ equal to $\Delta_{pqr} \cup \Delta_{pqs}$, since the edge $\langle rs\rangle$ does not occur in $\mathfrak{m}^*$ because $\mathfrak{m}$ is not a simplex. We exclude from $\mathfrak{m}^*$ the edge $\langle pq\rangle$. We obtain a quadrangular cell which we divide again into two triangles, but by means of the edge $\langle rs\rangle$. We shall call the transformation obtained in this way a reidentification and denote it by π_{pq}. Thus $\mathfrak{m}' = \pi_{pq}\mathfrak{m}$. We summarize:

Lemma 11. *Let $\mathfrak{m}$ be a simple polyhedron and $\mathfrak{m}' = \pi_{pq}\mathfrak{m}$; then if $W_0(\mathfrak{m}) \neq \emptyset$, we also have $W_0(\mathfrak{m}') \neq \emptyset$. More generally, let $\mathfrak{m}$ be a (not necessarily simple) polyhedron such that in $\Omega_0(\mathfrak{m})$ there is a sequence $\alpha(m)$ with the coordinates $\alpha_{pq}(m) = 1/m$, $\alpha_{pr}(m) = \alpha_{ps}(m) = \alpha_{qr}(m) = \alpha_{qs}(m) = \pi/2$ and the others $\alpha_{ij}(m) < \pi/2 - \xi$, $\xi > 0$; then if $W_0(\mathfrak{m}) \neq \emptyset$ and $\mathfrak{m}' = \pi_{pq}\mathfrak{m}$, then also $W_0(\mathfrak{m}') \neq \emptyset$.*

If $\mathfrak{m}' = \pi_{pq}\mathfrak{m}$ then $\mathfrak{m} = \pi_{rs}\mathfrak{m}'$. Thus if $\mathfrak{m}$ and $\mathfrak{m}'$ are simple then $W_0(\mathfrak{m})$ and $W_0(\mathfrak{m}')$ are not empty at the same time.

Lemma 12. *If $\mathfrak{m}$ is a k-gonal prism, $k \geq 3$, then $W_0(\mathfrak{m}) \neq \emptyset$.*

Proof. In the plane H_1 we construct a regular k-gon P with angles $\pi/5$. Let Λ_1^- be one of the halfspaces bounded by H_1. We draw planes $H_2, \cdots, H_{k+1}$ through the sides of the polygon P orthogonal to H_1, and let $\Lambda_i^- \supset P$, $i = 2, \cdots, k+1$.

Let O be the center of P. From O we draw the perpendicular to $H_1 - F$ and let $X \in F \cap (\Lambda_1^- \setminus H_1)$. Through X we draw a plane orthogonal to $F - H_{k+2}$ and let $O \in \Lambda_{k+2}^-$.

Consider the polyhedron $M = \bigcap_{i=1}^{k+2} \Lambda_i^-$. To prove the lemma it is sufficient to show that the angles at the "upper" base are acute. Through F draw the plane $H \perp H_2$. Let $H \cap H_{12} = Y$, $H \cap H_{2,k+2} = Z$. The angle $\angle Z$ of the quadrangle $OXYZ$ equals the required dihedral angle. It is acute because three angles of the quadrangle equal $\pi/2$.

We conclude that $(\pi_{pq}\mathfrak{m})^* = \pi_{pq}\mathfrak{m}^*$.

Theorem 6. *If $\mathfrak{m}$ is a simple polyhedron, then $W_0(\mathfrak{m}) \neq \emptyset$.*

By Lemmas 11 and 12 the theorem will be established if we construct a sequence of reidentifications $\pi_{i_s j_s}$, $s = 1, \cdots, k$ such that $\pi_{i_s j_s} \cdots \pi_{i_1 j_1}\mathfrak{m}$ is a simple polyhedron if $s \leq k$, and an s-gonal prism if $s = k - N - 2$. Unfortunately this is not exactly true, but it is "almost" by this method that we prove the theorem.

We note that to verify the simplicity of $\pi_{pq}\mathfrak{m}$ it is sufficient to show that there is no vertex $\langle i\rangle \in \mathfrak{m}^*$, $i \neq p, q$, identified with $\langle r\rangle$ and $\langle s\rangle$.

