STAT 850
SPRING 2018
MIDTERM

Q1	2/1/1/2/2	= 8
Q2	1/1/1/1/2/4	= 10
Q3	4	= 4
Q4	1/1/1/1/3	= 8

1) y_{ijk} : k^{th} mouse on diet D_i and exercise treatment j.

a) $E(y_{211}) = E(y_{212}) = \begin{bmatrix} 1 & -1 & 1 \end{bmatrix} \begin{bmatrix} \beta_1 \\ \beta_2 \\ \beta_3 \end{bmatrix}$

$= \beta_1 - \beta_2 + \beta_3$

b) $X^T X = \begin{bmatrix} 8 & 0 & 0 \\ 0 & 8 & 0 \\ 0 & 0 & 8 \end{bmatrix}$ $(X^T X)^{-1} = \frac{1}{8} I_{3\times3}$

$Y_{...} = \sum_i \sum_j \sum_k y_{ijk}$

$X^T y = \begin{bmatrix} Y_{...} \\ Y_{1..} - Y_{2..} \\ Y_{.1.} - Y_{.2.} \end{bmatrix}$

$\hat{\beta} = (X^T X)^{-1} X^T Y = \begin{bmatrix} \bar{Y}_{...} \\ (\bar{Y}_{1..} - \bar{Y}_{2..})/2 \\ (\bar{Y}_{.1.} - \bar{Y}_{.2.})/2 \end{bmatrix}$

c) $[1\ -1\ 1]\begin{pmatrix}\hat\beta_1\\ \hat\beta_2\\ \hat\beta_3\end{pmatrix} = \hat\beta_1 - \hat\beta_2 + \hat\beta_3$

$$= \bar{Y}_{\cdot\cdot\cdot} - \left(\frac{\bar{Y}_{1\cdot\cdot} - \bar{Y}_{2\cdot\cdot}}{2}\right) + \left(\frac{\bar{Y}_{\cdot1\cdot} - \bar{Y}_{\cdot2\cdot}}{2}\right)$$

$$= \frac{\bar{Y}_{1\cdot\cdot}}{2} + \frac{\bar{Y}_{2\cdot\cdot}}{2} - \frac{\bar{Y}_{1\cdot\cdot}}{2} + \frac{\bar{Y}_{2\cdot\cdot}}{2} + \frac{\bar{Y}_{\cdot1\cdot}}{2} - \frac{\bar{Y}_{\cdot2\cdot}}{2}$$

$$= \bar{Y}_{2\cdot\cdot} + \frac{\bar{Y}_{\cdot1\cdot}}{2} - \frac{\bar{Y}_{\cdot2\cdot}}{2} - \frac{\bar{Y}_{\cdot1\cdot}}{2} + \frac{\bar{Y}_{\cdot1\cdot}}{2}$$

$$= \bar{Y}_{2\cdot\cdot} + \bar{Y}_{\cdot1\cdot} - \bar{Y}_{\cdot\cdot\cdot}$$

d)

	No exercise	Exercise	
Diet 1	$\beta_1+\beta_2+\beta_3$	$\beta_1+\beta_2-\beta_3$	$\beta_1+\beta_2$
Diet 2	$\beta_1-\beta_2+\beta_3$	$\beta_1-\beta_2-\beta_3$	$\beta_1-\beta_2$

$H_0: \beta_1+\beta_2 = \beta_1-\beta_2 \iff \boxed{H_0: \beta_2 = 0}$

e) No diet effect implies $2\beta_1 + 2\beta_3 = 2\beta_1 \Rightarrow \beta_3 = 0$

$A\beta = C$ $\boxed{\begin{array}{l} A = [0\ \ 0\ \ 1] \\ C = 0 \end{array}}$

2

a) μ_1 is the mean for treatment 1.

The design matrix

$$X = \begin{bmatrix} 1 & 0 & 0 & 0 & 0 \\ 1 & 0 & 0 & 0 & 0 \\ 1 & 1 & 0 & 0 & 0 \\ 1 & 1 & 0 & 0 & 0 \\ 1 & 0 & 1 & 0 & 0 \\ 1 & 0 & 1 & 0 & 0 \\ 1 & 0 & 0 & 1 & 0 \\ 1 & 0 & 0 & 1 & 0 \\ 1 & 0 & 0 & 0 & 1 \\ 1 & 0 & 0 & 0 & 1 \end{bmatrix} \Rightarrow \hat{\mu}_1 = 348.007$$

b) μ_2 is the mean for treatment 2.

$$\hat{\mu}_2 = \widehat{\text{intercept}} + \widehat{\text{dose}}\,2 = 343.418$$

c) $\hat{\mu}_2 = \frac{y_{21} + y_{22}}{2} = \bar{y}_{2\cdot}$

$$Var(\hat{\mu}_2) = \frac{\sigma^2}{2} \qquad \hat{\sigma}^2 = \frac{SSE}{n-r} = \frac{353.5}{5} = 8.40$$

Thus, $SE(\hat{\mu}_2) = \sqrt{\widehat{Var}(\hat{\mu}_2)} = \sqrt{\frac{\hat{\sigma}^2}{2}} = \boxed{5.94}$

$\Rightarrow$ You could have also used

$SE(\hat{\mu}_1) = \cdots = SE(\hat{\mu}_5)$ due to balanced design.

