STAT 710
Second Exam, March 16, 2018

Please show all your work for full credits.

1. Let $X_1, ..., X_n$ be i.i.d. positive random variables with Lebesgue p.d.f. $\theta f(\theta x)$, where $\theta > 0$ is unknown and f is a known Lebesgue p.d.f. that is positive and twice continuously differentiable on $(0, \infty)$ with f' and f'' being the first and second order derivatives.

   (a) (3 points) Assume that
   (C1) $\int_0^\infty t|f'(t)|dt < \infty$ and $\int_0^\infty t^2|f''(t)|dt < \infty$.
   Show that $\frac{d}{d\theta}\int_0^\infty f(\theta x)dx = \int_0^\infty xf'(\theta x)dx$ and $\frac{d}{d\theta}\int_0^\infty xf'(\theta x)dx = \int_0^\infty x^2 f''(\theta x)dx$ for any $\theta > 0$.

   (b) (2 points) Assume (C1) and
   (C2) $\int_0^\infty t^2[f'(t)]^2/f(t)dt < \infty$.
   Show that the Fisher information about θ contained in X_1 is
   $$I_1(\theta) = \frac{1}{\theta^2}\left[1 + \int_0^\infty \frac{t^2[f'(t)]^2}{f(t)}dt - \int_0^\infty t^2 f''(t)dt\right]$$

   (c) (2 points) Assume (C1)-(C2) and
   (C3) $\sup_{\frac{1}{2}\theta<\gamma<\frac{3}{2}\theta} |\frac{d^2}{d\gamma^2}\log(\gamma f(\gamma x))| \le h_\theta(x)$ with $\int_0^\infty h_\theta(x)f(\theta x)dx < \infty$.
   Derive the asymptotic distribution of the MLE $\hat{\theta}$ of θ.

   (d) (2 points) When $f(t) = 2/[\pi(1+t^2)]$, show that (C3) is satisfied and identify the function $h_\theta(x)$.

   (e) (2 points) Assume
   (C4) $\mu_2 = \int_0^\infty t^2 f(t)dt < \infty$.
   Let $\bar{X} = n^{-1}\sum_{i=1}^n X_i$. Show that a moment estimator $\tilde{\theta}$ of θ is $\mu_1/\bar{X}$, where $\mu_1 = \int_0^\infty tf(t)dt$.

   (f) (3 points) Assume (C1)-(C4). Obtain the asymptotic relative efficiency of the MLE $\hat{\theta}$ with respect to the moment estimator $\tilde{\theta}$.

   (g) (3 points) Let a be the median of f, i.e., $\int_0^a f(t)dt = 1/2$. Then the median of X_1 is $m = a/\theta$. Let $\hat{m}$ be the sample median and $\check{\theta} = a/\hat{m}$. Obtain the asymptotic relative efficiency of the MLE $\hat{\theta}$ with respect to $\check{\theta}$.

2. (3 points) Let $X_1, ..., X_n$ be i.i.d. random variables having the Lebesgue p.d.f.
   $$f(x) = (3/4)[1 - (x-\theta)^2]I_{(0,1)}(|x-\theta|),$$
   where $\theta \in (-\infty, \infty)$ is an unknown parameter. Calculate the asymptotic relative efficiency of the sample mean with respect to the sample median.

Solution:

1. (a)

$$\frac{d}{d\theta}\int_0^\infty f(\theta x)dx = \lim_{\delta\to 0}\int_0^\infty \frac{f((\theta+\delta)x)-f(\theta x)}{\delta}dx$$

$$\left|\frac{f((\theta+\delta)x)-f(\theta x)}{\delta}\right| \le |xf'(\xi x)|$$

which is integrable under the condition. Hence the first result follows. The proof of the second result is similar.

(b)

$$\frac{d}{d\theta}\log(\theta f(\theta x)) = \frac{1}{\theta} + \frac{xf'(\theta x)}{f(\theta x)}$$

$$\frac{d^2}{d\theta^2}\log(\theta f(\theta x)) = -\frac{1}{\theta^2} + \frac{x^2 f''(\theta x)}{f(\theta x)} - \left[\frac{xf'(\theta x)}{f(\theta x)}\right]^2$$

Taking the expectation gives the result.

(c) Apply Theorem 4.17 and use the $I_1(\theta)$ in part (b).

(d) When $f(t) = 2/[\pi(1+t^2)]$, $f'(t) = -4t/[\pi(1+t^2)^2]$ and $f''(t) = 4(3t^2-1)/[\pi(1+t^2)^3]$. Use these result, $(\gamma x)^2 f''(\gamma x)/f(\gamma x) = 2(\gamma x)^2[3(\gamma x)^2-1]/[1+(\gamma x)^2]^2$ and $(\gamma x)f'(\gamma x)/f(\gamma x) = -2(\gamma x)^2/[1+(\gamma x)^2]$ are bounded functions. Then there is a constant $C > 0$ such that

$$\sup_{\frac{1}{2}\theta<\gamma<\frac{3}{2}\theta} |\frac{d^2}{d\gamma^2}\log(\gamma f(\gamma x))| \le C \sup_{\frac{1}{2}\theta<\gamma<\frac{3}{2}\theta} \gamma^{-2} = 4C/\theta^2$$

We can take $h_\theta(x) = 4C/\theta^2$.

(e) The result follows from

$$E(X_1) = \theta\int_0^\infty xf(\theta)dx = \mu_1/\theta$$

(f) By CLT,

$$\sqrt{n}(\bar{X} - EX_1) \to_d N(0, EX_1^2 - (EX_1)^2) = N(0, \theta^{-2}(\mu_2 - \mu_1^2))$$

By delta-method,

$$\sqrt{n}(\tilde{\theta} - \theta) = \sqrt{n}(\mu_1/\bar{X} - \mu_1/EX_1) \to_d N(0, \theta^2(\mu_2 - \mu_1^2)/\mu_1^2)$$

The relative efficiency is

$$\theta^2(\mu_2 - \mu_1^2)I_1(\theta)/\mu_1^2$$

(g) By Theorem 5.9,

$$\sqrt{n}(\hat{m} - m) \to_d N(0, 1/4\theta^2 f(a)^2)$$

By delta-method,

$$\sqrt{n}(\check{\theta} - \theta) = \sqrt{n}(a/\hat{m} - a/m) \to_d N\left(0, \theta^2/4a^2 f(a)^2\right)$$

The asymptotic relative efficiency is $\theta^2 I_1(\theta)/4a^2 f(a)^2$.

2. For the sample median $\hat{\theta}$, Theorem 5.9 gives

$$\sqrt{n}(\hat{\theta} - \theta) \to_d N\left(0, \frac{1}{4(3/4)^2}\right) = N(0, 4/9)$$

For the sample mean, we calculate

$$\mathrm{Var}(X_1) = \frac{3}{2}\int_0^1 t^2(1 - t^2)dt = \frac{1}{5}$$

Hence the ARE is $(4/9)/(1/5) = 20/9$.