# STAT 710
## Third Exam, April 18, 2018

Please show all your work for full credits.

1. Let $X_{ij}$, $i = 1, ..., n_j$, $j = 1, 2$, be independent random observations from the exponential distributions on $(0, \infty)$ with scale parameters $\theta_j$, i.e., $P(X_{ij} \le t) = \theta_j^{-1} \int_0^t e^{-s/\theta_j} ds$, $j = 1, 2$. We would like to test $H_0 : \theta_2 = \lambda\theta_1$ versus $H_1 : \theta_2 \neq \lambda\theta_1$, where $\lambda > 0$ is a known constant.

   (a) (2 points) Show that Theorem 6.4 is applicable with $Y = X_1$ and $U = \lambda X_1 + X_2$, where $X_j = \sum_{i=1}^{n_j} X_{ij}$, and give the form of the UMPU test in Theorem 6.4 with size $\alpha \in (0, \frac{1}{2})$.

   (b) (2 points) Show that $X_1/X_2$ is independent of $U$ under $H_0$.

   (c) (3 points) Using the results in (b) and Lemma 6.7, show that the test in (a) is equivalent to the test that rejects $H_0$ when $W < b_1$ or $W > b_2$, where $W = \frac{X_2/\lambda}{X_1+X_2/\lambda}$, and $b_1$ and $b_2$ are chosen so that, under $H_0$, $P(b_1 < W < b_2) = 1 - \alpha$ (for size $\alpha$) and $E[W I_{(b_1,b_2)}(W)] = (1 - \alpha)E(W)$ (for unbiasedness).

   (d) (2 points) Using the fact (without proof) that, under $H_0$, $W$ has the beta distribution with p.d.f. $f_{k,l}(w) = \frac{\Gamma(k+l)}{\Gamma(k)\Gamma(l)} w^{k-1}(1 - w)^{l-1} I_{(0,1)}(w)$, where $k = n_1$ and $l = n_2$, show that $b_1$ and $b_2$ in (c) satisfy

$$\int_{b_1}^{b_2} f_{n_1,n_2}(w)dw = 1 - \alpha = \int_{b_1}^{b_2} f_{n_1+1,n_2}(w)dw$$

2. Let $X_1, ..., X_n$ be i.i.d. observations from the discrete distribution with

$$P(X_i = x) = (1 - p)^{x-1}, \qquad x = 1, 2, ...,$$

   where $p \in (0, 1)$ is unknown, and let $Y_1, ..., Y_n$ be i.i.d. observations from the discrete distribution with

$$P(Y_i = y) = (1 - q)^{y-1}, \qquad y = 1, 2, ...,$$

   where $q \in (0, 1)$ is unknown. Assume that $X_i$'s and $Y_j$'s are independent. Consider testing $H_0 : p = q$ versus $H_1 : p \neq q$.

   (a) (3 points) Obtain the likelihood ratio test statistic.

   (b) (3 points) Show that Rao's score test statistic is $R_n = n(X-Y)^2/[(X+Y)^2(1-\tilde{p})]$, where $X = \sum_{i=1}^n X_i$, $Y = \sum_{i=1}^n Y_i$, and $\tilde{p} = (2n)/(X + Y)$.

   (c) (3 points) Show that Wald's test statistic is $W_n = n(X - Y)^2/[Y^2(1 - n/X) + X^2(1 - n/Y)]$.

   (d) (2 points) Show directly (without applying any theorem) that, under $H_0$, $R_n \to_d \chi_1^2$ and $W_n \to_d \chi_1^2$, where $\chi_1^2$ is the chi-square distribution with degree of freedom 1.

Solution:

1. (a) The joint density of $X_1 = \sum_{i=1}^{n_1} X_{i1}$ and $X_2 = \sum_{i=1}^{n_2} X_{i2}$ is

$$\frac{X_1^{n_1-1}X_2^{n_2-1}}{\Gamma(n_1)\Gamma(n_2)\theta_1^{n_1}\theta_2^{n_2}}\exp\left\{-\frac{X_1}{\theta_1}-\frac{X_2}{\theta_2}\right\}$$

$$=\frac{X_1^{n_1-1}X_2^{n_2-1}}{\Gamma(n_1)\Gamma(n_2)\theta_1^{n_1}\theta_2^{n_2}}\exp\left\{-X_1\left(\frac{1}{\theta_1}-\frac{\lambda}{\theta_2}\right)-(\lambda X_1+X_2)\frac{1}{\theta_2}\right\}.$$

Hence, by Theorem 6.4, a UMPU test of size $\alpha$ rejects $H_0$ when $X_1 < c_1(U)$ or $X_1 > c_2(U)$.

(b) The result follows from Basu's theorem.

(c) By Lemma 6.7, the UMPU test is equivalent to the test that rejects $H_0$ when $X_1/X_2 < d_1$ or $X_1/X_2 > d_2$, which is equivalent to the test that rejects $H_0$ when $W < b_1$ or $W > b_2$, where $W = \frac{Y/\lambda}{1+Y/\lambda}$, $Y = X_2/X_1$, and $b_1$ and $b_2$ satisfy $P(b_1 < W < b_2) = 1-\alpha$ (for size $\alpha$) and $E[WI_{(b_1,b_2)}(W)] = (1-\alpha)E(W)$ (for unbiasedness) under $H_0$.

(d) $E(W) = n_1/(n_1+n_2)$ and $wf_{n_1,n_2}(w) = [n_1/(n_1+n_2)]f_{n_1+1,n_2}(w)$.

2. (a) The MLE under $H_0$ is $\hat\theta = N/T$, where $T = X+Y$, $X = \sum_{i=1}^{n} X_i$, $Y = \sum_{j=1}^{m} Y_j$, $N = n+m$. The MLE of $p$ in general is $n/X$ and MLE of $q$ in general is $m/Y$. Then

$$\lambda = \frac{(1-N/T)^{T-N}(N/T)^N}{(1-n/X)^{X-n}(n/X)^n(1-m/Y)^{Y-m}(m/Y)^m}$$

(b)

$$s(\theta) = \left(\frac{n}{p}-\frac{X-n}{1-p},\ \frac{m}{q}-\frac{Y-m}{1-q}\right)$$

$$\frac{\partial s(\theta)}{\partial\theta} = -\begin{pmatrix}\frac{n}{p^2}+\frac{X-n}{(1-p)^2} & 0\\ 0 & \frac{m}{q^2}+\frac{Y-m}{(1-q)^2}\end{pmatrix}$$

$$I(\theta) = \begin{pmatrix}\frac{n}{p^2(1-p)} & 0\\ 0 & \frac{m}{q^2(1-q)}\end{pmatrix}$$

Let $\tilde p = N/T$. Then

$$R = \left(\frac{n}{\tilde p}-\frac{X-n}{1-\tilde p},\ \frac{m}{\tilde p}-\frac{Y-m}{1-\tilde p}\right)\begin{pmatrix}\frac{\tilde p^2(1-\tilde p)}{n} & 0\\ 0 & \frac{\tilde p^2(1-\tilde p)}{m}\end{pmatrix}\begin{pmatrix}\frac{n}{\tilde p}-\frac{X-n}{1-\tilde p}\\ \frac{m}{\tilde p}-\frac{Y-m}{1-\tilde p}\end{pmatrix}$$

(c) Let $\hat p = n/X$ and $\hat q = m/Y$. Then

$$W = \frac{(\hat p-\hat q)^2}{\hat p^2(1-\hat p)/n+\hat q^2(1-\hat q)/m}$$

(d) By the CLT, $\sqrt{n}(X/n - p^{-1}) \to_d N(0,(1-p)/p^2)$. Hence, under $H_0$, $n^{-1}(X-Y)^2 \to_d 2(1-p)p^{-2}\chi_1^2$. By the LLN, $n^{-2}(X+Y)^2(1-\tilde p) \to_p 2(1-p)/p^2$ and $n^{-2}[Y^2(1-n/X)+X^2(1-n/Y)] \to_p 2(1-p)/p^2$.