COMP 335: Introduction to Theoretical Computer Science

Assignment 3

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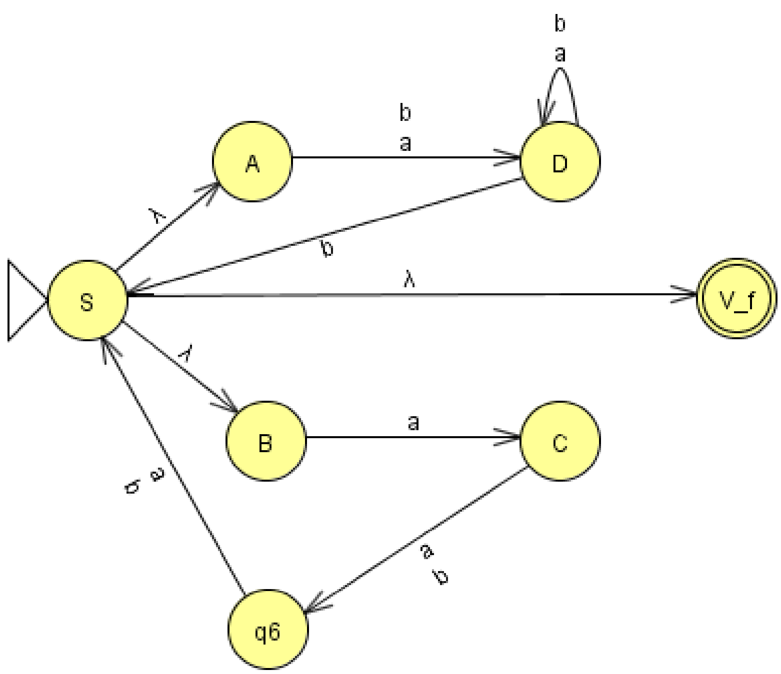
1. [20 Points] For each of the following languages over $\Sigma = \{a, b\}$, write a regular grammar and then convert it into an equivalent NFA using the procedure described in class.

   (a) (10 Points) $L(r)$ where $r = ((a+b)(a+b))^*b + a((a+b)(a+b))^*$

   G:

   - $S \rightarrow A \mid B \mid \lambda$
   - $A \rightarrow aD \mid bD$
   - $D \rightarrow aD \mid bD \mid bS$
   - $B \rightarrow aC$
   - $C \rightarrow aaS \mid abS \mid baS \mid bbS \mid$

   Equivalent NFA of G:

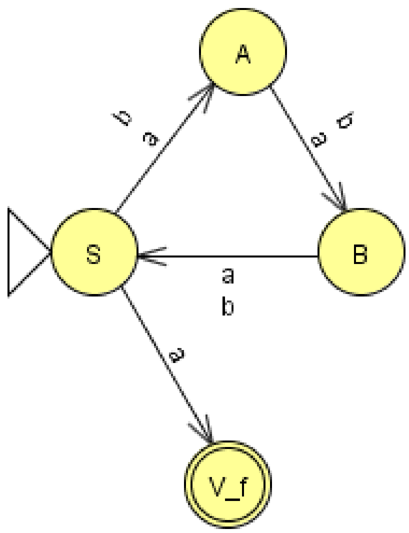


(b) (10 Points) $\{w \in \{a, b\}^* : w \text{ ends in } a \text{ and } | w | \equiv 1 \ (\mathsf{mod} 3)\}$

G:

- $S \to aA \mid bA \mid a$
- $A \to aB \mid bB$
- $B \to aS \mid bS$

Equivalent NFA of G:



2. [25 Points] Fix an alphabet Σ. For any string w with $\mid w \mid \geq 2$, let $skip(w)$ be the string obtained by removing the first two symbols of w. Define 2 operators on languages:

$$f_1(L) = \{w \in \Sigma^* : skip(w) \in L\}$$

$$f_2(L) = \{skip(w) \in \Sigma^* : w \in L\}$$

(a) (5 Points) Consider $L' = L(bba^*)$ over tha alphabet $\Sigma = \{a, b\}$. Write a regular expression representing $f_1(L')$. Write another regular expression representing $f_2(L')$.

