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Gravitational potential of a homogeneous circular torus: a new approach

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ABSTRACT

The integral expression for the gravitational potential of a homogeneous circular torus composed of infinitely thin rings is obtained. Approximate expressions for the torus potential in the outer and inner regions are found. In the outer region, the torus potential is shown to be approximately equal to that of an infinitely thin ring of the same mass; it is valid up to the surface of the torus. It is shown in a first approximation that the inner potential of the torus (inside a torus body) is a quadratic function of the coordinates. A method of sewing together the inner and outer potentials is proposed. This method provides a continuous approximate solution for the potential and its derivatives, working throughout the region.

Key words: galaxies: general – gravitation.

1 INTRODUCTION

Toroidal structures are currently detected in astrophysical objects of various types. Examples of such objects are ring galaxies, in which a ring of stars is observed. In some galaxies, the ring-like distribution of stars is believed to be due to collisions of galaxies, for example in M31 (Block et al. 2006) and Arp 147 (Gerber, Lamb & Balsara 1992). Analysis of Sloan Digital Sky Survey (SDSS) data (Ibata et al. 2003) indicates the existence of a star ring in the Milky Way on scales of about 15–20 kpc, which is believed to originate from the capture of a dwarf galaxy. Obscuring tori are observed in central regions of active galactic nuclei (AGN) (Jaffe et al. 2004) and play an essential role in the unified scheme (Antonucci 1993; Urry & Padovani 1995). Ring-like structures are exhibited in dark matter as well. An example might be the galaxy cluster C10024+17, where a ring-like structure has been found in the distribution of dark matter with the use of gravitational lensing methods (Jee et al. 2007). In the Milky Way, the rotation curves, together with the EGRET data, can be explained by the existence of two rings of dark matter located at distances of about 4 and 14 kpc from the Galactic Centre (de Boer et al. 2005). Such toroidal structures can possess a significant mass, and thus gravitationally affect the motion of matter.

B. Riemann devoted one of his last works to the gravitational potential of a homogeneous torus (Riemann 1948). However, this work remained unfinished. For over a century, no attention has been paid to the torus gravitational potential.[1] Kondratyev (2003) returned to this problem for the first time. In this work an exact expression for the potential of a homogeneous torus on the axis of symmetry was obtained. In Kondratyev (2007) the integral expression for a homogeneous torus potential was found using a disc as a primordial gravitating element. Stacking up such discs will result in a torus with potential equal to the sum of the potentials of the component discs. However, it is evident that any integral expressions are problematic to use, both in analytic studies and in the numerical integration of motion equations and also in solving the problems of gravitational lensing. Kondratyev et al. (2009) and Kondratyev & Trubitsina (2010) have obtained an expansion of the torus potential in terms of Laplace series, but showed, however, that such an expansion is impossible inside some spherical shells.

In this paper we propose a new approach to the investigation of the gravitational potential of a torus. Special attention has been paid to finding approximate expressions for the potential, which would simplify the investigation of astrophysical objects with gravitating tori as structural elements. In contrast to Kondratyev (2007), we used an infinitely thin ring as a torus component. Such a ring is actually a realization of a torus, with its minor radius tending to zero and its major one equalling the ring radius. Using such an approach, we obtained an integral expression for the potential of a homogeneous circular torus (Section 2) and approximate expressions for the potential in the outer (Section 3) and inner (Section 4) regions. In Section 5, a method of determining a torus potential for the entire region is suggested.

2 GRAVITATIONAL POTENTIAL OF A HOMOGENEOUS TORUS

Compose a torus with mass M, outer (major) radius R and minor radius R_0 from a set of infinitely thin rings, hereafter component rings (see Fig. 1), with their planes being parallel to the torus symmetry plane. Select a central ring with mass M_c and radius R from the set

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[1]In electrostatics, the potential of a conducting torus shell is used (Smythe 1950), which is much easier than the case in which the torus density is uniform inside its volume.

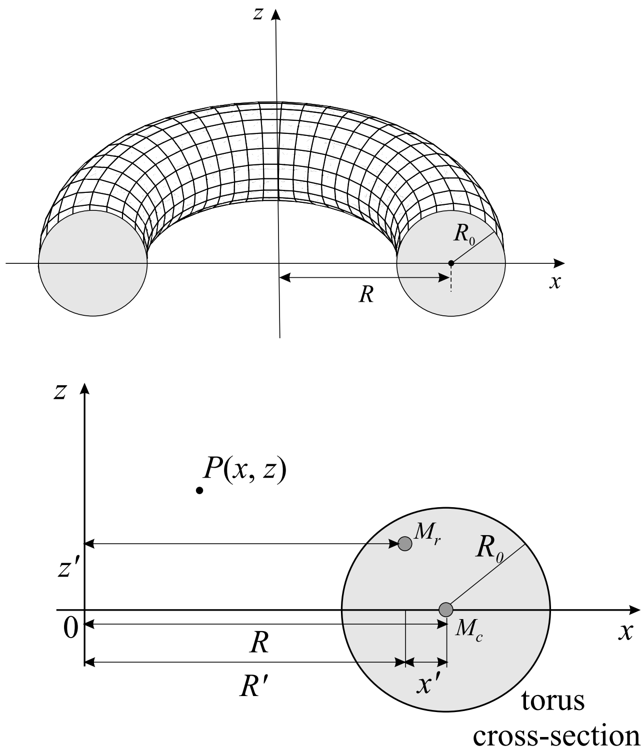

Figure 1. Top: 3D scheme of a torus configuration; bottom: schematic cross-section of a torus as a set of infinitely thin rings.

