STAT 709 Final Exam
10:00am-12:00pm, Dec 20, 2016

Please show all your work for full credits.

1. Let $X_1, X_2, ...$ and X be integrable random variables on a probability space, and Y be another random variable on the same space. Assume that $0 \le X_1 \le X_2 \le \cdots \le X$.

   (a) (3 points) Show that the sequence $\{E(X_n|Y)\}$ is uniformly integrable.

   (b) (3 points) If $\lim_{n\to\infty} X_n = X$ a.s., show that $E(X|Y) = \lim_{n\to\infty} E(X_n|Y)$ a.s. Prove it directly, without using Proposition 1.10(x), the dominated convergence theorem for conditional expectations; you can use the general dominated convergence theorem or monotone convergence theorem.

2. Let $X_1, ..., X_n$ be i.i.d. from a population with Lebesgue p.d.f.

$$f_\theta(x) = \begin{cases} \frac{1}{\sqrt{2\pi}\sigma_1} e^{-x^2/(2\sigma_1^2)} & x > 0 \\ \frac{1}{\sqrt{2\pi}\sigma_2} e^{-x^2/(2\sigma_2^2)} & x \le 0 \end{cases}$$

   where $\theta = (\sigma_1, \sigma_2)$ and $\sigma_j > 0$, $j = 1, 2$, are unknown parameters.

   (a) (5 points) Derive estimators of σ_1 and σ_2, using the method of moments.

   (b) (3 points) Show that (N, T_1, T_2) is a minimal sufficient statistic for θ, where $N = \sum_{i=1}^n I_{(0,\infty)}(X_i)$, $T_1 = n^{-1}\sum_{i=1}^n X_i^2 I_{(0,\infty)}(X_i)$, and $T_2 = n^{-1}\sum_{i=1}^n X_i^2 I_{(-\infty,0]}(X_i)$.

   (c) (4 points) Let $\bar{X}$ be the sample mean. Suppose that $\sigma_2 = 1$. Show that $\sqrt{2\pi}\bar{X} + 1$ is the moment estimator of σ_1 and derive the asymptotic distribution of $\sqrt{n}(\sqrt{2\pi}\bar{X} + 1 - \sigma_1)$.

   (d) (4 points) Assume the conditions in (c). Let $S = \sqrt{2T_1}$ be an estimator of σ_1. Obtain the asymptotic relative efficiency of S with respect to $\sqrt{2\pi}\bar{X} + 1$.

3. Let $X_1, ..., X_n$, $n \ge 2$, be i.i.d. random variables with finite 4th moments, $\mu = E(X_1)$, $\sigma^2 = \text{Var}(X_1)$, $\tau = E|X_1|$, and $\gamma = E(X_1|X_1|)$.

   (a) (3 points) Find a U-statistic U_n such that $E(U_n) = \tau\mu$ and obtain the asymptotic distribution of $\sqrt{n}(U_n - \tau\mu)$ in terms of μ, σ, τ, and γ.

   (b) (3 points) Obtain the V-statistic V_n corresponding to U_n in part (a). Show directly that $n(U_n - V_n) \to_p$ a constant so that $\sqrt{n}(V_n - \tau\mu)$ has the same asymptotic distribution as $\sqrt{n}(U_n - \tau\mu)$.

   (**There is another problem on page 2**)

4. Consider a linear model

$$X = Z\beta + \varepsilon,$$

where $E(\varepsilon) = 0$ and $\mathrm{Var}(\varepsilon)$ is finite.

(a) (3 points) Suppose that ε is multivariate normal and $\mathrm{Var}(\varepsilon) = \sigma^2 I_6$ (the identity matrix of order 6), and

$$Z = \begin{pmatrix} 1 & 0 & 1/2 \\ 1 & 1 & -1/2 \\ 1 & 2 & -3/2 \\ 1 & -1 & 3/2 \\ 1 & 1/2 & 0 \\ 1 & -1/2 & 1 \end{pmatrix}$$

Obtain the forms of all $l \in \mathcal{R}^3$ such that $l^\tau \beta$ is estimable. If

$$l = \begin{pmatrix} 1 \\ -1 \\ 1 \end{pmatrix}$$

Is $l^\tau \beta$ estimable?

(b) (3 points) Suppose that $Z = (A \;\; B)$, where A and B are submatrices of Z having full ranks and satisfying $A^\tau B = 0$. If $\mathrm{Var}(\varepsilon) = I + AA^\tau + BB^\tau$, where I is the identity matrix, show that the LSE is UMVUE.

(c) (3 points) Suppose that, for each $n = 3, 4, ...,$

$$Z = \begin{pmatrix} 1 & t_{n1} \\ 1 & t_{n2} \\ \vdots & \vdots \\ 1 & t_{nn} \end{pmatrix}$$

where t_{ni}'s are real numbers such that $\sum_{i=1}^n t_{ni} = 0$ and $\sum_{i=1}^n t_{ni}^2 > 0$. Obtain an explicit form of the LSE of β (a 2-dimensional vector).