Proof of Theorem 6. Let Γ_1 be a face of the polyhedron $\mathfrak{m}$ and let $\Gamma_2, \cdots, \Gamma_k$ be all its adjacent faces numerated successively. The subscheme with the vertices $\langle 2\rangle, \cdots, \langle k\rangle$ is a cyclic graph, since $\mathfrak{m}$ is a simple polyhedron. Indeed, if $\langle 2, s\rangle \in \mathfrak{m}^*$ and $3 < s < k$ then $\Gamma_1, \Gamma_2, \Gamma_3$ generate a t.p.e. We denote by $\mathcal{J}_1$ the subscheme with the vertices $\langle 2\rangle, \cdots, \langle k\rangle$. The set of the vertices united with the edges of $\mathcal{J}_1$ and lying outside $\mathcal{J}_1 \cup \langle 1\rangle$ will be denoted by $\mathcal{J}_2$. The subscheme $\mathcal{J}_1$ divides $\mathfrak{m}^*$ into two closed regions intersecting along $\mathcal{J}_1$, each of which is homeomorphic to a closed circle. Let Θ be the one which contains $\langle 1\rangle$.

With every vertex $\langle i\rangle \in \mathcal{J}_2$ we consider the set of vertices of $\mathcal{J}_1$ which are joined to it by an edge. Also let $\mathcal{J}_1^{\langle i\rangle}$ be the subscheme spanned over these vertices. The graph $\mathcal{J}_1^{\langle i\rangle}$ may consist of several connected components, but the connected components of different vertices cannot alternate in a circuit of $\mathcal{J}_1$. Indeed the edges connecting the vertex $\langle i\rangle$ with the vertices of $\mathcal{J}_1$ divide Θ into open connected regions such that the edges emanating from some other vertex may be contained only in one connected component. We shall show that there is a sequence of reidentifications as a result of which, without changing $Э_1$, the subscheme $Э_1^{\langle i\rangle}$ for every $\langle i\rangle \in Э_2$ remains connected. Indeed suppose that $Э_1^{\langle i\rangle}$ is not connected and $\langle 2\rangle \cdots \langle s\rangle$ are successively enumerated vertices of one of the connected components $Э_1^{\langle i\rangle}$. There are two distinct vertices $\langle p\rangle, \langle q\rangle$ such that the edges

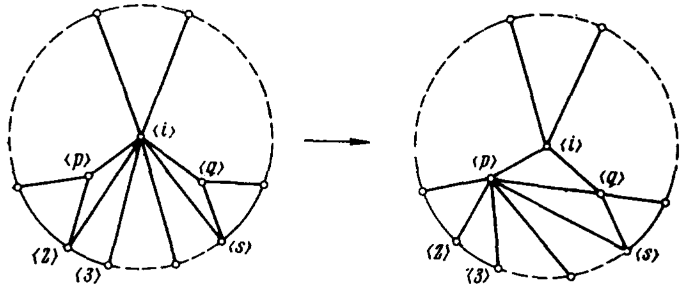

Figure 4

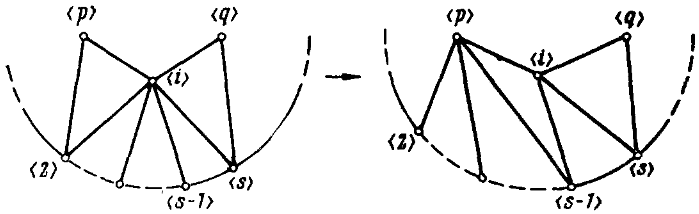

Figure 5

$\langle pr\rangle, \langle qs\rangle, \langle pi\rangle, \langle qi\rangle$ are edges of the scheme $\mathfrak{m}$ (see Figure 4) [1] and the vertices $\langle p\rangle$ and $\langle q\rangle$ are not joined by an edge. We carry out the reidentifications $\pi_{2i}, \pi_{3i}, \cdots, \pi_{si}$. It is readily seen that all the polyhedra obtained this way are simple and that the number of connected components of $Э_1^{\langle i\rangle}$ has been reduced by one and that of the other $Э_1^{\langle j\rangle}$ has not been increased. Applying an analogous construction a finite number of times, we conclude that all the $Э_1^{\langle i\rangle}$ are connected.

