Final Exam Review

May 19, 2018

1 Dirac Notation

1.1 Essential Math

1.1.1 Vectors

If you and I see a wild vector w we can each measure its length and direction without being in the same place. If you set up your coordinate system with respect to the eigenbasis $e_i^* = \{e_1^*, e_2^*, e_3^*\}$ and I make mine $e_i' = \{e_1', e_2', e_3'\}$. The components of w with respect to your basis will be $w_i^* = \langle e_i^* | w \rangle$

$$|w\rangle = \sum_i w_i^* \, |e_i^*\rangle = [w \cdot e_1^*, w \cdot e_2^*, w \cdot e_3^*] = [w_1^*, w_2^*, w_3^*]$$

and mine will be $w_i' = \langle e_i' | w \rangle$

$$|w\rangle = \sum_i w_i' \, |e_i'\rangle = [w \cdot e_1', w \cdot e_2', w \cdot e_3'] = [w_1', w_2', w_3']$$

Even though we disagree on the components of w we still measure the same vector w.

1.1.2 Matrices

The transpose of a product of two matrices

$$(AB)^T = B^T A^T,$$

and the complex transpose is simply the complex conjugate of the transpose result.

1.1.3 Eigenvalues and Eigenvectors

To find the eigenvalues and eigenvectors of a matrix A solve the equation

$$\det(A - \lambda I)$$

go on... I dare you. If you have indeed found the eigenvectors say $v_1, v_2, .., v_i$ and eigenvalues $\lambda_1, \lambda_2, ..., \lambda_i$ than it must be true for each vector and value that

$$Av_i = \lambda_i v_i.$$

Does it satisfy this condition? Good. These techniques extend to function's, there called eigenfunction's. One example of an eigenfunction is in section 1.5.

1.2 Dual Space

The ket is more intuitive with what we think of as a vector from our physics one days, but the bra is a bit different. The bra is a linear function of vectors like the inner product remember than in function space the inner product tells you to integrate. When it acts on a vector it produces a complex number.

$$\langle f| = \int f^*[\ldots] d\mu$$

For a finite vector $|\alpha\rangle$ it's corresponding bra is the complex transpose of $|\alpha\rangle$ and is now written $\langle\alpha|$. If you have a space of kets than there is a corresponding space of bras what is called the dual space.

1.3 Operators

Operators are observables like the Hamiltonian $\hat{H}$ can be thought of as matrices that transform states. Operators like states need to be written with respect to an eigenbasis, so if you're looking at a true wild Hamiltonian than it must be possible to decompose it into it's energy eigenfunctions. In general if you have an operator $\hat{Q}$ with eigenbasis $|e_n\rangle$ and $\langle e_m|$ than the elements of the matrix $Q_{mn} = \langle e_m|\hat{Q}|e_n\rangle$. This operator can transform a state $|\alpha\rangle = \Sigma_n a_n |e_n\rangle$ into a state $|\beta\rangle = \Sigma_n b_n |e_n\rangle$ by

$$\hat{Q}|\alpha\rangle = |\beta\rangle$$

$$\sum_n a_n \hat{Q}|e_n\rangle = \sum_n b_n |e_n\rangle$$

$$\langle e_m| \left[\sum_n a_n \hat{Q}|e_n\rangle\right] = \langle e_m| \left[\sum_n b_n |e_n\rangle\right]$$

$$\sum_n a_n \langle e_m|\hat{Q}|e_n\rangle = \langle e_m|\beta\rangle$$

$$\sum_n a_n Q_{mn} = b_m.$$

The expectation value of an observable, $Q(x,p)$, is the average value that you would get from making a measurement of that observable. It can be written in Dirac and integral form as

$$\langle Q\rangle = \langle\psi|Q\psi\rangle = \int \psi^* Q\psi \, dx.$$

1.4 Hermitian

We require that the expectation value be a real number. That is to say

$$\langle Q\rangle = \langle Q\rangle^*$$

$$\langle\psi|Q\psi\rangle = \langle\psi|Q\psi\rangle^* = \langle Q\psi|\psi\rangle$$

This property of an operator to move from acting on the ket to acting on the bra is called **hermitian**. You can show that the momentum operator is hermitian along with position using integration by parts and remembering that we require the wave function to go to zero at infinity. In fact all observables are represented by hermitian operators. The **hermitian conjugate** $Q^\dagger$ or **adjoint** of an operator is the complex transpose of the operator.