(d) (3 points) Assume the conditions in (c). Let β_2 be the second component of β and $\hat{\beta}_2$ be its LSE. Suppose that components of ε are i.i.d. and

$$\lim_{n\to\infty} \max_{i\le n} t_{ni}^2 \Big/ \sum_{i=1}^n t_{ni}^2 = 0.$$

Prove

$$\frac{\hat{\beta}_2 - \beta_2}{\sqrt{\mathrm{Var}(\hat{\beta}_2)}} \to_d N(0,1)$$

using Corollary 1.3 or Lindeberg's CLT directly (but not using Theorem 3.12).

Solution:

1. (a) Since $0 \le X_n \le X$, $0 \le E(X_n|Y) \le E(X|Y)$ a.s. and hence

$$E(|E(X_n|Y)|I_{\{|E(X_n|Y)|\ge t\}}) \le E(|E(X|Y)|I_{\{|E(X|Y)|\ge t\}})$$

for all n and $t > 0$. Since $E|E(X|Y)| \le E|X| < \infty$, the right hand side $\to 0$ as $t \to \infty$. The result follows.

(b) This is exercise 1.85. Let $\sigma(Y) = \mathcal{A}$. Since each $E(X_n|\mathcal{A})$ is measurable from $(\Omega, \mathcal{A})$ to $(\mathcal{R}, \mathcal{B})$, so is the limit $\lim_n E(X_n|\mathcal{A})$. We need to show that

$$\int_A \lim_n E(X_n|\mathcal{A}) dP = \int_A X dP$$

for any $A \in \mathcal{A}$. By Exercise 35, $0 \le E(X_1|\mathcal{A}) \le E(X_2|\mathcal{A}) \le \cdots \le E(X|\mathcal{A})$ a.s. By the monotone convergence theorem, for any $A \in \mathcal{A}$,

$$\int_A \lim_n E(X_n|\mathcal{A}) dP = \lim_n \int_A E(X_n|\mathcal{A}) dP = \lim_n \int_A X_n dP = \int_A \lim_n X_n dP = \int_A X dP.$$

2. (a) Note that

$$E(X_1 I_{(0,\infty)(X_1)}) = \frac{1}{\sqrt{2\pi}\sigma_1} \int_0^\infty x e^{-x^2/(2\sigma_1^2)} dx = \frac{\sigma_1}{\sqrt{2\pi}}$$

$$E(X_1 I_{(-\infty,0](X_1)}) = \frac{1}{\sqrt{2\pi}\sigma_2} \int_{-\infty}^0 x e^{-x^2/(2\sigma_2^2)} dx = -\frac{\sigma_2}{\sqrt{2\pi}}$$

Hence

$$E(X_1) = \frac{\sigma_1 - \sigma_2}{\sqrt{2\pi}}$$

Similarly,

$$E(X_1^2 I_{(0,\infty)(X_1)}) = \frac{1}{\sqrt{2\pi}\sigma_1} \int_0^\infty x^2 e^{-x^2/(2\sigma_1^2)} dx = \frac{\sigma_1^2}{2}$$

$$E(X_1^2 I_{(-\infty,0](X_1)}) = \frac{1}{\sqrt{2\pi}\sigma_2} \int_{-\infty}^0 x^2 e^{-x^2/(2\sigma_1^2)} dx = \frac{\sigma_2^2}{2}$$

Let $Y = T_1 + T_2 = n^{-1} \sum_{i=1}^n X_i^2$. Then we set

$$\bar{X} = \frac{\sigma_1 - \sigma_2}{\sqrt{2\pi}} \qquad Y = \frac{\sigma_1^2 + \sigma_2^2}{2}$$

Solving these equations, we get

$$\hat{\sigma}_1 = \frac{\sqrt{4Y - 2\pi\bar{X}^2} + \sqrt{2\pi}\bar{X}}{2} \qquad \hat{\sigma}_2 = \frac{\sqrt{4Y - 2\pi\bar{X}^2} - \sqrt{2\pi}\bar{X}}{2}$$

(b) The joint p.d.f. of $(X_1, ..., X_n)$ is

$$\frac{1}{(2\pi)^{n/2}} \exp\left\{ -\frac{nT_1}{2\sigma_1^2} - \frac{nT_2}{2\sigma_2^2} + N\log(\sigma_1/\sigma_2) + n\log\sigma_2 \right\}$$

This is a curved exponential family. Using the result for curved exponential family, a minimal sufficient statistic for θ is (N, T_1, T_2).

