Department of Statistics
University of Wisconsin, Madison
PhD Qualifying Exam Part I
Tuesday, January 18, 2011
12:30-4:30pm, Room 133 SMI

- There are a total of FOUR (4) problems in this exam. Please do a total of THREE (3) problems.
- Each problem must be done in a separate exam book.
- Please turn in THREE (3) exam books.
- Please write your code name and **NOT** your real name on each exam book.

1. Let $X_1, ..., X_n$ be independent and identically distributed with the Lebesgue probability density $e^{-(x-\theta)} I_{[\theta,\infty)}(x)$, where $I_A(x)$ is the indicator function of the set A and θ is an unknown parameter with parameter space $(-\infty, 1]$. Let $Y = \min(X_1, ..., X_n)$ be the smallest order statistic and let

$$Z_c = \begin{cases} Y & \text{if } Y \leq 1 \\ c & \text{if } Y > 1 \end{cases}$$

where c is a known constant.

(a) Show that Z_c is a sufficient statistic if and only if $c \geq 1$.

(b) Show that Z_c is a complete statistic for any c.

(c) Show that Y is sufficient but not complete.

(d) Let $g(\theta)$ be a known differentiable function of θ. Derive a uniformly minimum variance unbiased estimator of $g(\theta)$.

(e) For testing

$$H_0 : \theta \geq 0 \quad \text{versus} \quad H_1 : \theta < 0, \tag{1}$$

show that

$$\psi(Y) = \begin{cases} 1 & Y < -n^{-1}\log(1-\alpha) \\ 0 & Y \geq -n^{-1}\log(1-\alpha) \end{cases}$$

is a uniformly most powerful test of size $\alpha \in (0, 1)$.

(f) Show that, for any $c \geq 1$,

$$T_c = \begin{cases} 1 & \text{if } Z_c < 0 \\ \alpha & \text{if } Z_c \geq 0 \end{cases}$$

is another uniformly most powerful test of size α for the hypotheses in (1).

2. Suppose that $X_1, \cdots, X_n, Y_1, \cdots, Y_n$ are independent, and $X_1, \cdots, X_n \sim N(\alpha, \sigma^2)$ and $Y_1, \cdots, Y_n \sim N(\beta, \tau^2)$, where means α and β are non-negative unknown parameters, and variances σ^2 and τ^2 are known. Let $\theta = \alpha\beta$.

(a) Find the MLE, $\widehat{\theta}_1$, of θ.

(b) Consider an improper prior

$$\pi(\alpha, \beta) = \begin{cases} 1, & \text{if } \alpha \geq 0, \beta \geq 0 \\ 0, & \text{otherwise.} \end{cases}$$

Find the posterior mean, $\widehat{\theta}_2$, of θ.

(c) Derive the limiting distributions of $\widehat{\theta}_1$ and $\widehat{\theta}_2$ under $\theta > 0$ as $n \to \infty$.

3. Suppose we have n independent paired binary observations: (Y_{i1}, Y_{i2}), $i = 1, \ldots, n$ (Y_{i1} and Y_{i2} take values from 0 or 1). Within each pair, Y_{i1} and Y_{i2} are independent. Assume that

$$\begin{aligned}
\log \frac{P(Y_{i1} = 1)}{P(Y_{i1} = 0)} &= \alpha_i, \; i = 1, \ldots, n; \\
\log \frac{P(Y_{i2} = 1)}{P(Y_{i2} = 0)} &= \alpha_i + \beta, \; i = 1, \ldots, n.
\end{aligned}$$

(a) Write down the likelihood function, and derive the MLE for α_i, $i = 1, \ldots, n$ (denoted as $\widehat{\alpha}_i$) and the MLE for β (denoted as $\widehat{\beta}$).

(b) Does $\widehat{\beta}$ converge or diverge? If you think it converges, derive its limit β^* such that $\widehat{\beta} \xrightarrow{P} \beta^*$. If you think it diverges, prove it.

(c) Find the sufficient statistic T_i for α_i. Write down the conditional likelihood conditioning on T_i. Derive the MLE for β (denoted as $\widehat{\beta}_1$) based on the conditional likelihood. Does $\widehat{\beta}_1$ converge or diverge? If you think it converges, derive its limit β_1^* such that $\widehat{\beta}_1 \xrightarrow{P} \beta_1^*$. If you think it diverges, prove it.

4. This problem concerns consistency of parametric estimators (like MLE) which are formulated as maximizers of certain criterion functions $M_n(\theta)$. Deterministic $M_n(\theta)$ is in parts (a) and (b), and random $M_n(\theta)$ is in part (c). In all parts (a)–(c), $M(\theta)$ denotes a deterministic function.

   (a) Consider

$$M_n(\theta) = \begin{cases} \frac{\theta^2}{\theta^2+(1-n\theta)^2}, & \text{if } 0 \le \theta < 1, \\ \frac{1}{2}, & \text{if } \theta = 1, \end{cases} \qquad M(\theta) = \begin{cases} 0, & \text{if } 0 \le \theta < 1, \\ \frac{1}{2}, & \text{if } \theta = 1. \end{cases}$$

   Let $\widehat{\theta}_n = \arg\max_{\theta \in [0,1]} M_n(\theta)$ and $\theta_0 = \arg\max_{\theta\in[0,1]} M(\theta)$.

   i. Determine whether $\widehat{\theta}_n \to \theta_0$ as $n \to \infty$.
   ii. Determine whether $\sup_{\theta\in[0,1]} |M_n(\theta) - M(\theta)| \to 0$ as $n \to \infty$.

   (b) Consider

$$M_n(\theta) = \begin{cases} n\theta, & \text{if } \theta \in [0, 1/n), \\ 2 - n\theta, & \text{if } \theta \in [1/n, 2/n), \\ 0, & \text{if } \theta \in [2/n, 1), \\ \theta - 1, & \text{if } \theta \in [1, 3/2), \\ 2 - \theta, & \text{if } \theta \in [3/2, 2], \end{cases} \qquad M(\theta) = \begin{cases} 0, & \text{if } \theta \in [0,1), \\ \theta - 1, & \text{if } \theta \in [1, 3/2), \\ 2-\theta, & \text{if } \theta \in [3/2, 2]. \end{cases}$$

   Let $\widehat{\theta}_n = \arg\max_{\theta\in[0,2]} M_n(\theta)$ and $\theta_0 = \arg\max_{\theta\in[0,2]} M(\theta)$.

   i. Determine whether $\widehat{\theta}_n \to \theta_0$ as $n \to \infty$.
   ii. Determine whether $\sup_{\theta\in[0,2]} |M_n(\theta) - M(\theta)| \to 0$ as $n \to \infty$.

   (c) Let Θ be the parameter space. Define $\theta_0 = \arg\max_{\theta\in\Theta} M(\theta)$. Suppose $\sup_{\theta\in\Theta} |M_n(\theta) - M(\theta)| \xrightarrow{P} 0$, $\sup_{\theta:\, |\theta-\theta_0|\ge\varepsilon} M(\theta) < M(\theta_0)$ for any $\varepsilon > 0$, and $M_n(\widehat{\theta}_n) \ge M_n(\theta_0) - \delta_n$ with $0 \le \delta_n \xrightarrow{P} 0$.

   i. Show that $M(\widehat{\theta}_n) \xrightarrow{P} M(\theta_0)$.
   ii. Show that $\widehat{\theta}_n \xrightarrow{P} \theta_0$.