$$\hat{Q} = \hat{Q}^\dagger$$

1.5 Determinate States

A determinate states satisfy the eigenvalue equation for the operator,

$$\hat{Q}\psi = q\psi.$$

Where ψ is an **eigenfunction** of the operator $\hat{Q}$ and q is the eigenvalue. The spectra of eigenvalues be it discrete or continuous tells you whether or not your wave function is normalizable in the traditional sense. For example the states of free particles or wave packets are non-normalizable and thus its eigenvalues were continuous. If you recall finding the eigenfunction $f_p(x)$ for the momentum operator yields

$$f_p(x) = Ae^{ipx/\hbar},$$

the only way to "orthonormalize" this function is to introduce the dirac delta function.

$$\int_{-\infty}^{\infty} f^*_{p'}(x) f_p(x)\, dx = |A|^2 2\pi\hbar\delta(p-p')$$

$$\langle f_{p'}|f_p\rangle = \delta(p-p')$$

This is **Dirac orthonormality** The infinite square well had a normalized wave function and discrete separated energy levels or **eigenvalues**.

Normalizable wave functions have real eigenvalues and there eigenfunctions are orthogonal. Degenerate states have multiple eigenvalues for the same eigenfunction.

1.6 State Vectors

The state of a system denoted $|\delta(t)\rangle$ can be expressed in any basis we choose. If we want to know the position state of a quantum system then we need to write the state $|\delta(t)\rangle$ in the position basis. To do this we need the eigenfunction $|x\rangle$ in the same way that we require eigenvectors to represent the basis for a wild vector. The eigenfunction of $\hat{x}$ with eigenvalues x that represents the measurement obtained.

$$\psi(x,t) = \langle x|\delta(t)\rangle$$

These games can be played with respect to any physically observable quantity like energy or any form of momentum you so choose.

Momentum

$$\Phi(p,t) = \langle p|\delta(t)\rangle$$

Discrete Energy spectrum: $|n\rangle$ stands for the n^{th} eigenfunction of the Hamiltonian $\hat{H}$.

$$c_n(t) = \langle n|\delta(t)\rangle$$

These are three representations of the same state with respect to three different eigenbasis.

$$\psi(x,t) = \int \psi(y,t)\delta(x-y)dy = \int \Phi(p,t)\frac{1}{\sqrt{2\pi\hbar}}e^{ipx/\hbar}dp = \sum c_n e^{-iE_n t/\hbar}\psi_n(x).$$

2 Harmonic Oscillator

Let's assume that a particle find itself in a potential field that is well approximated by a parabolic function $V(x) = \frac{1}{2}m\omega x^2$. The hamiltonian operator for this system is

$$\hat{H} = \frac{\hat{p}^2}{2m} + \frac{1}{2}m\omega\hat{x}^2$$

$$= \frac{1}{2m}[\hat{p}^2 + (m\omega\hat{x})^2]$$

The raising operator for the harmonic oscillator is $a_+ = \frac{1}{2\hbar m\omega}[-ip + m\omega x]$ and the lowering operator is the complex conjugate of the raising operator $a_- = \frac{1}{2\hbar m\omega}[+ip + m\omega x]$. We can represent the hamiltonian for the SHO in terms of the raising and lowering operators like so

$$a_- a_+ = \frac{1}{2\hbar m\omega}[\hat{p}^2 + (m\omega x)^2] - \frac{i}{2\hbar}[x,p].$$

Where $[x,p] = i\hbar$ is the commutator of the operators of position and momentum. It should be noted that $[a_-, a_+] = 1$.

$$a_- a_+ == \frac{1}{\hbar\omega}\hat{H} + \frac{1}{2}$$

$$\hat{H} = \hbar\omega\left[a_- a_+ + \frac{1}{2}\right].$$

3 Orbital Angular Momentum

To fully describe the state of a hydrogen atom we require three quantum numbers n, ℓ, m.

$$L = r \times \hat{p}$$

The components of angular momentum do not commute, their relations are provided below in condensed notation ϵ is the levi-civita symbol. Since these components do not commute there are no states that are eigenfunctions for two components at the same time.

$$[L_i, L_j] = i\hbar\sum_{k=1}^{3}\epsilon_{ijk}\,L_k$$

Similar to the raising and lowering operators from the HO we can use a ladder operator to move between different angular momentum states.

$$L_\pm = L_x \pm iL_y$$

However the magnitude squared of the angular momentum,

$$L^2 = L_x^2 + L_y^2 + L_z^2$$

does commute with each component. This means that given a particular state we can simultaneously describe the magnitude of its angular momentum along with one of its components. Which component we choose is arbitrary traditionally it is denoted z.

