STAT 709 Second Exam
8:25am-9:15pm, Oct 26, 2010

Please show all your work for full credits.

1. Let $X_1, ..., X_n$ be i.i.d. random variables with Lebesgue p.d.f.
$$\frac{x}{\theta^2} e^{-x/\theta} I_{(0,\infty)}(x),$$
where $\theta = 1$ or 2 is an unknown parameter.
   (a) (3 points) Show that $\bar{X} = n^{-1} \sum_{i=1}^n X_i$ is minimal sufficient for $\theta = 1, 2$.
   (b) (3 points) Show that $\bar{X}$ is not complete.
   Hint: compute $E(\bar{X})$ and $E(\bar{X}^2)$.

2. Let $X_1, X_2, ...$ be a sequence of i.i.d. random variables.
   (a) (2 points) Let $\phi(t)$ be the characteristic function of X_1. Assume that X_1 and $-X_1$ have the same distribution. Show that $n^{-1} \sum_{i=1}^n \phi(X_i)$ converges a.s. to $E[\cos(X_1 X_2)]$.
   (b) (2 points) Let g be a function on $\mathcal{R}$ such that $g \geq 0$ and $-g$ is convex. Assume that $E|X_1| < \infty$. Show that $n^{-1} \sum_{i=1}^n g(X_i)$ converges a.s. to a constant.

3. Let $X_1, X_2, ...$ be a sequence of independent random variables with
$$P(X_j = 0) = 1 - 2p_j, \quad P(X_j = j^a) = p_j, \quad P(X_j = -j^a) = p_j, \quad p_j = \begin{cases} \frac{1}{2} & j^{-b} \geq \frac{1}{2} \\ j^{-b} & j^{-b} < \frac{1}{2} \end{cases}$$
where a and b are positive constants. Let $T_n = \sum_{j=1}^n X_j$.
   (a) (2 points) Suppose that $b > a$. Show that the sequence $\{|X_j|\}$ is uniformly integrable.
   (b) (2 points) Suppose that $2a - 1 < b$. Show that
   $$\frac{T_n}{n} \to_{a.s.} 0.$$
   (c) (3 points) Suppose that $b < 1$ and $b \leq 2a$. Show that
   $$\frac{T_n}{\sqrt{\mathrm{Var}(T_n)}} \to_d N(0,1)$$
   and
   $$a_n \log\left(1 + \frac{T_n}{a_n \sqrt{\mathrm{Var}(T_n)}}\right) \to_d N(0,1)$$
   for any sequence $\{a_n\}$ with $\lim_{n\to\infty} a_n = \infty$ and $a_n > 0$.
   Hint: $\lim_{n\to\infty} n^{-(t+1)} \sum_{j=1}^n j^t = (t+1)^{-1}$ for any $t \geq 0$.
   (d) (3 points) Suppose that $b < 1$ and $2a - 1 > b$. Show that
   $$\frac{n}{T_n} \to_p 0.$$