STAT 710 Final Exam
10:00am-12:00noon, May 8, 2018

1. Let $X_1, ..., X_n$ be i.i.d. having the Lebesgue p.d.f.

$$f_{\theta_1,\theta_2}(x) = \begin{cases} \frac{1}{\theta_1+\theta_2} e^{-x/\theta_1} & x \geq 0 \\ \frac{1}{\theta_1+\theta_2} e^{x/\theta_2} & x < 0 \end{cases}$$

where $\theta_1 > 0$ and $\theta_2 > 0$ are unknown parameters and $n \geq 2$. Let $X_{i+} = X_i I_{(0,\infty)}(X_i)$, $X_{i-} = -X_i I_{(-\infty,0]}(X_i)$, $\bar{X}_+ = n^{-1} \sum_{i=1}^n X_{i+}$, and $\bar{X}_- = n^{-1} \sum_{i=1}^n X_{i-}$.

   (a) (4 points) Show that a UMPU test of size $\alpha \in (0, 0.5)$ for testing $H_0 : \theta_1 = \theta_2$ versus $H_1 : \theta_1 \neq \theta_2$ rejects H_0 if and only if $V < c_1$ or $V > c_2$, where $V = n\bar{X}_+ / \sum_{i=1}^n |X_i|$ and show how to determine c_1 and c_2 in terms of the p.d.f of V.

   (b) (3 points) Show that the MLE of (θ_1, θ_2) is $(\bar{X}_+ + \sqrt{\bar{X}_+\bar{X}_-}, \bar{X}_- + \sqrt{\bar{X}_+\bar{X}_-})$. (You need to verify this is indeed an MLE.)

   (c) (3 points) Derive the MLE of $\theta_1 - \theta_2$ and obtain the non-degenerate asymptotic distribution of this MLE.

   (d) (3 points) For testing $H_0 : \theta_1 - \theta_2 = \phi_0$ versus $H_1 : \theta_1 - \theta_2 \neq \phi_0$, where ϕ_0 is a constant, find a function $R(\theta_1, \theta_2)$ such that H_0 is equivalent to $R(\theta_1, \theta_2) = 0$. Then, construct Wald's test statistic for H_0 versus H_1.

   (e) (3 points) For testing the hypotheses in part (a) derive the likelihood ratio $\lambda(X)$ and show that it is a function of $\bar{X}_+/\bar{X}_-$ (defined to be ∞ if $\bar{X}_- = 0$). Show that the likelihood ratio test is equivalent to the UMPU test in part (a).

   (f) (4 points) Let $\hat{m}$ be the sample median based on $X_1, ..., X_n$. Derive the non-degenerate asymptotic distribution of $\hat{m}$. When $\theta_1 = \theta_2$, obtain the asymptotic relative efficiency of $\hat{m}$ w.r.t. the MLE in part (c).

2. Let $X_1, ..., X_n$ be i.i.d. having the Lebesgue p.d.f.

$$f_\theta(x) = \begin{cases} a\theta x^{a-1} e^{-\theta x^a} & x > 0 \\ 0 & x \leq 0 \end{cases}$$

where $\theta > 0$ is unknown, $a > 0$ is known and $n \geq 2$.

   (a) (4 points) Let the prior for θ to be the gamma distribution with known shape parameter $\alpha > 0$ and scale parameter $\gamma > 0$. Under the squared error loss, obtain the Bayes estimators of θ and θ^{-1}.

   (b) (4 points) Show that θT is a pivotal quantity, where $T = \sum_{i=1}^n X_i^a$, and derive the shortest length confidence interval of the form $(c_1 T^{-1}, c_2 T^{-1})$ and confidence coefficient $1 - \alpha$, where c_1 and c_2 are positive constants.

   (c) (6 points) For a given α, derive $1-\alpha$ asymptotically correct confidence intervals of θ by inverting acceptance regions of likelihood ratio tests, Wald tests, and Rao's score tests.

   (d) (4 points) Obtain a UMAU confidence interval for θ with confidence coefficient $1 - \alpha$.

   (e) (2 points) Which interval in part (b) and part (d) has shorter expected length? Does your conclusion contradict to UMAU and Theorem 7.6 (Pratt's theorem)?

