STAT 709 Final Exam
9:30am-11:30am, Dec. 17, 2010

Please show all your work for full credits.

1. (a) (2 points) Let ξ and η be random variables. Using the definition of the conditional expectation, show that if ξ and η are independent, then $E(I_A|\xi) = P(A)$ a.s. for any $A \in \sigma(\eta)$.

   (b) (2 points) Let ξ, ζ, and η be random variables. Show that if η is independent of (ξ, ζ), then η and ζ are conditionally independent given ξ.

   (c) (4 points) Let $Y_1, Y_2, \ldots$ be independent random variables with $E(Y_n) = 0$ and $\text{Var}(Y_n) = \sigma^2$ for all n. Define $X_n = (Y_1 + \cdots + Y_n)^2 - n\sigma^2$, $n = 1, 2, \ldots$ Show that

$$E(X_{n+1}|X_n) = X_n \quad \text{a.s.} \quad n = 1, 2, \ldots$$

2. Let $X_1, \ldots, X_n$ be i.i.d. with Lebesgue p.d.f.

$$\sqrt{\frac{2}{\pi}} e^{-(x-\theta)^2/2} I_{(\theta,\infty)}(x),$$

   where $\theta \in \mathcal{R}$ is unknown and $n \geq 2$.

   (a) (3 points) Find a two-dimensional minimal sufficient statistic for θ and show why it is minimal sufficient.

   (b) (3 points) Let $X_{(1)}$ be the minimum order statistic. Show that $X_{(1)} - a$ is an unbiased estimator of θ, where

$$a = 2^{n-1} n \sqrt{\frac{2}{\pi}} \int_0^\infty y[1 - \Phi(y)]^{n-1} e^{-y^2/2} dy$$

   and Φ is the standard normal c.d.f.

3. Let $X_1, \ldots, X_n$ be i.i.d. from a population with Lebesgue p.d.f.

$$\frac{1-\epsilon}{\sqrt{2\pi}\theta} e^{-x^2/2\theta^2} + \frac{\epsilon}{2\theta} e^{-|x|/\theta},$$

   where $\theta > 0$ is unknown and $\epsilon \in (0, 1)$ is known.

   (a) (2 points) Using the method of moments and the moment $E(X_1^2)$, obtain an estimator of θ^2.

   (b) (2 points) Using the method of moments and the moment $E|X_1|$, obtain an estimator of θ^2.

   (c) (4 points) Derive the asymptotic relative efficiency of the estimator in (a) w.r.t. the estimator in (b).

   (Two problems on page 2)

4. Let $X_1, ..., X_n$ be i.i.d. random variables with a Lebesgue p.d.f. and $E|X_1|^6 < \infty$, where $n \geq 2$. Let $\mu_j = E(X_1^j)$, $j = 1, ..., 6$.

   (a) (2 points) Derive the UMVUE of $\gamma = \mu_1 \mu_3$.

   (b) (4 points) Obtain the asymptotic distribution of $\sqrt{n}(T_n - \gamma)$ in terms of μ_j, $j = 1, ..., 6$, where T_n is the UMVUE in (a).

   (c) (4 points) Obtain the variance of T_n, in terms of μ_j, $j = 1, ..., 6$.

5. Consider a linear model

$$X = Z\beta + \varepsilon,$$

   where ε is normal with mean 0 and covariance matrix V.

   (a) (4 points) Suppose that $V = \sigma^2 I_4$, where I_4 is the identity matrix of order 4, and that

$$Z = \begin{bmatrix} 1 & 0 & 2 \\ 1 & 2 & 0 \\ 1 & -1 & 3 \\ 1 & 1 & 1 \end{bmatrix}$$

   Obtain the forms of all $l \in \mathcal{R}^3$ such that $l^\tau \beta$ is estimable.

   (b) (4 points) Suppose that

$$Z = \begin{bmatrix} 1 & 0 \\ 1 & -1 \\ -1 & 1 \end{bmatrix} \qquad \text{and} \qquad V = \begin{bmatrix} a & 0 & 0 \\ 0 & b & c \\ 0 & c & d \end{bmatrix}$$

   Show that any LSE of $l^\tau \beta$ is the UMVUE if and only if $b = d$.