COMP 335: Introduction to Theoretical Computer Science

Assignment 1

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1. [15 Points] For each of the following statements write if the statement is TRUE or FALSE. If the statement is TRUE then provide a proof. If the statement is FALSE then provide a counter-example.

   (a) For every language L we have $L^2 \subseteq L^3$

   Answer: FALSE

   Counter Example:

   $\Sigma = \{a, b\}$

   $L = \{a, ab\}$

   $L^2 = LL = \{aa, aab, aba, abab\}$

   $L^3 = L^2 L = \{aaa, aaaba, aaba, aabab, abaa, abaab, ababa, ababab\}$

   $\{aa, aab, aba, abab\} \not\subseteq \{aaa, aaaba, aaba, aabab, abaa, abaab, ababa, ababab\}$

   $\therefore L^2 \not\subseteq L^3$

   (a) For every language L we have $L^2 \subseteq L^3$

   This statement is TRUE.

   Proof:
   Let L be any language over some alphabet Σ.
   Consider any string $w \in L^2$. This means w can be written as w = xy, where $x, y \in L$.
   Now, we can form a string in L^3 by concatenating w with any string $z \in L$.
   So, $xyz \in L^3$, where $x, y, z \in L$.
   But notice that xy (which is w) is a prefix of xyz.
   This means that every string in L^2 is also contained in L^3 as a prefix.
   Therefore, $L^2 \subseteq L^3$.

   This proof shows that for any language L, every string in L^2 is also in L^3, which establishes the subset relationship $L^2 \subseteq L^3$.

   (b) For every two languages L_1 and L_2 we have $(L_1 \cup L_2)^* \subseteq (L_1 L_2)^*$

   Answer:

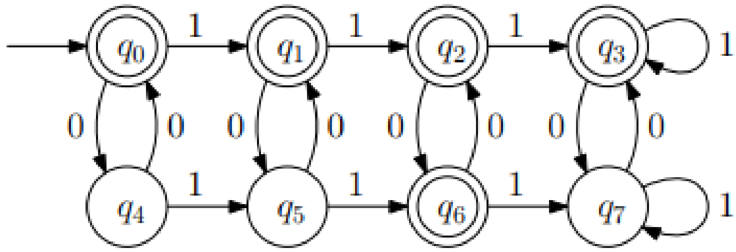
   Proof:

   (c) Let L_1 and L_2 be two languages such that $\lambda \in L_1 \cap L_2$. Then it holds that $(L_1 L_2)^* = (L_2 L_1)^*$

Answer:

Proof:

2. [10 Points] The following is a transition diagram for a DFA over the alphabet $\Sigma = \{0, 1\}$. Answer the following questions about this automaton:



(a) What is the start state? What is the set of accept states?

Answer:

Start State = q_0

Accept States = $\{q_0, q_1, q_2, q_3, q_6\}$

(b) What is the sequence of states the DFA goes through on input 101100?

Answer: $(q_0, q_1, q_5, q_6, q_7, q_3, q_7)$

(c) Does the machine accept every string w that contains exactly two 1s? Why or why not?

Answer: $L = \{0^*10^*10^*\}$

Yes. Since the string always contains exactly two 1s, the machine traverses to one of these states $\{q_2, q_6\}$. Since $q_2, q_6 \in F$ and only 0 appear in the string after consuming both 1s, the machine can only transition between these acceptance states.

(d) Does the machine reject every string w that has odd number of 0s? Why or why not?

Answer: $L = \{w \in \Sigma^* : \text{Where } n_0(w) \bmod 2 \neq 0\}$

No. Since there's an odd number of 0s, the any string will always terminate in the following states $\{q_4, q_5, q_6, q_7\}$. However, not all of these states are acceptance states (only $q_6 \in F$). Therefore, depending on how many 1s are present in the string, we may or may not reach an acceptance state.

Counter Example: $w = 01$, Final State = q_5

(e) Describe the language accepted by the machine using the set builder notation.

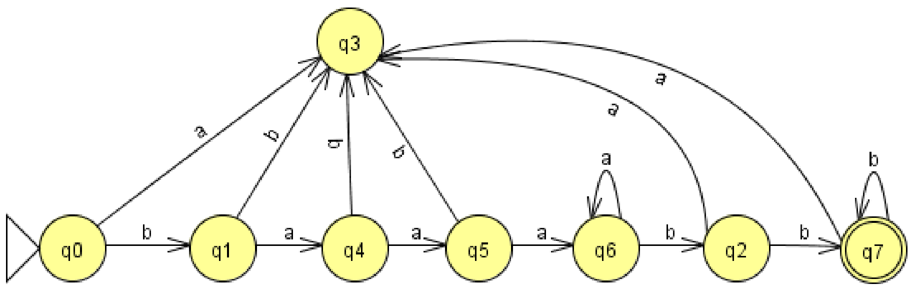
Answer:

$$\begin{aligned} L = &\{w \in \Sigma^* : \text{Where } n_0(w) \bmod 2 = 0\} \vee \\ &\{w \in \Sigma^+ : \text{Where } n_0(w) \bmod 2 \neq 0 \wedge n_1(w) = 2\} \end{aligned} \tag{1}$$

3. [30 Points] For each of the following languages, give a DFA that accepts it.