1. (a) The likelihood is

$$\frac{1}{(\theta_1+\theta_2)^n}\exp\left(-\frac{n}{\theta_1}\bar{X}_+ - \frac{n}{\theta_2}\bar{X}_-\right) = \frac{1}{(\theta_1+\theta_2)^n}\exp\left\{-\left(\frac{n}{\theta_1}-\frac{n}{\theta_2}\right)\bar{X}_+ - \frac{n}{\theta_2}(\bar{X}_+ + \bar{X}_-)\right\}$$

Then the result follows from Theorem 6.4 and Lemma 6.7, since V is independent of $U = \bar{X}_+ + \bar{X}_- = n^{-1}\sum_i |X_i|$ when $\theta_1 = \theta_2$. The constants c_1 and c_2 satisfy

$$\int_{c_1}^{c_2} h(v)dv = 1-\alpha \quad \text{and} \quad \int_{c_1}^{c_2} vh(v)dv = (1-\alpha)\int_{c_1}^{c_2} vh(v)dv$$

where h is the p.d.f. of V.

(b) The score function is

$$s_n(\theta_1,\theta_2) = \begin{pmatrix} -\frac{n}{\theta_1+\theta_2} + \frac{n\bar{X}_+}{\theta_1^2} \\ -\frac{n}{\theta_1+\theta_2} + \frac{n\bar{X}_-}{\theta_2^2} \end{pmatrix}$$

If $\bar{X}_+ \neq 0$ and $\bar{X}_- \neq 0$, then we get solution

$$\hat{\theta} = (\hat{\theta}_1, \hat{\theta}_2) = (\bar{X}_+ + \sqrt{\bar{X}_+\bar{X}_-}, \bar{X}_- + \sqrt{\bar{X}_+\bar{X}_-})$$

Since the likelihood is bounded and the score equation has a unique solution, the solution must be the MLE.
If $\bar{X}_+ = 0$, then as a function of θ_1 the likelihood is strictly decreasing and thus 0 is the MLE of θ_1. Similarly, if $\bar{X}_- = 0$, then 0 is the MLE of θ_2. In any case, $\hat{\theta}$ is the MLE.

(c) The MLE of $\theta_1 - \theta_2$ is

$$\hat{\theta}_1 - \hat{\theta}_2 = \bar{X}_+ - \bar{X}_- = \bar{X}$$

by part (b), where $\bar{X}$ is the sample mean. To get the asymptotic distribution of $\bar{X}$, we can either apply Theorem 4.17 with the Fisher information

$$I_n(\theta_1,\theta_2) = \frac{n}{(\theta_1+\theta_2)^2}\begin{pmatrix} 1+\frac{2\theta_2}{\theta_1} & -1 \\ -1 & 1+\frac{2\theta_1}{\theta_2} \end{pmatrix}$$

or directly apply the CLT to $\bar{X}$,

$$\sqrt{n}(\bar{X} - (\theta_1-\theta_2)) \to_d N\left(0, \frac{2(\theta_1^3+\theta_2^3)}{\theta_1+\theta_2} - (\theta_1-\theta_2)^2\right)$$

(d) $R(\theta_1,\theta_2) = \theta_1 - \theta_2 - \phi_0$. Wald's test is

$$W_n = n(\bar{X}-\phi_0)^2 \Big/ \left(\frac{2(\hat{\theta}_1^3+\hat{\theta}_2^3)}{\hat{\theta}_1+\hat{\theta}_2} - \bar{X}^2\right)$$

(e) The MLE under H_0 is $\tilde{\theta} = n^{-1}\sum_{i=1}^n |X_i| = \bar{X}_+ + \bar{X}_-$. Then the likelihood ratio is

$$\lambda(X) = \frac{1}{(2\tilde{\theta})^n}\exp\left(-\frac{n}{\tilde{\theta}}\bar{X}_+ - \frac{n}{\tilde{\theta}}\bar{X}_-\right) \Big/ \frac{1}{(\hat{\theta}_1+\hat{\theta}_2)^n}\exp\left(-\frac{n}{\hat{\theta}_1}\bar{X}_+ - \frac{n}{\hat{\theta}_2}\bar{X}_-\right)$$

From the definitions of $\tilde{\theta}$, $\hat{\theta}_1$ and $\hat{\theta}_2$, $\lambda(X)$ is a function of $\bar{X}_+/\bar{X}_-$.

