. 2010

1.

a) Consider $A \in \sigma(\eta)$.

WTS

$$\int_B E(I_A|\xi)\,dP = \int_B P(A)\,dP \quad , \forall B \in \sigma(\xi)$$

Prof $E(I_A|\xi)$ is meas. by $(\Omega, \sigma(\xi))$.

$$\int_B E(I_A|\xi)\,dP = \int_B I_A\,dP \quad \text{(by def}^n \text{ of cond. exp)}$$

$$= \int I_A I_B\,dP$$

$$= \int I_A I_B f_{\xi,\eta}(x,y)\,d x{\times}y$$

$$= \left(\int I_A f_\eta(y)dy\right) \int I_B f_\xi(x)dx \quad \text{(Fubini)}$$

$$= P(A) \int_B dP$$

$$= \int_B P(A)\,dP \quad \forall B \in \sigma(\xi).$$

$\Rightarrow E(I_A|\xi) = P(A)$ a.s. $\forall A \in \sigma(\eta)$.

b)

WTS

$$P(B|\eta,\xi) = P(B|\eta) \quad \text{a.s. } \forall B \in \sigma(\xi)$$

Put $X = I_B$, $Y_1 = \eta$, $Y_2 = \xi$. by Prop. 1.1

the results follow.

c)

$$X_{n+1} = (Y_1 + \dots + Y_n + Y_{n+1})^2 - (n+1)\sigma^2 .$$

$$= (Y_1 + \dots + Y_n)^2 + Y_{n+1}^2 + 2(Y_1 + \dots + Y_n)Y_{n+1} - (n+1)\sigma^2 .$$

$$= X_n + Y_{n+1}^2 + 2 Y_{n+1} \sum_{i=1}^{n} Y_i - \sigma^2$$

$$\Rightarrow E(X_{n+1} | X_n) = E(X_n | X_n) + E(Y_{n+1}^2 + 2 Y_{n+1} \sum_{i=1}^{n} Y_i \,|\, X_n) - \sigma^2$$

$$= X_n + \underbrace{E(Y_{n+1}^2)}_{\sigma^2} + \underbrace{E(Y_n)}_{0} E(\sum_{i=1}^{n} Y_i | X_n) - \sigma^2$$

$$= X_n \quad a.s. \quad , \quad n = 1, 2, \dots$$

2.

a)

$$f_\theta(x_1 \dots x_n) = \left(\frac{2}{\pi}\right)^{n/2} \exp\left\{ -\frac{1}{2} \sum_{i=1}^{n} (x_i - \theta)^2 \right\} I_{(\theta, \infty)}(x_{(1)})$$

$$= \left(\frac{2}{\pi}\right)^{n/2} \exp\left\{ -\frac{1}{2} \sum x_i^2 + \theta \sum x_i + \frac{n\theta^2}{2} \right\} I_{(\theta,\infty)}(x_{(1)}) .$$

$\Rightarrow T = (\sum X_i , X_{(1)})$ is suff. by Factorization theorem.

Let $X = (X_1 \dots X_n)$, $Y = (Y_1 \dots Y_n)$ s.t.

$$\frac{f_\theta(x_1 \dots x_n)}{f_\theta(y_1 \dots y_n)} = \phi(x, y)$$

then

$$\phi(x,y) = \exp\left\{ -\frac{1}{2}\left[\sum x_i^2 - \sum y_i^2\right] + \theta \sum_{i=1}^{n} (x_i - y_i) \right\} \frac{I_{(\theta,\infty)}(x_{(1)})}{I_{(\theta,\infty)}(y_{(1)})}$$

$$\Rightarrow \quad \sum (x_i - y_i) = 0 \quad \text{and} \quad \frac{I_{(\theta,\infty)}(x_{(1)})}{I_{(\theta,\infty)}(y_{(1)})} \perp \theta$$

$\Rightarrow (\sum x_i, X_{(1)}) = (\sum Y_i, Y_{(1)})$ $\therefore$ T is minimal sufficient b)

$$F_\theta(x) = \int_\theta^x \sqrt{\tfrac{2}{\pi}}\, e^{-\frac{1}{2}(y-\theta)^2}\, dy \quad , \text{if} \quad x > \theta$$

$$= 2 \int_\theta^x \frac{1}{\sqrt{2\pi}} e^{-\frac{1}{2}(y-\theta)^2}\, dy$$

$$= 2\left[\Phi(x) - \Phi(\theta)\right]$$

$$F_\theta(x) = 0 \ , \text{if} \quad x \leq \theta .$$

$$f_{X_{(1)}}(x) = n \ F_\theta(x)^{n-1} f(x)$$

$$= 2^{n-1} n\left[\Phi(x) - \Phi(\theta)\right]^{n-1} \sqrt{\tfrac{2}{\pi}}\, e^{-\frac{1}{2}(x-\theta)^2} I_{(\theta,\infty)}(x) .$$