$$\hat{L}^2 f = \lambda f \ , \ \hat{L}_z f = \mu f$$

We can use this the ladder operator along with these eigenfunctions to determine how the raising operator changes the state. The commutator of a component of the angular momentum and the total angular momentum with the ladder operator are needed.

$$[L_z, L_\pm] = \pm\hbar L_\pm \quad [L^2, L_\pm] = 0$$

These relations imply,

$$L^2(L_\pm f) = \lambda(L_\pm f) \quad L_z(L_\pm f) = (\mu \pm \hbar)(L_\pm f)$$

Using the principle of a bottom and top most eigenfunction we find that,

$$\lambda = \hbar^2 \ell(\ell+1)$$

which gives us the eigenvalue equation for the total angular momentum operator

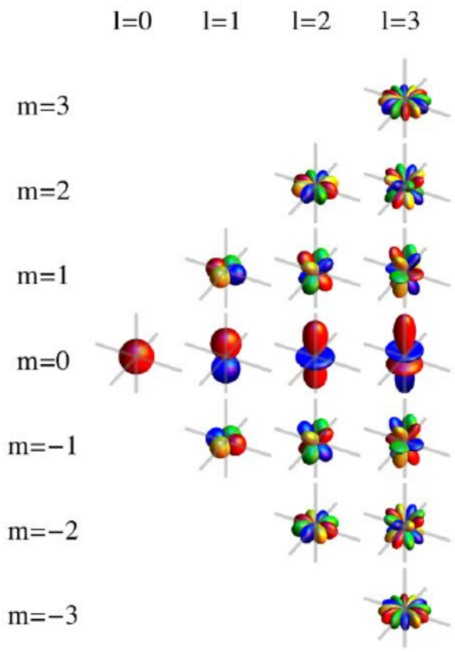
$$\hat{L}^2 f_\ell^m = \hbar^2 \ell(\ell+1) f_\ell^m.$$

Where $\ell = [0, \frac{1}{2}, 1, \frac{3}{2}]$ $\hbar\ell$ denotes the top rung of the ladder and $m = [-\ell, -\ell+1, ..., \ell-1, \ell]$ gives you the eigenvalue of L_z acting on an eigenfunction,

$$L_z f_\ell^m = \hbar m f_\ell^m.$$

What are the eigenfunctions of angular momentum, then? Well they are the beautifully orthogonal family of functions called the spherical harmonics. Below is the normalization constant A_ℓ^m for the eigenfunctions of the ladder operator.

$$L_\pm f_\ell^m = (A_\ell^m) f_\ell^{m\pm1}$$
$$A_\ell^m = \hbar\sqrt{\ell(\ell+1) - m(m\pm1)}$$



4 Spin

The spin of a particle is analogous to an objects intrinsic rotation independent to it's motion through space. This spin has all of the same operator properties of orbital angular momentum all that changes is the quantum number ℓ to s. The value of s for a particle never changes the most common particles that we are used to such as the electron neutron and proton are spin $\frac{1}{2}$.

$$\hat{S}^2 |sm\rangle = \hbar^2 s(s+1) |sm\rangle$$
$$S_z |sm\rangle = \hbar m |sm\rangle$$
$$S \pm |sm\rangle = \hbar\sqrt{s(s+1) - m(m+1)}\, |s(m\pm1)\rangle$$

For spin $\frac{1}{2}$ particles there are only down choices for m that is $\frac{1}{2}$ and $-\frac{1}{2}$. These two states are referred to as spin up and spin down respectively and so a linear combination of these two states is all that you need to define the spin state of a quantum system. These basis vectors are called **spinors**.

$$\chi = a\chi_+ + b\chi_-$$

$$S^2\chi_+ = \frac{3}{4}\hbar^2\chi_+ \quad S^2\chi_- = \frac{3}{4}\hbar^2\chi_-$$

$$S^2 = \frac{3}{4}\hbar^2\begin{bmatrix}1 & 0\\ 0 & 1\end{bmatrix}$$

$$S_z = \frac{\hbar}{2}\begin{bmatrix}1 & 0\\ 0 & -1\end{bmatrix} \quad S_y = \frac{\hbar}{2}\begin{bmatrix}0 & -i\\ i & 0\end{bmatrix} \quad S_x = \frac{\hbar}{2}\begin{bmatrix}0 & 1\\ 1 & 0\end{bmatrix}$$

4.1 Clebsch-Gordan Coefficients

$$|sm\rangle = \sum_{m_1+m_2=m} C^{s_1 s_2 s}_{m_1 m_2 m}\,|s_1 m_1\rangle\,|s_2 m_2\rangle$$

$$|s_1 m_1\rangle\,|s_2 m_2\rangle = \sum_s C^{s_1 s_2 s}_{m_1 m_2 m}\,|sm\rangle$$

Here is an example of using the table.