Let $\langle i\rangle \in Э_2$ and $\langle 2\rangle \cdots \langle s\rangle$ be the successively enumerated vertices of $Э_1^{\langle i\rangle}$. First we shall assume that $N - k > 2$. Evidently, since $\mathfrak{m}$ is simple, in this case $Э_2$ also contains more than two vertices, and, in particular, contains two distinct vertices, say $\langle p\rangle$ and $\langle q\rangle$, such that $\langle p2\rangle, \langle pi\rangle, \langle qi\rangle, \langle qs\rangle$ are edges of $\mathfrak{m}^*$. Using the sequence π_{il}, $l = 2, \cdots, s-1$ of reidentifications, we obtain the polyhedron $\mathfrak{m}'$ where $Э_2^{\langle i\rangle}$ consists of two vertices $\langle s\rangle$ and $\langle s-1\rangle$ and the concluding polyhedron as well as all intermediate ones are simple, because the condition stated before the proof is satisfied. We note that the condition $\langle p\rangle \neq \langle q\rangle$ is essential (cf. Figure 5).

The polyhedron $\mathfrak{m}'$ has the property that two and only two vertices, namely $\langle s\rangle$ and $\langle s-1\rangle$, are joined by an edge to $\langle 1\rangle$ and $\langle i\rangle$. Let us carry out the identification $\pi_{s,s-1}$. The polyhedron $\pi_{s,s-1}\mathfrak{m}'$ is simple. After the reidentification the vertex $\langle i\rangle$ is carried into a number of vertices $Э_1$ of the polyhedron $\mathfrak{m}'' = \pi_{s,s-1}\mathfrak{m}'$. In this way the magnitude $N - k$ for the polyhedron $\mathfrak{m}''$ is smaller than for the original polyhedron (cf. Figure 6).

1) In all figures relating to Theorem 6 the region in question is realized in a plane.

Applying one after another all the three types of identifications of the sequence, we conclude that $N - k = 2$. *Acting as above we shall also obtain the result that* $\partial_1^{<N>}$ *contains the vertices* $<2>$, $<3>$, $<4>$ and $\partial_1^{<N-1>}$ the vertices $<4>, \cdots, <s>, <2>$. After carrying out the identification π_{2N} we shall have a nonsimple polyhedron, and the identification π_{34} turns it into an $(N-2)$-gonal prism. Now let

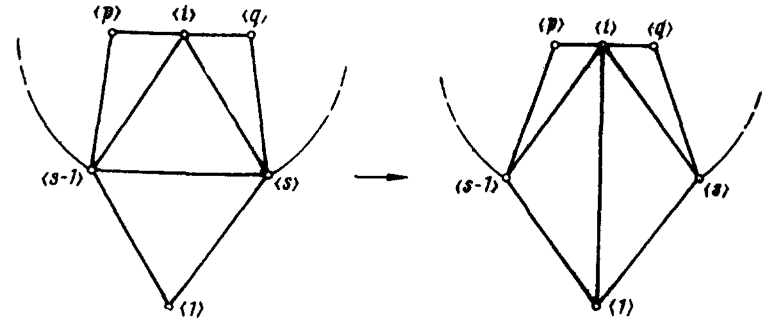

Figure 6

us describe this process in the opposite direction: The identification π_{N1} turns the prism $\mathfrak{p}$ into a nonsimple polyhedron $\pi_{N1}\mathfrak{p}$ with a single t.p.e. $\nabla_{3,4,N-1}$; then the identification $\pi_{3,N-1}$ turns $\pi_{N1}\mathfrak{p}$ into a simple polyhedron $\pi_{3,N-1}\pi_{N1}\mathfrak{p} = \mathfrak{U}$, so that the theorem is proved if we can show that $W_0(\mathfrak{U}) \neq \emptyset$.

