STAT 710
First Exam, Feb 21, 2018

Please show all your work for full credits.

1. Consider the estimation of an unknown $\theta \in \mathcal{R}$ under the loss $L(\theta, a)$ which is strictly convex in a for any θ. Let $\hat{\theta}_1$ and $\hat{\theta}_2$ be two estimators of θ with finite risks.

   (a) (2 points) Show that if both $\hat{\theta}_1$ and $\hat{\theta}_2$ are minimax, then $c\hat{\theta}_1 + (1-c)\hat{\theta}_2$ is also minimax for any constant $c \in (0, 1)$.

   (b) (2 points) Show that if $\hat{\theta}_2$ is minimax and $\hat{\theta}_1$ has constant risk, then $\hat{\theta}_1$ is inadmissible.

2. Let $X = (X_1, ..., X_n)$ and $X_1, ..., X_n$ be i.i.d. with Lebesgue p.d.f.

$$f_\theta(x) = \alpha\beta(1 - e^{-\beta x})^{\alpha-1}e^{-\beta x}, \quad x > 0,$$

   where $\theta = (\alpha, \beta)$ and $\alpha > 0$, $\beta > 0$ are two unknown parameters.

   (a) (4 points) Obtain the likelihood $\ell(\theta)$ and show that the likelihood equation $\frac{\partial \log \ell(\theta)}{\partial \theta} = 0$ can be expressed as $\alpha = h(\beta, X)$ and $g(\beta, X) = 0$ for two functions h and g. Obtain the explicit expressions of h and g.

   (b) (2 points) Show that $g(\beta, X) = 0$ has a solution in $(0, \infty)$.

   (c) (2 points) Show that $g(\beta, X) = 0$ has a unique solution in $(0, \infty)$.

3. Let $X_1, ..., X_n$ be i.i.d. having the Lebesgue p.d.f.

$$f_\theta(x) = \frac{1}{2\theta}e^{-|x|/\theta}, \qquad x \in \mathcal{R}$$

   where $\theta > 0$ is unknown.

   (a) (2 points) Suppose that the prior (Lebesgue) p.d.f. of $1/\theta$ is the gamma distribution $\Gamma(\alpha, \gamma)$ with known hyperparameters $\alpha > 0$ and $\gamma > 0$. Show that this is a conjugate prior and derive the Bayes estimator of θ under the squared error loss.

   (b) (3 points) Suppose that the hyperparameter γ is unknown and α is known, and that the prior p.d.f. for γ is the improper prior $\pi(\gamma) = \gamma^{-1}$. Under the hierarchical Bayes approach, derive the posterior of θ and the generalized Bayes estimator of θ under the squared error loss.

   (c) (3 points) Show that $\sum_{i=1}^n |X_i|/(n+1)$ is admissible under the squared error loss.

Solution:

1. (a) Let c be a known constant in $(0,1)$. By Jensen's inequality,

$$R_{c\hat{\theta}_1+(1-c)\hat{\theta}_2}(\theta) < cR_{\hat{\theta}_1}(\theta) + (1-c)R_{\hat{\theta}_2}(\theta) \le c\sup_\theta R_{\hat{\theta}_1}(\theta) + (1-c)\sup_\theta R_{\hat{\theta}_2}(\theta)$$

If both $\hat{\theta}_1$ and $\hat{\theta}_2$ are minimax, then the two sup's are the same. Hence,

$$\sup_\theta R_{c\hat{\theta}_1+(1-c)\hat{\theta}_2}(\theta) = \sup_\theta R_{\hat{\theta}_1}(\theta)$$

(b) Suppose $R_{\hat{\theta}_1}(\theta) = R$, a constant. Then

$$R_{c\hat{\theta}_1+(1-c)\hat{\theta}_2}(\theta) < cR + (1-c)R_{\hat{\theta}_2}(\theta) \le R$$

because $\hat{\theta}_2$ is minimax. This shows that $\hat{\theta}_1$ is worse than $c\hat{\theta}_1 + (1-c)\hat{\theta}_2$.

2. (a) $\log \ell(\theta) = n\log\alpha + n\log\beta - \beta\sum_i X_i + (\alpha-1)\sum_i \log(1-e^{-\beta X_i})$

$$\frac{\partial \ell(\theta)}{\partial \alpha} = \frac{n}{\alpha} + \sum_i \log(1-e^{-\beta X_i})$$

$$\frac{\partial \ell(\theta)}{\partial \beta} = \frac{n}{\beta} - \sum_i X_i + (\alpha-1)\sum_i \frac{X_i}{e^{\beta X_i}-1}$$

Thus,

$$h(\beta, X) = -\frac{n}{\sum_i \log(1-e^{-\beta X_i})}$$

$$g(\beta, X) = \frac{n}{\beta} - \sum_i X_i - \left[\frac{n}{\sum_i \log(1-e^{-\beta X_i})} + 1\right]\sum_i \frac{X_i}{e^{\beta X_i}-1}$$

(b) The result follows from $g(\beta, X)$ is continuous in β, $\lim_{\beta\to 0} g(\beta, X) = \infty$ and $\lim_{\beta\to\infty} g(\beta, X) = -\sum_i X_i < 0$.
(c) It can be shown that $\frac{\partial g(\beta, X)}{\partial \beta} < 0$, so $g(\beta, X)$ is a decreasing function of β. The result follows.

3. (a) The posterior of $1/\theta$ is a gamma distribution. The Bayes estimator of θ is $(\sum_i |X_i| + \gamma^{-1})/(\alpha+n-1)$.
(b) The prior of θ given γ is

$$\frac{1}{\Gamma(\alpha)\theta^{\alpha+1}\gamma^\alpha} e^{-1/(\gamma\theta)}$$

Then prior of θ is

$$\int_0^\infty \frac{1}{\Gamma(\alpha)\theta^{\alpha+1}\gamma^\alpha} e^{-1/(\gamma\theta)} \frac{1}{\gamma} d\gamma = \int_0^\infty \frac{r^{\alpha-1}}{\Gamma(\alpha)\theta^{\alpha+1}} e^{-r/\theta} dr = \frac{1}{\theta}$$

which is an improper prior. The posterior of θ is proportional to

$$\frac{T^n}{\Gamma(n)\theta^{n+1}} e^{-T/\theta}, \qquad T = \sum_i |X_i|$$

The generalized Bayes estimator is

$$\int_0^\infty \frac{T^n}{\Gamma(n)\theta^n} e^{-T/\theta} d\theta = \frac{T}{n-1}$$

(c) We apply Theorem 4.14. The joint p.d.f. is $(2\theta)^{-n} e^{-T/\theta} = (-2n\eta)^{-n} e^{\eta T_1}$, where $\eta = -1/(\theta n) \in (-\infty, 0)$ and $T_1 = T/n$ with $E(T_1) = \theta$. Note that $\sum |X_i|/(n+1) = (T_1 + \gamma\lambda)/(\lambda + 1)$ with $\gamma = 0$ and $\lambda = n^{-1}$. The result follows from

$$\int_{-\infty}^t \frac{1}{(-2n\eta)^{n\lambda}} d\eta = \int_t^0 \frac{1}{(-2n\eta)^{n\lambda}} d\eta = \infty.$$