STAT 710 Second Exam
8:50am-9:40am, March 16, 2012

Please show all your work for full credits.

1. Let $X_1, ..., X_n$ be i.i.d. with the Lebesgue p.d.f. $f(x-\mu)$, where μ is a real-valued unknown parameter and $f(x)$ is a known Lebesgue p.d.f., $f(x) > 0$ for all $x \in \mathcal{R}$, and $f''(x)$ exists and is continuous.
   (a) (2 points) Obtain the log-likelihood equation and show that it has at most one solution if
   $$\frac{f''(x)}{f(x)} - \left[\frac{f'(x)}{f(x)}\right]^2 < 0 \quad \text{for all } x$$
   (b) (2 points) Under what condition the log-likelihood equation in (a) has a solution?
   (c) (3 points) Assume that all regularity conditions in Theorem 4.16 are satisfied. Find the asymptotic distribution of the MLE $\hat{\mu}$.
   (d) (3 points) Assume that $f(x-\mu) = c_k e^{-(x-\mu)^k/k}$, where k is a known even integer and $c_k^{-1} = \int e^{-x^k/k} dx$. Show that all regularity conditions in Theorem 4.16 are satisfied. In addition, find an explicit function $h_\mu(x) > 0$ such that
   $$\int h_\mu(x) f(x-\mu) dx < \infty \qquad \text{and} \qquad \sup_{\gamma: |\gamma-\mu|<1} \left| \frac{d^2}{d\gamma^2} \log f(x-\gamma) \right| \le h_\mu(x)$$
   (e) (3 points) Assume that $f(x-\mu) = c_k e^{-(x-\mu)^k/k}$ as in part (d). Let $\bar{X}$ be the sample mean. Show that the asymptotic relative efficiency of the MLE $\hat{\mu}$ with respect to $\bar{X}$ is
   $$E(X_1-\mu)^{2(k-1)} E(X_1-\mu)^2.$$

2. Let $X_1, ..., X_n$ be i.i.d. with the Lebesgue p.d.f.
   $$\frac{(e+1)e^{-|x-\mu|}}{(e-1)(1+e^{-|x-\mu|})^2} I_{[0,1]}(|x-\mu|)$$
   where μ is a real-valued unknown parameter.
   (a) (4 points) Find the αth quantile of the population, where $0 < \alpha < \frac{1}{2}$, and the asymptotic distribution of the αth sample quantile.
   (b) (3 points) Find the asymptotic relative efficiency of the α-trimmed sample mean with respect to the sample median.