In Lobačevskiĭ space there is a prism none of whose angles exceeds $\pi/2$, and consequently in the combinatorial type $\pi_{N1}\mathfrak{p}$ there is a polyhedron with acute angles. The point $\alpha(m) \in \Omega(\pi_{N1}\mathfrak{p})$ with the coordinates $\alpha_{3,N-1}(m) = \pi/m$, $\alpha_{1,N-2}(m) = \alpha_{3,N-1}(m) = \alpha_{N-2,N-1}(m) = \alpha_{N-1,N}(m) = \pi/2$ and with its other coordinates $\alpha_{ij} = 2\pi/5$ lies in $\Omega_0(\phi_{N1}\mathfrak{p})$. This can easily be verified because $\Gamma_3, \Gamma_4, \Gamma_{N-1}$ is the only t.p.e. By Lemma 7 $W_0(\mathfrak{U}) \neq \emptyset$.

§6. Conclusion of the proof of the existence theorem

Let $\mathfrak{m}$ be a truncated polyhedron and consider together with $\mathfrak{m}$ the polyhedron $\mathfrak{pm}$ defined in §2. We enumerate the triangular faces of $\mathfrak{m}$ from $k+1$ to N. Consider the subspace Ω' of the space $\Omega(\mathfrak{m})$ which is distinguished by the condition $\omega_{ij} = 0$, $i > k$ or $j > k$. Then Ω' can be identified with the space $\Omega(\mathfrak{pm})$.

If the system of inequalities $\mathfrak{m}\,0-4$ is compatible, then in $\Omega_0(\mathfrak{m})$ there are points arbitrarily near to the point $(\cdots, \pi/3, \cdots)$. Consider the projection of such a point onto the subspace Ω'. If the projected point is sufficiently near to $(\pi/3, \cdots, \pi/3)$ then its projection will be near to $(\pi/3, \cdots, \pi/3, 0 \cdots 0)$. In the set $W_\alpha(\mathfrak{pm})$ one can find a polyhedron of finite volume with angles $\pi/3$; if $\mathfrak{pm}$ is simple this was shown in the Supplement to Theorem 4; if $\mathfrak{pm}$ is a simplex it was proved in the paper [6] and can also be obtained as an evident consequence of [2]. Applying Lemma 4, we conclude that there is $P \in W_\alpha(\mathfrak{pm})$, $|\alpha_{ij}(P) - \pi/3| < \epsilon$ where according to $\mathfrak{m}\,1$ and $\mathfrak{m}\,2$ the sum of the angles at distinguished vertices equals π, and at the others is greater than π. This implies that unmarked vertices $\widetilde{P}$ lie inside the sphere $\widetilde{V}_+$ and the distinguished ones outside.

Let $\widetilde{\Gamma}_{123}$ be a distinguished vertex. Through the intersections of the edges $\widetilde{\Gamma}_{12}, \widetilde{\Gamma}_{13}, \widetilde{\Gamma}_{23}$ with the sphere $\widetilde{V}_0$ we draw the plane $\widetilde{H}_{k+1}$ and we suppose that the halfspace $\widetilde{\lambda}^-_{k+1}$ does not contain $\widetilde{\Gamma}_{123}$. We carry out this operation for all distinguished vertices and consider $M = P \cap (\bigcap_{i=k+1}^N \lambda_i^-)$.

By construction M is an α-polyhedron of finite volume and $\widetilde{M} \sim \mathfrak{m}$. We shall show that $\alpha_{ij}(M) < \pi/2$. Indeed, $\alpha_{ij}(M) < \pi/2$, if $i, j < k$. Consider $\alpha_{1,k+1}(M)$, $\alpha_{2,k+1}(M)$ and $\alpha_{3,k+1}(M)$. They are found from the relations $\alpha_{1,k+1} + \alpha_{2,k+1} + \alpha_{12} = \pi$; $\alpha_{2,k+1} + \alpha_{3,k+1} + \alpha_{23} = \pi$, $\alpha_{1,k+1} + \alpha_{3,k+1} + \alpha_{13} = \pi$. If $\alpha_{12} = \alpha_{13} = \alpha_{23} = \pi/3$ then the angles to be found equal $\pi/3$. In the given case the solutions of the system depend continuously on the right side (the rank of the system equals 3); if we select $\epsilon > 0$ sufficiently small we can state that $\alpha_{ij}(M) < \pi/2$ for all $(ij) \in I(\mathfrak{m})$.

