STAT 709 Third Exam
9:55am-10:45pm, Nov 23, 2016

Please show all your work for full credits.

1. Let X be a sample from $P \in \mathcal{P}$ and ϑ be a functional of P.

   (a) (4 points) Suppose that $T(X)$ is a sufficient statistic for P. Show (without using the Rao-Blackwell Theorem) that, to find the best unbiased estimator of ϑ in terms of the mean squared error, it is enough to consider functions of T.

   (b) (4 points) Suppose that $T(X)$ is a minimal sufficient statistic for P, and that there exists a $\mathcal{P}_0 \subset \mathcal{P}$ such that $\mathcal{P}_0$ has countably many elements and a.s. $\mathcal{P}_0$ is the same as a.s. $\mathcal{P}$. Show that, if there is a UMVUE of ϑ, then it must be a Borel function of T a.s. $\mathcal{P}$.

2. Let $X_1, ..., X_n$ be i.i.d. observations having a Lebesgue p.d.f.

$$f_\theta(x) = \begin{cases} \phi(x)/c(\theta) & 0 < x < \theta \\ 0 & x \leq 0 \quad \text{or} \quad x \geq \theta \end{cases}$$

   where $\phi(x)$ is a positive integrable function on $(0, \infty)$, $\theta > 0$ is an unknown parameter, and $c(\theta) = \int_0^\theta \phi(t)dt$.

   (a) (4 points) Let $g(\theta)$ be a differentiable function of θ. Show that the UMVUE of $g(\theta)$ is $T_n = g(X_{(n)}) + n^{-1}g'(X_{(n)})c(X_{(n)})/\phi(X_{(n)})$, where $X_{(n)}$ is the largest order statistic.

   (b) Assume that g' and ϕ are continuous at θ.

      (b1) (2 points) Find the asymptotic bias of $S_n = g(X_{(n)})$ as an estimator of $g(\theta)$. (Hint: Find the limiting distribution of $n[S_n - g(\theta)]$. You may use the result proved in class: $n(\theta - X_{(n)}) \to_d c(\theta)Z/\phi(\theta)$, where Z has the exponential distribution with location parameter 0 and scale parameter 1.)

      (b2) (2 points) Find the asymptotic relative efficiency of $S_n = g(X_{(n)})$ w.r.t. T_n in part (a).

   (c) (4 points) Let $s > 0$ be a fixed constant and $\vartheta = P(X_1 \leq s)$. Obtain the UMVUE of ϑ.

Solution

1. (a) Let U be an unbiased estimator of ϑ. Then $E(U|T)$ is also an unbiased estimator of θ and
$$\mathrm{Var}(U) = \mathrm{Var}(E(U|T)) + E[\mathrm{Var}(U|T)] \geq \mathrm{Var}(E(U|T))$$

   (b) Let U be a UMVUE of ϑ. From the previous expression, we know that $E[\mathrm{Var}(U|T)] = 0$ must be true for all P. Then, for every P, there exists an event N_P such that if $T \notin N_P$, $U = E(U|T)$. Since $\mathcal{P}_0$ is countable so $U = E(U|T)$ a.s. $\mathcal{P}_0$. Since a.s. $\mathcal{P}_0$ is the same as a.s. $\mathcal{P}$, U must be a Borel function of T a.s. $\mathcal{P}$.

