**STAT 710 Final Exam**
**10:00am-12:00noon, May 10, 2011**

Please show all your work for full credits.

1. Let $X_1, ..., X_n$ be iid observations with the Lebesgue p.d.f. $\theta\eta^\theta x^{-(\theta+1)} I_{[\eta,\infty)}(x)$, where $\theta > 0$ and $\eta > 0$.
   (a) (4 points) Assume that $\theta$ is known and $\eta$ is unknown. Obtain a $\Theta'$-UMA upper confidence bound of $\eta$ with confidence coefficient $1 - \alpha \in (1/2, 1)$, where $\Theta' = (\eta, \infty)$.
   (b) (3 points) Let $\eta_0$ and $\theta_0$ be positive constants. When both $\theta$ and $\eta$ are unknown, derive the likelihood ratio statistic for testing the null hypothesis $H_0 : \eta = \eta_0$ and $\theta = \theta_0$, in terms of a function of the MLE $(\hat{\theta}, \hat{\eta})$ of $(\theta, \eta)$.
   (c) (3 points) Show that $\sqrt{n}\log(\hat{\eta}/\eta) \to 0$ in probability as $n \to \infty$.
   (d) (3 points) Show that $\sqrt{n}(\frac{\theta}{\hat{\theta}} - 1)$ is asymptotically pivotal as $n \to \infty$. as $n \to \infty$.
   (e) (4 points) Let $\lambda(\theta_0, \eta_0)$ be the likelihood statistic in (b). Show that $\lambda(\theta, \eta)$ is asymptotically pivotal and discuss how to obtain a confidence set of $(\theta, \eta)$ with asymptotic level $1 - \alpha$.
   (f) (4 points) Consider the parameter $\phi = 2^{1/\theta}\eta$. Show that the sample median $\hat{\phi}$ is a consistent estimator of $\phi$ and derive the asymptotic distribution of $\sqrt{n}(\hat{\phi} - \phi)$.

2. Let $X$ be an observation with the Lebesgue p.d.f. $\theta f(\theta x)$, where $\theta > 0$ is unknown and $f$ is a known Lebesgue p.d.f., $f(-x) = 0$ and $f(x) > 0$ for any $x > 0$.
   (a) (4 points) Let the prior for $\theta$ to be the Gamma distribution with shape parameter $a > 0$ and scale parameter $\gamma > 0$ where $a$ and $\gamma$ are known. Under the squared error loss, obtain the Bayes estimators of $\theta$ and $\theta^{-1}$.
   (b) (3 points) Assume that $f$ has finite moment of any order. Obtain the generalized Bayes estimator of $\theta$ by letting $\gamma$ in (a) tend to $\infty$.
   (c) (3 points) Show that $\theta X$ is a pivotal quantity. If $f$ is unimodal at $x_0 > 0$, derive the shortest length confidence interval of the form $(aX^{-1}, bX^{-1})$ and confidence coefficient $1 - \alpha$.

3. Let $X_1, ..., X_n$ be iid from the Gamma distribution with known shape parameter $a > 0$ and unknown scale parameter $\theta > 0$, and $Y_1, ..., Y_n$ be iid from the Gamma distribution with known shape parameter $a > 0$ and unknown scale parameter $\varphi > 0$. Suppose that $X_i$'s and $Y_i$'s are independent.
   (a) (3 points) Obtain a $1 - \alpha$ asymptotically correct confidence interval for $\theta/\varphi$ by inverting the acceptance regions of likelihood ratio tests.
   (b) (3 points) Obtain a $1 - \alpha$ asymptotically correct confidence interval for $\theta/\varphi$ by inverting the acceptance regions of Wald's tests.
   (c) (3 points) Obtain a $1 - \alpha$ asymptotically correct confidence interval for $\theta/\varphi$ by inverting the acceptance regions of Rao's score tests.