$$E(X_{(1)} - a) = 2^{n-1} n \sqrt{\tfrac{2}{\pi}} \int_\theta^\infty y\left[\Phi(y) - \Phi(\theta)\right]^{n-1} e^{-\frac{1}{2}(y-\theta)^2} dy .$$

$$- 2^{n-1} n \sqrt{\tfrac{2}{\pi}} \int_0^\infty y\left[1 - \Phi(y)\right]^{n-1} e^{-y^2/2}\, dy$$

$$= 2^{n-1} n \sqrt{\tfrac{2}{\pi}} \left\{ \int_\theta^\infty y\left[\Phi(y) - \Phi(\theta)\right]^{n-1} e^{-\frac{1}{2}(y-\theta)^2} dy \right.$$

$$\left. - \int_\theta^\infty (x+\theta)\left[1 - \Phi(x-\theta)\right]^{n-1} e^{-\frac{1}{2}(x-\theta)^2}\, dx \right\}$$

$\vdots$

$$= \theta \ ?$$

3. a) $$E(X_1^2) = \int_{-\infty}^{\infty} \frac{1-\epsilon}{\sqrt{2\pi}\,\theta} x^2 e^{-x^2/2\theta^2}\,dx + \int_{-\infty}^{\infty} \frac{\epsilon}{2\theta} x^2 e^{-|x|/\theta}\,dx$$

$$= (1-\epsilon)\,\theta^2 + \frac{\epsilon}{2}\left[\int_0^\infty x^2 e^{-x/\theta}\,dx + \int_{-\infty}^0 x^2 e^{-|x|/\theta}\,dx\right]$$

$$= (1-\epsilon)\theta^2 + \frac{\epsilon}{\cancel{2}}\left[\cancel{2}\theta^2 + \underbrace{\int_0^\infty y^2 e^{-y/\theta}\,dy}_{\cancel{2}\theta^2}\right]$$

$$= (1+\epsilon)\theta^2.$$

$\Rightarrow$ θ^2 can be estimated by $\hat{\theta}^2_a = \left(\frac{1}{n}\sum_{i=1}^n X_i^2\right)/(1+\epsilon)$.

b)

$$E(|X_1|) = \frac{1-\epsilon}{\theta\sqrt{2\pi}} \int_{-\infty}^{\infty} |x|\, e^{-x^2/2\theta^2}\,dx + \frac{\epsilon}{2\theta}\int_{-\infty}^{\infty} |x|\, e^{-|x|/\theta}\,dx$$

$$= \frac{1-\epsilon}{\theta\sqrt{2\pi}}\left[2\int_0^\infty x e^{-x^2/2\theta^2}\,dx\right] + \frac{\epsilon}{\theta}\int_0^\infty y\, e^{-y/\theta}\,dy$$

$$= \frac{2(1-\epsilon)}{\cancel{\theta}\sqrt{\pi}}\int_0^\infty \cancel{\theta^2}\, \cancel{e^{-y}}\,dy + \epsilon\theta \qquad \text{let } y = \frac{x^2}{2\theta^2}$$

$$= \theta\left(\frac{2-\epsilon}{\sqrt{\pi}} + \epsilon\right) \qquad dy = \frac{2x}{2\theta^2}\,dx$$

$$\Rightarrow x\,dx = \theta^2\,dy$$

$$\Rightarrow \theta^2 = E(|X_1|)^2 \Big/ \left(\frac{2-\epsilon}{\sqrt{\pi}} + \epsilon\right)^2 = \frac{\pi E(|X_1|)^2}{[2+(\sqrt{\pi}-1)\epsilon]^2}$$

$$\Rightarrow \hat{\theta}^2_b = \left(\frac{1}{n}\sum |X_i|\right)^2 \times \frac{\pi}{[2+(\sqrt{\pi}-1)\epsilon]^2}.$$

c) let $Y_i = X_i^2$, by CLT.

$$\sqrt{n}\left(\bar{Y} - E(Y_1)\right) \xrightarrow{d} N(0, \sigma_Y^2)$$

$$\bar{Y} = \frac{1}{n}\sum Y_i = \frac{1}{n}\sum X_i^2. \quad , \quad E(Y_1) = (1+\epsilon)\theta^2.$$

$$\sigma_Y^2 = Var(Y_1). = E(X_1^4) - E(X_1^2)^2.$$

$$\Rightarrow \sqrt{n}\left(\hat{\theta}_a^2 - \theta^2\right) \xrightarrow{d} N\left(0, \frac{\sigma_Y^2}{(1+\epsilon)^2}\right)$$

