Statistics 709, Exam 1

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Be sure to show all relevant work !

1. (3 points) Consider a probability space $(\Omega, \mathcal{F}, P)$ and $A_1, \cdots, A_n \in \mathcal{F}$, where $A_i \cap A_j = \emptyset$ for $i \neq j$, $\cup_{j=1}^n A_j = \Omega$, and $P(A_j) > 0$. Let $\mathcal{G} = \sigma\{A_1, \cdots, A_n\}$ be the sigma field generated by $A_1, \cdots, A_n$. Assume that X is a random variable, and set $Y = P(X \geq 1|\mathcal{G})$. It can be shown that
$$Y = \sum_{j=1}^n \frac{P([X \geq 1] \cap A_j)}{P(A_j)} 1_{A_j} \quad \text{(you do not need to prove the result).}$$
Explain the meaning of $Y(\omega)$ for $\omega \in \Omega$.

2. Suppose that X and Y are two independent random variables on a probability space $(\Omega, \mathcal{F}, P)$. X is a Bernoulli random variable $P(X = 1) = P(X = 0) = 0.5$, and Y follows a uniform distribution on $(0, 1)$ with pdf $f(x)$ w.r.t. the Lebesgue measure m: $f(x) = 1$ for $x \in (0, 1)$ and zero otherwise. Let $\lambda = m + \delta_0$, where δ_0 denotes the point mass measure at 0.

   (a) (3 points) Show that the distribution of $\min(X, Y)$ is absolutely continuous w.r.t. λ.

   (b) (3 points) Specify the pdf of $\min(X, Y)$ w.r.t. λ. You do not need to provide the proof.

3. Let $\Omega = [0, 1]^2$, and $\mathcal{F} = \mathcal{B}^2_{[0,1]^2}$ be a Borel σ-field on the unit square Ω. Denote by m the one dimensional Lebesgue measure, and $\mathcal{B}_{[0,1]}$ the Borel σ-field on $[0, 1]$. For $B \in \mathcal{F}$, define $\mathcal{L}(B) = \{x : (x.x) \in B\}$ to be a subset of $[0, 1]$, that is, $\mathcal{L}(B)$ is the projection of $B \cap \Omega_0$ on each coordinate, where $\Omega_0 = \{(x, x) : x \in [0, 1]\}$ represents the diagonal line of Ω.

   (a) (2 points) Show that for any $B \in \mathcal{F}$, $\mathcal{L}(B) \in \mathcal{B}_{[0,1]}$.

   (b) (2 points) For $B \in \mathcal{F}$, define $\lambda(B) = m(\mathcal{L}(B))$. Prove that λ is a probability on $(\Omega, \mathcal{F})$.

   (c) (4 points) Prove that for any Borel function $f(x, y)$ on $(\Omega, \mathcal{F})$, $f(x, x)$ is a Borel function on $([0, 1], \mathcal{B}_{[0,1]})$. If $\int_0^1 |f(x, x)| m(dx) < \infty$, and $f(x, y)$ is integrable w.r.t λ, show
   $$\int_\Omega f(x, y) d\lambda = \int_0^1 f(x, x) m(dx).$$

   (d) (3 points) Suppose that $g_n(x, y)$, $n \geq 1$, are non-negative Borel functions on $(\Omega, \mathcal{F})$ satisfying $\int_\Omega g_n(x, y) d\lambda = 1$, and $h(x)$ is a positive Borel function on $([0, 1], \mathcal{B}_{[0,1]})$. Assume that as $n \to \infty$, $g_n(x, y)$ converges to $h(x)/h(y)$ for $(x, y) \in \Omega$. Prove
   $$\lim_{n\to\infty} \int_\Omega |g_n(x, y) - h(x)/h(y)| d\lambda = 0.$$

Solution 1

Since
$$P(X \geq 1|A_j) = \frac{P([X \geq 1] \cap A_j)}{P(A_j)},$$
For $\omega \in A_j$, $Y(\omega)$ is the conditional probability of $[X \geq 1]$ given event A_j

Solution 2

(a) For any $B \in \mathcal{B}$ and $\lambda(B) = 0$, we have $0 \notin B$, and $m(B) = 0$. Let $Z = \min(X, Y)$. $P_Z(B) = P(\min(X, Y) \in B) = P(X = 0 \in B) + P(X = 1, Y \in B) = 0 + P(X = 1)P(Y \in B) = \frac{1}{2}m(B \cap (0, 1)]) = 0$. So $P_Z \ll \lambda$.