(f) Let m be the true median of the distribution of X_1. By Theorem 5.10,

$$\sqrt{n}(\hat{m}-m) \to_d N\left(0, \frac{1}{4f^2_{\theta_1,\theta_2}(m)}\right)$$

where

$$f_{\theta_1,\theta_2}(m) = \begin{cases} \frac{1}{\theta_1+\theta_2}e^{-m/\theta_1} & \theta_1 < \theta_2 \\ \frac{1}{\theta_1+\theta_2}e^{-m/\theta_2} & \theta_1 > \theta_2 \\ \frac{1}{\theta_1+\theta_2} & \theta_1 = \theta_2 \end{cases}$$

When $\theta_1 = \theta_2$, the ARE is 2.

2. (a) The posterior is the gamma distribution with shape parameter $n+\alpha$ and scale parameter $(\sum_{i=1}^n X_i^a + \gamma^{-1})^{-1}$. Thus the Bayes estimator of θ is $(n+a\alpha)/(\sum_{i=1}^n X_i^a + \gamma^{-1})$ and the Bayes estimator of θ^{-1} is $(\sum_{i=1}^n X_i^a + \gamma^{-1})/(n+\alpha-1)$.

(b) θX_i^a has the exponential distribution $E(0,1)$. Hence $\theta\sum_{i=1}^n X_i^a = \theta T$ has the gamma distribution with shape parameter n and scale parameter 1. Then we can apply Theorem 7.3.

(c) The likelihood ratio statistic is

$$\frac{\theta^n e^{-\theta T}}{(n/T)^n e^{-n}}$$

By Theorem 6.5, the confidence set based on the likelihood ratio is

$$\left\{\theta : \theta^n e^{-\theta T} \le e^{-\chi^2_{1,\alpha}/2}(n/T)^n e^{-n}\right\}$$

which is an interval. The score function is

$$s_n(\theta) = \frac{n}{\theta} - T$$

and

$$I_n(\theta) = \frac{n}{\theta^2}$$

Let $\hat{\theta} = n/T$ be the MLE of θ. The confidence interval based on Wald's tests is

$$\left\{\theta : I_n(\hat{\theta})(\hat{\theta}-\theta)^2 \le \chi^2_{1,\alpha}\right\} = \left\{\theta : n^{-1}T^2(n/T-\theta)^2 \le \chi^2_{1,\alpha}\right\}$$

The confidence interval based on Rao's tests is

$$\left\{\theta : [I_n(\theta)]^{-1}[s_n(\theta)]^2 \le \chi^2_{1,\alpha}\right\} = \left\{\theta : n^{-1}\theta^2(n/\theta-T)^2 \le \chi^2_{1,\alpha}\right\}$$

which is exactly the same as the interval from Wald's tests.

(d) By Theorem 6.4, the UMPU test of size α for testing $H_0 : \theta = \theta_0$ versus $H_1 : \theta \neq \theta_0$ has acceptance region

$$A(\theta_0) = \{c_1(\theta_0) \leq T \leq c_2(\theta_0)\}$$

or

$$A(\theta_0) = \{\theta_0 c_1(\theta_0) \leq \theta_0 T \leq \theta_0 c_2(\theta_0)\}$$

Since θT is pivotal, $\theta_0 c_j(\theta_0) = d_j$, with

$$\int_{d_1}^{d_2} h(t)dt = 1 - \alpha \qquad \int_{d_1}^{d_2} th(t)dt = (1 - \alpha)n$$

where h is the p.d.f. of the gamma distribution with shape parameter n and scale parameter 1. Then the confidence interval is $(d_1 T^{-1}, d_2 T^{-1})$.

(e) The interval in (b) is shorter and thus has shorter expected length. The interval in (b) is not unbiased and thus this does not contradict to Theorem 7.6.