# ECE368: Probabilistic Reasoning
# Lab 2: Bayesian Linear Regression

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**You should hand in:** 1) A scanned .pdf version of this sheet with your answers (file size should be under 2 MB); 2) four figures for Question 2 and three figures for Question 4 in the .pdf format; and 3) one Python file regression.py that contains your code. All these files should be uploaded to Quercus.

1. Express the posterior distribution $p(\mathbf{a}|x_1, z_1, \ldots, x_N, z_N)$ using $\sigma^2, \beta,\ x_1, z_1, x_2, z_2, \ldots, x_N, z_N$. (1 **pt**)

$$\mathcal{D} = \{(x_1, y_1), \ldots, (x_N, y_N)\} \tag{1}$$

$$y(\mathbf{x}) = \mathbf{a}^\top \mathbf{x} + w = \sum_{j=1}^{d} a_j x_j + w \tag{2}$$

$$z_i = a_i x + a_0 + w \tag{3}$$

$$\mathcal{Z} = \{z_1, \ldots, z_N\} \tag{4}$$

$$a = (a_0, a_1)^\top \tag{5}$$

$$w \sim \mathcal{N}(0, \sigma^2) \tag{6}$$

$$p(\mathbf{a}) = \mathcal{N}\left(\begin{bmatrix}0\\0\end{bmatrix}, \begin{bmatrix}\beta & 0\\0 & \beta\end{bmatrix}\right) \tag{7}$$

$$p(z_1, ..., z_N|a, x_1, ..., x_N) = \prod_{i=1}^{N} p(z_i|a, x_i) \tag{8}$$

$$p(a|x_1, z_1, ..., x_N, z_N) = p(z_1, ..., z_N|a, x_1, ..., x_N)p(a) \tag{9}$$

$$p(a|x_1, z_1, ..., x_N, z_N) = \mathcal{N}([\mu_a|x_1, z_1, ..., x_N, z_N], [\Sigma_a(a|x_1, z_1, ..., x_N, z_N)]) \tag{10}$$

$$\Sigma_a^{-1} = \begin{bmatrix}\frac{1}{\beta} & 0\\0 & \frac{1}{\beta}\end{bmatrix} \tag{11}$$

$$\mathcal{A} = \begin{bmatrix}\vdots & z_1\\ \vdots & \vdots\\ \vdots & z_N\end{bmatrix} \tag{12}$$

$$\Sigma_w^{-1} = \frac{1}{\sigma^2} \tag{13}$$

$$[\Sigma_a(a|x_1, z_1, ..., x_N, z_N)] = (\Sigma_a^{-1} + \mathcal{A}^\top \Sigma_w^{-1} \mathcal{A})^{-1} \tag{14}$$

$$[\mu_a|x_1, z_1, ..., x_N, z_N] = (\Sigma_a^{-1} + \mathcal{A}^\top \Sigma_w^{-1} \mathcal{A})^{-1}(\mathcal{A}^\top \Sigma_w^{-1} \mathcal{Z}) \tag{15}$$

2. Let $\sigma^2 = 0.1$ and $\beta = 1$. Draw four contour plots corresponding to the distributions $p(\mathbf{a})$, $p(\mathbf{a}|x_1, z_1)$, $p(\mathbf{a}|x_1, z_1,\ ldots, x_5, z_5)$, and $p(\mathbf{a}|x_1, z_1, \ldots, x_{100}, z_{100})$. In all contour plots, the x-axis represents $a_0$, and the y-axis represents $a_1$. Please save the figures with names **prior.pdf, posterior1.pdf, posterior5.pdf, posterior100.pdf**, respectively. (1.5 **pt**)

