Statistics 709, Exam 1

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Be sure to show all relevant work !

1. Suppose that X is a random variable on probability space $(\Omega, \mathcal{F}, P)$ and has a pdf $e^{-x}1(x > 0)$ w.r.t. Lebesgue measure m. Define $Y = min(X, 1)$. Y has a pdf $h(x)$ w.r.t. $m + \delta_1$, where δ_1 is a point mass measure at 1.

   (a) (2 points) Provide an expression for $h(x)$. No need to prove that the provided expression is the pdf.

   (b) (2 points) Prove
   $$\int_\Omega Y dP = \int_R yh(y)d(m + \delta_1),$$
   where $R = (-\infty, \infty)$. State the formula(s) used in your proof.

2. Suppose that X_i is a random variable on probability space $(\Omega_i, \mathcal{F}_i, P_i)$, $\int_{\Omega_i} |X_i| dP_i < \infty$, $i = 1, 2$. Assume $\Omega_1 \cap \Omega_2 = \emptyset$, $\Omega = \Omega_1 \cup \Omega_2$, $\mathcal{F} = \{A_1 \cup A_2, A_i \in \mathcal{F}_i, i = 1, 2\}$.

   (a) (3 points) Show that $\mathcal{F}$ is a σ-field on Ω.

   (b) (3 points) For $A = A_1 \cup A_2$, $A_i \in \mathcal{F}_i$, define $P(A) = [P_1(A_1) + P_2(A_2)]/2$. Show that P is a probability measure on $(\Omega, \mathcal{F})$.

   (c) (3 points) Let
   $$X(\omega) = \begin{cases} X_1(\omega) & \omega \in \Omega_1 \\ X_2(\omega) & \omega \in \Omega_2 \end{cases}$$
   Show that X is a random variable on $(\Omega, \mathcal{F}, P)$.

   (d) (4 points) Assume that $\mathcal{A}_i$ is a σ-field on Ω_i, and $\mathcal{A}_i \subset \mathcal{F}_i$, $i = 1, 2$. Let $\mathcal{A} = \{A_1 \cup A_2, A_i \in \mathcal{A}_i, i = 1, 2\}$. Show
   $$E_P[X|\mathcal{A}] = \begin{cases} E_{P_1}[X_1|\mathcal{A}_1] & \text{on } \Omega_1 \\ E_{P_2}[X_2|\mathcal{A}_2] & \text{on } \Omega_2 \end{cases} \qquad a.s. P(\text{almost surely w.r.t.} P)$$
   Here for a given probability measure Q, we denote by E_Q the (conditional) expectation taken w.r.t. Q.

3. (3 points) Suppose that X is a random variable on a probability space $(\Omega, \mathcal{F}, P)$,$E|X| < \infty$, and events $A_1, \cdots, A_n \in \mathcal{F}$, $A_i \cap A_j = \emptyset$ for $i \neq j$, and $\cup_{i=1}^n A_i = \Omega$. Let $\mathcal{A} = \sigma\{A_1, \cdots, A_n\}$, and $Y = E[X|\mathcal{A}]$. Show that if X and Y have the same distribution, then $X = Y$ a.s.

Solution 1 (a)

$$h(x) = \begin{cases} 0, & y \le 0 \text{ or } y > 1 \\ e^{-x}, & 0 < y < 1 \\ e^{-1}, & y = 1 \end{cases}$$

(b) Two formulas are used to prove

$$\int_\Omega Y dP = \int_R y d(P \circ Y^{-1}) = \int_R y h(y) d(m + \delta_1),$$

where the first equality is due to the change variable formula (thm 1.2): with $f = Y, g(u) = u, \nu = P, \nu \circ f^{-1} = P \circ Y^{-1}$,

$$\int_\Omega g \circ f d\nu = \int_\Lambda g d(\nu \circ f^{-1}),$$

and the second equality is due to the calculus with Radon-Nikodym derivative (prop 1.7 (i)): with $\nu = m + \delta_1$, $\lambda = P \circ Y^{-1}$, and $\frac{d\lambda}{d\nu} = h$,

$$\int f d\lambda = \int f \frac{d\lambda}{d\nu} d\nu.$$

Solution 2 (a) (i) $\emptyset \in \mathcal{F}$, (ii) $A = A_1 \cup A_2$, $A^c = (\Omega_1 - A_1) \cup (\Omega_2 - A_2) \in \mathcal{F}$, (iii) $A_n = A_{1n} \cup A_{2n}$, $\cup_n A_n = (\cup_n A_{1n}) \cup (\cup_n A_{2n}) \in \mathcal{F}$.