Let $Z_i = |X_i|$, by CLT.

$$\sqrt{n}\left(\bar{Z} - E(Z)\right) \xrightarrow{d} N(0, \sigma_Z^2)$$

$$\sigma_Z^2 = E(X_i^2) - E(|X_i|)^2.$$

By Delta's method, $g(y) = y^2$, $g'(y) = 2y$.

$$\Rightarrow \sqrt{n}\left(\bar{Z}^2 - E(Z_1)^2\right) \xrightarrow{d} N(0, 4E(Z_1)^2 \sigma_Z^2)$$

$$\Rightarrow \sqrt{n}\left(\hat{\theta}_b^2 - \theta^2\right) \xrightarrow{d} N\left(0, 4E(|X_1|)^2\left[E(X_i^2) - E(|X_1|)^2\right] \times \frac{\pi^2}{[2+(\sqrt{\pi}-1)\epsilon]^4}\right)$$

$\therefore$ asy. relative eff of $\hat{\theta}_a^2$ w.r.t. $\hat{\theta}_b^2$ is

$$\frac{Var(\hat{\theta}_b^2)}{Var(\hat{\theta}_a^2)} = \frac{E(X_1^4) - E(X_1^2)^2}{(1+\epsilon)^2} \times \frac{[2+(\sqrt{\pi}-1)\epsilon]^4}{4E(|X_1|)^2[E(X_1^2) - E(|X_1|)^2]}$$

4.

a) Order statistics is complete & sufficient.
$(X_{(1)} \dots X_{(n)})$

Consider the kernel.
$h(x_1, x_2) = \frac{1}{2}(x_1 x_2^3 + x_1^3 x_2)$, which is sym. for argument.

$E[h(X_1, X_2)] = \frac{1}{2}[E(X_1 X_2^3) + E(X_1^3 X_2)]$
$= \mu_1 \mu_3 = \gamma.$

$\therefore$ the U-statistic

$$T_n = \binom{n}{2}^{-1} \sum_{1 \le i < j \le n} h(X_i, X_j)$$

$$= \frac{2}{n(n-1)}\left[\frac{1}{2}\sum_{1 \le i < j \le n}(x_i x_j^3 + x_i^3 x_j)\right]$$

$$= \frac{1}{n(n-1)} \sum_{1 \le i \ne j \le n} x_i x_j^3$$ is UMVUE.

b) $h_1(x_1) = E[h(x_1, X_2)]$

$= E\left[\frac{1}{2}(x_1 X_2^3 + x_1^3 X_2)\right]$

$= \frac{1}{2}(x_1 \mu_3 + x_1^3 \mu_1)$

$\zeta_1 = Var(h_1(X_1)) = \frac{1}{4}[Var(X_1 \mu_3 + X_1^3 \mu_1)]$

$= \frac{1}{4}[Var(X_1 \mu_3) + Var(X_1^3 \mu_1) + 2Cov(X_1 \mu_3, X_1^3 \mu_1)]$

$$= \frac{1}{4}\left[\mu_3^2 Var(X_1) + \mu_1^2 Var(X_1^3) + 2\mu_1\mu_3 Cov(X_1, X_1^3) \right]$$

$$= \frac{1}{4}\left[\mu_3^2(\mu_2 - \mu_1^2) + \mu_1^2(\mu_6 - \mu_3^2) + 2\mu_1\mu_3(\mu_4 - \mu_1\mu_3) \right]$$

$$= \frac{1}{4}\left[\mu_2\mu_3^2 - \mu_1^2\mu_3^2 + \mu_1^2\mu_6 - \mu_1^2\mu_3^2 + 2\mu_1\mu_3\mu_4 - 2\mu_1^2\mu_3^2 \right]$$

$$= \frac{1}{4}\left[2\mu_1\mu_3\mu_4 - 4\mu_1^2\mu_3^2 + \mu_1^2\mu_6 + \mu_2\mu_3^2 \right]$$

$$= \frac{1}{2}\mu_1\mu_3\mu_4 - \mu_1^2\mu_3^2 + \frac{1}{4}(\mu_1^2\mu_6 + \mu_2\mu_3^2) \quad //$$

By Thm. 3.5 (i.)

$$\sqrt{n}(T_n - \gamma) \xrightarrow{d} N(0, 4\zeta_1) \quad \text{if } \zeta_1 > 0.$$

c) $\zeta_2 = Var(h_2(X_1, X_2)) = \frac{1}{4} Var(X_1X_2^3 + X_1^3X_2)$

$$= \frac{1}{4}\Big\{ \underbrace{E\left[(X_1X_2^3 + X_1^3X_2)^2\right]}_{①} - \Big[\underbrace{E(X_1X_2^3 + X_1^3X_2)}_{②}\Big]^2 \Big\}$$

