Estimation Laboratory 2

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1 Introduction

Question 1 :

For a given data set stored in vector $\underline{y}$, the power mean is given by eq. 1 :

$$P_y = \frac{\underline{y}^T.\underline{y}}{N} \tag{1}$$

The definition for the power mean of the noise is much the same, but simplifies under the hypothesis of an independent, identically distributed, centered noise :

$$\begin{aligned} P_\epsilon = & \frac{\underline{\epsilon}^T.\underline{\epsilon}}{N} \\ = & \left\langle \epsilon_0^2 \right\rangle \\ = & \sigma^2 \end{aligned} \tag{2}$$

Hence, we can expect a Signal to Noise Ratio of 20dB if $\sigma = \frac{1}{10}\sqrt{\frac{\underline{y}^T.\underline{y}}{N}}$.

2 Spectral analysis with the Fourier transform

2.1 Estimation for known frequencies

Question 1 :

We denote as $\underline{\epsilon}$ the noise vector of size N, as $\underline{c}$ the complex amplitudes vector of size $2K+1$, and as $\underline{y}$ the data vector of size N. We also define the matrix $\underline{\underline{W}}$ of size $N \times (2K+1)$ whose coefficients are :

$$W_{n,k} = e^{2j\pi f_k t_n} \tag{3}$$

From all these definitions, it is clear that :

$$\underline{y} = \underline{\underline{W}}\,\underline{c} + \underline{\epsilon} \tag{4}$$

Question 2 :

We are considering a linear problem with additive, independent, centered, Gaussian noise, hence the Maximum Likelihood Estimator is simply :

$$\overset{\circ}{\underline{c}}_{ML}(\underline{y}) = (\underline{\underline{W}}^\dagger \underline{\underline{W}})^{-1} \underline{\underline{W}}^\dagger \underline{y} \tag{5}$$

Question 3 :

We implement in MatLab the preceeding equations, and add a method to retrieve the real amplitudes and phase angles from the complex amplitudes :

$$\begin{aligned} \overset{\circ}{\underline{A}}_{ML} = & \quad 2|\overset{\circ}{\underline{c}}_{ML}| \\ \overset{\circ}{\underline{\phi}}_{ML} = & \quad angle(\overset{\circ}{\underline{c}}_{ML}) \end{aligned} \tag{6}$$

We get the results in Tab. 2.1, from which we find that the estimation of the real amplitudes are very good (the relative error is lower than the 1 % noise in the data). However, the estimates of the phases are good, but not as much, with a relative error up to 7 times higher than the noise in the data. Still, the estimates yield a 10 dB Signal-to-Noise Ratio in the worst case, which is acceptable.

Parameter	Value	Estimation	Relative error
A_0	0	0.008	N/A
A_1	0.25	0.253	1.2 %
ϕ_1	0.393	0.364	7.0 %
A_2	0.75	0.748	0.2 %
ϕ_2	0.996	0.975	2.1 %
A_3	1	0.994	0.6 %
ϕ_3	0.493	0.487	1.2 %
A_4	0.75	0.752	0.2 %
ϕ_4	0.281	0.290	3.2 %
A_5	1	1.001	0.1 %
ϕ_5	0.596	0.591	0.8 %

Table 1: Table of comparison between original and estimated parameters

2.2 Estimation for unknown frequencies

Question 1 :

In the irregular sampling case, we cannot simply use the Shannon-Nyquist criteria, because there is no "sampling frequency". We have to study the shape of the spectra to deduce for which frequencies it is reliable, and where the sampling lost information.

The highest frequency in our data in around 35 Hz, and we chose to draw a spectrum up to about 3 times that frequency, hence $\nu_{max.} = 100$ Hz. The smallest difference between 2 frequencies in our data is 0.3 Hz, and we chose to draw a spectrum with a frequency comb 10 times thinner than that, hence $M = 3000$.

Question 2 :

We plot the time and frequency representations of the single-frequency signal in fig. 1, in both the noisy and non-noisy case.

- For the non-noisy signal, we read easily that the amplitude is close to 0.25 on the time representation, and that the frequency is close to 31 Hz on the frequency representation. We could also read that the amplitude is close to $2 \times 0.12 = 0.24$ on the frequency representation. The true values where $\nu = 31.012$ Hz and $A = 0.25$. Hence the parameters are well estimated visually.

- For the noisy signal, we can still estimate the frequency near 31 Hz from the frequency representation. However, we cannot estimate the amplitude from the time representation anymore, we have to rely on the frequency representation to estimate an amplitude around 0.24. But on the whole, the estimated parameters seem as good as in the non-noisy case.

- We did not find a reliable way to estimate the phase of the signal from the plots. We tried to look at the phase of the Fourier transform at the frequency we just found, but the phase of the Fourier transform is too noisy. We tried to look at the time delays of the maximums in the time representation, and to link them to the phase, but even in the non noisy case the imprecision on the frequency makes this task impossible. The only solution left is to use the frequency we found and re-do the analysis of sec. 2.1.

- In the frequency domain, we observe a background noise around 0.4, hence we will not be able to detect a frequency with an amplitude lower than this. We also observe high secondary peaks about 1.5 Hz on both side of the main peak. We believe they are due to the gaps in the sampling, since we see on the time representation that these gaps are periodic with a period close to 1 s. They will clearly be a big issue when we will want to estimate the frequencies in a multiple-frequency signal.

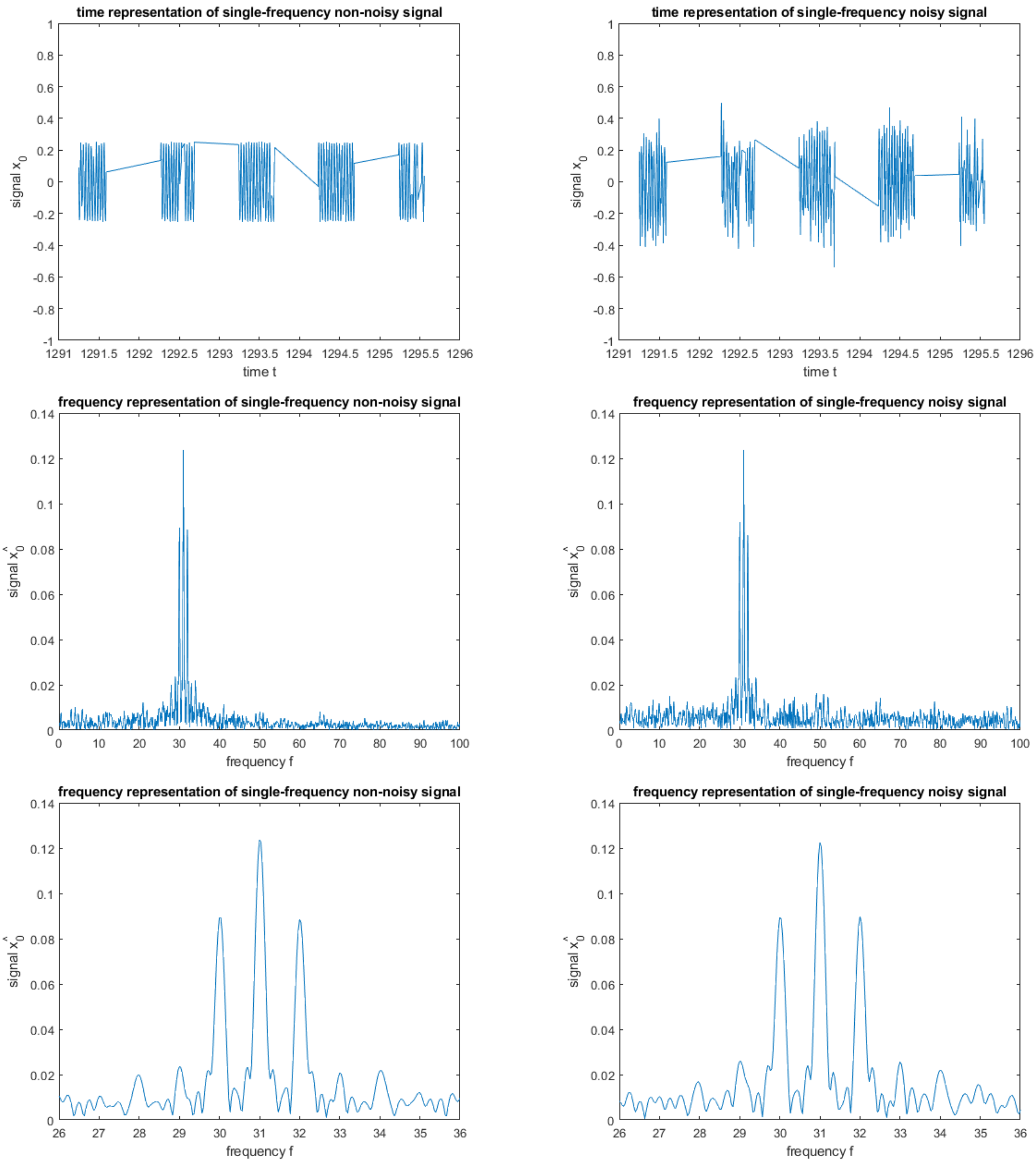

Figure 1: Time and frequency representation of the irregularly sampled, single-frequency signal.

- We observe that there is no Shannon-Nyquist mirror peak due to sampling. This was expected in the irregular sampling case.

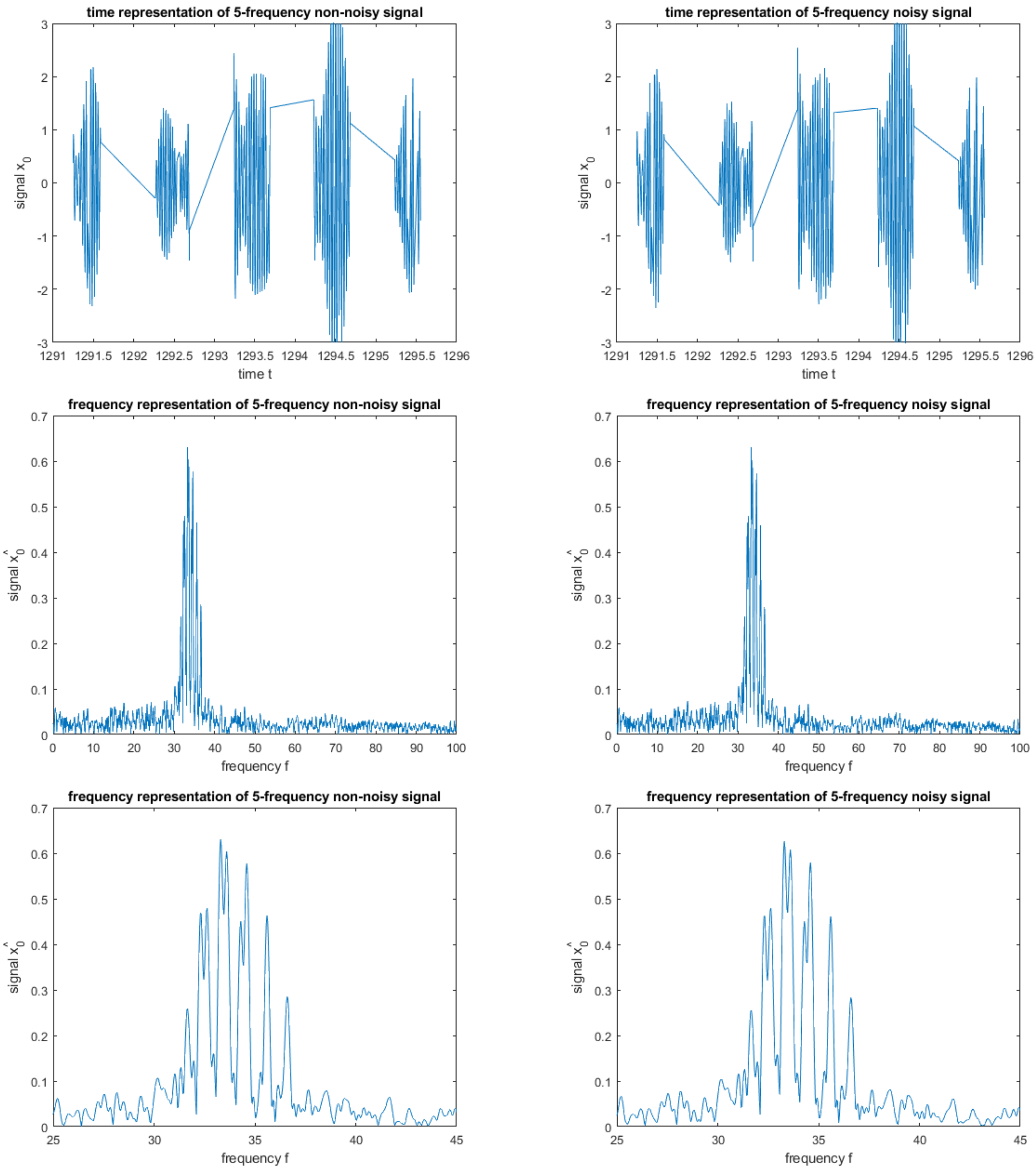

Figure 2: Time and frequency representation of the irregularly sampled, multiple-frequencies signal.

Question 3 :

We plot the time and frequency representations of the multiple-frequencies signal in fig. 2, in both the noisy and non-noisy case.

In both cases, we find very similar results : nothing can be deduced from the time representation because the frequencies mix. And in the frequency representation there is a peak between 30 Hz and 40 Hz, but when we zoom we find that this peak is divided in many secondary peaks that do not correspond to any of the expected frequencies. In particular there is no clear peak at the position of the smallest frequency at 31.012 Hz.

What happens is that because of the secondary peaks observed earlier, we cannot distinguish the five frequencies from each other, or from the effects of the sampling. Hence we cannot deduce anything from these Fourier transforms.

Question 4 :

We find the spectral window from fig. 3 :

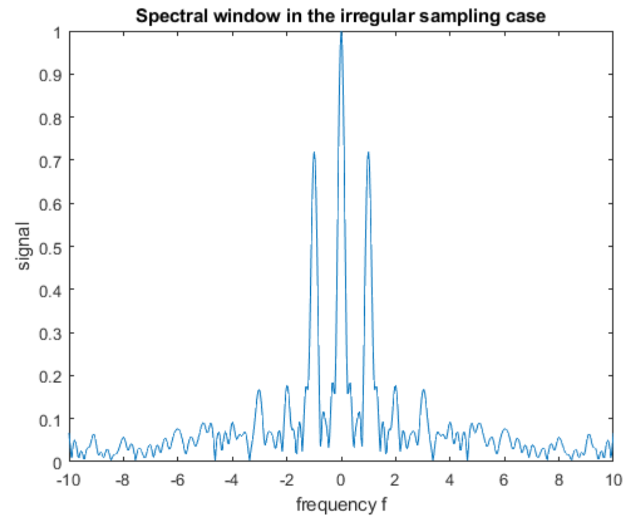

Figure 3: Spectral window for irregular sampling times. We observe symmetric secondary peaks at around 70% amplitude and about 1 Hz around the main peak.

We observe secondary peaks on both sides of the main peak. This is what we predicted in question II.2.2 : The irregular sampling leave an important footprint on the frequency representation. This explains the secondary peaks we observed in question II.2.2, and it also explain why we could not differentiate the frequencies in the multiple-frequency signal : the signal's difference in frequencies was close the secondary peaks' distance to the main peak.

3 Sparse representation with greedy algorithms

3.1 *Matching Pursuit* (MP) algorithm

Question 2 :

For `x_0`, the *Matching Pursuit* algorithm detect 2 frequencies : $31.02 days^{-1}$ and its opposite $-31.02 days^{-1}$. This was easily predictable since $cos(\theta) = \frac{1}{2}(e^{i\theta} + e^{-i\theta})$. The small error is due to the sampling of the frequency grid and the result is really satisfying. The amplitude has been nearly equally divided between the positive frequency and its opposite one (0.1272 and 0.1279 respectively). The sum of the amplitude is a little bit higher than the original one (0.25) but it is also pretty satisfying - relative error of 2%-.
For `x`, if we consider only the different frequencies in absolute value, we have 28 different frequencies between $30.67\, days^{-1}$ and $41.7\, days^{-1}$. with some which are really near to one another. To get a better idea, of the distribution of this frequencies and their amplitudes compared to the 5 original ones, we have plotted the estimated spectrum in figure (4). The 5 true frequencies are plotted in red and the 28 estimated one are in blue. We note that the estimated frequencies are not completely false, they are in the good range of values $(30-40)days^{-1}$ and 4 of them are quite correctly estimated. But the second one $(32.675\, days^{-1})$ is badly recovered, and even if most of the false frequencies have small amplitudes some of them could be wrongly interpreted as characteristic frequencies (especially $31.67\, days^{-1}$). Furthermore, the amplitudes have a mean 25% relative error.

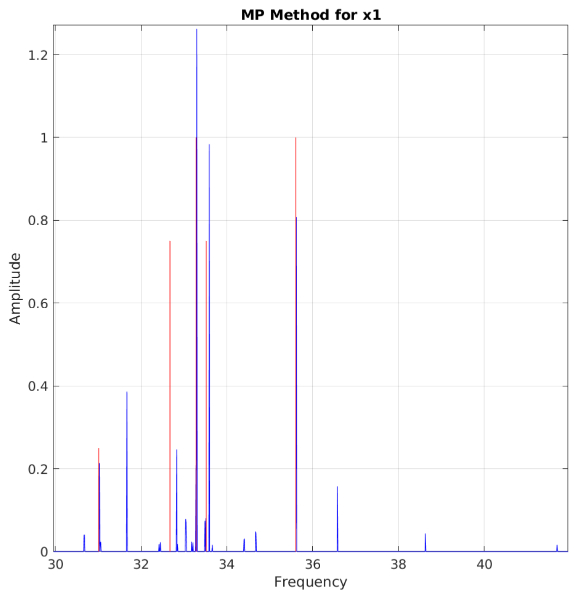

Figure 4: Spectrum of the data x estimated by *Matching Pursuit* algorithm - in blue - compared to the true one - in red -

Question 3 :

3.2 *Orthogonal Matching Pursuit* (OMP) algorithm

Question 2 :

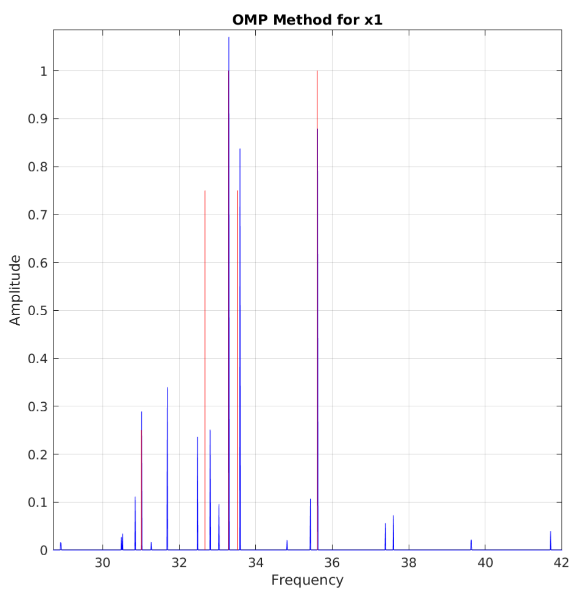

Figure 5: Spectrum of the data x estimated by *Orthogonal Matching Pursuit* algorithm - in blue - compared to the true one - in red -

For x_0, we obtain the same result (same frequencies) with *Orthogonal Matching Pursuit* as with *Matching Pursuit*. The amplitude is equally distributed between the positive frequency and its opposite (which is quite better than the *Matching Pursuit*), the relative error is a little bit higher (2.3%) but still is very good.
For x, we have 21 different frequencies (in absolute values) which is better than the results obtained using *Matching Pursuit* algorithm. But this frequencies are ranged between $28.9\,days^{-1}$ and $41.71\,days^{-1}$ which is more spread than with *Matching Pursuit*. We have plotted in figure (5) the estimated spectrum vs the true one as for *Matching Pursuit* before. We can see that there is less noise around the third ($33.283\,days^{-1}$)and fourth ($33.521\,days^{-1}$) frequencies, but a little

bit more around the first one ($31.012\,days^{-1}$). The second one is still not satisfyingly approached but there is now two pics around it and there still is an issue at $31.69\,days^{-1}$ frequency (was $31.67\,days^{-1}$ for *Matching Pursuit*). Finally, the amplitude estimation are better with a mean 11.5% relative error on the 4 out of 5 well recovered frequencies.

Question 3 :

3.3 *Orthogonal Least Squares* (OLS) algorithm

Question 1 :

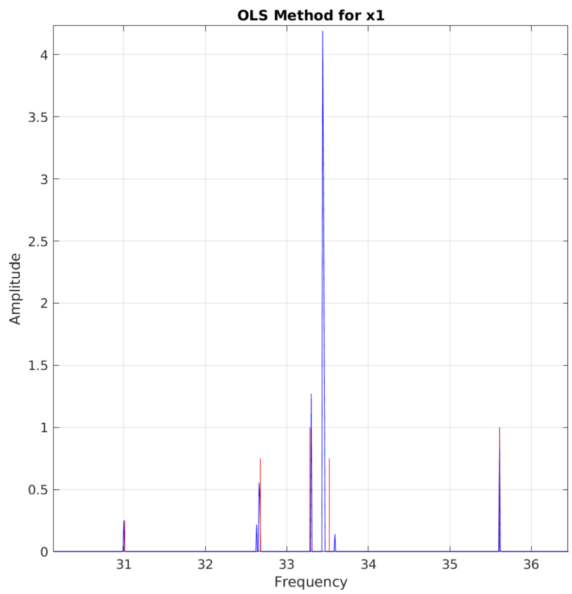

Figure 6: Spectrum of the data `x` estimated by *Orthogonal Least Square* algorithm - in blue - compared to the true one - in red -

For `x_0`, we obtain the exact same results very satisfying as with *Orthogonal Matching Pursuit*.
For `x`, we have 10 different frequencies (in absolute values) which is really better than the results obtained using *Matching Pursuit* or *Orthogonal Matching Pursuit* algorithms. These frequencies are ranged between $31.0\,days^{-1}$ and $35.61\,days^{-1}$ which is a truly good result. There is still one poorly estimated frequency but now it is the fourth one ($33.521\,days^{-1}$) and no more the second one($32.675\,days^{-1}$). Furthermore, close to this fourth frequency there is an estimated pic spreading on 3 frequencies of the grid (33.44 , 33.45 and $33.46\,days^{-1}$) with an amplitude of nearly 4.2, 3 and 1.5 respectively. This is very surprising compared to the quality of the estimation apart from this error and it could be truly misleading in a study were the true state is unknown. The results have been summarized in tab (2).

True frequency	True amplitude	Estimated frequencies	Estimated amplitudes
31.012	0.25	31.00	0.2514
32.675	0.75	32.63	0.2177
		32.66	0.5536
		32.67	0.4171
33.283	1	33.30	1.2722
33.521	0.75	33.44	4.1929
		33.45	2.9772
		33.46	1.5354
		33.59	0.1407
35.609	1	35.61	0.9991

Table 2: Spectrum of the data `x` estimated by *Orthogonal Least Squares* vs true spectrum

Question 2 :

4 Sparse representation with convex relaxation

Question 1 :

The amplitude in the data is of order 0.25. Hence, in the Fourier domain, the lowest spectral amplitude will be $0.125 \approx 0.1$. We want a Signal-to-Noise Ratio of 20dB, Hence the maximal spectral amplitudes of the residuals should be lower than 10^{-3}. Since at convergence we can prove that the maximal spectral amplitude of the residuals is lower or equal to λ/N, with $N \approx 500$, we chose $\lambda = 0.5$.

For this value of λ, we find that we need 100 iterations of the algorithm to reach convergence. More precisely, as shown on fig. 7, we would need a little bit more than 100 iterations for pure convergence, but the residuals are already acceptable for 100 iterations so we keep this value and reduce the computation time.

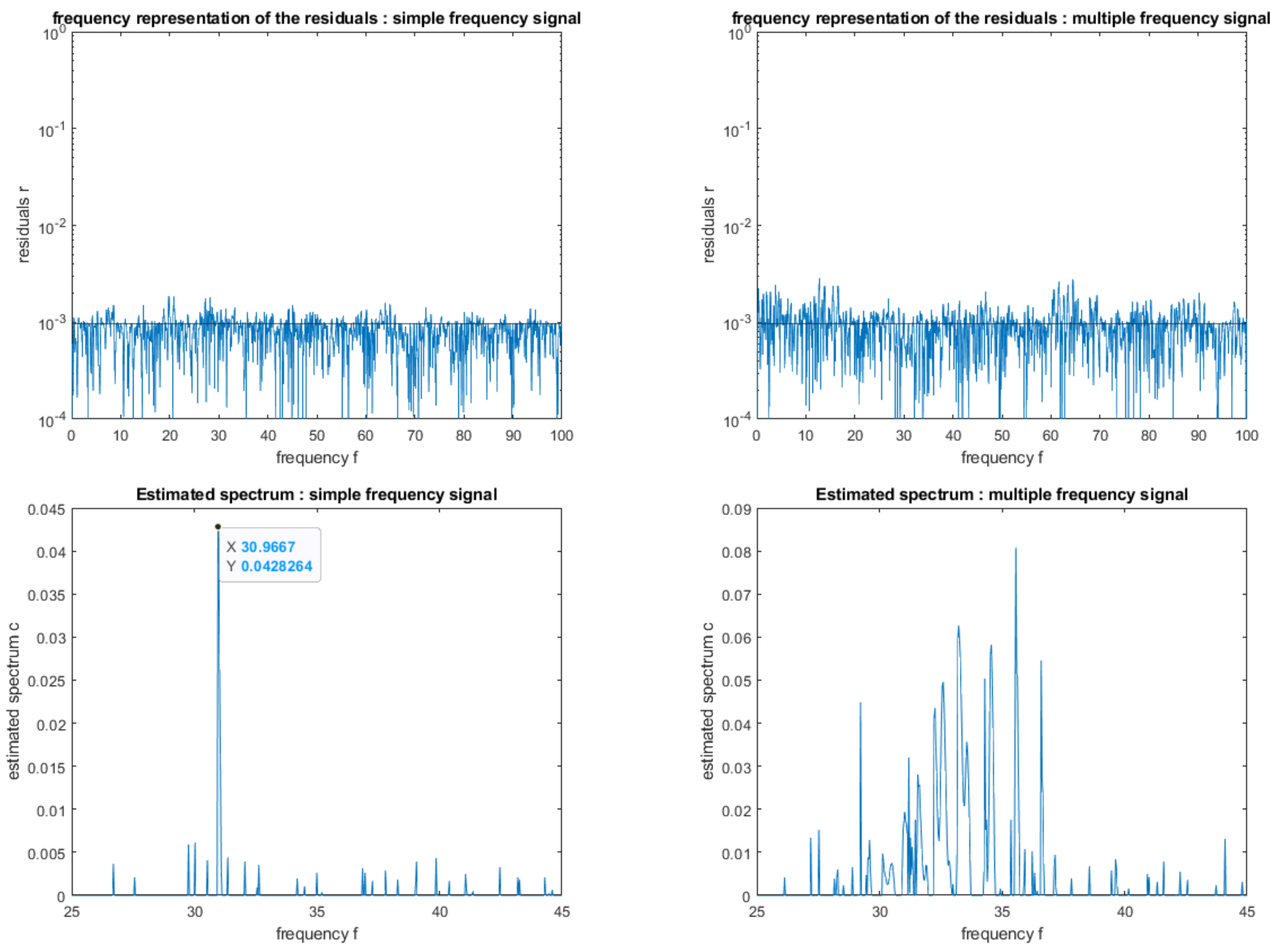

Figure 7: Spectrum of the residuals and estimated spectrum with the convex relaxation method, in the simple-frequency case and the multiple-frequency case. We notice that the residuals are low enough for the signal-to-noise ratio to be at 20dB. But they are slightly above the black line which represents the convergence level, so we do not have pure convergence, only approximate convergence.

Question 2 :

With the simple frequency signal, we see that the estimated spectrum exhibits a single sharp peak at 31 ± 0.2 Hz. The expected value being 31.012 Hz, the estimated frequency is good. However, the estimated amplitude of $2 \times 0.04 = 0.08$ is

far from the expected value of 0.25. Hence we will have to estimate the amplitude (and the phase) with the method from sec. 2.1.
Let's also notice that this issue was expected, is is caused by the relaxation : once the right frequency is detected, changing the height of the peak will not produce a big difference is the norm L2 of the residuals, because the norm L2 is integral and the peak is very sharp. But the height of the peak will have a big effect on the norm L1 of the estimated spectrum. Hence the algorithm reduces the height of the peak to lower the L1 norm of the spectrum.

With the multiple frequency signal, the picture is shadier. We see 6 or 7 peaks in the estimated spectrum, several of them being double peaks. By decreasing order of amplitude, the frequencies are 35.5 Hz, 33.2 Hz, 34.6 Hz, 36.6 Hz, 32.3 Hz, 29.2 Hz and 31.2 Hz. We expected only 5 frequencies (all single-peaked), at different places. And, once again, wildly different amplitudes.

Question 3 :

We run the algorithm of sec. 2.1 with the 7 frequencies found just above, and we get the amplitudes stored in Tab. 4. But since the estimated frequencies were not well correlated with the correct frequencies, we cannot extract much meaning from the estimated amplitudes : they are as incorrect as expected.

Frequency	Amplitude	Frequency	Amplitude	Frequency	Amplitude
0 Hz	0.005	32.3 Hz	0.92	35.5	0.24
29.2 Hz	0.21	33.2 Hz	0.50	36.6	0.65
31.2 Hz	0.57	34.6 Hz	1.17		

Table 3: Maximum Likelihood amplitudes, assuming the frequencies given by the convex relaxation method.

5 Conclusion