(b) It has a pdf w.r.t. λ
$$f_\lambda(z) = \frac{1}{2}1_{[z=0]} + \frac{1}{2}1_{[0<z<1]} = \frac{1}{2}1_{[0\leq z<1]}.$$

Solution 3

(a) Notice that $\mathcal{L}$ preserves set operations. Define $\mathcal{P} = \{B \in \mathcal{B}^2_{[0,1]^2}, \mathcal{L}(B) \in \mathcal{B}_{[0,1]}\}$. Then $\mathcal{P}$ is a σ-field that contains all open rectangles in $[0, 1]^2$ (check). Then $\mathcal{F} = \mathcal{B}^2_{[0,1]^2} \subset \mathcal{P}$.

(b) To prove λ is a probability measure, just notice that if $B_1, B_2, .., ..$are disjoint, then $\mathcal{L}(B_1), \mathcal{L}(B_2), .., ..$are also disjoint and $\mathcal{L}(\cup B_i) = \cup \mathcal{L}(B_i)$.

(c) Define $\mathcal{C} = \{B : B = \{(x, y) : 0 \leq x \leq t, 0 \leq y \leq s\}, t, s \in [0, 1]\}$which are all closed rectangles start from left corner and is a π-system. We also define
$$\mathcal{D} = \left\{B \in \mathcal{B}^2_{[0,1]^2} : \int_{[0,1]^2} 1_B d\lambda = \int_0^1 1_{\mathcal{L}(B)} dx\right\}$$
which is a $\lambda-$ system that contains $\mathcal{C}$ (check). Note that $\mathcal{B}^2_{[0,1]^2}$ can be generated from $\mathcal{C}$, so from $\pi - \lambda$ theorem, all statements are true for $f = 1_B$, $B \in \mathcal{B}^2_{[0,1]^2}$. By the property of Borel function and integration, all statements are also true for simple functions. For nonnegative Borel function $f(x, y)$, there exists monotone increasing simple functions $f_n(x, y)$ such that $f_n(x, y) \to f(x, y)$, $n \to \infty$ point-wise. Since $f_n(x, x)$ is a Borel function and $f(x, x) = \lim_{n\to\infty} f_n(x, x)$, we conclude $f(x, x)$ is a Borel function. $\int_{[0,1]^2} f(x, y)d\lambda = \int_0^1 f(x, x)dx$ follows from MCT. For general Borel function $f(x, y)$,write $f(x, y) = f^+(x, y) - f^-(x, y)$. Then we conclude $f(x, x)$ is a Borel function and $\int_{[0,1]^2} f^+(x, y)d\lambda = \int_0^1 f^+(x, x)dx$, $\int_{[0,1]^2} f^-(x, y)d\lambda = \int_0^1 f^-(x, x)dx$. Then $f(x, y)$ is integrable w.r.t λ if and only if $\int_0^1 |f(x, x)|dx < \infty$, in which case we have
$$\int_{[0,1]^2} f(x, y)d\lambda = \int_0^1 f(x, x)dx$$

(d) From (c) we know $1 = \int_\Omega g_n(x,y)d\lambda = \int_0^1 g_n(x,x)m(dx)$, and

$$\int_\Omega |g_n(x,y) - h(x)/h(y)|d\lambda = \int_0^1 |g_n(x,x) - h(x)/h(x)|m(dx) = \int_0^1 |1 - g_n(x,x)|m(dx)$$

$$= 2\int_0^1 [1 - g_n(x,x)]_+ m(dx) - \int_0^1 [1 - g_n(x,x)]m(dx) = 2\int_0^1 [1 - g_n(x,x)]_+ m(dx),$$

which converges to zero by DCT (as $[1 - g_n(x,x)]_+ \le 1$).