Thus for truncated polyhedra we have proved

Theorem 7. *Let* $\mathfrak{m}$ *be an abstract three-dimensional polyhedron, not a simplex. If* $\Omega_0(\mathfrak{m}) \neq \emptyset$, *then also* $W_0(\mathfrak{m}) \neq \emptyset$.

Proof. It remains to prove the theorem in the case of a composite polyhedron. We give the proof by induction with respect to the number of truncated components. Let $\Gamma_1, \Gamma_2, \Gamma_3$ be some t. p. e. of $\mathfrak{m}$. We decompose $\mathfrak{m}$ with respect to this t. p. e. into two polyhedra $\mathfrak{m}'$ and $\mathfrak{m}''$, to which evidently the induction assumption can be applied. Let us assume that the enumeration of the faces has been chosen so that the first k coordinates of the space $\Omega(\mathfrak{m})$ refer to $\mathfrak{m}'$. It is readily seen that if $\alpha \in \Omega_0(\mathfrak{m})$, then

$$\alpha' = \left(\alpha_{12}, \ \ldots, \ \alpha_{i_k j_k}, \left(\frac{\pi}{2}\right)_{1,N+1} \left(\frac{\pi}{2}\right)_{2,N+1} \left(\frac{\pi}{2}\right)_{3,N+1}\right) \in \Omega_0(\mathfrak{m}).$$

In the case of $\mathfrak{m}''$ we proceed in the same way.

In view of the induction assumption and Theorem 5 there exist polyhedra M' and M'' in $W_0(\mathfrak{m}')$ and $W_0(\mathfrak{m}'')$ respectively such that $\alpha(M') = \alpha'$ and $\alpha(M'') = \alpha''$.

Consider the triangles $\Gamma_{N+1}(M')$ and $\Gamma_{N+1}(M'')$. By construction their angles are equal, but in the Lobačevskiĭ plane there is only one triangle with given angles. We superpose the polyhedra M' and M'' so that the faces $\Gamma_{N+1}(M')$ and $\Gamma_{N+1}(M'')$ coincide and the polyhedra themselves lie on the same side of the plane H_{N+1}. Then the planes $H_i(M')$ and $H_i(M'')$, $i = 1, 2, 3$, coincide because the angles adjacent to $\Gamma_{N+1}(M')$ and $\Gamma_{N+1}(M'')$ are right angles. Let $M = M' \cup M''$. From the construction it follows that the polyhedron M is convex and bounded. It is easy to see that $M \sim \mathfrak{m}$ and $\alpha(M) = \alpha$. Thus Theorem 7 is completely established.

So we have concluded the proof of the existence theorem. Finally, we point out those places at which the proof fails for simplexes. In the first place there is Theorem 4, since the conditions of type $\mathfrak{m}\,0 - 4$ give no security against the degeneration of the faces of a simplex. (The simplex is the only polyhedron where triangular faces do not correspond to a t. p. e.) Also Theorem 5 is not satisfied. On the other hand, in the case of a simplex of arbitrary dimension with dihedral angles the Gram matrix is uniquely defined.

In the space Λ^n there is a bounded simplex with given angles if and only if all principal minors of dimension $n - 1$ are positive definite and the determinant of the matrix is negative.

§7. Criterion of compatibility of the inequalities $\mathfrak{m}\,0-4$

We have pointed out in §2 *that the inequalities* $\mathfrak{m}\,0-4$ *are compatible if and only if the inequalities* $\mathfrak{m}\,1$ and $\mathfrak{m}\,2$ are compatible. Put $\zeta_{ij}=\alpha_{ij}-\pi/3$. In terms of $\mathfrak{m}^*$ the inequalities $\mathfrak{m}\,1$ and $\mathfrak{m}\,2$ are as follows:

$\zeta_{ij}+\zeta_{ik}+\zeta_{jk}>0$ for every Δ_{ijk}.

$-\zeta_{ij}-\zeta_{ik}-\zeta_{jk}>0$ for every ∇_{ijk}.

