Name: ____________________________________

Show sufficient work to make *very clear* your method of solution.

1. Assume $X_1, X_2, X_3, ...,$ be a sequence of independent Bernoulli(p) random variables. Let n denote the sample size.

   (a) (4 points) If $p = \frac{1}{2}$, use Chebyshev's inequality to find a lower bound for $P(0.4 < \frac{1}{100}\sum_{i=1}^{100} X_i < 0.6)$.

   (b) (4 points) If $p = \frac{1}{2}$, use the normal approximation to approximate the probability $P(0.4 < \frac{1}{100}\sum_{i=1}^{100} X_i < 0.6)$.

   (c) (4 points) If $p = 10^{-4}$ and $n = 10^4$, use the Poisson approximation to find $P(\sum_{i=1}^{10^4} X_i >= 3)$.

   (d) (4 points) Show that the PMF for each X_i belongs to an exponential family and find the natural parameter θ in terms of p and the log-partition function $A(\theta)$.

   (e) (4 points) For a sample size n, find a complete and sufficient statistic with respect to the Bernoulli(p) distribution.

   (f) (4 points) Let $\bar{X} = \frac{1}{n}\sum_{i=1}^{n} X_i$ and $p = \frac{1}{2}$. Find ν such that $n^{\nu}(2\bar{X} - 1)$ converges in distribution to a non-trivial limit. For this value of ν, what distribution does it converge to?

2. Assume $X_1, X_2, X_3, ...,$ be a sequence of independent Uniform(θ_ℓ, θ_u) random variables where $\theta_\ell < \theta_u$. Let n denote the sample size.

   (a) (4 points) Assume $\theta_\ell = 0$ and $\theta_u > 0$ and sample size n . Find a minimal sufficient statistic.

   (b) (4 points) Using your answer to part (a), find the PDF for your minimal sufficient statistic.

   (c) (4 points) Using your answer to part (a) and still assuming $\theta_\ell = 0$ and $\theta_u > 0$, is this minimal sufficient statistic complete? Either way, prove your result.

   (d) (4 points) Assume $\theta_\ell = -\frac{\theta}{2}$ and $\theta_u = \frac{\theta}{2}$ where $\theta > 0$ and sample size n. Find a minimal sufficient statistic.

   (e) (4 points) Assume $\theta_\ell = 0$ and $\theta_u = 1$ and sample size n. Let $X_{(n)} := \max(X_1, X_2, X_3, ..., X_n)$. Find a ν such that $n^{\nu}(1 - X_{(n)})$ converges in distribution to a non-trivial limit. What distribution does it converge to?

3. Assume $X_1, X_2, X_3, ...,$ be a sequence of independent MVN(μ, Σ) where $\mu \in R^p$ and $\Sigma \in R^{p\times p}$ and Σ is symmetric positive definite.

   (a) (4 points) Show that X_i belongs to an exponential family and state the sufficient statistics, natural parameters in terms of μ and Σ, and log-partition function.

   (b) (4 points) Find a matrix $A \in R^{p\times p}$ such that AX_i has mutually independent components. For this choice of A the $E[AX_i]$.

   (c) (4 points) Consider $p = 1$ and let $\bar{X} = \frac{1}{n}\sum_{i=1}^{n} X_i$ and $S^2 = \frac{1}{n-1}\sum_{i=1}^{n}(X_i - \bar{X})^2$. Prove that $\bar{X}$ and S^2 are independent.

   (d) (4 points) Consider $p = 1$ and let $\mu = 1$ and $\sigma^2 = 1$. Find a ν such that $n^{\nu}(\bar{X}^2 - 1)$ converges in distribution to a non-trivial limit. What distribution does it converge to?

   (e) (4 points) Consider $p = 1$ and let $\mu = 0$ and $\sigma^2 = 1$. Find a ν such that $n^{\nu}\bar{X}^2$ converges in distribution to a non-trivial limit. What distribution does it converge to?

4. Let $X_1, X_2, ..,$ be independent random variables with $X_i \sim$ Exponential(1), and let $N \sim$ Poisson(λ). Let $X_{(1)}^N = \min_{1 \le i \le N} X_i$. Let $S_N = \sum_{i=1}^N X_i$.

   (a) (4 points) Find a transformation $g(.)$ such that $g(X_i) \sim$ Uniform(0, 1).

   (b) (4 points) Find the conditional PDF for $X_{(1)}^N \mid N = n$.

   (c) (4 points) Find $P(X_{(1)}^N > x)$.

   (d) (4 points) Find $E[S_N]$.

   (e) (4 points) Find $E[e^{tS_N} | N = n]$.

   (f) (4 points) Find the MGF for S_N.

5. Let $X_1, ..., X_n$ be independent random variables having the double-exponential(μ, σ) distribution with PDF

$$f(x) = \frac{1}{2\sigma} e^{-|x-\mu|/\sigma}, \quad -\infty < x < \infty,$$

   where $\mu \in (-\infty, \infty)$ and $\sigma > 0$ are fixed constants. In the following, the limiting process is with respect to $n \to \infty$.

   (a) (4 points) Let $T_n = X_1 + \cdots + X_n$. Show that

$$\frac{T_n - n\mu}{\sqrt{2n}\sigma} \text{ converges in distribution to } N(0, 1)$$

   (b) (4 points) If $\mu \neq 0$, show that

$$\sqrt{n}\left[\log(T_n/n)^2 - \log \mu^2\right] \text{ converges in distribution to } N(0, 8\sigma^2/\mu^2)$$

   (c) (4 points) If $\mu = 0$, show that

$$\log(T_n^2/n) \text{ converges in distribution to } \log Y + \log(2\sigma^2),$$

   where Y is a random variable having the chi-square distribution with one degree of freedom.

   (d) (4 points) Suppose that $\mu = 0$ and $\sigma = 1$. Let $W_n = \min_{i=1,\ldots,n} |X_i|$. Show that W_n converges in probability to 0, but nW_n does not converge in probability to 0.

   (e) (4 points) Assume σ is fixed and μ is unknown. Let $X = (X_1, X_2, ..., X_n)$. Find an ancillary statistic.

   (f) (4 points) Assume both μ and σ are unknown. Let $X = (X_1, X_2, ..., X_n)$. Find an ancillary statistic.