STAT 710 Final Exam
10:00am-12:00noon, May 15, 2012

Please show all your work for full credits.

1. Let $X_1, ..., X_n$ be iid observations with the Lebesgue p.d.f. $(\sqrt{2\pi}\sigma)^{-1}e^{-x^2/(2\sigma^2)}$, and let $Y_1, ..., Y_n$ be iid observations with the Lebesgue p.d.f. $(2\tau)^{-1}e^{-|y|/\tau}$, where $\sigma > 0$ and $\tau > 0$ are unknown parameters and X_i's and Y_i's are independent.
   (a) (4 points) Obtain a $1-\alpha$ asymptotically correct confidence set for the 2-dimensional parameter (σ, τ) by inverting the acceptance regions of likelihood ratio tests.
   (b) (4 points) Obtain a $1 - \alpha$ asymptotically correct confidence interval for σ/τ by inverting the acceptance regions of Wald's tests.
   (c) (4 points) Obtain a $1 - \alpha$ asymptotically correct confidence interval for σ/τ by inverting the acceptance regions of Rao's score tests.
   (d) (4 points) Assume that $\tau = \sigma^2$ and consider $H_0 : \sigma^2 \leq \sigma_0^2$ versus $H_1 : \sigma^2 > \sigma_0^2$, where $\sigma_0^2 > 0$ is a constant. Show that the UMP test of size α rejects H_0 if and only if $T > c_\alpha$, where $\alpha \in (0, 1)$, c_α is a constant depending on α and T is a statistic. Show that T has a gamma distribution and provide a formula to determine c_α.
   (e) (4 points) Derive a (θ, ∞)-UMAU upper confidence bound with confidence coefficient $1 - \alpha$ for $\theta = \sigma^2/\tau$. You do not need to derive the explicit form of the confidence bound.

2. Let $(X_1, Y_1), ..., (X_n, Y_n)$ be iid bivariate random vectors having the Lebesgue p.d.f.
$$f_\theta(x, y) = \frac{1}{\pi\theta^2} I_{(0,\theta)}\left(\sqrt{x^2 + y^2}\right),$$
   where $\theta > 0$ is unknown.
   (a) (4 points) Assume that the prior for θ has a Lebesgue p.d.f. $\pi(\theta) = ac^a\theta^{-(a+1)}I_{[c,\infty)}(\theta)$, where $a > 0$ and $c > 0$ are known constants. Derive the Bayes estimator of θ under the squared error loss.
   (b) (4 points) Find a sufficient and necessary condition under which the Bayes estimator of θ in part (a) is consistent as $n \to \infty$. What is the reason why the Bayes estimator may be inconsistent?
   (c) (2 points) Consider the improper prior $\pi(\theta) = \theta^{-1}I_{(0,\infty)}(\theta)$. Derive the Bayes estimator of θ under the squared error loss and show that it is consistent.

3. Let X be an observation having the Lebesgue p.d.f. $e^{-(x-\theta)}/(1 + e^{-(x-\theta)})^2$, where $\theta \in (-\infty, \infty)$ is unknown.
   (a) (4 points) Let α_1 and α_2 be positive constants with $\alpha_1 + \alpha_2 = \alpha < 0.5$. Using the c.d.f. of X, derive a confidence interval for θ with confidence coefficient $1 - \alpha$.
   (b) (4 points) Derive the shortest confidence interval in the class of intervals of the form $(X - c, X - d)$ with confidence coefficient $1 - \alpha$.
   (c) (2 points) What is the relationship between the confidence interval in (a) and the confidence interval in (b)?