① : $E\left[(X_1X_2^3 + X_1^3X_2)^2\right] = E(X_1^2X_2^6 + X_1^6X_2^2 + 2X_1^4X_2^4)$

$$= 2\mu_2\mu_6 + 2\mu_4^2$$

② : $2\mu_1\mu_3 = 2\gamma.$

$\Rightarrow \zeta_2 = \frac{1}{2}(\mu_2\mu_6 + \mu_4^2) - \mu_1^2\mu_3^2.$

By Thm 3.4.

$$Var(T_n) = \binom{n}{2}^{-1}\left[\binom{2}{1}\binom{n-2}{2-1}\zeta_1 + \binom{2}{2}\binom{n-2}{2-2}\zeta_2\right]$$

$$= \frac{2}{n(n-1)}\left[2(n-2)\zeta_1 + \zeta_2\right] \quad //$$

5.

a)

$$Z: \begin{bmatrix} 1 & 0 & 2 \\ 1 & 2 & 0 \\ 1 & -1 & 3 \\ 1 & 1 & 1 \end{bmatrix} \xrightarrow[R_3-R_1]{R_4-R_2} \begin{bmatrix} 1 & 0 & 2 \\ 1 & 2 & 0 \\ 0 & -1 & 1 \\ 0 & -1 & 1 \end{bmatrix} \xrightarrow[R_2-R_1]{R_4-R_3} \begin{bmatrix} 1 & 0 & 2 \\ 0 & 2 & -2 \\ 0 & -1 & 1 \\ 0 & 0 & 0 \end{bmatrix}$$

$$\rightarrow \begin{bmatrix} 1 & 0 & 2 \\ 0 & 1 & -1 \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{bmatrix} \qquad \therefore \mathcal{R}(Z) = \left\{ c_1\begin{bmatrix} 1 \\ 0 \\ 2 \end{bmatrix} + c_2\begin{bmatrix} 0 \\ 1 \\ -1 \end{bmatrix} \,\middle|\, c_1, c_2 \in \mathbb{R}^2 \right\}.$$

$$l = \begin{bmatrix} a \\ b \\ c \end{bmatrix} \quad \text{s.t.} \quad \begin{aligned} a &= c_1 \\ b &= c_2 \\ c &= 2c_1 - c_2 \end{aligned}$$

$$\overset{\text{if}}{\Leftrightarrow} \quad l = \begin{bmatrix} c_1 \\ c_2 \\ 2c_1 - c_2 \end{bmatrix}, \quad c_1, c_2 \in \mathbb{R},$$

then $l^T\beta$ is estimable. by Thm 3.6.

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b)

$$Z^T Z = \begin{bmatrix} 1 & 1 & -1 \\ 0 & -1 & 1 \end{bmatrix} \begin{bmatrix} 1 & 0 \\ 1 & -1 \\ -1 & 1 \end{bmatrix}$$

$$= \begin{bmatrix} 3 & -2 \\ -2 & 2 \end{bmatrix}$$

$$(Z^T Z)^{-1} = \frac{1}{6-4} \begin{bmatrix} 2 & 2 \\ 2 & 3 \end{bmatrix} = \begin{bmatrix} 1 & 1 \\ 1 & \frac{3}{2} \end{bmatrix}$$

$$Z(Z^T Z)^{-1} Z^T V = \begin{bmatrix} 1 & 0 \\ 1 & -1 \\ -1 & 1 \end{bmatrix} \begin{bmatrix} 1 & 1 \\ 1 & \frac{3}{2} \end{bmatrix} \begin{bmatrix} 1 & 1 & -1 \\ 0 & -1 & 1 \end{bmatrix} \begin{bmatrix} a & 0 & 0 \\ 0 & b & c \\ 0 & c & d \end{bmatrix}$$

$$= \begin{bmatrix} 1 & 1 \\ 0 & -\frac{1}{2} \\ 0 & \frac{1}{2} \end{bmatrix}_{3\times 2} \begin{bmatrix} a & b-c & c-d \\ 0 & c-b & d-c \end{bmatrix}_{2\times 3}$$

$$= \begin{bmatrix} a & 0 & 0 \\ 0 & \frac{1}{2}(b-c) & \frac{1}{2}(c-d) \\ 0 & \frac{1}{2}(c-b) & \frac{1}{2}(d-c) \end{bmatrix}$$

by Corollary 3.3 with condition (e).

$l^T \hat{\beta}$ is UMVUE iff

$Z(Z^T Z)^{-1} Z^T V$ is sym.

$\Leftrightarrow \frac{1}{2}(c-d) = \frac{1}{2}(c-b) \Leftrightarrow b = d$ //

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