Consider the matrix A of the given system of linear inequalities. The system is compatible if and only if there is no nontrivial linear dependence with nonnegative coefficients between the rows of the matrix A. This has been proved in the paper [4] (Corollary 5, Russian p. 227).

Let us assume that there is such a dependence, and let $\Delta_1\cdots\Delta_k$ and $\nabla_{k+1}\cdots\nabla_m$ correspond to those rows of the matrix A which occur in the dependence with positive coefficients. The vertices, edges and faces belonging to the system $\Delta_1\cdots\Delta_k\nabla_{k+1}\cdots\nabla_m$ generate a subcomplex of the scheme $\mathfrak{m}^*$ which we denote by D.

We shall say that the system $\Delta_{i_1}\cdots\Delta_{i_s}\nabla_{j_1}\cdots\nabla_{j_t}$ is complete when the corresponding complex D has the following property: For every edge $\langle pq\rangle\in D$ there are Δ_i and Δ_j in the system such that $\langle pq\rangle=\Delta_i\cap\nabla_j$.

From the form of the inequalities $\mathfrak{m}\,1$ and $\mathfrak{m}\,2$ it follows that the system $\Delta_1\cdots\Delta_k\nabla_{k+1}\cdots\nabla_m$ is complete.

A complete system is said to be irreducible if it does not contain proper complete subsystems.

For a moment let us forget about the compatibility problem; we rather consider an arbitrary complete system $\Delta_1\cdots\Delta_k\nabla_{k+1}\cdots\nabla_m$. Clearly, by definition, from every complete system one can separate an irreducible subsystem.

We shall assume that the scheme $\mathfrak{m}^*$ is realized in the form of a chart (map) on the two-dimensional sphere. Let us assume $\nabla_j\in\mathfrak{m}^*$ and exclude it, and then decompose the sphere into two homeomorphic plane regions $\mathfrak{A}_{1j}$ and $\mathfrak{A}_{2j}$.

Lemma 13. *Let* $\Delta_1\cdots\nabla_m$ *be a complete irreducible system, and let* $\nabla_j\in\mathfrak{m}^*$ *be such that* $\mathfrak{A}_{ij}\cap D\neq\emptyset$, $i=1,2$. *If we split the original system in two by the criterion of belonging or not to* $\overline{\mathfrak{A}}_{ij}$ *and adjoin to each subsystem an element* ∇_j, *then one of the two subsystems is complete.*

Proof. Assume first that ∇_j lies in the system. If $\overline{\mathfrak{A}}_{1j}$ contain some elements different from j, then $\overline{\mathfrak{A}}_{1j}$ contains elements of both types, Δ as well as ∇. Therefore one can choose an element ∇_{j_1} such that $\overline{\mathfrak{A}}_{2j_1}$ contains only elements ∇_p of the system which do not contain points of the subcomplex D in $\mathfrak{A}_{1p}$. Let $D_q=\overline{\mathfrak{A}}_{qj_1}\cap D$, $q=1,2$ and suppose that $\Delta_{i_1}\cdots\Delta_{i_s}\nabla_{j_1}\cdots\nabla_{j_t}$ are all elements of the system contained in $\overline{\mathfrak{A}}_{2l}$. If we adjoin to the complex D_1 the cells $\mathfrak{A}_{1j_1}\cdots\mathfrak{A}_{1j_s}$ and also denote $\overline{\mathfrak{A}}_{1j_p}$ by $\widetilde{\Delta}_{j_p}$, then we obtain a two-dimensional simplicial complex which satisfies the conditions C1–4. Indeed C1, C3 and C4 are valid by construction and because D_2 is a subcomplex of the scheme. The condition C2 is satisfied for all edges not in ∇_{j_1} since the system is complete, and for the edges of the element ∇_{j_1} because the cell $\mathfrak{A}_{1j_1}$ has been adjoined.