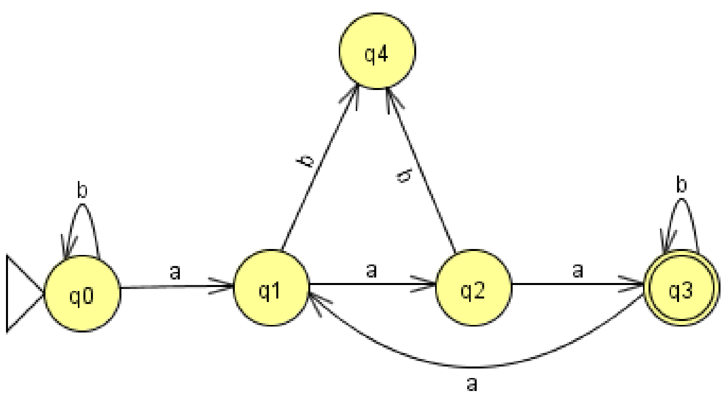
(a) $\{ba^n b^m : n \geq 3, m \geq 2\}$

Answer:



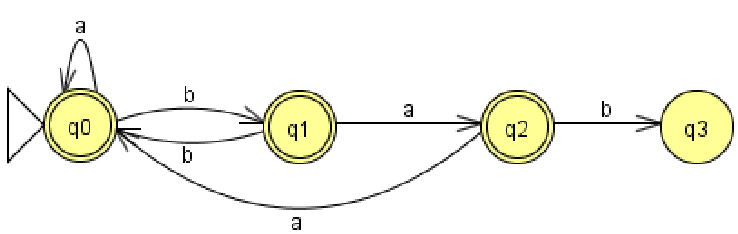
(b) $\{w \in \{a, b\}^* :$ every maximal substring w consisting entirely of symbols a is of length exactly $3\}$

Answer:



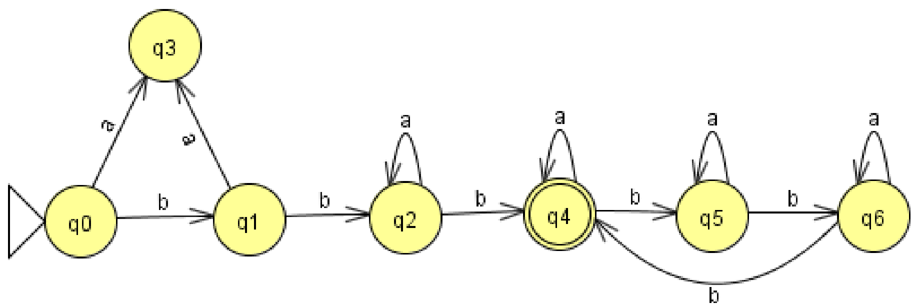
(c) $\{w \in \{a, b\}^* : w$ does not contain bab as a substring$\}$

Answer:



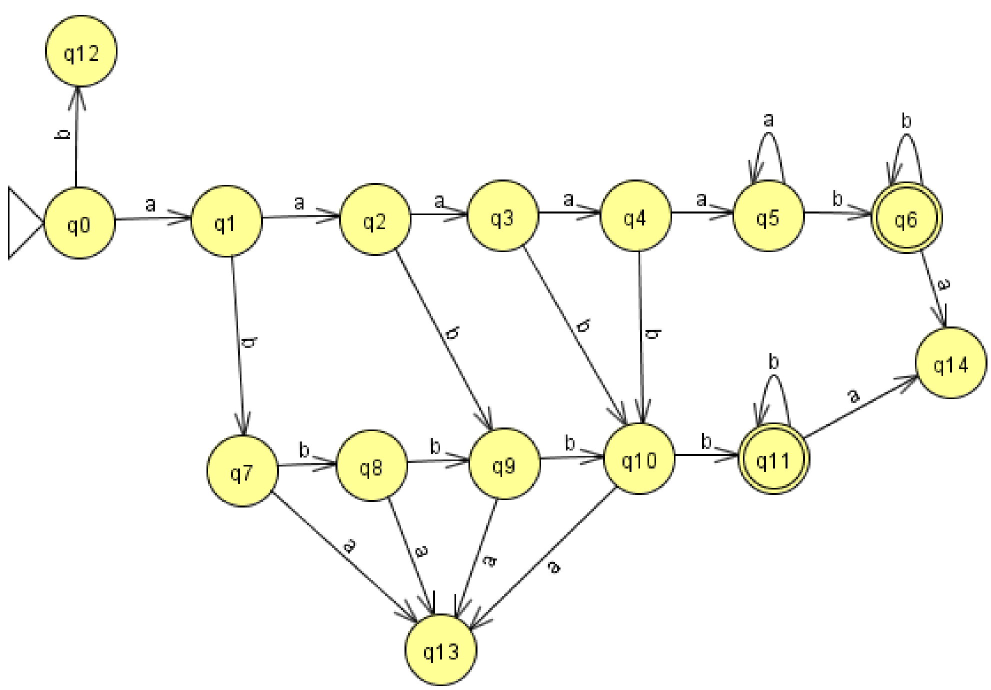
(d) $\{w \in \{a, b\}^* : w \text{ begins with bb and } n_b(w) \bmod 3 = 0\}$

Answer:



(e) $\{a^m b^n : mn > 4\}$

Answer:



(f) $\{vwv^R : v, w \in \{a, b\}^* \text{ and } | v |= 2\}$

Answer: