**STAT 710 Final Exam**
**10:00am-12:00noon, May 12, 2016**

Please show all your work for full credits.

1. Let $X$ be a single observation having a discrete p.d.f.

$$f_\theta(x) = P(X = x) = -(x \log \theta)^{-1}(1-\theta)^x, \quad x = 1, 2, ...,$$

where $\theta \in (0, 1)$ is an unknown parameter. Let $U$ be another single observation from the uniform distribution on (0,1). Assume that $X$ and $U$ are independent.

   (a) (3 points) Show that the Lebesgue p.d.f. of the random variable $X + U$ is

$$g_\theta(t) = -([t] \log \theta)^{-1}(1-\theta)^{[t]} I_{(1,\infty)}(t),$$

   where $[t]$ is the integer part of $t$.

   (b) (3 points) Show that the family $\{g_\theta : \theta \in (0, 1)\}$ has monotone likelihood ratio in $-(X + U)$.

   (c) (3 points) Consider testing $H_0 : \theta \geq \theta_0$ versus $H_1 : \theta < \theta_0$ for a given $\theta_0$. Show that the UMP test of size $\alpha \in (0, 1)$ is of the form $T = I_{(c(\theta_0),\infty)}(X + U)$ with $c(\theta_0)$ satisfies

$$\alpha = \int_{c(\theta_0)}^{\infty} g_{\theta_0}(t)dt.$$

   (d) (4 points) Use the c.d.f. of $X + U$ to construct a pivotal quantity for $\theta$, and obtain a lower confidence bound for $\theta$ with confidence coefficient $1 - \alpha$.

   (e) (4 points) Obtain a $\Theta'$-UMA lower confidence bound for $\theta$ with confidence coefficient $1 - \alpha$, where $\Theta' = (0, \theta)$.

2. (4 points) Let $f$ be an unknown Lebesgue p.d.f. on $(0, \infty)$ with finite second order moment and $f(0) > 0$. Let $X_1, ..., X_n$ be i.i.d. observations having the Lebesgue p.d.f. $\frac{1}{2\sigma} f(\frac{|x-\mu|}{\sigma})$, where $\mu \in \mathcal{R}$ and $\sigma > 0$ are unknown parameters. Find the asymptotic relative efficiency of the sample mean with respect to the sample median, as a function of $f(0)$ and $\int_0^\infty t^2 f(t)dt$.

3. Let $X_1, ..., X_n$ be i.i.d. having the Lebesgue p.d.f. $\frac{1}{2}(1-\theta^2)e^{\theta x - |x|}$, where $\theta \in (-1, 1)$ is unknown.

   (a) (4 points) Show that the likelihood equation has one sequence of solution that is consistent for $\theta$ as $n \to \infty$, and obtain the asymptotic distribution of this sequence.

   (b) (4 points) For testing $H_0 : \theta = 0$ versus $H_1 : \theta \neq 0$, derive a UMPU test with size $\alpha \in (0, 1)$. Discuss how to obtain the rejection region but do not try to obtain an explicit form.

**(continued on the next page)**

(c) (6 points) For a given $\alpha$, derive $1-\alpha$ asymptotically correct confidence intervals of $\theta$ by inverting acceptance regions of likelihood ratio tests, Wald tests, and Rao's score tests.

(d) (5 points) Consider the uniform distribution on $(-1, 1)$ as a prior for $\theta$. When $n = 1$, derive the Bayes estimator of $\theta$ under the squared error loss. You need to use integration by parts:

$$\int_{-1}^{1} \theta^p e^{\theta x} d\theta = \frac{e^x - (-1)^p e^{-x}}{x} - p \int_{-1}^{1} \theta^{p-1} e^{\theta x} d\theta$$