

STAT 710
Second Exam, March 16, 2018

Please show all your work for full credits.

1. Let X_1, \dots, X_n be i.i.d. positive random variables with Lebesgue p.d.f. $\theta f(\theta x)$, where $\theta > 0$ is unknown and f is a known Lebesgue p.d.f. that is positive and twice continuously differentiable on $(0, \infty)$ with f' and f'' being the first and second order derivatives.

- (a) (3 points) Assume that

$$(C1) \int_0^\infty t|f'(t)|dt < \infty \text{ and } \int_0^\infty t^2|f''(t)|dt < \infty.$$

Show that $\frac{d}{d\theta} \int_0^\infty f(\theta x)dx = \int_0^\infty x f'(\theta x)dx$ and $\frac{d}{d\theta} \int_0^\infty x f'(\theta x)dx = \int_0^\infty x^2 f''(\theta x)dx$ for any $\theta > 0$.

- (b) (2 points) Assume (C1) and

$$(C2) \int_0^\infty t^2 [f'(t)]^2 / f(t) dt < \infty.$$

Show that the Fisher information about θ contained in X_1 is

$$I_1(\theta) = \frac{1}{\theta^2} \left[1 + \int_0^\infty \frac{t^2 [f'(t)]^2}{f(t)} dt - \int_0^\infty t^2 f''(t) dt \right]$$

- (c) (2 points) Assume (C1)-(C2) and

$$(C3) \sup_{\frac{1}{2}\theta < \gamma < \frac{3}{2}\theta} \left| \frac{d^2}{d\gamma^2} \log(\gamma f(\gamma x)) \right| \leq h_\theta(x) \text{ with } \int_0^\infty h_\theta(x) f(\theta x) dx < \infty.$$

Derive the asymptotic distribution of the MLE $\hat{\theta}$ of θ .

- (d) (2 points) When $f(t) = 2/[\pi(1+t^2)]$, show that (C3) is satisfied and identify the function $h_\theta(x)$.

- (e) (2 points) Assume

$$(C4) \mu_2 = \int_0^\infty t^2 f(t) dt < \infty.$$

Let $\bar{X} = n^{-1} \sum_{i=1}^n X_i$. Show that a moment estimator $\tilde{\theta}$ of θ is μ_1/\bar{X} , where $\mu_1 = \int_0^\infty t f(t) dt$.

- (f) (3 points) Assume (C1)-(C4). Obtain the asymptotic relative efficiency of the MLE $\hat{\theta}$ with respect to the moment estimator $\tilde{\theta}$.

- (g) (3 points) Let a be the median of f , i.e., $\int_0^a f(t) dt = 1/2$. Then the median of X_1 is $m = a/\theta$. Let \hat{m} be the sample median and $\check{\theta} = a/\hat{m}$. Obtain the asymptotic relative efficiency of the MLE $\hat{\theta}$ with respect to $\check{\theta}$.

2. (3 points) Let X_1, \dots, X_n be i.i.d. random variables having the Lebesgue p.d.f.

$$f(x) = (3/4)[1 - (x - \theta)^2]I_{(0,1)}(|x - \theta|),$$

where $\theta \in (-\infty, \infty)$ is an unknown parameter. Calculate the asymptotic relative efficiency of the sample mean with respect to the sample median.

Solution:

1. (a)

$$\frac{d}{d\theta} \int_0^\infty f(\theta x) dx = \lim_{\delta \rightarrow 0} \int_0^\infty \frac{f((\theta + \delta)x) - f(\theta x)}{\delta} dx$$

$$\left| \frac{f((\theta + \delta)x) - f(\theta x)}{\delta} \right| \leq |x f'(\xi x)|$$

which is integrable under the condition. Hence the first result follows. The proof of the second result is similar.

(b)

$$\frac{d}{d\theta} \log(\theta f(\theta x)) = \frac{1}{\theta} + \frac{x f'(\theta x)}{f(\theta x)}$$

$$\frac{d^2}{d\theta^2} \log(\theta f(\theta x)) = -\frac{1}{\theta^2} + \frac{x^2 f''(\theta x)}{f(\theta x)} - \left[\frac{x f'(\theta x)}{f(\theta x)} \right]^2$$

Taking the expectation gives the result.

(c) Apply Theorem 4.17 and use the $I_1(\theta)$ in part (b).

(d) When $f(t) = 2/[\pi(1+t^2)]$, $f'(t) = -4t/[\pi(1+t^2)^2]$ and $f''(t) = 4(3t^2-1)/[\pi(1+t^2)^3]$. Use these result, $(\gamma x)^2 f''(\gamma x)/f(\gamma x) = 2(\gamma x)^2 [3(\gamma x)^2 - 1]/[1 + (\gamma x)^2]^2$ and $(\gamma x) f'(\gamma x)/f(\gamma x) = -2(\gamma x)^2/[1 + (\gamma x)^2]$ are bounded functions. Then there is a constant $C > 0$ such that

$$\sup_{\frac{1}{2}\theta < \gamma < \frac{3}{2}\theta} \left| \frac{d^2}{d\gamma^2} \log(\gamma f(\gamma x)) \right| \leq C \sup_{\frac{1}{2}\theta < \gamma < \frac{3}{2}\theta} \gamma^{-2} = 4C/\theta^2$$

We can take $h_\theta(x) = 4C/\theta^2$.

(e) The result follows from

$$E(X_1) = \theta \int_0^\infty x f(\theta x) dx = \mu_1/\theta$$

(f) By CLT,

$$\sqrt{n}(\bar{X} - EX_1) \rightarrow_d N(0, EX_1^2 - (EX_1)^2) = N(0, \theta^{-2}(\mu_2 - \mu_1^2))$$

By delta-method,

$$\sqrt{n}(\tilde{\theta} - \theta) = \sqrt{n}(\mu_1/\bar{X} - \mu_1/EX_1) \rightarrow_d N(0, \theta^2(\mu_2 - \mu_1^2)/\mu_1^2)$$

The relative efficiency is

$$\theta^2(\mu_2 - \mu_1^2)I_1(\theta)/\mu_1^2$$

(g) By Theorem 5.9,

$$\sqrt{n}(\hat{m} - m) \rightarrow_d N(0, 1/4\theta^2 f(a)^2)$$

By delta-method,

$$\sqrt{n}(\check{\theta} - \theta) = \sqrt{n}(a/\hat{m} - a/m) \rightarrow_d N(0, \theta^2/4a^2 f(a)^2)$$

The asymptotic relative efficiency is $\theta^2 I_1(\theta)/4a^2 f(a)^2$.

2. For the sample median $\hat{\theta}$, Theorem 5.9 gives

$$\sqrt{n}(\hat{\theta} - \theta) \rightarrow_d N\left(0, \frac{1}{4(3/4)^2}\right) = N(0, 4/9)$$

For the sample mean, we calculate

$$\text{Var}(X_1) = \frac{3}{2} \int_0^1 t^2(1-t^2)dt = \frac{1}{5}$$

Hence the ARE is $(4/9)/(1/5) = 20/9$.