Name: ____________________________________

Show sufficient work to make *very clear* your method of solution.

1. Let A, B and C be three events on a probability space.
   (a) (5 points) If $P(A) = 0.4$, $P(B) = 0.6$, $P(C) = 0.5$, and A, B, and C are mutually independent, calculate $P(A \cup B \cup C)$.
   (b) (5 points) If $P(A|C) = 0.7$, $P(B|C) = 0.4$, and A and B are conditionally independent given C (i.e. $P(A \cap B|C) = P(A|C).P(B|C)$), calculate $P(A \cup B|C)$.
   (c) (5 points) Suppose that $P(B \cap C) > 0$. Prove that if A and B are conditionally independent givn C, then $P(A|B \cap C) = P(A|C)$.
   (d) (5 points) Suppose that A, B, and C are pairwise independent, and that A and B are conditionally independent given C. Show that A, B, and C are mutually independent.

2. You are waiting at a bus stop and can take any one of three buses Bus 1, Bus 2 or Bus 3. Bus 1 comes every 5 minutes, Bus 2 every 10 minutes and Bus 3 every 15 minutes. Further assume that the waiting times are memoryless in the sense that the amount of time since the previous bus arrived does not affect how much time to wait until the next bus comes and that the waiting times for each of the three buses are independent.
   (a) (10 points) What distribution would you use to model how long it it will take for Bus 1 to arrive (specify all related parameters)? Do the same for Buses 2 and 3. Using these distributions, how long do you expect to wait for either of the 3 buses, 1, 2 or 3 arrive?
   (b) (10 points) What distribution would you use to model the *total* number of buses 1, 2 or 3 arrive over a 30 minute period? Using these distributions, what is the chance that 2 or fewer buses (out of any of Bus 1, 2 or 3) arrive over a 30 minute period?

3. Consider a continuous random variable X with PDF $f(x) = cxe^{-(\theta.x)^2}$, for $x > 0$ and 0 otherwise, where $0 < \theta < \infty$.
   (a) (5 points) Find c and the CDF for X.
   (b) (5 points) If you want to simulate the waiting time for each individual, what tranformation would you apply to the standard uniform random variable U. That is for what g, does $X = g(U)$ where X is the waiting time for a single individual.
   (c) (5 points) Let $W = X^2$. Find the distribution of W. Name this distribution, specifying all relevant parameters.
   (d) (5 points) Show that X is an exponential family distribution. Find its sufficient statistics, natural parameter, base measure and log-partition function.

4. Suppose that $X_1, X_2, X_3, \ldots$ are mutually independent random variables with density $f(x) = 0.5e^{-|x|}$ for $-\infty < x < \infty$. Let $S_n = X_1 + \cdots + X_n$.
   (a) (5 points) Show that the MGF for each X_i is $\frac{1}{1-t^2}$ for $|t| < 1$.
   (b) (5 points) Find $E(X_i)$ and $\mathrm{Var}(X_i)$ for all i.
   (c) (5 points) Find the MGF for S_n.
   (d) (5 points) Let Y and Z be independent Exponential(1) random variables. Find the MGF for the random variable $Y - Z$, specifying the range of t. Find the PDF for the random variable $Y - Z$.

5. Let X and Z be independent Normal$(0, 1)$ random variables and $Y = X + Z$.

   (a) (5 points) Find Corr(X, Y). Are X and Y independent? If yes, provide a proof. If no, provide an explanation.

   (b) (5 points) Find the MGF for Y, $E[e^{tY}]$.

   (c) (5 points) Do (X, Y) form a bivariate normal distribution? If yes, find the mean vector and covariance matrix.

   (d) (5 points) Repeat parts (a)-(c) if $Y = X^2 + Z$ instead of $Y = X + Z$.