$$r_1 = (a + b)(a + b)bba^*$$

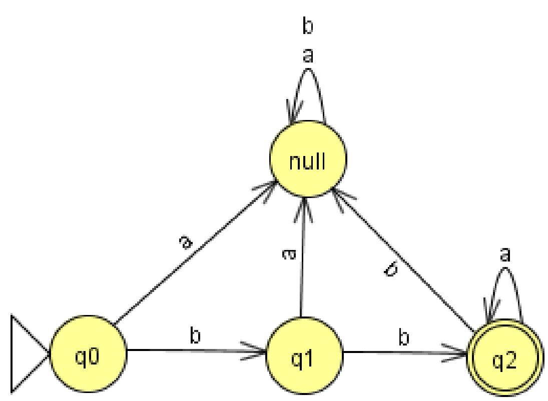
$$r_2 = a^*$$

(b) (10 Points) Claim: For every regular language L the language $f_1(L)$ is regular. Clearly state whether the claim is TRUE or FALSE, and then prove your answer.

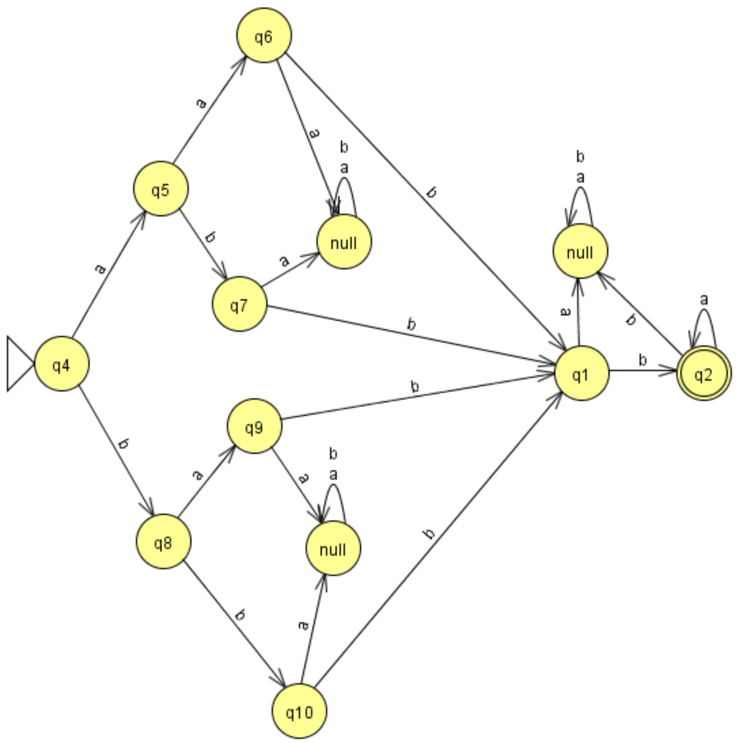
Answer: TRUE

Proof: Since L is regular, there exists a DFA that accepts L. We define a new DFA M' based on M, and will show that $L(M') = f_1(L(M)) = f_1(L)$. This will show that $f_1(L)$ is regular.

We start by creating a DFA for L($bba?*$):



Then, we create a new DFA M' that accepts $f_1(L)$. We do this by adding states that accept any 2 characters from the alphabet Σ at the start of the string in any word w:

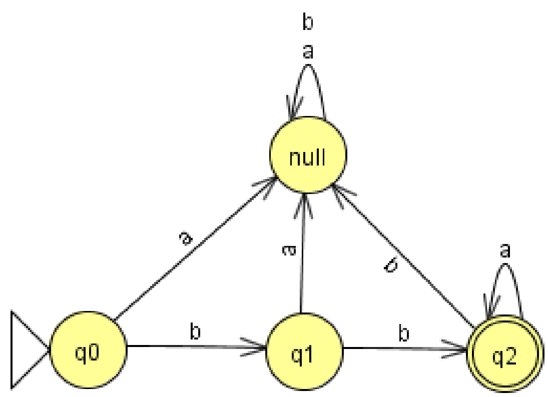


(c) (10 Points) Claim: For every regular language L the language $f_2(L)$ is regular. Clearly state whether the claim is TRUE or FALSE, and then prove your answer.