   **All figures start on page 3**

3. Suppose that there is a new input $x$, for which we want to predict the corresponding target value $z$. Write down the distribution of the prediction $z$, i.e, $p(z|x, x_1, z_1, \ldots, x_N, z_N)$. (1 **pt**)

$$\mathcal{A} = \begin{bmatrix} 1 & x \end{bmatrix} \tag{16}$$

$$\Sigma_w = \sigma^2 \tag{17}$$

$$[\mu_a|x_1, z_1, ..., x_N, z_N] = (\Sigma_a^{-1} + \mathcal{A}^\top \Sigma_w^{-1} \mathcal{A})^{-1} (\mathcal{A}^\top \Sigma_w^{-1} \mathcal{Z}) \tag{18}$$

$$p(z|x, x_1, z_1, ..., x_N, z_N) = \mathcal{N}([\mu_z|x, x_1, z_1, ..., x_N, z_N)], [\Sigma_z(z|x, x_1, z_1, ..., x_N, z_N)]) \tag{19}$$

$$[\Sigma_z(z|x, x_1, z_1, ..., x_N, z_N)] = (\Sigma_w + \mathcal{A}[\Sigma_a(a|x_1, z_1, ..., x_N, z_N)]\mathcal{A}^\top) \tag{20}$$

$$[\mu_z|x, x_1, z_1, ..., x_N, z_N] = (\mathcal{A}[\mu_a|x_1, z_1, ..., x_N, z_N]) \tag{21}$$

4. Let $\sigma^2 = 0.1$ and $\beta = 1$. Given a set of new inputs $\{-4, -3.8,\ ldots, 3.8, 4\}$, plot three figures, whose x-axis is the input and y-axis is the prediction, corresponding to three cases:

   (a) The predictions are based on one training sample, i.e., based on $p(z|x, x_1, z_1)$.

   (b) The predictions are based on 5 training samples, i.e., based on $p(z|x, x_1, z_1, \ldots, x_5, z_5)$.

   (c) The predictions are based on 100 training samples, i.e., based on $p(z|x, x_1, z_1, \ldots, x_{100}, z_{100})$.

   The range of each figure is set as $[-4, 4] \times [-4, 4]$. Each figure should contain the following three components: 1) the new inputs and the corresponding predicted targets; 2) a vertical interval at each predicted target, indicating the range within one standard deviation; 3) the training sample(s) that are used for the prediction. Use `plt.errorbar` for 1) and 2); use `plt.scatter` for 3). Please save the figures with names **predict1.pdf, predict5.pdf, predict100.pdf**, respectively. (1.5 **pt**)

