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Job Matching and the Theory of Turnover

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A long-run equilibrium theory of turnover is presented and is shown to explain the important regularities that have been observed by empirical investigators. A worker's productivity in a particular job is not known ex ante and becomes known more precisely as the worker's job tenure increases. Turnover is generated by the existence of a nondegenerate distribution of the worker's productivity across different jobs. The nondegeneracy is caused by the assumed variation in the quality of the worker-employer match.

The objective of this paper is to construct and to interpret a model of permanent job separations. A permanent job separation involves a change of employers for the worker. Temporary separations (consisting mainly of temporary layoffs) have been the subject of recent theoretical work by Baily (1974), Azariadis (1975), and Feldstein (1976), and are not considered here.

Recent evidence on labor turnover falls into two categories: the cross-sectional industry studies (Stoikov and Ramon 1968; Burton and Parker 1969; Pencavel 1970; Parsons 1972; Telser 1972), and the more recent studies using longitudinal data on individuals (Bartel 1975; Bartel and Borjas 1976; Freeman 1976; Jovanovic and Mincer 1978). The strongest and most consistent finding of all these studies is a negative relationship between quits and layoffs on the one hand, and job tenure on the other. This finding is equally strong for quits as it is for layoffs. Jovanovic and Mincer (1978) find that roughly one-

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half of this negative relationship is explained by the negative structural dependence of the separation probability on job tenure. The rest of the observed dependence is only apparent and is caused by the fact that within any nonhomogeneous group a negative correlation between job tenure and the separation probability will exist, simply because those people with a lower propensity to change jobs will tend to have longer job tenure and vice versa. Other observed relationships are as follows: women, young workers, production workers, those with less schooling, and those in the private sector tend to turn over more, as do those workers not covered by a pension plan and those who work in industries with lower concentration ratios or with smaller average firm size. None of these relationships is nearly as strong as that between job tenure and separation probabilities.

Existing models of turnover (that is the "permanent separations" component of turnover) all involve imperfect information. New information arrives either about one's current match or about a possible alternative match that leads to a job change. In fact, a natural distinction among the models can be made along these lines. In one category are models in which turnover occurs as a result of the arrival of information about the current job match, and the present model falls into this category, as do the models of Viscusi (1976), Wilde (1977), and Johnson (1978).[1] These are models in which a job is an "experience good" in the terminology of Nelson (1970); that is, the only way to determine the quality of a particular match is to form the match and "experience it." In the second category are "pure search-good" models of job change (Kuratani 1973; Lucas and Prescott 1974; Burdett 1977; Jovanovic 1978*b*; Mortensen 1978; Wilde 1978). In those models, jobs are pure search goods and matches dissolve because of the arrival of new information about an alternative prospective match. Hirshleifer (1973) introduces the more appropriate designation "inspection goods." *Inspection* is evaluation that can take place prior to purchase, *experience* only after purchase.

In this paper, a job match is treated as a pure experience good. The paper makes two separate contributions. First, it is the only explicitly equilibrium treatment of turnover in its category. An equilibrium wage contract is proved to exist and a particular wage contract is demonstrated to be an equilibrium one. This particular wage contract has the property that at each moment in time the worker is paid his marginal product conditional upon all the available information at that time.

Second, the characterization of the model's implications regarding

[1] I became aware of the work of these three authors after the present work was largely finished.

the tenure-turnover relationship and the tenure-wage relationship is more explicit than that of earlier models, and the predictions are largely consistent with the evidence. The model predicts that workers remain on jobs in which their productivity is revealed to be relatively high and that they select themselves out of jobs in which their productivity is revealed to be low. Since wages always equal expected marginal products for all workers, the model generates (on average) wage growth as tenure increases. Since job tenure and labor market experience are correlated across workers, this also implies wage growth over the life cycle. The model also predicts that each worker's separation probability is a decreasing function of his job tenure. Loosely speaking, this is because a mismatch between a worker and his employer is likely to be detected early on rather than late. The learning mechanism is such that longer job tenure has a negative structural effect on the worker's separation probability. After correcting for the regression bias that arises because of the spurious correlation between job tenure and the separation probability in a heterogeneous group of workers, Jovanovic and Mincer (1978) find that this structural dependence is very strong.

Before developing the model we summarize the major assumptions of the job-matching approach to turnover. First it is assumed that for each worker a nondegenerate distribution of productivities exists across different jobs. The same is true for the employer—workers differ in their productivities in a given task that the employer needs to have performed. The problem is one of optimally assigning workers to jobs.

The second assumption is that employers can contract with workers on an individual basis. The employer is then able to reward a worker with whom he matches well by paying the worker relatively more. Individual contracting creates a structure of rewards that provides proper signals for the attainment of optimal matches. An extreme example of individual contracting is a piece-rate wage scale. A less extreme and a widely prevalent example is a system of promotion or delayed pay increases based on the quality of the worker's performance on the job over a past period of time of some given length. These are examples where workers' pay is contingent on their performance.

The third major assumption of the job-matching approach is that imperfect information exists on both sides of the market about the exact location of one's optimal assignment. Following an initial assignment, new information becomes available, and reassignment becomes optimal in certain cases. The job-matching model generates turnover as the phenomenon of optimal reassignment caused by the accumulation of better information with the passage of time.

The Model

Assume that firms' production functions exhibit constant returns to scale and that labor is the only factor of production. Under competitive conditions, the size of firm is then indeterminate. Each worker's output is assumed to be observed instantaneously by the worker and by the employer so that informational asymmetries do not arise. Let $X(t)$ be the contribution by a worker to the total output of the firm over a period of length t, and let

$$X(t) = \mu t + \sigma z(t) \qquad (\text{for each } t > 0) \tag{1}$$

where μ and σ are constants and $\sigma > 0$, and where $z(t)$ is a standard normal variable with mean zero and variance t (a standard Wiener process with independent increments so that cov $[z(t), z(t')] = \min [t, t']$). Then $X(t)$ is normally distributed with mean μt and with variance $\sigma^2 t$. Assume that σ is the same for each firm-worker match while in general μ differs across matches. The interpretation of μ is not one of worker ability but a measure of the quality of the match. When the match is formed, μ is unknown. As the match continues, further information (in the form of output as given by eq. [1]) is generated. A "good match" is one possessing a large μ. Let μ be normally distributed across matches, with mean m and with variance s, and assume that job changing involves drawing a new value of μ from this distribution and the successive drawings are independent. The latter assumption guarantees that the worker's prior history is of no relevance in assessing his μ on a newly formed match. The only way to learn about μ is to observe the worker on the job for a period of time. This independence assumption also means that the informational capital thus generated is completely match specific and is analogous to the concept of firm-specific human capital.[2]

