STAT 709 Second Exam
9:55am-10:45am, Oct 26, 2016

Please show all your work for full credits.

1. Let $Z, Y, X_1, X_2, ...$ be random variables. Suppose that $X_n \to_d Z$ as $n \to \infty$.
   (a) (3 points) Suppose that Y is independent of X_n for any n. Show that the sequence of c.d.f. of $X_n + Y$ converges weakly to a c.d.f., and identify that c.d.f.
   (b) (2 points) Suppose that $E|X_n| \to E|Z| < \infty$ and $E|Y| < \infty$. Show that the sequence $\{|X_n + Y|, n = 1, 2, ...\}$ is uniformly integrable.

2. Let $X_1, X_2, ...$ be a sequence of independent random variables. Suppose that for each i, X_i has Lebesgue p.d.f.

$$f_i(x) = \begin{cases} \frac{1}{\Gamma(i^s)} x^{i^s - 1} e^{-x} & x > 0 \\ 0 & x \leq 0 \end{cases}$$

   $i = 1, 2, ...$, where s is a constant, $-1 < s < 1$.
   (a) (2 points) Show that

$$\frac{1}{n} \sum_{i=1}^{n} (X_i - i^s) \to_{a.s.} 0.$$

   (b) (4 points) Show that

$$\sum_{i=1}^{n} (X_i - i^s) \Big/ \left(\sum_{i=1}^{n} i^s \right)^{1/2} \to_d N(0, 1).$$

   (Hint: $|X_i - i^s|^3 \leq (X_i - i^s)^2 (X_i + i^s)$ and $n^{-(t+1)} \sum_{i=1}^{n} i^t \to (t+1)^{-1}$ for $t > -1$.)
   (c) (2 points) Let $Y_n = \sum_{i=1}^{n} X_i / \sum_{i=1}^{n} i^s$. Show that $n^{(s+1)/2} \log Y_n$ converges in distribution to a random variable Z and identify the distribution of Z.

3. Let $X_1, ..., X_n$ ($n \geq 2$) be i.i.d. random variables with Lebesgue p.d.f.

$$f_{\alpha,\beta}(x) = \begin{cases} \beta \alpha^{\beta} x^{-(\beta+1)} & x > \alpha \\ 0 & x \leq \alpha \end{cases}$$

   where $\alpha > 0$ and $\beta > 0$ are parameters.
   (a) (2 points) When both α and β are unknown, show that the transformed variables $Y_i = \log X_i$, $i = 1, ..., n$, has a distribution in a location-scale family, and identify the location and scale parameters.
   (b) (3 points) When β is known and α is unknown, show that $X_{(1)} = \min(X_1, ..., X_n)$ is minimal sufficient for α.
   (c) (2 points) When both α and β are unknown, obtain a 2 dimensional minimal suffcient statistic for (α, β), and show the proof.

Solution:

1. (a) Using the characteristic functions, we can show that $X_n + Y \to_d Z + \tilde{Y}$, where $\tilde{Y}$ has the same distribution as Y but is independent of Z. Note that Y is not necessarily independent of Z.

   (b) From Theorem 1.8, $E|X_n| \to E|Z| < \infty$ implies that $\{|X_n|, n = 1, 2, ...\}$ is uniformly integrable. Since $|X_n + Y| \leq |X_n| + |Y|$ and $|Y|$ is integrable, $\{|X_n + Y|, n = 1, 2, ...\}$ is uniformly integrable.

2. (a) Apply Theorem 1.14(b). $EX_i = i^s$. Consider random variable $X_i - EX_i = X_i - i^s$. $E(X_i - i^s)^2 = i^s$. If $s < 1$,

$$\sum_{i=1}^{\infty} \frac{i^s}{i^2} < \infty$$

   (b) Apply Liapounov's condition. $EX_i = i^s$, $EX_i^2 = i^s(i^s+1)$, $EX_i^3 = i^s(i^s+1)(i^s+2)$, $E(X_i - i^s)^2 = i^s$, $\sigma_n^2 = \sum_{i=1}^n i^s$, Then

$$\begin{aligned} E|X_i - i^s|^3 &\leq E(X_i - i^s)^2(X_i + i^s) \\ &= E(X_i^3 - 2i^s X_i^2 + i^{2s} X_i) + i^s E(X_i - i^s)^2 \\ &= i^s(i^s+1)(i^s+2) - 2(i^s)^2(i^s+1) + i^{3s} + i^{2s} \\ &= 2(i^{2s} + i^s) \end{aligned}$$

   Then

$$\frac{1}{\sigma_n^{3/2}} \sum_{i=1}^n E|X_i - i^s|^3 \leq \frac{2\sum_{i=1}^n (i^s)^2 + i^s)}{(\sum_{i=1}^n i^s)^{3/2}} \approx 2\left(\frac{n^{2s+1}}{2s+1} + \frac{n^{s+1}}{s+1}\right) \Big/ \left(\frac{n^{s+1}}{s+1}\right)^{3/2} \to 0$$

   if $-1 < s < 1$.

   (c) Apply the delta method to

$$\left(\sum_{i=1}^n i^s\right)^{1/2} (Y_n - 1) = \left(\sum_{i=1}^n i^s\right)^{1/2} \left(\frac{\sum_{i=1}^n X_i}{\sum_{i=1}^n i^s} - 1\right) \to_d N(0,1)$$

   and $g(y) = \log y$, $g(1) = 0$, $g'(y) = 1/y$, $g'(1) = 1$. Then $Z \sim N(0, s+1)$.

3. (a) Y_i has the two parameter exponential distribution, location parameter $\log \alpha$ and scale parameter $1/\beta$.

   (b) Apply Theorem 2.3(iii). For $x = (x_1, ..., x_n)$ and $y = (y_1, ..., y_n)$,

$$\frac{f_{\alpha,\beta}(x)}{f_{\alpha,\beta}(y)} = \frac{\prod_i y_i^{\beta+1}}{\prod_i x_i^{\beta+1}} \frac{I_{(\alpha,\infty)}(x_{(1)})}{I_{(\alpha,\infty)}(y_{(1)})}$$

   The first factor does not depend on α. The 2nd factor does not depend on α implies that $x_{(1)} = y_{(1)}$.

   (c) The minimal sufficient statistic is $(\prod_i X_i, X_{(1)})$. The proof is similar to that of (b).