$SE(\hat{\mu}_2) = SE(\hat{\mu}_1) = SE(\widehat{\text{intercept}}) = \boxed{5.94}$

Read off from R output

d) $\widehat{Dose}$ is an estimate of $\mu_2 - \mu_1$.
Thus, a t-statistic for testing $H_0: \mu_1 = \mu_2$
is given as $\boxed{-0.546}$ with $\boxed{\text{p-value} = 0.608663}$
from t statistic with $\boxed{5}$ degrees of freedom.
$\Rightarrow$ There is no significant evidence of a difference
between μ_1 & μ_2.

e) The t-statistic for testing
$H_0: \mu_3 = \mu_4$ is $t = \dfrac{\bar{y}_{3\cdot} - \bar{y}_{4\cdot}}{\sqrt{MSE\left(\frac{1}{2}+\frac{1}{2}\right)}} = \dfrac{\bar{y}_{3\cdot} - \bar{y}_{4\cdot}}{\sqrt{MSE}}$

From the R output

$$t = \frac{-6.217 - (-9.491)}{\sqrt{\frac{353.5}{5}}} = 0.389$$

Thus, the F stat is $(0.389)^2 = \boxed{\underline{0.152}} \sim F_{1,5}$

$\Rightarrow$ Cannot reject the null

Or from the R output

$$\left[\left(-6.217 - (-9.491)\right)/8.408\right]^2 = \boxed{0.152}$$

Under M2

7) $\mu_1 = \beta_0 + \beta_1 \cdot 0$

$\mu_2 = \beta_0 + \beta_1 \cdot 2$

$\mu_3 = \beta_0 + \beta_1 4$

$\mu_4 = \beta_0 + \beta_1 \cdot 8$

$\mu_5 = \beta_0 + \beta_1 \cdot 16$

$\Longleftrightarrow$

$\mu_2 - \mu_1 = 2\beta_1$

$\mu_3 - \mu_2 = 2\beta_1$

$\mu_4 - \mu_3 = 4\beta_1$

$\mu_5 - \mu_4 = 8\beta_1$

$\Rightarrow \mu_2 - \mu_1 = \mu_3 - \mu_2$ and $\mu_4 - \mu_3 = 2(\mu_3 - \mu_2)$

$\Rightarrow \boxed{-\mu_1 + 2\mu_2 - \mu_3 = 0}$ $\Rightarrow \boxed{2\mu_2 - 3\mu_3 + \mu_4 = 0}$

$\Rightarrow \mu_5 - \mu_4 = 2(\mu_4 - \mu_3) \Leftrightarrow 2\mu_3 - 3\mu_4 + \mu_5 = 0$

$$A = \begin{bmatrix} -1 & 2 & -1 & 0 & 0 \\ 0 & 2 & -3 & 1 & 0 \\ 0 & 0 & 2 & -3 & 1 \end{bmatrix} \quad C = \begin{bmatrix} 0 \\ 0 \\ 0 \end{bmatrix}$$

3) – Doubling the sample size will double the noncentrality parameter. Halving σ will cut σ^2 by a factor of 4 and quadruple the noncentrality parameter.

– We've enough degrees of freedom in both cases that the difference in error df will not have much effect on the result. Thus, halving σ will give the greater increase in power

(4)

a) $SSE = (5-1)\times 2 + (5-1)\times 2 + (5-1)\times 1 + (5-1)\times 2 + (5-1)\times 3$

$= 40$

$MSE = \frac{SSE}{20} = 2$

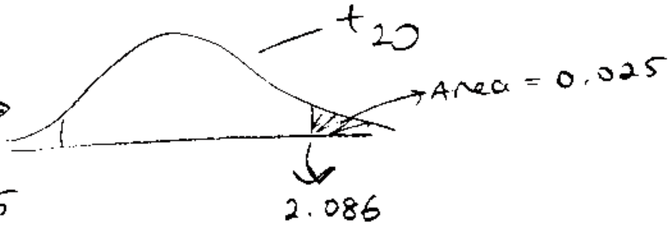
b) $c = [0 \ -1 \ 0 \ 1 \ 0]^T$

$\hat{L} = y_{4\cdot} - y_{2\cdot} = (3) - (-1) = 4$

$SS_L = \frac{n \hat{L}^2}{\sum_i c_i^2} = \frac{5 \times 4^2}{0 + (-1)^2 + 0 + 1^2} = 40$

$F = \frac{SSL}{MSE} = \frac{40}{2} = 20 \sim F_{1,20}$

$t = \sqrt{20} = 4.472 \longleftrightarrow$



$\boxed{\text{Reject Ho}}$ at $\alpha = 0.05$

c) By Scheffe's

Compare $\underbrace{|y_{4\cdot} - y_{2\cdot}|}_{4}$ with $\underbrace{\sqrt{\underbrace{(t-1)}_{4}\underbrace{\hat{\sigma}^2}_{2}\underbrace{\left(\frac{1}{n_4} + \frac{1}{n_2}\right)}_{2/5}\underbrace{F_{0.05,4,20}}_{2.87}}}_{3.03}$

$4 > 3.03$

$\boxed{\text{Reject Ho}}$ at $\alpha = 0.05$

d) Tukey's critical value

$\frac{1}{\sqrt{2}}\underbrace{q(0.05, 5, 20)}_{4.232}\ \underbrace{\hat{\sigma}}_{\sqrt{2}}\ \underbrace{\sqrt{\frac{1}{n_4} + \frac{1}{n_2}}}_{\sqrt{2/5}} \cong 2.67$

$4 > 2.67$

$\boxed{\text{Reject Ho at } \alpha = 0.05}$

e) Note that $t_{\frac{0.05}{2\times 10}, 20} = 3.153$

$$\underbrace{t_{\frac{0.05}{2\times 10}, 20}}_{3.153} \cdot \hat{\sigma}\sqrt{\frac{1}{n_2}+\frac{1}{n_4}} = 2.82$$

$4 > 2.82$

$\boxed{\text{Reject } H_0}$ at $\alpha = 0.05$

f) Tukey is most powerful when we compare the critical values.