STAT 709 First Exam
9:55am-10:45pm, Sept 28, 2016

Please show all your work for full credits.

1. Let f, f_n, $n = 1, 2, ...,$ be Borel functions on a measurable space and assume that $f = \lim_{n\to\infty} f_n$ and $\sigma(f_n) \subset \sigma(f)$ for any n. Show that

$$\sigma(f) = \bigcap_{n=1}^{\infty} \sigma\left(\bigcup_{k=n}^{\infty} \sigma(f_k)\right)$$

2. Let X and Y be two independent random variables, and X be integer-valued.

   (a) Show that
   $$F_{X+Y}(t) = \int F_Y(t-x)dF_X(x)$$

   (b) Assume that F_Y has a Lebesgue p.d.f. f_Y. Show that $X+Y$ has a Lebesgue p.d.f. given as
   $$f_{X+Y}(t) = \int f_Y(t-x)dF_X(x)$$

3. Let X and Y be random variables such that $Y > 0$ and given $Y = y$, X is distributed as $N(\mu, y^{-2})$, where μ is a fixed constant. Assume that the m.g.f. of Y is finite in a neighborhood of 0.

   (a) Using the properties of conditional expectation, show that $E(XY) = \mu E(Y)$.

   (b) Derive the ch.f. of XY. Show that by differentiating the ch.f., you can obtain the same result as in (a) for $E(XY)$.

4. Let X, Y, Z be three random variables defined on a probability space and let $F_{Y|Z}(\cdot)$ be the conditional c.d.f. of Y given Z corresponding to the conditional probability of Y given Z defined in Theorem 1.7. Assume that $E|X| < \infty$. From the definition of the conditional expectation, show that

$$E[E(X|Y)|Z] = \int E(X|Y=y)dF_{Y|Z}(y) \quad a.s.$$

Solution:

1. Note that $f = \limsup_n f_n$. It was shown in homework problem 1.19 that

$$\sigma(f) \subset \bigcap_{n=1}^{\infty} \sigma\left(\bigcup_{k=n}^{\infty} \sigma(f_k)\right)$$

Because $\sigma(f_n) \subset \sigma(f)$ for any n, $\sigma(f) \supset \sigma(f_1, ..., f_k)$ for any k. Also, by home work 1.19, $\sigma(f) \supset \sigma(f_n, f_{n+1}, ...)$ for any n. The result follows.

2.

$$P(X+Y \le t) = \sum_x P(X=x, X+Y \le t) = \sum_x P(X=x, Y \le t-x)$$

$$= \sum_x P(X=x)P(Y \le t-x) = \int P(Y \le t-x)dF_X(x)$$

For any s,

$$\int_{-\infty}^{s} \int f_Y(t-x)dF_X(x)dt = \int \int_{-\infty}^{s} f_Y(t-x)dtdF_X(x)$$

$$= \int \int_{-\infty}^{s-x} f_Y(u)dudF_X(x) = \int F_Y(s-x)dF_X(x) = F_{X+Y}(s)$$

3. $E(XY) = E[E(XY|Y)] = E[YE(X|Y)] = E(Y\mu) = \mu E(Y)$.

$$Ee^{\sqrt{-1}tXY} = E[E(e^{\sqrt{-1}tXY}|Y)] = E[e^{\mu tY - t^2/2}] = e^{-t^2/2}Ee^{\mu tY} = e^{-t^2/2}\psi(\mu t)$$

where ψ is the m.g.f. of Y.

4. From Theorem 1.7 and Fubini Theorem, $\int E(X|Y=y)dF_{Y|Z}(y)$ is $\sigma(Z)$ measurable. Let $Z^{-1}(B) \in \sigma(Z)$. Then

$$\int_{Z^{-1}(B)} \int E(X|Y=y)dF_{Y|Z}(y)dP_Z = \int_{Z^{-1}(B)} \int E(X|Y)dP_{Y|Z}dP_Z = \int_{Z^{-1}(B)} E(X|Y)dP$$

so that the 2nd requirement is satisfied.