STAT 710
Second Exam, March 9, 2016

All 8 parts of the problem are equally weighted. Please show all your work for full credits.

Let $X_1, ..., X_n$ be i.i.d. with normal distribution $N(\theta, \theta^2)$, where θ is unknown and $\theta \neq 0$.

1. Let $\ell(\theta)$ be the likelihood function. Show that

$$\frac{d \log \ell(\theta)}{d\theta} = -\frac{n}{\theta^3}\left(\theta^2 + \theta\bar{X} - \bar{X}^2 - T^2\right),$$

   where $\bar{X} = n^{-1}\sum_{i=1}^n X_i$ and $T^2 = n^{-1}\sum_{i=1}^n (X_i - \bar{X})^2$.

2. Show that the likelihood equation $\frac{d\log\ell(\theta)}{d\theta} = 0$ has two solutions given by

$$\hat{\theta}_+ = \frac{1}{2}\left(-\bar{X} + \sqrt{5\bar{X}^2 + 4T^2}\right) \quad \text{and} \quad \hat{\theta}_- = \frac{1}{2}\left(-\bar{X} - \sqrt{5\bar{X}^2 + 4T^2}\right)$$

3. When $\theta > 0$, show that $\hat{\theta}_+$ is a consistent sequence and $\hat{\theta}_-$ is inconsistent. Obtain a similar result for the case of $\theta < 0$.

4. Obtain the asymptotic distribution of $\hat{\theta}_+$ when $\theta > 0$.

5. Obtain the asymptotic relative efficiency of $\hat{\theta}_+$ with respect to the sample mean $\bar{X}$ when $\theta > 0$.

6. Obtain the asymptotic relative efficiency of $\hat{\theta}_+$ with respect to the sample median when $\theta > 0$.

7. Suppose that $\theta > 0$, $E(X_i) = \theta$, and $\text{Var}(X_i) = \theta^2$, but the distribution of X_i is not necessarily normal. Show that

$$\sum_{i=1}^n \left(\theta^2 + \theta X_i - X_i^2\right) = 0$$

   is a generalized estimation equation (GEE) and $\hat{\theta}_+$ is a solution to the GEE and is still consistent.

8. Suppose that $\text{Var}(X_i^2) = a\theta^4$ and $\text{Cov}(X_i, X_i^2) = b\theta^3$ for some constant $a > 0$ and b. Obtain the asymptotic distribution of the GEE estimator $\hat{\theta}_+$ in the previous part.