(b) $P(A) \ge 0$, $P(\emptyset) = 0$, $P(\Omega) = 1$. Given $A_n = A_{1n} \cup A_{2n}$, $A_n \cap A_m = \emptyset$ for $m \neq n$, then $A_{in} \cap A_{im} = \emptyset$, $i = 1, 2$, and

$$P_i\left(\cup_n A_{in}\right) = \sum_n P_i(A_{in}),$$

add up to obtain

$$P\left(\cup_n A_n\right) = \sum_n P(A_n).$$

(c) For Borel set B, $X^{-1}(B) = X_1^{-1}(B) \cup X_2^{-1}(B) \in \mathcal{F}$, since $X_i^{-1}(B) \in \mathcal{F}_i$.

(d) The same argument in (a) proves that $\mathcal{A}$ is a sigma field on Ω. Denote the right hand side by W. First as $E_{P_i}[X_i|\mathcal{F}_i]$ is $\mathcal{A}_i$-measurable, $i = 1, 2$, the same method in (c) shows that W is $\mathcal{A}$-measurable. For $A = A_1 \cup A_2 \in \mathcal{A}$, $A_i \in \mathcal{A}_i$,

$$\int_A X dP = \int_{A_1} X dP + \int_{A_2} X dP = \frac{1}{2}\left[\int_{A_1} X_1 dP_1 + \int_{A_2} X_2 dP_2\right]$$

$$= \frac{1}{2}\left[\int_{A_1} E_{P_1}[X_1|\mathcal{A}_1] dP_1 + \int_{A_2} E_{P_2}[X_2|\mathcal{A}_2] dP_2\right]$$

$$= \frac{1}{2}\left[\int_{A_1} W dP_1 + \int_{A_2} W dP_2\right] = \int_{A_1} W dP + \int_{A_2} W dP = \int_A W dP,$$

so W is the conditional expectation of X given $\mathcal{A}$ w.r.t. P.

Solution 3

$$Y = \sum_{i=1}^n \frac{E[X 1_{A_i}]}{P(A_i)} 1_{A_i}$$

Let $a_i = E[X1_{A_i}]/P(A_i)$. Without loss of generality, we may assume $a_1 < a_2 < \cdots <$ (otherwise we may merge some A_i and re-order a_i), and

$$Y = \sum_j a_j 1_{A_j},$$

which has a discrete distribution, taking value a_j with probability $p_j = P(A_j)$, $j = 1, 2 \cdots$. Thus X also takes value a_j with probability $p_j = P(X = a_j)$. Let $B_j = [X = a_j]$, then

$$X = \sum_j a_j 1_{B_j}.$$

As $Y = E[X|\mathcal{A}]$,

$$\int_{A_1} X dP = \int_{A_1} Y dP,$$

$$\sum_j a_j P(A_1 \cap B_j) = a_1 P(A_1) \Longrightarrow \sum_j (a_j - a_1) P(A_1 \cap B_j) = 0,$$

Since $a_1 < a_2 < \cdots$, $P(A_1 \cap B_j) = 0$ for $j = 2, 3, \cdots$, which together with $P(A_1) = P(B_1)$ implies $P(A_1 \Delta B_1) = 0$, that is, $1_{A_1} = 1_{B_1}$ a.s. Applying the same argument to $A_2, A_3, \cdots$, we can show that $1_{A_j} = 1_{B_j}$ a.s., which indicates $X = Y$ a.s.

Alternative method, X is discrete and takes values a_j with probability p_j, then $E(X^2) = \sum_{j=1}^n a_j^2 p_j < \infty$, $E(Y^2) = E(X^2) < \infty$, then $E[XY] = E[E(XY|\mathcal{A})] = E[YE(X|\mathcal{A})] = E[Y^2]$, and

$$E[X - Y]^2 = E(X^2) + E(Y^2) - 2E(XY) = E[X^2] - E[Y^2] = 0, \qquad X = Y a.s.$$

However, for $n = \infty$, $E(X^2) = \sum_{j=1}^\infty a_j^2 p_j$ can not be proved to be finite.