STAT 710
Third Exam, April 15, 2016

Please show all your work for full credits.

1. Let X be one observation with double exponential distribution p.d.f. $\frac{1}{2}e^{-|x-\mu|}$, where $\mu \in \mathcal{R}$ is unknown.

   (a) (3 points) We would like to test $H_0 : \mu = 0$ versus $H_1 : \mu = \mu_0$, where $\mu_0 > 0$ is a known constant. Show that the most powerful test has the rejection region $X > c$ with a constant c.

   (b) (2 points) Let $\alpha \in (0, \frac{1}{2})$ be a fixed level. Obtain the value of c in the previous part for the most powerful test with size α.

   (c) (2 points) Show that the test in part (b) is a UMP test of size α for testing $H_0 : \mu = 0$ versus $H_1 : \mu > 0$.

   (d) (2 points) Obtain the power function of the UMP test in part (b) for testing $H_0 : \mu = 0$ versus $H_1 : \mu > 0$.

   (e) (2 points) Show that the test in (b) is a UMP test of size α for testing $H_0 : \mu \leq 0$ versus $H_1 : \mu > 0$.

2. Let $X_1, ..., X_n$ be i.i.d. observations from the discrete distribution with

$$P(X_i = x) = \frac{\theta^x}{x!(e^\theta - 1)}, \qquad x = 1, 2, ...,$$

   where $\theta > 0$ is unknown. Consider testing $H_0 : \theta = 1$ versus $H_1 : \theta \neq 1$.

   (a) (2 points) Show that the MLE of θ is $\hat{\theta} = g(\bar{X})$, where $\bar{X}$ is the sample mean and g is a strictly increasing function.

   (b) (3 points) Let $\ell(\theta)$ be the likelihood function. Show that $d \log \ell(\hat{\theta})/d\bar{X} = n \log \hat{\theta}$ so that $\log \ell(\hat{\theta})$ is strictly increasing in $\bar{X}$ when $\hat{\theta} > 1$ and strictly decreasing in $\bar{X}$ when $\hat{\theta} < 1$. Use this result to show that the likelihood ratio test is equivalent to the test rejecting H_0 when $\bar{X} < c_1$ or $\bar{X} > c_2$ for some constants c_1 and c_2.

   (c) (2 points) Derive the form of Wald's test.

   (d) (2 points) Derive the form of Rao's score test.