Now suppose that $\nabla_{j_1}=\nabla_{pqr}$ and consider the polyhedron P whose scheme coincides with the complex. We consider those vertices $\Gamma_{ijk}(P)$ to which correspond the ∇_j. Because the system is complete every unmarked vertex is connected by an edge with marked ones, and every marked one, except Γ_{pqr}, only with unmarked ones. We shall show that all vertices connected with $\Gamma_{pqr}(P)$ are simultaneously either marked or unmarked. Let v_1 be the number of marked vertices. Assume the opposite; then every edge of P, except one or two, leads from a marked edge to an unmarked one and therefore the number of edges is $R = 3v_1 - 1$ or $3v_1 - 2$. But we have shown that $R = 3v/2$, where v is the number of vertices of the polyhedron P; but $3v/2 \neq 3v_1 - 1$, $3v_1 - 2$ since the left side is divisible by 3, but the right side is not so.

The statement which has just been proved implies that three elements of the system which lie in $\mathfrak{A}_{1j_1}$ and each contain at least one edge of the element ∇_{j_1} are either elements of the type ∇ or of the type Δ, all at the same time. In the first case, excluding from the system $\Delta_1 \cdots \nabla_m$ the element ∇_{j_1}, we obtain again a complete subsystem, and in the second case the subsystem $\Delta_{i_1} \cdots \nabla_{j_t}$ itself is complete. Thus we have shown that ∇_j cannot enter the system and that for every ∇_j of the system $D_1 \setminus \nabla_j$ or $D_2 \setminus \nabla_j$ is empty.

Assume that ∇_j does not enter the system; then by what has been shown three elements of the system lying in $\mathfrak{A}_{rj}$, $r = 1, 2$ and containing each at least one edge of the element ∇_j are elements of one and the same type. Thus since the system is complete these elements in different $\mathfrak{A}_{rj}$ are of different types.

Let $\nabla_{s_1}, \nabla_{s_2}, \nabla_{s_3}$ be these three elements in $\mathfrak{A}_{1j}$. Then the system formed by the elements of the original system living in $\overline{\mathfrak{A}}_{2j}$ and ∇_j will be complete.

Corollary 1. *If the system* $\Delta_1 \cdots \nabla_m$ *is irreducible then one of the two regions* $\mathfrak{A}_{rj}$, $r = 1, 2$, *contains no points of the complex* D.

Corollary 2. *In an irreducible system every edge belongs to exactly two elements of the system, and* D *is a subscheme.*

Corollary 3. *If the scheme* $\mathfrak{m}^*$ *contains a complete system, then the inequalities* $\mathfrak{m}\,1$ *and* $\mathfrak{m}\,2$ *are incompatible.*

Proofs. The proof of Corollary 1 is contained in the proof of the lemma. Corollary 2 is true; indeed, if some edge occurs in three elements of the system, then Corollary 1 is not valid.

To prove Corollary 3 we add the rows of the matrix A which correspond to the elements of a complete irreducible system. In every column two elements are different from zero; then one equals $+1$ and the other -1, so that as a sum we obtain a row of zeros. Thus between the rows of the matrix A we have a nontrivial linear dependence with nonnegative coefficients.

Let $\Delta_1 \cdots \nabla_m$ be a complete irreducible system; we construct a subscheme of it with respect to a two-dimensional simplicial complex, as has been done in the proof of Lemma 9. If P is simple then $\mathfrak{m}$ contains an element ∇_j which satisfies the conditions of Lemma 9, and we can reduce the system. By doing this we eventually obtain the simplicity of the polyhedron P. It is not hard to see that the construction of the scheme of the polyhedron P is dual to the construction of simple components. Together with the completeness of the system this yields

Corollary 4. *If the scheme* $\mathfrak{m}$ *contains a complete system* $\Delta_1 \cdots \nabla_m$ *then there is a simple component* $\mathfrak{pm}_l$ *of the polyhedron* $\mathfrak{m}$ *where every edge has at one end a marked vertex, and at the other end an unmarked one.*

We sum up the results of this section in

Theorem 9. *The system of inequalities* $\mathfrak{m}\,0 - 4$ *is compatible if* $\mathfrak{m}$ *is a simple polyhedron or a triangular prisma. Let* $\mathfrak{m}$ *be a truncated polyhedron; the system* $\mathfrak{m}\,0 - 4$ *is compatible if and only if* *in one of its simple components there is an edge the endpoints of which are either both marked or both unmarked vertices. If* $\mathfrak{m}$ *is a composite polyhedron then the system* $\mathfrak{m}\,0-4$ *is compatible if and only if the corresponding systems of inequalities for all its truncated components are compatible.*