2. (a) The joint p.d.f. of $X_1, ..., X_n$ is
$$[c(\theta)]^{-n} \prod_{i=1}^{n} \phi(x_i) I_{(0,\theta)}(x_{(n)})$$
Hence, $X_{(n)}$ is sufficient. With the same argument as the case where $\phi(x) \equiv 1$, we can show that $X_{(n)}$ is complete. The p.d.f. of $X_{(n)}$ is $n[c(\theta)]^{-n}[\int_0^x \phi(t)dt]^{n-1}\phi(x)I_{(0,\theta)}(x)$. Thus, we solve for an h such that $g(\theta) = E[h(X_{(n)})]$ for all θ, i.e.,
$$g(\theta) = n[c(\theta)]^{-n} \int_0^\theta h(x)[\int_0^x \phi(t)dt]^{n-1}\phi(x)dx$$
or
$$[c(\theta)]^n g(\theta) = n \int_0^\theta h(x)[\int_0^x \phi(t)dt]^{n-1}\phi(x)dx$$
Differentiating with respect to θ gives
$$g'(\theta)[c(\theta)]^n + n[c(\theta)]^{n-1}c'(\theta)g(\theta) = nh(\theta)[\int_0^\theta \phi(t)dt]^{n-1}\phi(\theta) = nh(\theta)[c(\theta)]^{n-1}\phi(\theta)$$
i.e.,
$$g'(t)c(t) + nc'(t)g(t) = nh(t)\phi(t)$$
Note that $c'(t) = \phi(t)$. Hence
$$h(t) = g(t) + n^{-1}g'(t)c(t)/\phi(t)$$

   (b1) In class, we stated that
$$n(\theta - X_{(n)}) \to_d c(\theta)Z/\phi(\theta)$$
By the delta method,
$$n[g(X_{(n)}) - g(\theta)] \to_d -g'(\theta)c(\theta)Z/\phi(\theta)$$
Since $EZ = 1$, the asymptotic bias of S_n is $-g'(\theta)c(\theta)/\phi(\theta)/n$.

(b2) Note that

$$n[T_n - g(\theta)] = n[g(X_{(n)}) - g(\theta)] + g'(X_{(n)})c(X_{(n)})/\phi(X_{(n)})$$

it follows from continuous mapping and Sluzsky theorems that

$$n[T_n - g(\theta)] \to_d -g'(\theta)c(\theta)Z/\phi(\theta) + g'(\theta)c(\theta)/\phi(\theta) = g'(\theta)c(\theta)(1-Z)/\phi(\theta)$$

Hence, the asymptotic relative efficiency of S_n w.r.t. T_n is

$$\frac{E[g'(\theta)c(\theta)(1-Z)/\phi(\theta)]^2}{E[g'(\theta)c(\theta)Z/\phi(\theta)]^2} = \frac{E(1-Z)^2}{EZ^2} = \frac{1}{2}$$

(c) A reasonable guess is that the UMVUE of ϑ is

$$T_n = \begin{cases} 1 & X_{(n)} \le s \\ h(X_{(n)}) & X_{(n)} > s \end{cases}$$

Then, if $s < \theta$,

$$\begin{aligned} E(T_n) &= \frac{n}{[c(\theta)]^n}\left\{\int_s^\theta h(x)\left[\int_0^x \phi(t)dt\right]^{n-1}\phi(x)dx + \int_0^s \left[\int_0^x \phi(t)dt\right]^{n-1}\phi(x)dx\right\} \\ &= \frac{n}{[c(\theta)]^n}\left\{\int_s^\theta h(x)\left[c(x)\right]^{n-1}\phi(x)dx + \int_0^s \left[c(x)\right]^{n-1}\phi(x)dx\right\} \end{aligned}$$

Since $P(X_1 \le s) = \int_0^s \phi(x)dx/c(\theta) = c(s)/c(\theta)$,

$$\frac{c(s)}{c(\theta)} = \frac{n}{[c(\theta)]^n}\left\{\int_s^\theta h(x)\left[c(x)\right]^{n-1}\phi(x)dx + \int_0^s \left[c(x)\right]^{n-1}\phi(x)dx\right\}$$

or

$$n^{-1}c(s)[c(\theta)]^{n-1} = \int_s^\theta h(x)\left[c(x)\right]^{n-1}\phi(x)dx + \int_0^s \left[c(x)\right]^{n-1}\phi(x)dx$$

Differentiating w.r.t. θ,

$$n^{-1}c(s)(n-1)[c(\theta)]^{n-2}\phi(\theta) = h(\theta)\left[c(\theta)\right]^{n-1}\phi(\theta)$$

This shows that

$$h(t) = \frac{n-1}{n}\frac{c(s)}{c(x)}, \quad s < x$$