STAT 709 Final Exam
10:05am-12:05pm, Dec. 18, 2009

Please show all your work for full credits.

1. Let $X_1, ..., X_n$ be i.i.d. random variables with finite $E(X_1^2)$. Define $T = \sum_{i=1}^n X_i$ and $S = n^{-1} \sum_{i=1}^n X_i^2$.
   (a) (3 points) Show that $E(X_i|T) = T/n$ for any i.
   (b) (3 points) Show that $E(X_iX_j|T,S) = [n(n-1)]^{-1} \sum_{k\neq l} X_kX_l$ for any $i \neq j$.
   (c) (3 points) Show that, when $n \to \infty$ and $i \neq j$, $E(X_iX_j|T,S)$ and $E(X_i|T,S)E(X_j|T,S)$ converge in probability to the same limit.

2. Let $X_1, ..., X_n$ be i.i.d. from a population with Lebesgue p.d.f.
$$\frac{x-a}{\theta^2} e^{-(x-a)/\theta} I_{(a,\infty)}(x),$$
   where $\theta > 0$ and $a \in \mathcal{R}$ are unknown parameters.
   (a) (3 points) Derive estimators $\hat{a}$ and $\hat{\theta}$ of a and θ, respectively, using the method of moments.
   (b) (4 points) Derive the asymptotic distributions of $\sqrt{n}(\hat{\theta} - \theta)$ and $\sqrt{n}(\hat{a} - a)$.
   (c) (3 points) Obtain a sufficient statistic T for θ and a and $T \neq (X_1, ..., X_n)$. You need to show why T is sufficient.
   (d) (3 points) Assume that $n = 2$. Find a minimal sufficient statistic for θ and a. You need to show why it is minimal sufficient.
   (e) (2 points) Assume that $\theta = 1$ is known. Show that the bias of $X_{(1)} = \min_{i=1,...,n} X_i$ as an estimator of a is
$$n \int_0^\infty \left(\int_x^\infty te^{-t}dt \right)^{n-1} x^2e^{-x}dx.$$

3. Let $X_1, ..., X_n$ be i.i.d. random variables with finite $\mu = E(X_1)$ and $\sigma^2 = \mathrm{Var}(X_1)$, $n > 2$. Suppose that there is a known function $g(x)$ such that $E[g(X_1)] = \sigma^{-1}$.
   (a) (2 points) Find a U-statistic U_n such that $E(U_n) = \mu/\sigma$.
   (b) (2 points) Find the $h_1(x)$ function for U_n.
   (c) (3 points) Assume that h_1 is not constant. Obtain the asymptotic distribution of $\sqrt{n}(U_n - \mu/\sigma)$ in terms of μ, σ, $\mathrm{Var}(g(X_1))$, and $\mathrm{Cov}(X_1, g(X_1))$.

   (**There is another problem on page 2**)

4. Consider a linear model

$$X = Z\beta + \varepsilon,$$

where $E(\varepsilon) = 0$ and ε has a finite covariance matrix.

(a) (3 points) Suppose that ε is normal, $\mathrm{Var}(\varepsilon) = \sigma^2 I_4$ (the identity matrix of order 4), and

$$Z = \begin{bmatrix} 1 & 0 & 1 \\ 1 & 1/2 & 1/2 \\ 1 & 1 & 0 \\ 0 & -1 & 1 \end{bmatrix}$$

Obtain the forms of all $l \in \mathcal{R}^3$ such that $l^\tau \beta$ is estimable.

(b) (3 points) Suppose that

$$Z(Z^\tau Z)^{-1} Z^\tau = \begin{bmatrix} a & u^\tau \\ u & C \end{bmatrix} \qquad \text{and} \qquad \mathrm{Var}(\varepsilon) = \begin{bmatrix} r^2 & 0^\tau \\ 0 & \sigma^2 I \end{bmatrix},$$

where $a \in \mathcal{R}$, $r^2 \neq \sigma^2$, I is the identity matrix of the same order as C, u is a vector of appropriate order, and 0 is the zero vector of the same order as u. Obtain a sufficient and necessary condition for the LSE to be BLUE.

(c) (3 points) Suppose that $\mathrm{Var}(\varepsilon) = \sigma^2 I_4$ and

$$Z = \begin{bmatrix} 1 & 1 \\ 1 & 1 \\ 0 & -1 \\ 0 & -1 \end{bmatrix}$$

Obtain the covariance matrix of the LSE $\hat{\beta}$.