STAT 710 Second Exam
8:25am-9:15am, March 10, 2011

Please show all your work for full credits.

1. Let X be a discrete observation satisfying

$$P(X=-1)=\theta, \quad P(X=x)=(1-\theta)^2\theta^x, \quad x=0,1,2,..., \tag{1}$$

where $\theta \in (0,1)$ is an unknown parameter.

(a) (3 points) Obtain the likelihood function $\ell(\theta)$ when $X=x$, and derive the MLE of θ and show it is a unique MLE.

(b) (2 points) Suppose that we know $\theta \in [0.5, 1)$. Derive the MLE of θ and show why it is the MLE.

From now on we suppose that there are iid observations $X_1, ..., X_n$ from the distribution given by (1) with $\theta \in (0,1)$.

(c) (3 points) Obtain the likelihood function in terms of $T=\sum_{i=1}^n X_i I_{\{X_i \geq 0\}}$ and $N=$ the number of i's with $X_i=-1$, and provide a sufficient and necessary condition under which there is a unique MLE inside $(0,1)$.

(d) (2 points) Let $\hat{\theta}_{1n}$ be the MLE of θ, $\hat{\theta}_{2n}=N/n$, and $\hat{\theta}_{3n}=T/n$. Show that each $\hat{\theta}_{kn}$ is a consistent estimator of θ.

(e) (2 points) Derive the asymptotic distribution of the MLE.

(f) (2 points) Obtain the asymptotic relative efficiency of the MLE w.r.t. $\hat{\theta}_{2n}$.

2. Let $X_1, ..., X_n$ be i.i.d. observations having the Lebesgue p.d.f.

$$f_\theta(x)=\frac{1+|x-\theta|}{3} I_{(0,1)}(|x-\theta|),$$

where $\theta \in \mathcal{R}$ is an unknown parameter.

(a) (3 points) Obtain the asymptotic distribution of the pth sample quantile when $p=0.75$.

(b) (3 points) Calculate the asymptotic relative efficiency of the sample mean with respect to the sample median.