   **All figures start on page 3**

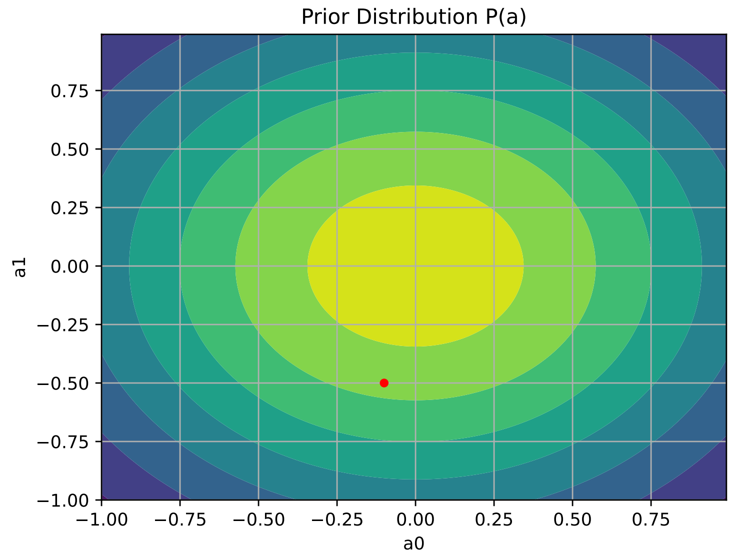


Figure 1: prior.pdf

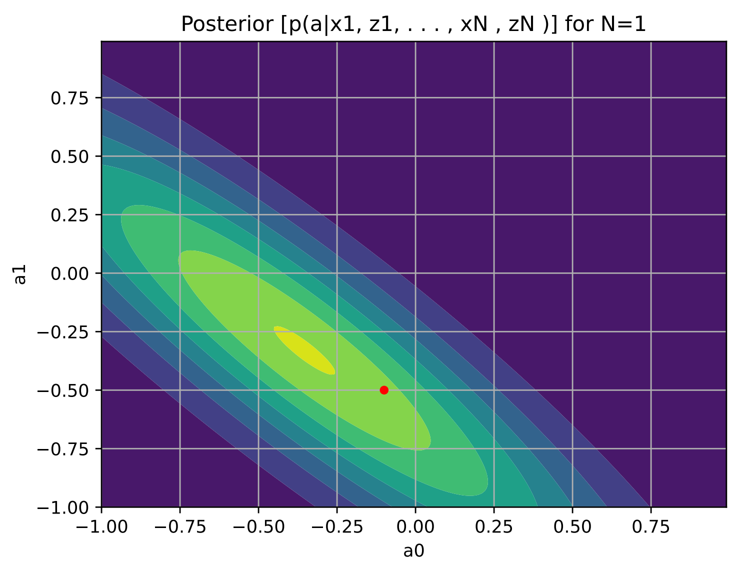


Figure 2: posterior1.pdf

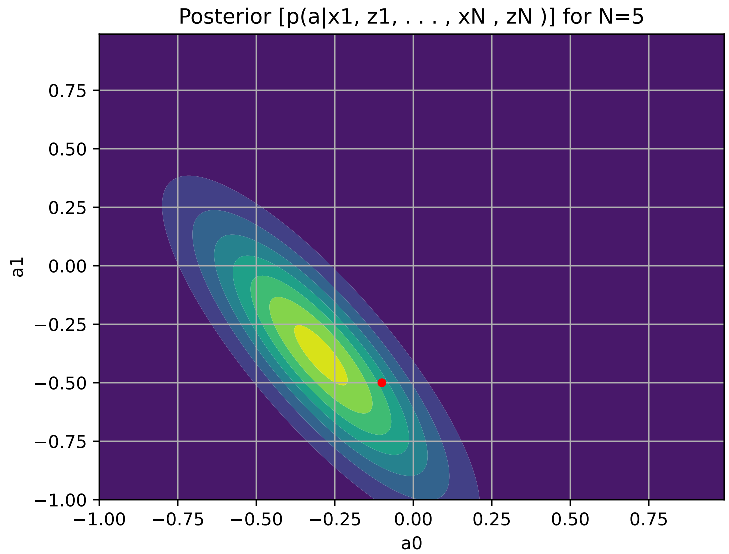


Figure 3: posterior5.pdf

