ECE368: Probabilistic Reasoning
Lab 3: Hidden Markov Model

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You should hand in: 1) A scanned .pdf version of this sheet with your answers (file size should be under 2 MB); 2) one Python file inference.py that contains your code. The files should be uploaded to Quercus.

1. (a) Write down the formulas of the forward-backward algorithm to compute the marginal distribution $p(\mathbf{z}_i|(\hat{x}_0, \hat{y}_0), \ldots, (\hat{x}_{N-1}, \hat{y}_{N-1}))$ for $i = 0, 1, \ldots, N-1$. Your answer should contain the initializations of the forward and backward messages, the recursion relations of the messages, and the computation of the marginal distribution based on the messages. (1 **pt**)

$$\alpha : \text{Forward Messages} \tag{1}$$

$$\beta : \text{Backward Messages} \tag{2}$$

$$p(\mathbf{z}_i|(\hat{x}_0, \hat{y}_0), \ldots, (\hat{x}_{N-1}, \hat{y}_{N-1})) = \gamma(z_i) = \alpha(z_i)\beta(z_i) \tag{3}$$

$$p(\mathbf{z}_i|(\hat{x}_0, \hat{y}_0), \ldots, (\hat{x}_{N-1}, \hat{y}_{N-1})) = \frac{\alpha(z_i)\beta(z_i)}{\Sigma_{z_i}\alpha(z_i)\beta(z_i)} \tag{4}$$

$$\alpha(z_i) = p(\hat{x}_i, \hat{y}_i|\mathbf{z}_i)\Sigma_{(z_i-1)}p(z_i|z_{i-1})\alpha(z_i-1) \tag{5}$$

$$\alpha(z_0) = p(\hat{x}_0, \hat{y}_0|\mathbf{z}_0)p(z_0) \tag{6}$$

$$\beta(z_{N-1}) = 0 \tag{7}$$

$$\beta(z_i) = \Sigma_{(z_i+1)}p(\hat{x}_{i+1}, \hat{y}_{i+1}|\mathbf{z}_{i+1})p(z_{i+1}|z_i)\beta(z_i+1) \tag{8}$$

$$p(\hat{x}_i, \hat{y}_i|\mathbf{z}_i) = 1 \text{ if } \hat{x}_i, \hat{y}_i \text{ missing} \tag{9}$$

(b) After you run the forward-backward algorithm on the data in test.txt, write down the obtained marginal distribution of the state at $i = 99$ (the last time step), i.e., $p(\mathbf{z}_{99}|(\hat{x}_0, \hat{y}_0), \ldots, (\hat{x}_{99}, \hat{y}_{99}))$. Only include states with non-zero probability in your answer. (2 **pt**)

$$p(\mathbf{z}_{99}|(\hat{x}_0, \hat{y}_0), \ldots, (\hat{x}_{99}, \hat{y}_{99})) = \begin{cases} 0.010128291694800 & \text{if } 0,\ z_{99} = (10, 1, \text{down}) \text{ if otherwise} \\ 0.17960837272113 & \text{if } 0,\ z_{99} = (11, 0, \text{right}) \text{ if otherwise} \\ 0.8102633355840648 & \text{if } 0,\ z_{99} = (11, 0, \text{stay}) \text{ if otherwise} \end{cases} \tag{10}$$

2. Modify your forward-backward algorithm so that it can handle missing observations. After you run the modified forward-backward algorithm on the data in test_missing.txt, write down the obtained marginal distribution of the state at $i = 30$, i.e., $p(\mathbf{z}_{30}|(\hat{x}_0, \hat{y}_0), \ldots, (\ hatx_{99}, \hat{y}_{99}))$. Only include states with non-zero probability in your answer. (1 **pt**)

$$p(\mathbf{z}_{30}|(\hat{x}_0,\hat{y}_0),\ldots,(\hat{x}_{99},\hat{y}_{99})) = \begin{cases} 0.04347826086956523 & \text{if } 0,\ z_{30} = (5,7,\text{right}) \text{ if otherwise} \\ 0.04347826086956522 & \text{if } 0,\ z_{30} = (5,7,\text{stay}) \text{ if otherwise} \\ 0.9130434782608696 & \text{if } 0,\ z_{30} = (6,7,\text{right}) \text{ if otherwise} \end{cases} \tag{11}$$

3. (a) Write down the formulas of the Viterbi algorithm using $\mathbf{z}_i$ and $(\ hatx_i, \hat{y}_i), i = 0, 1, \ldots, N-1$. Your answer should contain the initialization of the messages and the recursion of the messages in the Viterbi algorithm. (1 **pt**)

$$w_0(z_0) = \log(p(\hat{x}_0, \hat{y}_0|\mathbf{z}_0)) = \log(p(\hat{x}_0, \hat{y}_0|\mathbf{z}_0)) + \log(p(z_0)) \tag{12}$$

$$w_i(z_i) = \log(p(\hat{x}_i, \hat{y}_i|\mathbf{z}_i)) + \max_{zi-1}(\log(p(z_i|z_{i-1}) + w_{i-1}(z_{i-1}))) \tag{13}$$

$$\mathbf{z}_i^* = \text{argmax}_{z_i}(w_i(z_i)) \tag{14}$$

$$\mathbf{z}_{i-1}^* = \phi_i(z_i^*) = \text{argmax}_{z_{i-1}}(w_{i-1}(z_{i-1}^*)) \tag{15}$$

$$p(\hat{x}_i, \hat{y}_i|\mathbf{z}_i) = 1 \text{ if } \hat{x}_i, \hat{y}_i \text{ missing} \tag{16}$$

(b) After you run the Viterbi algorithm on the data in test missing.txt, write down the last 10 hidden states of the most likely sequence (i.e., $i = 90, 91, 92, \ldots, 99$) based on the MAP estimate. (3 **pt**)

$$z_{90} = (11, 5, \text{down}) \tag{17}$$

$$z_{91} = (11, 6, \text{down}) \tag{18}$$

$$z_{92} = (11, 7, \text{down}) \tag{19}$$

$$z_{93} = (11, 7, \text{stay}) \tag{20}$$

$$z_{94} = (11, 7, \text{stay}) \tag{21}$$

$$z_{95} = (10, 7, \text{left}) \tag{22}$$

$$z_{96} = (9, 7, \text{left}) \tag{23}$$

$$z_{97} = (8, 7, \text{left}) \tag{24}$$

$$z_{98} = (7, 7, \text{left}) \tag{25}$$

$$z_{99} = (6, 7, \text{left}) \tag{26}$$

4. Compute and compare the error probabilities of $\{\tilde{\mathbf{z}}_i\}$ and $\{\check{\mathbf{z}}_i\}$ using the data in test_missing.txt. The error probability of $\{\tilde{\mathbf{z}}_i\}$ is $\boxed{0.03}$. The error probability of $\{\check{\mathbf{z}}_i\}$ is $\boxed{0.02}$. (1 **pt**)

5. Is sequence $\{\check{\mathbf{z}}_i\}$ a valid sequence? If not, please find a small segment $\check{\mathbf{z}}_i, \check{\mathbf{z}}_{i+1}$ that violates the transition model for some time step i. You answer should specify the value of i as well as the corresponding states $\check{\mathbf{z}}_i,\ ma\check{t}hbfz_{i+1}$. (1 **pt**)

$$z_{64} = (3, 7, \text{stay}) \tag{27}$$

$$z_{65} = (2, 7, \text{stay}) \tag{28}$$

$$\text{If the rover went from } (3,7) \text{ to } (2,7) \text{ it should be left at } (3,7) \text{ at } i = 65 \tag{29}$$