STAT 710 Final Exam
10:05am-12:05pm, May 12, 2010

Please show all your work for full credits.

1. Let $X_1, ..., X_n$ be iid observations with the Lebesgue p.d.f. $e^{-(x-\theta)}I_{(\theta,\infty)}(x)$, where $\theta > 0$ is an unknown parameter. Let $\pi(\theta) = ae^{-a\theta}I_{(0,\infty)}(\theta)$ be the Lebesgue p.d.f. of the prior for θ, where $a > 0$ is a hyperparameter.
   (a) (3 points) Obtain the posterior Lebesgue p.d.f. of θ given the data.
   (b) (4 points) Under the squared error loss, obtain the Bayes estimator of θ when a is known.
   (c) (4 points) Show that the Bayes estimator in (b) is consistent when $n \to \infty$.
   (d) (3 points) Obtain the empirical Bayes estimator of θ under the squared error loss, using the moment method.
   (e) (4 points) Obtain a $1 - \alpha$ HPD credible interval for θ when a is known.

2. Let $(X_1, Y_1), ..., (X_n, Y_n)$ be iid random vectors from the bivariate normal distribution with density

$$\frac{1}{2\pi\sigma_x\sigma_y\sqrt{1-\rho^2}} \exp\left\{-\frac{x^2}{2\sigma_x^2(1-\rho^2)} + \frac{\rho xy}{\sigma_x\sigma_y(1-\rho^2)} - \frac{y^2}{2\sigma_y^2(1-\rho^2)}\right\},$$

   where $\sigma_x > 0$, $\sigma_y > 0$ and $\rho \in (-1, 1)$ are unknown parameters. Let $S_x^2 = n^{-1}\sum_{i=1}^n X_i^2$, $S_y^2 = n^{-1}\sum_{i=1}^n Y_i^2$, $S_{xy} = n^{-1}\sum_{i=1}^n X_iY_i$, and $R = S_{xy}/\sqrt{S_x^2S_y^2}$.
   (a) (4 points) Consider testing $H_0 : \rho = 0$ vs $H_1 : \rho \neq 0$. Show that a UMPU test of size α rejects H_0 if and only if $|R| > c$ (you do not need to give the value of c).
   (b) (3 points) Let $\theta = \sigma_y/\sigma_x$. Show that

$$W(\theta) = |\theta^2S_x^2 - S_y^2| \Big/ \sqrt{(\theta^2S_x^2 + S_y^2)^2 - 4\theta^2S_{xy}^2}$$

   is a pivotal quantity for constructing a confidence interval for θ. Hint: consider the transformation

$$\begin{pmatrix} A_i \\ B_i \end{pmatrix} = \begin{pmatrix} \theta & 1 \\ \theta & -1 \end{pmatrix} \begin{pmatrix} X_i \\ Y_i \end{pmatrix}$$

   (c) (3 points) Show that the confidence set $C = \{W(\theta) \leq c\}$ with $W(\theta)$ given in (b) is UMAU.
   (d) (4 points) Assume that $\sigma_x = \sigma_y = \sigma$. Obtain the MLE of (σ^2, ρ).
   (e) (3 points) Assume that $\sigma_x = \sigma_y = \sigma$. Show that the likelihood ratio test for $H_0 : \rho = 0$ vs $H_1 : \rho \neq 0$ is the same as the UMPU test.
   (f) (5 points) Assume that $\sigma_x = \sigma_y = 1$. Derive Rao's score test statistic for testing $H_0 : \rho = 0$ vs $H_1 : \rho \neq 0$.