Clebsch-Gordan (C-G) coefficients

$$|\,SM\,\rangle = \sum_{m_1+m_2=M} |s_1 s_2; m_1 m_2\rangle\langle s_1 s_2; m_1 m_2\,|SM\rangle \quad = \sum_{m_1+m_2=M} C^{s_1 s_2 S}_{m_1 m_2 M}\,|\,s_1 m_1\,\rangle|\,s_2 m_2\,\rangle$$

2 × 1

/3	3/5			
/6	−3/10	2	2	1
/2	1/10	0	0	0
+1	−1	1/5	1/2	3/10
0	0	3/5	0	−2/5
−1	+1	1/5	−1/2	3/10

Shaded column gives

$$|\,30\,\rangle = \frac{1}{\sqrt{5}}|\,21\,\rangle|\,1-1\,\rangle + \sqrt{\frac{3}{5}}|\,20\,\rangle|\,10\,\rangle + \frac{1}{\sqrt{5}}|\,2-1\,\rangle|\,11\,\rangle$$

$$|\,s_1 s_2; m_1 m_2\,\rangle = \sum_S |SM\rangle\langle SM|s_1 s_2; m_1 m_2\rangle \qquad M = m_1 + m_2$$

$$|\,s_1 m_1\,\rangle|\,s_2 m_2\,\rangle = \sum_S C^{s_1 s_2 S}_{m_1 m_2 M}\,|\,SM\,\rangle$$

3/2 × 1

2/5	3/5	5/2	3/2	1/2
3/5	−2/5	+1/2	+1/2	+1/2
+3/2	−1	1/10	2/5	1/2
+1/2	0	3/5	1/15	−1/3
−1/2	+1	3/10	−8/15	1/6

Shaded row gives

$$\left|\,\frac{3}{2}\frac{1}{2}\,\right\rangle|\,10\,\rangle = \sqrt{\frac{3}{5}}\left|\,\frac{5}{2}\frac{1}{2}\,\right\rangle + \frac{1}{\sqrt{15}}\left|\,\frac{3}{2}\frac{1}{2}\,\right\rangle - \frac{1}{\sqrt{3}}\left|\,\frac{1}{2}\frac{1}{2}\,\right\rangle$$

Sum of the squares of each row or column is 1.

Do Prob 4.36

5 Two Identical Particles

Given a system of two particles described by the state $\psi(r_1, r_2, t)$, which satisfies the Schrödinger equation. The hamiltonian would be

$$H = -\frac{\hbar^2}{2m_1}\nabla_1^2 - \frac{\hbar^2}{2m_2}\nabla_2^2 + V(r_1, r_2, t),$$

all of the same statistical rules will apply to this state. Ignoring spin suppose particle one is in state $\psi_a(r)$ and the other is in state $\psi_b(r)$. Assuming that the particles are distinguishable the system state can be expressed like so,

$$\psi(r_1, r_2) = \psi_a(r_1)\psi_b(r_2)$$

However, no particles are distinguishable and quantum mechanics has a way of describing the states of indistinguishable particles.

$$\psi_{\pm} = A[\psi_a(r_1)\psi_b(r_2) \pm \psi_b(r_1)\psi_a(r_2)]$$

The plus sign is used for bosons which have integer spin, the minus sign is used for fermions with half integer spin. Next we define the **exchange operator**, P which exchanges two particles

$$Pf(r_1, r_2) = f(r_2, r_1).$$

It follows that $P^2 = 1$ and the eigenvalues of P are ± 1 and that the exchange operator commutes with the hamiltonian, $[P, H] = 0$. The **symmetrization requirement** is that for identical particles the wave function if required to satisfy,

$$\psi(r_1, r_2) = \pm\psi(r_2, r_1)$$

where the plus sign is for bosons and the minus is for fermions.

6 Perturbation Theory

6.1 Nondegenerate Perturbation

We begin with a state that we have solved using Schrödinger equation, this will be our unperturbed solution. This gives us a complete set of orthonormal eigenfunctions with corresponding eigenvalues E_n^0.

$$H^0\psi_n^0 = E_n^0\psi_n^0$$
$$\langle\psi_n^0|\psi_m^0\rangle = \delta_{mn}$$

We would now like to solve Schroedinger equation again now with a slightly changed potential field. The adjust Hamiltonian can be written as

$$H = H^0 + \lambda H'.$$

So our new equation is

$$H\psi_n = E_n\psi_n$$

We can write the perturbed energies and wave functions in a power series expansion.

