STAT 710 First Exam
9:55am-10:45am, Feb. 14, 2014

Please show all your work for full credits.

1. Let $Y_1, ..., Y_n$ be independent binary observations such that $P(Y_i = 1) = e^{\theta x_i}/(1+e^{\theta x_i})$, where θ is a real-valued unknwon parameter and x_i's are known covariate values.

   (a) (2 points) Write down the likelihood function $\ell(\theta)$ for a given set of $Y_i = y_i$, x_i, $i = 1, ..., n$.

   (b) (2 points) Show that the score function $s(\theta) = d \log \ell(\theta)/d\theta$ is a strictly monotone function of θ.

   (c) (2 points) Suppose that the values of x_i are all positive. Show that $s(\theta) = 0$ has a unique solution, provided that y_i's are not all equal to 1 or all equal to 0.

   (d) (2 points) Suppose that, for all i, $x_i = 0$ or 1, and y_i's with x_i's =1 are not all equal to 1 or all equal to 0. Obtain explicitly the MLE of θ.

2. Let $X_1, ..., X_n$ be i.i.d. observations having the discrete distribution

$$P(X_1 = x) = \begin{cases} \theta & x = 1 \\ \theta^{x-2}(1-\theta)^2 & x = 2, 3, ..., \end{cases}$$

   where $0 < \theta < 1$ is an unknown parameter.

   (a) (3 points) Consider a prior p.d.f. $\pi(\theta) = \frac{\Gamma(\alpha+\beta)}{\Gamma(\alpha)\Gamma(\beta)}\theta^{\alpha-1}(1-\theta)^{\beta-1}I_{(0,1)}(\theta)$ (w.r.t. Lebesgue measure) with known hyperparameters $\alpha > 0$ and $\beta > 0$. Under the squared error loss, derive the Bayes estimator of θ in terms of an explicit function of the hyperparameters, the sample mean $\bar{X}$ and K/n, where K is the number of X_i's that are equal to 1.

   (b) (3 points) Show that the generalized Bayes estimator of θ under the squared error loss and improper prior $\pi(\theta) = \theta^{-1}(1-\theta)^{-1}I_{(0,1)}(\theta)$ w.r.t. the Lebesgue measure is equal to the MLE of θ.

   (c) (2 points) Comment on whether you can conclude that the estimators in (a) and (b) are admissible under the squared error loss. Give your reason if there is a definite answer.

3. Consider the estimation of a two dimensional $\theta = (\theta_1, \theta_2)$ such that the risk for any estimator $T = (T_1, T_2)$ is $R_T(\theta) = R_{T_1}(\theta_1) + R_{T_2}(\theta_2)$, where $R_{T_j}(\theta_j)$ is the risk function of T_j as an estimator of θ_j, $j = 1, 2$.

   (a) (2 points) Show that $\sup_\theta R_T(\theta) = \sup_{\theta_1} R_{T_1}(\theta_1) + \sup_{\theta_2} R_{T_2}(\theta_2)$ for any estimator $T = (T_1, T_2)$.

   (b) (2 points) Suppose that $\sup_{\theta_j} R_{T_j^*}(\theta_j) < \infty$, $j = 1, 2$, for an estimator $T^* = (T_1^*, T_2^*)$. Show that T^* is minimax as an estimator of θ if and only if T_j^* is minmax as an estimator of θ_j, $j = 1, 2$.

Solution

1. (a)

$$\ell(\theta) = \prod_{i=1}^{n} \frac{e^{\theta x_i y_i}}{1 + e^{\theta x_i}}$$

(b)

$$s(\theta) = \frac{d \log \ell(\theta)}{d\theta} = \sum_{i=1}^{n} x_i \left(y_i - \frac{e^{\theta x_i}}{1 + e^{\theta x_i}} \right)$$

$$s'(\theta) = \frac{d^2 \log \ell(\theta)}{d\theta^2} = -\sum_{i=1}^{n} x_i^2 \frac{e^{-\theta x_i}}{(1 + e^{-\theta x_i})^2} < 0$$

(c)

$$\lim_{\theta \to \infty} s(\theta) = \sum_{i=1}^{n} x_i (y_i - 1) < 0 \quad \text{since } y_i\text{'s are not all 1}$$

$$\lim_{\theta \to -\infty} s(\theta) = \sum_{i=1}^{n} x_i y_i > 0 \quad \text{since } y_i\text{'s are not all 0}$$

Hence, $s(\theta) = 0$ has a unique solution.