For a worker with job tenure t and cumulative output $X(t) = x$ the above assumptions imply that the available information on μ on his current job can be characterized by a posterior distribution that is normal (see Chernoff 1968, p. 266) with

$$\begin{aligned} &\text{posterior mean} \equiv E_{xt}(\mu) = (ms^{-1} + x\sigma^{-2})(s^{-1} + t\sigma^{-2})^{-1} \\ &\text{posterior variance} \equiv S(t) = (s^{-1} + t\sigma^{-2})^{-1}. \end{aligned} \tag{2}$$

The pair $[X(t), t]$ is therefore a sufficient statistic for the information contained in the entire posterior distribution. (This property is essentially due to the independent increments property of the Wiener

[2] To elaborate: When dealing with random variables the concept of information specificity is associated with the concept of independence while perfect informational generality is associated with perfect correlation.

process.) Furthermore, $E_{X(t)t}(\mu)$ is normally distributed with mean m and variance $s - S(t)$ (Chernoff 1968).

Firms are assumed to be risk neutral and to maximize the mathematical expectation of revenues discounted by the rate of interest, r. They compete for workers by offering wage contracts. In a long-run equilibrium the payments practices of each firm would be well understood and would not need to be explicitly written. An implicit contract equilibrium is studied here. The present model abstracts entirely from the consideration of shocks stemming from the product market. All firms face the same product price, normalized at unity, so that a maintained hypothesis of the model is that demand conditions are stationary. Assume that the firm's wage policy can be characterized by a wage function $w[X(t), t]$. This is the wage paid to the worker with tenure t if his cumulative output contribution is equal to $X(t)$. If the firm wishes to fire a certain worker, rather than doing so directly the firm is assumed to lower his wage by an amount sufficient to induce him to quit. All the job separations are therefore at the worker's initiative, but since some of the separations are disguised layoffs their empirical counterpart is really total separations (quits plus layoffs).

Workers are assumed to live forever,[3] and this assumption justifies the exclusion of age as an explicit argument from the wage function. As long as he remains with the firm, the worker receives payment according to the wage function $w(\cdot)$. He has the option of quitting at any time. Let Q be the present value of quitting a job and then pursuing the best alternative. The infinite horizon, constant discount rate, and the independence of the successive drawings of μ imply that Q is a constant.[4] Let $\alpha(Q,[w])$ be the present value to the worker of obtaining a job with a firm which offers $w(\cdot)$ as its wage contract and when the value of quitting is Q. Then if c represents the direct and the foregone earnings costs of job changing,

$$Q = \alpha(Q,[w]) - c. \tag{3}$$

The constant c is assumed to be parametrically given for each worker, although it may vary across workers. Let T be the quitting time and let $H(x,t \mid [w],Q) = \text{prob}\ (X[t] \leq x \text{ and } T > t \text{ given } [w] \text{ and } Q)$ and $F(t \mid [w], Q) = \text{prob}\ (T \leq t \text{ given } [w] \text{ and } Q)$. Then F is the probability that the worker quits before tenure t, while H is the probability that he does not quit before tenure t, and that by that time his cumulative output

[3] More generally, one could assume that workers' lifetimes are exponentially distributed implying the absence of aging—one would not make a different prediction about the length of the remaining life of a worker who has already lived a long time than for a worker who has only lived a short time.

[4] The constancy of Q over time means that the worker never returns to a job from which he once separated. In other words, if it existed, the option of recall would never be exercised by the worker.

does not exceed x. Then define the appropriate densities $h(x, t \mid [w], Q)$ and $f(t \mid [w], Q)$ by $h = \partial H/\partial x$ and $f = \partial F/\partial t$. Both f and h are chosen by the worker in response to a wage function $w(\cdot)$ and a present value of quitting Q. Then

$$\alpha(Q, [w]) = \int_0^\infty e^{-rt} \int_{-\infty}^\infty whdxdt + Q\int_0^\infty e^{-rt}fdt. \tag{4}$$

Equation (4) holds at the optimally chosen functions h and f. Since f integrates to a number not exceeding unity, $\partial\alpha/\partial Q = \int_0^\infty e^{-rt}fdt < 1$. Then it is easily seen that for given functions h, f, and w, equations (3) and (4) possess exactly one solution for the pair of scalars (α, Q).

All new workers look alike to the firm, and each worker is offered the same wage contract.[5] In differential form, equation (1) reads $dX(t) = \mu dt + \sigma dz(t)$. Letting E_{xt} be the mathematical expectation operation conditional on $X(t) = x$ at t, the discounted revenue from the output of a single worker is $E\int_0^\infty e^{-rt}dX(t) = E\int_0^\infty e^{-rt}E_{X(t)t}dX(t) = E\int_0^\infty e^{-rt} E_{X(t)t}(\mu)dt + E\int_0^\infty e^{-rt}\sigma E_{X(t)t}dz(t)$. The stochastic integrals are Itô integrals (see Kushner [1971] for their definition) and the last integral is therefore zero, by the independent increments property of the Wiener process, so that $E\int_0^\infty e^{-rt}dX(t) = E\int_0^\infty e^{-rt}E_{X(t)t}(\mu)dt = \int_0^\infty e^{-rt} \int_{-\infty}^\infty E_{xt}(\mu)\, h(x,t \mid [w], Q)dxdt \equiv \beta(Q, [w])$. Firms are aware of the worker's optimal quitting response to the wage contract $\{w\}$, and this is reflected in the above equation. Now let $\pi(Q, \{w\})$ be the discounted expected net revenue from the employment of a given worker who is offered the contract $\{w\}$ and who has a present value of quitting equal to Q. Then

$$\pi(Q, [w]) = \beta(Q, [w]) - \alpha(Q, [w]) + \gamma(Q, [w]) \tag{5}$$

where $\gamma = Q\int_0^\infty e^{-rt}f(t \mid [w], Q)dt$.

In maximizing $\pi(Q, [w])$ over functions $[w]$, the firm treats Q as given, since Q is determined by the wage policies of other firms.

Equilibrium

Let B be the set of competitive equilibrium wage contracts, and for any $w(\cdot)$ let $Q([w])$ denote the unique solution for Q from equation (3). Then, if $w(\cdot) \in B$, (E1) each worker follows his optimal quitting

[5] Similarly, all firms look alike to the worker ex ante. Straightforward extensions of the model to the case where there are observable differences in characteristics among workers are outlined at the end of the paper. Salop (1973) takes up the search problem when the worker is able to distinguish among firms ex ante and has partial information not only about the wage offered by the firm but also about the likelihood that he will receive an employment offer from the firm in the event that he samples it. In Salop's analysis the most attractive opportunities are sampled first, and the job seeker lowers his acceptance wage with his duration of unemployment as his remaining opportunities worsen.

policy in response to $w(\cdot)$ and to $Q([w])$; (E2) $\pi\{Q([w]), [w]\} \geqslant \pi\{Q([w]), [\hat{w}]\}$ for all $\hat{w}(\cdot) \neq w(\cdot)$, so that $w(\cdot)$ maximizes expected profits; (E3) $\pi\{Q([w]),[w]\} = 0$ (zero expected profit constraint). Let $w^*(x, t) = E_{xt}(\mu)$ for all (x, t). This wage contract states that the worker will be paid his expected (marginal) product at each moment in time. Let $Q^* = Q([w^*])$.

Theorem 1.—$w^* \in B$.

Proof.—E3 is clearly satisfied by w^*. To prove E1 and E2, suppose by contradiction that E2 is not satisfied by w^* so that there exists some $w \in B$ such that a deviant firm offers it

$$\pi(Q^*, [w^*]) < \pi(Q^*, [w]) \tag{6}$$

while the worker must be doing at least as well as under w^*:

$$\alpha(Q^*, [w^*]) \leqslant \alpha(Q^*, [w]). \tag{7}$$

(The value of quitting the deviant firm is unchanged at Q^*.) From (5),

$$\pi(Q^*, [w]) - \pi(Q^*, [w^*]) + \alpha(Q^*, [w]) - \alpha(Q^*, [w^*]) = \beta(Q^*, [w]) - \beta(Q^*, [w^*]) + \gamma(Q^*, [w]) - \gamma(Q^*, [w^*]). \tag{8}$$

Then equations (6) and (7) imply that the left-hand side of (8) is strictly positive. But the right-hand side of (8) is equal to $\int_0^\infty e^{-rt}\int_{-\infty}^\infty w^*(x,t)\{h(x,t\,|\,[w],Q^*) - h(x,t\,|\,[w^*], Q^*)\}dxdt + Q^*\int_0^\infty e^{-rt}\{f(t\,|\,[w], Q^*) - f(t\,|\,[w], Q^*)\}dt$, and this expression cannot be positive since the quitting policy implied by $\{h(x, t\,|\,[w^*], Q^*), f(t\,|\,[w^*], Q^*)\}$ is optimal for the workers when faced with the wage contract $w^*(x, t)$ and the present value of quitting Q^*. *Q.E.D.*

Since workers and firms are risk neutral, $w^*(x, t)$ is not a unique equilibrium contract, any random variable ξ possessing the property $E_{xt}(\xi) = w^*(x,t)$ would also qualify. A pure piece-rate wage involving a payment of $X(t + \Delta t) - X(t)$ over the interval $(t, t + \Delta t)$ therefore also qualifies as equilibrium since $E_{xt}dX(t) = w^*(x, t)dt + \sigma E_{xt}dz(t) = w^*(x, t)dt$. Any such contract leads to identical turnover behavior as under $w^*(x, t)$. Even within the class of functions of x and t alone, $w^*(x, t)$ may not be unique. The following theorem guarantees, however, that turnover behavior is unique.

Theorem 2.—If $w \in B$ then $h\{x, t\,|\,[w], Q([w])\} = h\{x, t\,|\,[w^*], Q([w^*])\}$, and $f\{t\,|\,[w], Q([w])\} = f\{t\,|\,[w], Q([w^*])\}$.

Proof.—See Jovanovic 1978*a*. The proof is lengthy and not particularly instructive. Theorem 2 states that the separation policy of the worker is unique even though the wage contract leading to it is not. This turnover behavior is identical with that which results in a situation in which each firm offers a wage contract $w^*(x, t) = E_{xt}(\mu)$.

Pareto optimality of turnover.—Since all the agents are risk neutral, the correct optimality criterion is the maximization of the discounted expectation of aggregate output. Theorem 2 implies that whatever the prevailing equilibrium wage contract, the worker behaves so as to maximize his own expected discounted output. He collects all of the rent associated with the match, and the decision about whether or not to terminate the match rests with him (although the firm is equally involved in the separation decision since it lowers the worker's wage to the point where it knows the worker will quit). Therefore, a separation occurs if and only if the rent associated with the match falls to zero. A central planner could improve on this situation only if he knew *which* workers and *which* firms would make good matches.

Assume that the worker is faced with the wage contract $w^*(x, t) = E_{xt}(\mu)$ and a present value of quitting Q. The sufficient statistics (state variables) are $X(t)$ and t. It is more convenient to use instead $w(t)$ and t as the two state variables, where $w(t) = E_{x(t)t}(\mu)$. Since $w(t)$ is normally distributed with mean m and variance $s - S(t)$ for all t, it satisfies the stochastic differential equation

$$dw(t) = S(t)\sigma^{-1}dz(t) \quad w(0) = m \tag{9}$$

so that the worker's wage follows a driftless random process with ever-decreasing incremental variance that tends to zero as tenure tends to infinity. Let $V(w, t)$ be the ("current") value of the game to the worker who has tenure t and wage $w(t) = w$. Then letting E_{wt} denote the mathematical expectation operator conditioned upon w and t,[6]

$$V(w, t) = w\Delta t + e^{-r\Delta t}E_{wt}V(w[t + \Delta t], t) + o(\Delta t). \tag{10}$$

Subtracting $V(w, t)$ from both sides, dividing through by Δt, taking the limit as Δt tends to zero, and applying Itô's Lemma (see Kushner 1971) yields

$$w - rV(w, t) + \frac{S(t)^2}{2\sigma^2}V_{ww}(w, t) + V_t(w, t) = 0. \tag{11}$$

As with most optimal stopping problems involving Markov processes, the space of points (w, t) can be divided into a continuation region and a stopping region (see Shiryaev 1973). The continuation region consists of those wage-tenure combinations at which it is optimal for the worker to remain with the firm. Equations (10) and (11) hold for all

[6] $o(\Delta t)$ represents terms tending to zero faster than Δt does. Note that the option of stopping on $(t, t + \Delta t)$ (in which case a reward Q is collected) is exercised with a probability that behaves essentially as does

$$\left\{1 - N\left[\frac{\hat{x}}{(\Delta t)^{1/2}}\right]\right\} < \frac{(\Delta t)^{1/2}}{\sqrt{2\pi}\hat{x}} \exp\left[-\frac{\hat{x}^2}{2(\Delta t)^{1/2}}\right] = o(\Delta t)$$

(see Feller 1966, p 171), where the inequality follows by a well-known result on the Mill's ratio and where $\hat{x}$ is equal to $w - \theta(t)$.

wage-tenure combinations that belong to the continuation region. Let $[\theta(t), t]$ be the boundary of the continuation region so that along the boundary $V[\theta(t), t] = Q$, and $\theta(t)$ may be thought of as the reservation wage at which the worker quits the firm. Evaluating equation (11) at $w = \theta(t)$, $\theta(t) = rQ - [S(t)^2/2\sigma^2]V_{ww}[\theta(t), t] - V_t[\theta(t), t]$. A well-known "smooth-fit" condition of optimal stopping (see Shiryaev 1973) states that along the boundary,[7] $V_t[\theta(t), t] = \partial Q/\partial t = 0$, implying that

$$\theta(t) = rQ - \frac{S(t)^2}{2\sigma^2}V_{ww}[\theta(t), t]. \tag{12}$$

In the interior of the continuation region $V(w, t) > Q$. Since at the reservation wage $V[\theta(t), t] = Q$,[8] and since $V_w[\theta(t), t] = 0$, this implies that $V_{ww}[\theta(t), t] \geq 0$. Note that $S(t)$ declines monotonically to zero which suggests that $\theta(t)$ should be monotonically increasing up to rQ. It is possible to prove (see the Appendix) that $\theta(t) < rQ$ for all t, that $d\theta/dt \geq 0$, and that $\lim_{t\to\infty} \theta(t) = rQ$ so that the reservation wage increases up to its limit from below. The reason for the increase in the reservation wage is the decrease of the incremental variance of the wage process as tenure increases. A large incremental variance implies a large dispersion in possible future wages. If wages turn out to be very high the worker does not quit. If they become very low, the worker partially avoids this adverse outcome by quitting and collecting Q. In the absence of the opportunity to quit, the risk-neutral worker's welfare would be unaffected by changes in the incremental variance. The limit of the reservation wage is rQ. This is because the wage tends to a constant as t tends to infinity. There is nothing further to be learned, and at the point of indifference between staying and quitting the capitalized value of this constant wage must be equal to the present value of quitting, Q.

To obtain an approximation to the probability of job separation by tenure, set $\theta(t) = rQ$ for all t. Then for this approximation to the reservation wage,[9]

[7] An informal proof is as follows: $V(w, t) = Q + \int_{\theta(t)}^{w} V_w(y, t)dy$ is maximized with respect to $\theta(t)$ (the reservation wage at t). Therefore differentiating both sides with respect to $\theta(t)$, setting the result equal to zero, and taking the limit as w tends to $\theta(t)$, one obtains that $V_w[\theta(t), t] = 0$, which in turn implies $V_t[\theta(t), t] = 0$ since $V[\theta(t), t] = Q =$ constant.

[8] In the Appendix it is shown that $\theta(t) < rQ$ for all t, implying that $V_{ww} > 0$ along the boundary, where it is also true that $V_w = 0$. So, if it was true that the continuation region was bounded from above, this would imply that $V < Q$ for some point in the interior of the continuation region sufficiently close to the boundary, which cannot be true. Therefore, $\theta(t)$ is single valued and it bounds the continuation region from below so that the optimal policy does have the reservation wage property. This is not surprising since it is known (Rothchild 1974, p. 709) that optimal search rules from normal distributions with unknown means and known variance have the reservation price property when the prior distribution is also normal.

[9] The wage is a standard Wiener process in the $s - S(t)$ scale (see the discussion

$$F(t \mid [w], Q) = 2\left\{1 - N\left[\frac{m - rQ}{p(t)^{1/2}}\right]\right\} = 2N\left[\frac{-m + rQ}{p(t)^{1/2}}\right]$$

with density

$$f(t \mid [w], Q) = \frac{(2\pi)^{-1/2}(m - rQ)}{p(t)^{3/2}}\exp\left[\frac{-(m - rQ)^2}{2p(t)}\right]$$

where $N(x) = (2\pi)^{-1/2}\int_{-\infty}^{x} e^{z^2/2}\, dz$ where $p(t) = s - S(t)$ is the precision. The unique mode of this distribution is $(m - rQ)^2$. After the mode, the probability of turnover declines rapidly to zero. Some workers never change jobs, since $\lim_{t\to\infty} F(t \mid \cdot) < 1$.

To determine the predicted behavior of the separation probability by tenure, consider the hazard rate, $\phi(t) \equiv f/(1 - F)$. Then $\phi(t)$ is the density of separation conditional upon an attained level of tenure, t. The model predicts a nonmonotonic relationship: first $[\phi'(t)] > 0$ and then $\phi'(t) < 0$ as t gets relatively large. That $\phi(t)$ must eventually decline follows since $\lim_{t\to\infty} f(t) = 0$, while $1 - F(t)$ is bounded away from zero. The precise mathematical expression for the tenure level t^* at which $\phi'(t)$ changes sign and finally becomes negative cannot be obtained in closed form, but since $f' > 0$ implies $\phi' > 0$ clearly $t^* \geqslant m - rQ$ = the model of f. If the mode of f is close to zero, $\phi'(t)$ is likely to become negative early on, as appears to be the case empirically (see Jovanovic and Mincer 1978).

The tenure-wage profile (defined as the conditional expectation of the wage given that the worker has attained tenure t) may also be calculated[10] and is equal to $\hat{w}(t) = (m + (m - rQ)2N\{-a[s - S(t)]^{-1/2}\}/1 - 2N\{-a[s - S(t)]^{1/2}\})$. Note that $\hat{w}(t)$ increases monotonically from m when tenure is zero up to $[m + (m - rQ)2N(-as^{-1/2})/1 - 2N(-as^{-1/2})]$ when tenure tends to infinity. Therefore, as low-wage workers quit and high-wage workers stay, the model implies that the average wage of a cohort of workers increases with tenure, eventually at a decreasing rate. In the limit, as tenure becomes indefinitely large, the average wage of those members of the cohort who have not quit approaches a constant as the wage of each worker becomes constant and equal to his true productivity. This then is an alternative explanation for wage growth on the job.

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preceding eq. [9]). Therefore the formula represents the first passage probability for a Wiener process through a linear boundary (Cox and Miller 1965, p. 221).

[10] The probability that a Wiener process will not cross a linear boundary by a particular time and that it will end up at a particular value at that time is also available in closed form (see Cox and Miller 1965, p. 221, eq. 71). After appropriate adjustment the conditional density of wages (by tenure level) is obtained, and $\hat{w}(t)$ is the mathematical expectation of this distribution.

A mismatch leads to a lower wage and an early separation. Thus, holding constant market experience, average past earnings are likely to be lower for a worker who has experienced many job separations.[11] This prediction appears to be consistent with evidence from the National Longitudinal Study (NLS) mature men's sample (see Bartel and Borjas 1976).

Job durations over the life cycle are identically and independently distributed random variables. The turnover generated by the model therefore forms a pure renewal process (see Feller 1966, chap. 11). Let y denote the worker's labor market experience and $R(y) + o(\Delta y)$ denote the probability that the worker experiences a job separation on the market experience interval $(y, y + \Delta y)$. Then $R(y)$ is the renewal density which satisfies the equation

$$R(y) = f(y) + \int_0^y f(t)R(y - t)dt. \tag{13}$$

Jovanovic and Mincer (1978) prove that a monotonically declining $\phi(t)$ implies a monotonically declining $R(y)$. In other words, a monotonically declining separation probability by tenure is *by itself* sufficient to cause turnover to decline monotonically over the life cycle.[12]

Last, the model generalizes straightforwardly to incorporate permanent differences in workers' characteristics such as level of schooling, ability, race, sex, and so on. The parameters of the model (S, m, σ^2, r) can then be regarded as functions of these variables, with each distinct group of workers treated as though they belonged to a distinct market of workers of that type. The entire analysis remains valid so long as informational symmetry between workers and employers is maintained, so that issues of signaling and self-selection are sidestepped. The nature of the assumed functional dependence between σ, r, m, and s on the one hand, and the workers' personal characteristics on the other, will determine the predicted relationships between turnover and these personal characteristics. This is not pursued here, but is an interesting problem for future research.

[11] Holding everything else constant. This statement should not be interpreted as saying that within a group of observationally equivalent people those that have changed jobs often in the past have had lower average past earnings than those that have not changed jobs often. In other words, the model does not imply that "movers" should do worse than "stayers" even though empirically this appears to be true.

[12] A similar relationship holds for wages: Let $L(y)$ be the mathematical expectation of the wage at a given level of labor market experience y. Then $L(y)$ satisfies the equation $L(y) = \hat{w}(y)[1 - F(y)] + \int_0^y f(t)L(y - t)dt$. Eq. (13) is known as the renewal equation which, for any given continuous density $f(t)$, possesses a unique solution $R(y)$ (Feller 1966) such that $R(0) = f(0)$ and $\lim_{y\to\infty} R(y) = [\int_0^\infty tf(t)dt]^{-1}$.

Appendix

We now prove the assertions made in the text following equation (12) about $\theta(t)$, the boundary of the optimal continuation region. We prove that $\theta(t) < rQ$ for all t, that $\theta(t)$ is nondecreasing, and that it approaches rQ as t tends to infinity. Some transformations of the original problem were necessary before these assertions could be proved, and since these transformations move one away from the economics of the problem, it seemed preferable to include these proofs in the Appendix.

Suppose that a probability space (Ω, F, P) is given, with ω being the elementary events $(\omega \in \Omega)$. For any real-valued F-measurable function $f(\omega)$, the mathematical expectation operator E is defined as $E[f(\omega)] = \int f(\omega)dP$. Let $X(t) \in R^1$ be a Markov process defined on the above space. A particular sample path of the process is written as $[X(t, \omega)]_{t=0}^{t=\infty}$. Let E_{xt} be the expectation operator conditional upon $X(t) = x$.

Consider the following problem of optimally stopping $X(t)$: Let a utility function $u(x)$ be given, when $u(x)$ denotes the instantaneous payoff to the player at time t if the game is still in progress at t and if $X(t) = x$. Let $G(x)$ be the terminal payoff function denoting the utility to the player if the game is stopped exactly at t and $X(t) = x$. The player's objective is to maximize his expected discounted utility from playing the game (with r = the discount rate)

$$E\left(\int_0^{T(\omega)} e^{-rt}u[X(t, \omega)]dt + e^{-rT(\omega)}G\{X[T(\omega), \omega]\}\right) \tag{A1}$$

over F-measurable stopping time functions $T(\omega)$. A further restriction on $T(\omega)$ is that it must not anticipate the future. A rigorous discussion of this requirement appears in Shiryaev (1973). For most stopping problems, and certainly for the problems discussed below, this requirement means that the solution to the optimal stopping problem can be characterized by a continuation region for the process $X(t)$ so that the first exit time from the region is the optimal stopping time for $X(t)$. Let $C(x, t)$ be the value of the game to the player at t, conditional upon $X(t) = x$. Then

$$C(x, t) = E_{xt}\left(\int_t^{T^*(\omega)} e^{-r(s-t)}u[X(s, \omega)]ds + e^{-r[T^*(\omega)-t]}G\{X[T^*(\omega), \omega]\}\right) \tag{A2}$$

where $T^*(\omega)$ is the optimal stopping policy and $C(x, t)$ is the current value function. Let

$$U(x, t) = E_{xt}\int_t^{\infty} e^{-r(s-t)}u[X(s, \omega)]ds. \tag{A3}$$

Let $g(x, t) = G(x) - U(x, t)$, for all (x, t), and let

$$\psi(x, t) = E_{xt}e^{-r[T^*(\omega)-t]}g\{X[T^*(\omega), \omega], T^*(\omega)\}, \tag{A4}$$

and consider the problem of maximizing

$$E\ e^{-rT(\omega)}\ g\{X[T(\omega), \omega], T(\omega)\} \equiv E\ g^*\{X[T(\omega), \omega], T(\omega)\} \tag{A5}$$

over stopping-time functions $T(\omega)$. Let $\hat{T}(\omega)$ be the optimal policy for this problem. Then the following theorem holds.

Theorem 3.—If $E\int_0^\infty e^{-rt}\left|u[X(t, \omega), t]\right|dt < \infty$, then $\hat{T}(\omega) = T^*(\omega)$, and

$$C(x, t) = U(x, t) + \psi(x, t). \tag{A6}$$

Proof.—Shiryaev 1973, p. 101. Theorem 3 asserts that stopping problems such as (A1) which involve an instantaneous utility obtainable while the game is played can be transformed into problems such as (A5) which involve only a terminal payoff function $g(x, t)$. Note that $U(x, t)$ is the current value of the policy "never stop the game no matter what happens to $X(t)$."

Let $X(t)$ satisfy the stochastic Itô equation

$$X(t) = X(0) + \int_0^t a[X(s), s]ds + \int_0^t b[X(s), s]dv(s) \tag{A7}$$

(or $dX(t) = a[X(t), t]dt + b[X(t), t]dv(t)$ in differential form). Here $v(t)$ is the standard Wiener process and $X(t)$ is a Markov process with instantaneous mean $a(\cdot)$ and instantaneous variance $[b(\cdot)]^2$.

The following theorem contains the basic results associated with the problem of optimally stopping $X(t)$ when $X(t)$ is defined by equation (A7).

Theorem 4.—Let $X(t)$ be defined by equation (A7), and let the stopping problem be given by equation (A5). Let $T_0 < \infty$ be given, and in addition to the other requirements on $T(\omega)$, let $T(\omega) \in [0, T_0]$ for all $\omega \in \Omega$. Let $J = \{(t, x): t \in [0, T_0], x \in R^1\}$, and let $V(x,t) = \sup E_{xt}\xi\{X[T(\omega),\omega], T(\omega)\}$, where the sup is taken over the admissable functions $T(\omega)$. Assume that the functions $a(\cdot)$, $b(\cdot)$ and $\xi(\cdot)$ are all twice continuously differentiable in x and once in t, and that for all $(x,t) \epsilon J$, $|\xi| + |\xi_t| + |\xi_x| + |\xi_{xx}| \leq \hat{k}(1 + |x|)^k$, $|a_{xx}| + |b_{xx}| + |a_x| + |b_x| \leq \hat{k}(1 + |x|)^k$, and that $|a_x| + |b_x| \leq \hat{k}$ where $\hat{k}$ and k are positive constants. Let $D = [(t,x): V > \xi]$ and $A = \{(t,x): \xi_t(\cdot) + a(\cdot)\xi_x(\cdot) + (1/2)[b(\cdot)]^2\xi_{xx}(\cdot) > 0\}$. Then the following propositions hold: (1) $V \geq \xi$ on J. (2) If V is differentiable, then $V_t(\cdot) + a(\cdot)V_x(\cdot) + (1/2)[b(\cdot)]^2V_{xx}(\cdot) = 0$ for $(t,x) \in J$. (3) The first exit time of the process $[t, X(t)]$ from D is an optimal stopping time. Therefore D is the region of the continued observations, and along its boundary, $V = \xi$. (4) $A \subset D$. (5) If A is connected, so is D.

Proof.—Miroshnichenko 1975, p. 387. Consider now the worker's problem. Let $w^*[X(t),t] = E_{X(t)t}(\mu) \equiv W^*(t)$ be the basic Markov process defined on (Ω, F, P). The worker maximizes discounted expected earnings. His instantaneous utility is $W^*(t)$, while the terminal payoff function is a constant, Q. Therefore the counterpart of equation (A1) is

$$E\left[\int_0^{T(\omega)} e^{-rt}W^*(t, \omega)dt + e^{-rT(\omega)}Q\right]. \tag{A8}$$

The process $W^*(t)$ has zero drift. Therefore the counterpart of $U(x, t)$ is $E_{W^*t} \int_t^\infty e^{-r(s-t)}W^*(s, \omega)ds = r^{-1}W^*$. Therefore, $g(x, t) = Q - r^{-1}W^*$. Since $E\int_0^\infty e^{-rt}|W^*(t, \omega)|dt < \infty$, theorem 3 may be applied to the problem to conclude that the solution to the worker's problem of maximizing the expression in (A8) is identical with the solution to the problem of maximizing

$$Ee^{-rT(\omega)}\{Q - r^{-1}W^*[T(\omega), \omega]\}. \tag{A9}$$

If $T^*(\omega)$ is the optimal solution, then equation (A6) yields

$$C(W^*, t) = r^{-1}W^* + Ee^{-r[T^*(\omega)-t]}\{Q - r^{-1}W^*[T^*(\omega), \omega]\} \tag{A10}$$

where $C(W^*, t)$ is the worker's current value function. Now let $W(y)$ be the standard Wiener process, with $W(0) = m$; $W^*(t)$ is a standard Wiener process in the $s - S(t)$ scale (Chernoff 1968, p. 226). Letting $y \equiv s - S(t) \rightarrow t = \sigma^2[(s - y)^{-1} - s^{-1}]$, and $Y^*(\omega) \equiv s - S[T^*(\omega)]$,

$$Ee^{-r[T^*(\omega)-t]}\{Q - r^{-1}W^*[T^*(\omega),\omega]\} = Ee^{-r[T^*(\omega)-t]}(Q - r^{-1}W\{s - S[T^*(\omega)],\omega\})$$
$$= \{Q - r^{-1}W[Y^*(\omega),\omega]\} \exp(-r\sigma^2\{[s - Y^*(\omega)]^{-1} - (s - y)^{-1}\}) \tag{A11}$$

where $T^*(\omega)\in[0,\infty) \rightarrow Y^*(\omega)\in[0, s)$. The problem has therefore been transformed into one of stopping a standard Wiener process, $W(y)$, on the interval $[0, s)$, with only a terminal payoff function

$$\xi(W, y) \equiv (Q - r^{-1} W) \exp\{-r\sigma^2 [(s - y)^{-1} - s^{-1}]\}. \tag{A12}$$

Theorem 4 may now be applied to this problem with $a(\cdot) = 0, b(\cdot) = 1$. Let $V(W, y)$ be the present value function for this problem defined by

$$V(W, y) = E_{Wy}\, \xi\{W[Y^*(\omega), \omega],Y^*(\omega)\}. \tag{A13}$$

Since $\xi_{ww} = 0$,

$$A = [(W, y):\xi_y > 0] = [(W, y):y\in[0, s),W > rQ]. \tag{A14}$$

Proposition 4 of the theorem asserts that $A \subset D$ where D is the continuation region for the process $[W(y), y]$. Let $[\theta(y), y]$ be the boundary of the continuation region. Then $[\theta(y), y] \notin A \rightarrow$

$$\theta(y) \leq rQ \qquad \text{for } y\in[0, s). \tag{A15}$$

The $\theta(y)$ is the reservation wage in the (W, y) space. Let $\hat{\theta}(t)$ be the reservation wage in the (W, t) space. Then $\hat{\theta}(t) = \theta[s - S(t)]$.

Theorem 5.—$\theta(y) < rQ$ for $y\in[0, s)$.

Proof.—Along the boundary,

$$V[\theta(y), y] = \xi[\theta(y), y]. \tag{A16}$$

In view of (A15), it is sufficient to prove that $\theta(y) \neq rQ$ for any $y\in[0, s)$. By contradiction, suppose that for some $y°\in[0,s)$, $\theta(y°) = rQ$. Equation (A16) then implies that $V[\theta(y°), y°] = \xi(rQ, y°) = 0$. Consider now the value of the following policy at $(rQ, y°)$: For some δ such that $y° + \delta < s$, continue the game until $y° + \delta$. Then if $v(y° + \delta) < rQ$, stop the game at $y° + \delta$, and collect $\xi[v(y° + \delta), y° + \delta] > 0$. If $v(y° + \delta) > rQ$, continue the game until $y = s$, and collect a payoff equal to zero. But prob $[v(y° + \delta) < rQ$ given that $v(y°) = rQ] = 1/2$, and so there is a positive expected payoff under this policy. Since this policy is feasible, $V[\theta(y°), y°]$ must also be positive. This completes the proof of the theorem.

Let $F(y)$ be the probability that the worker's optimal policy will lead him to quit before y. Then $F(y°) = \text{prob}\{\inf_{0 \leq y \leq y°}[W(y) - \theta(y)] \leq 0\}$. Let $f(y)$ be the density. Then

$$E\xi\{W[Y^*(\omega), \omega], Y^*(\omega)\} = \int_0^s \xi[\theta(y), y]f(y)dy. \tag{A17}$$

The Envelope Theorem

Let α and β be two parameters. Assume that the evolution of $X(t)$ is not affected by α and β. Let $u(x, t, \alpha)$ be the instantaneous utility function in present value terms, and let $G(x, t, \beta)$ be the terminal payoff function also in present value terms. Let $[\theta(t; \alpha, \beta), t]$ be the optimally determined boundary of the continuation region for the process $[X(t), t]$. The function $\theta(t, \alpha, \beta)$ is assumed to be single valued. Let $h(x, t, \alpha, \beta)$ be the probability (density) that the game will not have been stopped before t, and that $X(t) = x$, and let $f(t, \alpha,$

β) be the probability (density) that the game will be stopped exactly at t. It is clear that $\theta(\cdot)$, $h(\cdot)$, and $f(\cdot)$ are in one-to-one correspondence with one another and should be thought of as decision variables. Let T be the horizon, $0 < T \leqslant \infty$. Let $V(\alpha, \beta)$ be the value of the game at time zero. Then,

$$V(\alpha, \beta) = \int_0^T \int_{-\infty}^{\infty} u(x, t, \alpha) h(x, t, \alpha, \beta) dx dt + \int_0^T G[\theta(t, \alpha, \beta), t, \beta] f(t, \alpha, \beta) dt. \tag{A18}$$

Theorem 6 (Envelope Theorem).—If α and β do not affect the evolution of $X(t)$, and if $u(\cdot)$, $G(\cdot)$, $h(\cdot)$, $f(\cdot)$, and $\theta(\cdot)$ are differentiable with respect to α and β, then $dV/d\alpha = \partial V/\partial\alpha = \int_0^T \int_{-\infty}^{\infty} u_\alpha(x, t, \alpha) h(x, t, \alpha, \beta) dx dt$ and $dV/d\beta = \partial V/\partial\beta = \int_0^T G_\beta[\theta(t, \alpha, \beta), t, \beta] f(t, \alpha, \beta) dt$.

Proof.—Unless stated otherwise, $u(\cdot)$, $G(\cdot)$, $h(\cdot)$, $f(\cdot)$, and $\theta(\cdot)$ are all evaluated at (x, t, α, β). Furthermore, since the proof for α is almost identical with the proof for β, only the latter is given:

$$\frac{dV}{d\beta} = \iint u h_\beta dx dt + \int (G_x \theta_\beta + G f_\beta) dt + \int G_\beta f dt. \tag{A19}$$

Therefore if it can be shown that

$$\iint u h_\beta dx dt + \int (G_x \theta_\beta + G f_\beta) dt = 0 \tag{A20}$$

the theorem will have been proved. Since the worker's policy in response to α and β is optimal,

$$V(\alpha, \beta) \geqslant \iint u h(x, t, \alpha, \beta + d\beta) dx dt + \int G[\theta(t, \alpha, \beta + d\beta), t, \beta] f(t, \alpha, \beta + d\beta) dt \tag{A21}$$

for all $d\beta \neq 0$. Subtracting $V(\alpha, \beta)$ from both sides of (A21), dividing through by $d\beta$, and taking the limit as $d\beta \to 0$, the result is

$$\iint u h_\beta dx dt + \int [G_x \theta_\beta + G f_\beta) dt \leqslant 0. \tag{A22}$$

A change in β implies, in general, a change in the optimal stopping policy. But the policy which was optimal prior to the shift in β remains a feasible policy. Therefore

$$\frac{dV}{d\beta} \geqslant \frac{\partial V}{\partial\beta} = \int G_\beta f dt. \tag{A23}$$

Equations (A23) and (A19) imply that

$$\iint u h_\beta dx dt + \int (G_x \theta_\beta + G f_\beta) dt \geqslant 0, \tag{A24}$$

and (A24) and (A19) imply that (A20) holds $\to$ $dV/d\beta = \partial V/\partial\beta$ and the theorem is proved.

The results of Theorem 6 are now used to obtain qualitative information about the derivatives of the worker's current value function $C(W^*, t)$. Since $W^*(0) = m$,

$$\begin{aligned} C(m, 0) &= r^{-1}m + \int_0^s \xi[\theta(y), y] f(y) dy \\ &= r^{-1}m + \int_0^s [Q - r^{-1}\theta(y)] \exp\{-r\sigma^2 [(s - y)^{-1} - s^{-1}]\} f(y) dy \\ &= \alpha[Q, (w^*)]. \end{aligned} \tag{A25}$$

By the envelope theorem, since $\xi[\theta(s), s] = 0$,

$$\frac{\partial C}{\partial s} = \int_0^s \frac{\partial \xi}{\partial s}[\theta(y), y]f(y)dy + \xi[\theta(s), s]f(s)$$
$$= r\sigma^2 \int_0^s [(s-y)^{-2} - s^{-2}]\xi[\theta(y), y]f(y)dy. \quad \text{(A26)}$$

Since $f(y)$ is a density, it is nonnegative, while theorem 5 implies that $\xi[\theta(y), y] > 0$ for $y \in [0, s)$. Therefore $(\partial C/\partial s)(m, 0) > 0$. But the state $(m, 0)$ is arbitrary. If the state is (W, y), where $y = s - S(t)$, the analogue of the right-hand side of equation (A25) would hold, with s replaced by $S(t)$. The only way in which the worker's welfare is affected by the mere passage of time is through the decrease in $S(t)$. Since $\partial C(W^*, t)/\partial S(t) > 0$,

$$\frac{\partial C(W^*, t)}{\partial t} = \frac{\partial C}{\partial S(t)} \frac{dS(t)}{dt} = \frac{-\partial C}{\partial S(t)} \sigma^{-2}[S(t)]^2 < 0. \quad \text{(A27)}$$

The envelope theorem cannot be directly applied in (A25) to calculate $\partial C/\partial m$ because m is the starting point of the standard Wiener process $W(y)$, and if it is changed it changes the probabilities of reaching a given boundary $\theta(y)$. However, $f(y^\circ)$ is the derivative of $F(y^\circ)$ which in turn is defined by

$$F(y^\circ) = \text{prob}\,\{\inf_{0 \le y \le y^\circ} [W(y) - \theta(y)] \le 0 \mid W(0) = m\}$$
$$= \text{prob}\,(\inf_{0 \le y \le y^\circ} \{W(y) - [\theta(y) + dm]\} \le 0 \mid W(0) = m + dm). \quad \text{(A28)}$$

This means that if $[\theta(y), f(y)]$ was a feasible policy pair prior to the change in m, then the new feasible policy pair is $[\theta(y) + dm, f(y)]$. In other words, after the change in m, the boundary $[\theta(y) + dm]$ induces the same first-passage density $f(y)$ as did the boundary $\theta(y)$ prior to the change in m, and this holds for all boundaries $\theta(y)$. Therefore, the change from m to $m + dm$ can be considered as having no effect on the feasibility of reaching a boundary, but simply as changing the form of the payoff function from $\xi(W, y)$ to $\xi(W + dm, y)$. Application of theorem 6 then yields

$$\frac{\partial}{\partial m}\left\{\int_0^s \xi[\theta(y), y]f(y)dy\right\} = -r^{-1}\int_0^s \exp\,\{-r\sigma^2[(s-y)^{-1} - s^{-1}]\}f(y)dy$$

and (A29)

$$\frac{\partial C}{\partial m}(m, 0) = r^{-1}\left(1 - \int_0^s \exp\,\{-r\sigma^2\,[(s-y)^{-1} - s^{-1}]\}f(y)dy\right),$$

and since $f(y)$ is a density, $\partial C/\partial m > 0$. Again, the state (m, o) is arbitrary, and a similar result holds for $(\partial C/\partial W)(W, y)$. Letting $\hat{f}(t) \equiv f(y)(dy/dt)$ be the first-passage probability in the original time scale,

$$\frac{\partial C}{\partial m} = r^{-1}\left[1 - \int_0^\infty e^{-rt}\hat{f}(t)dt\right]. \quad \text{(A30)}$$

Theorem 7.—$\hat{\theta}(t)$ is nondecreasing in t.

Proof.—By contradiction, suppose that at t^*, $\theta(t)$ is decreasing. Then there exists an $\epsilon > 0$ sufficiently small such that the points $[\theta(t^*), t^* + \tau]$ for $\tau \in [0, \epsilon]$ all lie in the continuation region. Therefore, since $C > Q$ in the continuation region,

$$C[\theta(t^*), t^* + \epsilon] > C[\theta(t^*), t^*] = Q. \quad \text{(A31)}$$

But

$$C[\theta(t^*), t^* + \epsilon] = C[\theta(t^*), t^*] + \int_0^\epsilon \frac{\partial C}{\partial t}[\theta(t^*), t^* + \tau]d\tau < C[\theta(t^*), t^*] \quad (A32)$$

in view of (A27). Since (A32) is a contradiction to (A31), the theorem is proved.

Theorem 8.—$\lim_{t\to\infty} \hat{\theta}(t) = rQ$.

Proof.—Since $\hat{\hat{\theta}}(t) = \theta[s - S(t)] = \theta(y)$, it is sufficient to prove that

$$\lim_{y\to s} \theta(y) = rQ. \quad (A33)$$

By contradiction, suppose that $\lim_{y\to s} \theta(y) = \eta$ and that $\eta < rQ$. Now choose $\delta > 0$ such that $\eta + \delta < rQ$. By theorem 7, $\theta(y)$ is nondecreasing in y. Therefore the point $(\eta + \delta, s - \epsilon)$ must lie in the continuation region for any $\epsilon > 0$. In terms of the present value function $V(W, y)$ and the present value of the payoff function $\xi(W, y)$, this means that

$$V(\eta + \delta, s - \epsilon) = \int_{s-\epsilon}^{s} \xi[\theta(y), y]f(\eta + \delta, s - \epsilon, y)dy > \xi(\eta + \delta, s - \epsilon) \quad (A34)$$

where $f(\eta + \delta, s - \epsilon, y)$ is the probability (density) that the game will end at $y \in [s - \epsilon, s)$ given that $W(s - \epsilon) = \eta + \delta$. Since ξ is decreasing in W and decreasing in y, and since $\theta(y)$ is nondecreasing, $\xi[\theta(s - \epsilon), s - \epsilon] > \xi[\theta(y), y]$ for $y \in (s - \epsilon, s)$. Therefore

$$V(\eta + \delta, s - \epsilon) < \xi[\theta(s - \epsilon), s - \epsilon]\int_{s-\epsilon}^{s} f(\eta + \delta, s - \epsilon, y)dy. \quad (A35)$$

Furthermore, $f(\eta + \delta, s - \epsilon, y)$ is the first-passage density of the standard Wiener process (originating at $\eta + \delta$ at $s - \epsilon$) through the boundary $\theta(y)$ on the interval $[s - \epsilon, s)$. Then the integral on the right-hand side of (A35) is smaller than the probability that the same standard Wiener process will cross the threshold $\sup_{y\in(s-\epsilon, s)} \theta(y) = \eta$. From Feller (1966, p. 171) this latter probability is equal to $2[1 - N(\epsilon^{-1/2}\delta)]$ where $N(x) = \int_{-\infty}^{x}(2\pi)^{-1/2}\exp[-1/2\, u^2]du^2$. Therefore

$$V(\eta + \delta, s - \epsilon) < \xi[\theta(s - \epsilon), s - \epsilon]2[1 - N(\epsilon^{-1/2}\delta)]. \quad (A36)$$

Equations (A36), (A34), and (A12) then imply that

$$[Q - r^{-1}\theta(s - \epsilon)]2[1 - N(\epsilon^{-1/2}\delta)] > [Q - r^{-1}(\eta + \delta)] > 0. \quad (A37)$$

But since $\theta(y)$ is nondecreasing, and since by assumption $\lim_{y\to s} \theta(y) = \eta < rQ$,

$$(Q - r\eta)2[1 - N(\epsilon^{-1/2}\delta)] > [Q - r^{-1}(\eta + \delta)]. \quad (A38)$$

The right-hand side of (A38) is positive and does not depend on ϵ. Therefore ϵ may be chosen sufficiently small such that the inequality in (A38) does not hold. The theorem is proved.

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