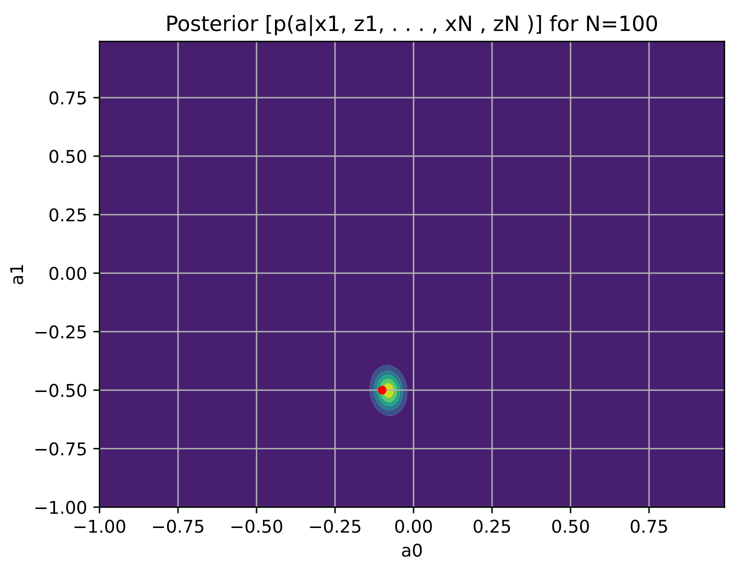


Figure 4: posterior100.pdf

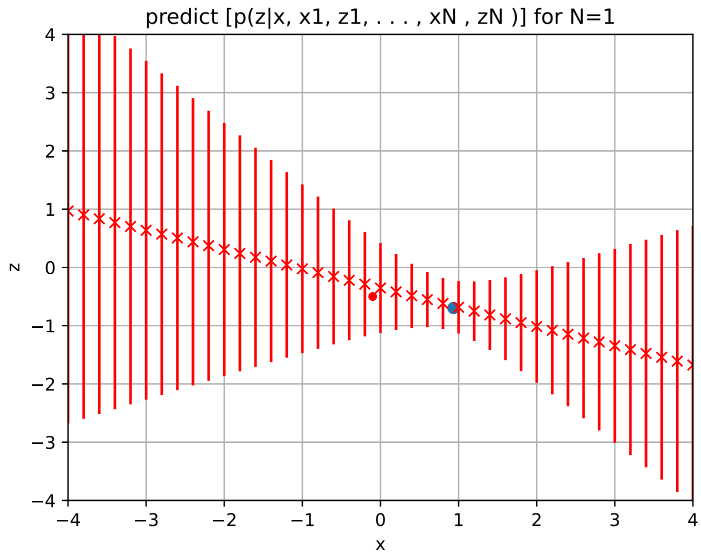


Figure 5: predict1.pdf

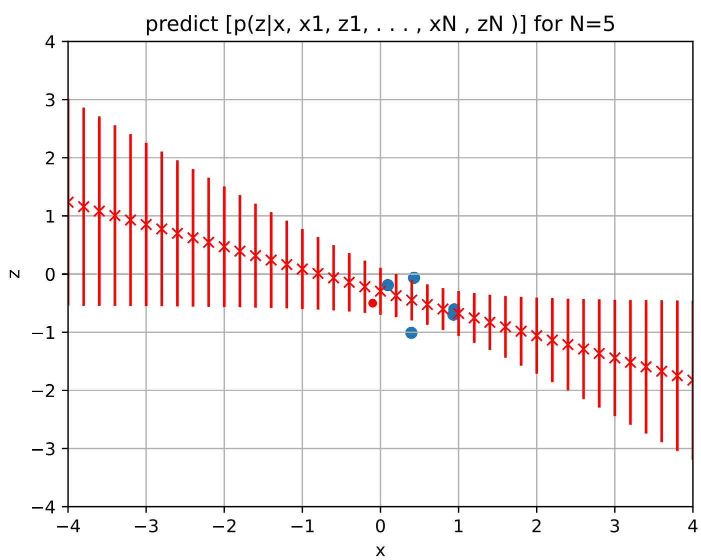


Figure 6: predict5.pdf

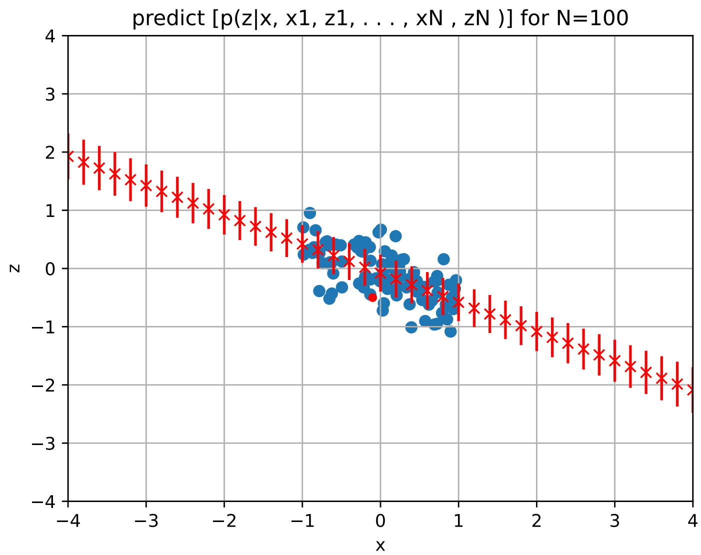


Figure 7: predict100.pdf