(c) Since $E(X_1) = (\sigma_1 - 1)/\sqrt{2\pi}$, $\sqrt{2\pi}\bar{X} + 1$ is the moment estimator of σ_1. From the CLT, $\sqrt{n}(\sqrt{2\pi}\bar{X} + 1 - \sigma_1) \to_d N(0, 2\pi\mathrm{Var}(X_1))$,

$$\mathrm{Var}(X_1) = E(X_1^2) - [E(X_1)]^2 = \frac{\sigma_1^2 + \sigma_2^2}{2} - \frac{(\sigma_1 - \sigma_2)^2}{2\pi}$$

$$2\pi\mathrm{Var}(X_1) = \pi(\sigma_1^2 + 1) - (\sigma_1 - 1)^2 = (\pi - 1)(\sigma_1^2 + 1) + 2\sigma_1$$

(d) By the CLT,

$$\sqrt{n}(2T_1 - \sigma_1^2) \to_d N(0, 4a),$$

$$\begin{aligned} a = \mathrm{Var}(X_1^2 I_{(0,\infty)}(X_1)) &= E[X_1^4 I_{(0,\infty)}(X_1)] - [E(X_1^2 I_{(0,\infty)}(X_1))]^2 \\ &= 3\sigma_1^4/2 - (\sigma_1^2/2)^2 = 5\sigma_1^4/4 \end{aligned}$$

By the delta method,

$$\sqrt{n}(S - \sigma_1) \to_d N(0, 4a/(2\sqrt{\sigma_1^2})^2) = N(0, 5\sigma_1^2/4)$$

Hence, the ARE is

$$\frac{(\pi - 1)(\sigma_1^2 + 1) + 2\sigma_1}{5\sigma_1^2/4}$$

3. (a) Consider kernel $h(x_1, x_2) = (x_1|x_2| + x_2|x_1|)/2$, the U-statistic is

$$U_n = \frac{1}{n(n-1)} \sum_{i \neq j} X_i|X_j|$$

with $h_1(X_1) = (\tau X_1 + \mu|X_1|)/2$ and

$$\begin{aligned} \zeta_1 = \mathrm{Var}(h(X_1)) &= \frac{\tau^2\mathrm{Var}(X_1) + \mu^2\mathrm{Var}(|X_1|) + 2\mu\tau\mathrm{Cov}(X_1, |X_1|)}{4} \\ &= \frac{\tau^2\sigma^2 + \mu^2(\sigma^2 + \mu^2 - \tau^2) + 2\mu\tau(\gamma - \mu\tau)}{4} \end{aligned}$$

Then $\sqrt{n}(U_n - \mu\tau) \to_d N(0, 4\zeta_1)$.

(b) Note that

$$V_n = \frac{1}{n^2}\sum_{i=1}^n \sum_{j=1}^n X_i|X_j| = \frac{n-1}{n}U_n + \frac{1}{n^2}\sum_{i=1}^n X_i|X_i|$$

Then

$$n(U_n - V_n) = U_n - \frac{1}{n}\sum_{i=1}^n X_i|X_i|$$

By part (a) and the WLLN, the right hand side $\to_p \mu\tau - \gamma$.

4. (a) Any linear combinations of $(1, 0, 1/2)$ and $(1, 1/2, 0)$. No, because if $c(1, 0, 1/2) + d(1, 1/2, 0) = (1, -1, 1)$, then $c + d = 1$, $d/2 = -1$, $c/2 = 1$, which is not possible.

(b) Note that

$$Z^\tau Z = \begin{pmatrix} A^\tau A & 0 \\ 0 & B^\tau B \end{pmatrix}, \qquad (Z^\tau Z)^{-1} = \begin{pmatrix} (A^\tau A)^{-1} & 0 \\ 0 & (B^\tau B)^{-1} \end{pmatrix},$$

$$Z(Z^\tau Z)^{-1}Z^\tau = A(A^\tau A)^{-1}A^\tau + B(B^\tau B)^{-1}B^\tau$$

Then

$$Z(Z^\tau Z)^{-1}Z^\tau \mathrm{Var}(\varepsilon) = [A(A^\tau A)^{-1}A^\tau + B(B^\tau B)^{-1}B^\tau](I + AA^\tau + BB^\tau)$$

$$A(A^\tau A)^{-1}A^\tau + B(B^\tau B)^{-1}B^\tau + AA^\tau + BB^\tau$$

which is symmetric.

(c) Note that

$$Z^\tau Z = \begin{pmatrix} n & 0 \\ 0 & \sum_i t_{ni}^2 \end{pmatrix}$$

Hence

$$\hat{\beta} = (Z^\tau Z)^{-1}Z^\tau X = \begin{pmatrix} n^{-1} & 0 \\ 0 & (\sum_i t_{ni}^2)^{-1} \end{pmatrix} \begin{pmatrix} \sum_i X_i \\ \sum_i t_{ni}X_i \end{pmatrix} = \begin{pmatrix} \bar{X} \\ \sum_i t_{ni}X_i / \sum_i t_{ni}^2 \end{pmatrix}$$

(d) Let $s = \sum_i t_{ni}^2$. Then $\hat{\beta}_2 = \sum_i t_{ni}X_i/s$, $\mathrm{Var}(\hat{\beta}_2) = \sigma^2/s$. Since

$$\frac{\hat{\beta}_2 - \beta_2}{\sqrt{\mathrm{Var}(\hat{\beta}_2)}} = \frac{\sum_i t_{ni}X_i/s - \beta_2}{\sqrt{\mathrm{Var}(\sum_i t_{ni}X_i/s)}}$$

we can apply Corollary 1.3 with $c_{ni} = t_{ni}/s$.