(d) Let n_{jk} be the number of i's such that $y_i = j$ and $x_i = k$, $j, k = 0, 1$. Then

$$s(\theta) = \frac{n_{11}}{1 + e^{\theta}} - \frac{n_{01} e^{\theta}}{1 + e^{\theta}}$$

Then $s(\theta) = 0$ is the same as $n_{11} = n_{01} e^{\theta}$. Under the given condition, $n_{11} \neq 0$ and $n_{01} \neq 0$. Thus, the MLE is $\hat{\theta} = \log(n_{11}/n_{01})$.

2. (a) Likelihood

$$\ell(\theta) = \theta^{n\bar{X} - 2(n-K)} (1 - \theta)^{2(n-K)}$$

$$\hat{\theta}_{Bayes} = \frac{\bar{X} - 2(1 - K/n) + \alpha/n}{\bar{X} + \alpha/n + \beta/n}$$

(b)

$$\hat{\theta}_{GBayes} = \frac{\bar{X} - 2(1 - K/n)}{\bar{X}} = MLE$$

(c) The Bayes estimator is admissible, since it is the unique Bayes estimator under squared error loss. We don't know whether the generalized Bayes estimator is admissible or not.

3. (a) Note that $\sup_{\theta_j} R_{T_j}(\theta_j) \le sup_\theta R_T(\theta)$. If either $\sup_{\theta_1} R_{T_1}(\theta_1) = \infty$ or $\sup_{\theta_2} R_{T_1}(\theta_2) = \infty$, then $sup_\theta R_T(\theta) = \infty$ and the equality holds.
Assume now that $\sup_{\theta_j} R_{T_j}(\theta_j) < \infty$, $j = 1, 2$. For any θ_1 and θ_2,

$$R_{T_1}(\theta_1) + R_{T_2}(\theta_2) \le \sup_{\theta_1} R_{T_1}(\theta_1) + \sup_{\theta_2} R_{T_2}(\theta_2)$$

Hence

$$\sup_\theta R_T(\theta) = \sup_\theta [R_{T_1}(\theta_1) + R_{T_2}(\theta_2)] \le \sup_{\theta_1} R_{T_1}(\theta_1) + \sup_{\theta_2} R_{T_2}(\theta_2).$$

Since $\sup_{\theta_1} R_{T_1}(\theta_1) < \infty$, for any $\epsilon > 0$, there is a $\theta_{1\epsilon}$ such that

$$R_{T_1}(\theta_{1\epsilon}) > \sup_{\theta_1} R_{T_1}(\theta_1) - \epsilon$$

Since $\sup_{\theta_2} R_{T_2}(\theta_2) < \infty$, for any θ_2,

$$\sup_{\theta_1} R_{T_1}(\theta_1) - \epsilon + R_{T_2}(\theta_2) < R_{T_1}(\theta_{1\epsilon}) + R_{T_2}(\theta_2) \le \sup_\theta R_T(\theta)$$

Since θ_2 is arbitrary,

$$\sup_{\theta_1} R_{T_1}(\theta_1) - \epsilon + \sup_{\theta_2} R_{T_2}(\theta_2) \le \sup_\theta R_T(\theta)$$

Since $\epsilon > 0$ is arbitrary,

$$\sup_{\theta_1} R_{T_1}(\theta_1) + \sup_{\theta_2} R_{T_2}(\theta_2) \le \sup_\theta R_T(\theta)$$

This proves the equality.

(b) Suppose that T_j^* is minimax, $j = 1, 2$. For any T,

$$\sup_\theta R_{T^*}(\theta) = \sup_{\theta_1} R_{T_1^*}(\theta_1) + \sup_{\theta_2} R_{T_2^*}(\theta_2) \le \sup_{\theta_1} R_{T_1^*}(\theta_1) + \sup_{\theta_2} R_{T_2^*}(\theta_2) = \sup_\theta R_T(\theta)$$

Hence T^* is minimax.
Suppose that T^* is minimax. Suppose that T_1^* is not minimax. Then there is a T_1 such that $\sup_{\theta_1} R_{T_1}(\theta_1) < \sup_{\theta_1} R_{T_1^*}(\theta_1)$. Consider $T = (T_1, T_2^*)$.

$$\sup_\theta R_T(\theta) = \sup_{\theta_1} R_{T_1}(\theta_1) + \sup_{\theta_2} R_{T_2^*}(\theta_2) < \sup_{\theta_1} R_{T_1^*}(\theta_1) + \sup_{\theta_2} R_{T_2^*}(\theta_2) = \sup_\theta R_{T^*}(\theta)$$

This contradicts the fact that T^* is minimax. This shows that T_1^* is minimax. Similarly, we can show that T_2^* is minimax.