§8. Groups generated by reflections

A group G with a finite system of generators $A_1, \cdots, A_N$ and defining relations

$$(A_i A_j)^{a_{ij}} = 1,$$

will be called an abstract Coxeter group if $a_{ij} = a_{ji}$, $a_{ii} = 1$, $2 \le a_{ij} \le \infty$ for $i \neq j$ ($a_{ij} = \infty$ means that the corresponding element has infinite order).

To describe such groups it is convenient to use schemes which are constructed as follows.

We take a number N of vertices, and to each one, denoted by $<i>$, we associate one of the generators A_i. We connect two of the vertices by an edge if $a_{ij} < \infty$, and to the edge we ascribe the number a_{ij}.

Our schemes differ from the usual ones [5]; the reason is that although both are constructed with maximal economy of the edges, in the case of Lobačevskiĭ space the given method is more economical. The correspondence between our schemes and the usual ones is readily established.

It is known that the discrete group H generated by reflections in hyperplanes, acting in n-dimensional Lobačevskiĭ space Λ^n, is an abstract Coxeter group. For the following we shall assume that the fundamental region of the group H is bounded; we shall briefly call H a group generated by reflections.

Theorem 10. *Suppose that* H_1 *and* H_2 *are two discrete groups generated by reflections, acting in n-dimensional Lobačevskiĭ space,* $n \geq 3$, *and that their schemes coincide. Then there is an element* $g \in G(\Lambda^n)$ *such that* $gH_1g^{-1} = H_2$.

Let G be an abstract Coxeter group. In its scheme we separate the triangular subschemes which correspond to finite Coxeter groups. By G^* we denote the two-dimensional simplicial complex obtained from the scheme of the group after adjoining the triangles which correspond to the separated subschemes.

Theorem 11. *Let* G *be an abstract Coxeter group with* $N > 4$ *generators. The following properties of the complex* G^* *are necessary and sufficient for the existence of a discrete group generated by reflections acting in the three-dimensional Lobačevskiĭ space and such that its scheme coincides with the scheme of the group* G.

1. *Every edge coincides with exactly two triangles.*

2. *Every vertex lies on at least one edge.*

3. *The Euler characteristic of the complex* G^* *equals* 2.

4. *The scheme does not contain any subschemes which correspond to discrete motion groups of the euclidean plane with a bounded fundamental region.*

5. *If* $N = 5$ *then the mark given to any edge emanating from a simplicial vertex of the scheme* G^* *is different from* 2.

The proof of Theorems 10 and 11 follows immediately from the existence and uniqueness theorems, since the group generated by reflections is uniquely defined by its fundamental region, i.e. a convex polyhedron M with dihedral angles π/a_{ij} between the faces which correspond to the ith and the jth reflecting hyperplanes. The proof of Theorem 10 is thus reduced to the application of the uniqueness theorem to the polyhedron M.

The two-dimensional simplicial complexes $\mathfrak{m}^*$ and G^* coincide if in the latter one gives no attention to the marks of the edges. Thus to prove Theorem 11 one has to show the equivalence of the conditions 1–5 of the theorem with the existence conditions of the polyhedron first with the scheme G^* and second with the angles $\alpha_{ij} = \pi/a_{ij}$. First: Taking into account the character of the construction of the complex G^*, it is readily seen that conditions 1–3 of the theorem are equivalent to conditions C1–4 of §2. Second: In view of the rule for the construction of two-dimensional cells the inequality $\mathfrak{m}\,1$ is always satisfied. The inequalities $\mathfrak{m}\,2$ and $\mathfrak{m}\,3$ are equivalent to condition 4. We also recall that for all polyhedra with more than 5 faces, except the triangular prism, condition $\mathfrak{m}\,4$ follows from the inequalities $\mathfrak{m}\,1$–3. If $N = 5$ condition 5 and the inequality $\mathfrak{m}\,4$ are equivalent.

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