$$\psi_n = \psi_n^0 + \lambda\psi_n^1 + \lambda^2\psi_n^2 + ... = \sum_i \lambda^i\psi_n^i$$

$$E_n = E_n^0 + \lambda E_n^1 + \lambda^2 E_n^2 + ... = \sum_i \lambda^i E_n^i$$

Substituting these equations into Schrödinger's and taking the inner product with ψ_n^0 we find that,

$$E_n^1 = \langle\psi_n^0|H'|\psi_n^0\rangle.$$

Assume that we have a constant perturbation V_0 throughout the entire space of the system. Intuitively we could imagine that this would increase the energies of all the states of the unperturbed system. Which is what we find since H' is V_0

$$E_n^1 = \langle\psi_n^0|V_0|\psi_n^0\rangle = V_0\langle\psi_n^0|\psi_n^0\rangle = V_0$$

the corrected energies are then

$$E_n \approx E_n^0 + E_n^1 = E_n^0 + V_0$$

How about the perturbed wavefunctions? Using the fact that our new wavefunctions will be some linear combinations of the old ones,

$$|\psi_n^1\rangle = \sum_{m\neq n} c_m^{(n)}|\psi_m^0\rangle$$

Taking the inner product of the following equation with ψ_l^0

$$\sum_{m\neq n}(E_m^0 - E_n^0)c_m^{(n)}\left|\psi_m^0\right\rangle = -(H' - E_n^1)\left|\psi_n^0\right\rangle.$$

We find that

$$\psi_n^1 = \sum_{m\neq n}\frac{\left\langle\psi_m^0\middle|H'\middle|\psi_n^0\right\rangle}{E_n^0 - E_m^0}\psi_m^0.$$

The preceding derivation requires that there is only one eigenvalue per eigenfunction that is that the system be nondegenerate.

6.2 Degenerate Perturbation

If two states share the same energy than nondegenerate perturbation is needed. Suppose we have two unperturbed states ψ_a^0 and ψ_b^0 such that,

$$H^0\psi_a^0 = E^0\psi_a^0, \quad H^0\psi_b^0 = E^0\psi_b^0, \quad \left\langle\psi_a^0\middle|\psi_b^0\right\rangle = 0.$$

Than any solution to to Schroedinger's equation can be written as a linear combination of those states like so,

$$\psi^0 = \alpha\psi_a^0 + \beta\psi_b^0$$
$$H^0\psi^0 = E^0\psi^0$$

Using the same perturbed Hamiltonian $H = H^0 + \lambda H'$ as before where,

$$E = E^0 + \lambda E^1 + \lambda^2 E^2 + ..., \quad \psi = \psi^0 + \lambda\psi^1 + \lambda^2\psi^2 + ...$$

we can substitute and collect like powers of λ gives us the following relation for the first order approximation to the perturbed energy. However, this is a set of equations.

$$\alpha\left\langle\psi_a^0\middle|H'\middle|\psi_a^0\right\rangle + \beta\left\langle\psi_a^0\middle|H'\middle|\psi_b^0\right\rangle = \alpha E^1$$

At this point we introduce the matrices W_{ij} these are the matrix elements of H'.

$$W_{ij} = \left\langle\psi_i^0\middle|H'\middle|\psi_j^0\right\rangle, \quad (i,j = a,b)$$

Using inner products with ψ_a^0 and ψ_b^0 we get the set of equations that we can use to solve for E_1

$$\alpha W_{aa} + \beta W_{ab} = \alpha E^1$$

$$\alpha W_{ba} + \beta W_{bb} = \beta E^1$$

Reducing the above equations and using the quadratic formula we get the two solutions for E_1.

$$E_\pm^1 = \frac{1}{2}\left[W_{aa} + W_{bb} \pm\sqrt{(W_{aa} - W_{bb})^2 + |W_{ab}|^2}\right]$$

The results for a two-fold degeneracy can be extended to n-fold degeneracy quite simply. Solving for the energy of the last problem amounted to finding the eigenvalues of the following matrix.

$$\begin{pmatrix} W_{aa} & W_{ab} \\ W_{ba} & W_{bb}\end{pmatrix}\begin{pmatrix}\alpha \\ \beta\end{pmatrix} = E^1\begin{pmatrix}\alpha \\ \beta\end{pmatrix}$$

7 Bell's Theorem

$P(a,b)$ is the average value of the product of the spins, it is either ± 1.

$$P(a,b) = -a\cdot b$$

Bell's Inequality

$$|P(a,b) - P(a,c)| \leq 1 + P(b,c)$$

Bell's inequality shows that hidden variables do not exist.