STAT 710 First Exam
8:50am-9:40am, Feb. 17, 2012

Please show all your work for full credits.

1. Let $X_1, ..., X_n$ be i.i.d. observations having the Lebesgue p.d.f.

$$f_\theta(x) = \theta x^{-(\theta+1)} I_{(1,\infty)}(x),$$

where $\theta > 0$ is an unknwon parameter.

(a) (2 points) Show that the family of Lebesgue p.d.f.'s

$$\frac{1}{\Gamma(\alpha)\gamma^\alpha}\theta^{\alpha-1}e^{-\theta/\gamma}I_{(0,\infty)}(\theta),$$

where $\alpha > 0$ and $\gamma > 0$ are hyperparameters, contains conjugate priors.

(b) (2 points) For the prior in (a) with any known hyperparameters, obtain the Bayes estimators of θ and $\varphi = \theta^{-1}$ under the squared error loss.

(c) (2 points) Show that the limit of the Bayes estimators of φ in part (b) as $\alpha \to 1$ and $\gamma \to \infty$ is equal to the generalized Bayes estimator of φ under the squared error loss with respect to an improper prior. Identify this improper prior.

(d) (3 points) Let $T = \log(X_1 \cdots X_n)$. Using Theorem 4.14 (next page), show that $T/(n+1)$ is admissible as an estimator of φ under the squared error loss. [Hint: You may use (without a proof) the fact $E(\log X_i) = \varphi$.]

(e) (2 points) Find a loss function under which $T/(n+1)$ is the unique minimax estimator of φ. [Hint: You may use (without a proof) the fact $\mathrm{Var}(\log X_i) = \varphi^2$.]

(f) (3 points) Show that the generalized Bayes estimator in (c) is equal to the MLE of φ and is in fact an inadimissible estimator under the squared error loss.

2. Let $X_1, ..., X_n$ be i.i.d. observations having the Lebesgue p.d.f.

$$f_\theta(x) = \frac{\theta}{c(1+\theta^2x^2/a)^{(a+1)/2}}I_{(0,\infty)}(x),$$

where $\theta > 0$ is an unknwon parameter, $a > 0$ is known, and

$$c = \int_0^\infty \frac{1}{(1+x^2/a)^{(a+1)/2}}dx.$$

(a) (3 points) Obtain the likelihood function $\ell(\theta)$ and show that $\partial \log \ell(\theta)/\partial\theta = 0$ is equivalent to

$$\sum_{i=1}^n \frac{1-X_i^2\theta^2}{a+X_i^2\theta^2} = 0$$

(b) (3 points) Show that $\partial \log \ell(\theta)/\partial\theta = 0$ has a unique solution in $(0,\infty)$.

Theorem 4.14. Suppose that X has the p.d.f. $c(\theta)e^{\theta T(x)}$ w.r.t. a σ-finite measure ν, where $T(x)$ is real-valued and $\theta \in (\theta_-, \theta_+) \subset \mathcal{R}$. Consider the estimation of $\vartheta = E[T(X)]$ under the squared error loss. Let $\lambda \geq 0$ and γ be known constants and let $T_{\lambda,\gamma}(X) = (T + \gamma\lambda)/(1 + \lambda)$. Then a sufficient condition for the admissibility of $T_{\lambda,\gamma}$ is that

$$\int_{\theta_0}^{\theta_+} \frac{e^{-\gamma\lambda\theta}}{[c(\theta)]^\lambda} d\theta = \int_{\theta_-}^{\theta_0} \frac{e^{-\gamma\lambda\theta}}{[c(\theta)]^\lambda} d\theta = \infty,$$

where $\theta_0 \in (\theta_-, \theta_+)$.