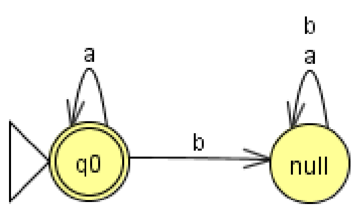
Answer: TRUE

Proof: Since L is regular, there exists a DFA that accepts L. We define a new DFA M' based on M, and will show that $L(M') = f_2(L(M)) = f_2(L)$. This will show that $f_2(L)$ is regular.

We start by creating a DFA for L(bba^*):



Then, we create a new DFA M' that accepts $f_2(L)$. Our new DFA M' will accept any string w that has the first two characters removed from the start of the string. If there is a valid path from the start state q_0 to another state q after reading any 2 characters, the state q is a valid starting state. The new DFA M' essentially starts from all states reachable after reading two symbols in M. After that, it follows the same transitions as M:



3. [20 Points] For each of the following languages, use the Pumping Lemma and/or closure properties of regular languages to show that the language is not regular.

   (a) (10 Points) $L_1 = \{0^k 1^l : k \geq l^4 \geq 0\}$

   Mapping of potential values:

   - $l = 0, k \geq 1$
   - $l = 1, k \geq 1$
   - $l = 2, k \geq 16$
   - $l = 3, k \geq 81$

   Let $w = 0^{m^4} 1^m$ (i.e $l = m$, $k = m^4$).

   - Check: If $l = m$ and $k = m^4$ then $0^{m^4} 1^m \in L$
   - Check: $\mid w \mid = m^4 + m \geq m$

$$\mid m \mid\mid m^4 - m \mid\mid m \mid$$

$$w = (00 \ldots 00)(00 \ldots 00)(11 \ldots 11)$$

$$\mid xy \mid\mid z \mid$$

   Therefore, $\exists j \geq 1, y = 0^j$

   When $i = 0$: $w_0 = 0^{(m-j)+(m^4-m)} 1^m = 0^{(m^4-j)} 1^m$

   - $w_0 \in L_1$ by PL
   - $w_0 \notin L_1$ because $m^4 - j < m^4 \therefore n_0(w_0) < n_1(w_0)^4$

   Therefore by contradiction, L_1 is not regular.

(b) (10 Points) $L_2 = \{a^n : n \text{ is not a perfect cube}\}$

Mapping of potential values:

- $n = 1^3 = 1$
- $n = 2^3 = 8$
- $n = 3^3 = 27$
- $n = 4^3 = 64$

By the property of complements on regular languages, if L_2 is regular then $\overline{L_2}$ is also regular.

Let $w = a^{m^3} = (a^m)(a^{(m^3-m)})$. (i.e $n = m^3$).

- Check: If $n = m^3$ then $a^{m^3} \in \overline{L_2}$
- Check: $\mid w \mid = m^3 \geq m$

$$\mid m \mid\mid m^3 - m \mid$$

$$w = (aa \ldots aa)(aa \ldots aa)$$

$$\mid xy \mid\mid z \mid$$

Therefore, $\exists\, k \geq 1, y = a^k$

When $i = 2$: $w_2 = xyyz = a^m\, a^k\, a^{m^3-m} = a^{m^3+k}$

- $w_2 \in \overline{L_2}$ by PL
- $w_2 \notin \overline{L_2}$ because for $k \geq 1,$ not all strings a^{m^3+k} are perfect cubes.

  In other words, $n_a(w_2) \neq m^3$ for $m \geq 0$

Therefore by contradiction, $\overline{L_2}$ is not regular and L_2 is also not regular by association.