of rings composing a torus. The potential produced by this ring at an arbitrary point $P(x, z)$ has the form

$$\varphi_c(x, z; R, M_c) = \frac{GM_c}{\pi R}\,\phi_c\left(\frac{x}{R}, \frac{z}{R}\right), \tag{1}$$

where the dimensionless potential of the infinitely thin ring is

$$\phi_c\left(\frac{x}{R}, \frac{z}{R}\right) = \sqrt{\frac{R}{x}\,m}\;K(m), \tag{2}$$

where

$$K(m) = \int_0^{\pi/2} \frac{\mathrm{d}\beta}{\sqrt{1 - m\sin^2\beta}}$$

is the complete elliptical integral of the first kind and its parameter is

$$m = \frac{4xR}{(x+R)^2 + z^2}. \tag{3}$$

The potential at a point $P(x, z)$ produced by an arbitrary ring with radius R' and mass M_r located in the torus at a height z' (Fig. 1) has the form

$$\varphi_r(x, z; M_r, z') = \frac{GM_r}{\pi R'}\,\phi_r, \tag{4}$$

where the expression for ϕ_r is obtained by substitution in (2) of the kind $x/R \to x/R'$ and $z/R \to (z - z')/R'$. We denote a ring coordinate[2] counted off the centre of a torus cross-section (Fig. 1) by $x' = R' - R$, and therefore its radius is $R' = R + x'$. We may conveniently introduce the dimensionless coordinates $\eta' = x'/R$, $\zeta' = z'/R$ and $\rho = x/R$, $\zeta = z/R$ that will result in an expression for the dimensionless potential of the form

[2]Here, by the ring coordinates we mean the coordinates of a point at which the ring intersects the plane of the torus cross-section. The position of this point is determined by the ring radius R' and the ring distance from the torus plane of symmetry z'.

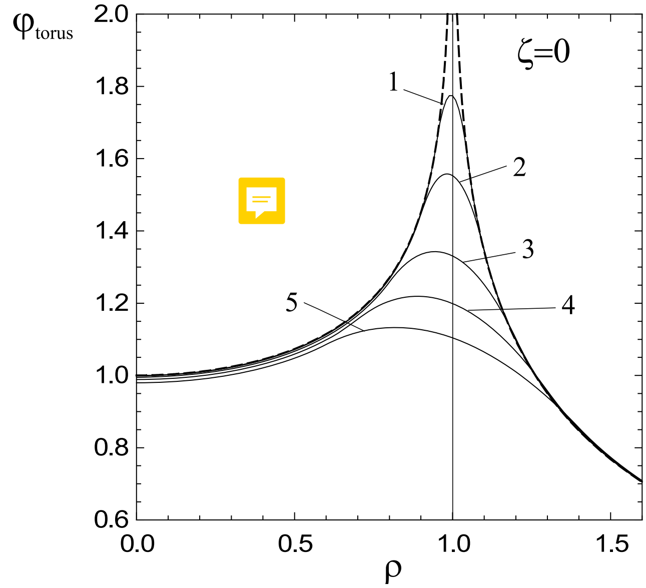

Figure 2. Dependence of the potential on the radial coordinate for $\zeta = 0$, for tori with various values of geometrical parameter: (1) $r_0 = 0.05$, (2) $r_0 = 0.1$, (3) $r_0 = 0.2$, (4) $r_0 = 0.3$, (5) $r_0 = 0.4$. The potential of an infinitely thin ring (1) with mass M equal to the mass of the torus is shown by the dashed line. In this plot and in all subsequent figures, $M = 1, R = 1, G = 1$.

$$\phi_r(\rho, \zeta; \eta', \zeta') = \sqrt{\frac{(1+\eta')\,m_r}{\rho}}\;K(m_r) \tag{5}$$

where

$$m_r = \frac{4\rho\,(1+\eta')}{(1+\eta'+\rho)^2 + (\zeta - \zeta')^2}. \tag{6}$$

From the condition of torus homogeneity, mass-to-radius ratios for the central and arbitrary component rings are the same and thus $M_c = M_r/(1+\eta')$. The expression for the potential of the component ring is then

$$\varphi_r(\rho, \zeta; \eta', \zeta') = \frac{GM_c}{\pi R}\phi_r, \tag{7}$$

where ϕ_r is determined by expressions (5) and (6). Due to additivity, the torus potential can be represented as the integral over the potentials of the component rings. To do this, we replace the discrete mass of ring M_c in (7) by a differential $\mathrm{d}M$, which for a homogeneous torus equals $\mathrm{d}M = [M/(\pi r_0^2)]\,\mathrm{d}\eta'\,\mathrm{d}\zeta'$, where M is the total mass of the torus, equal to the sum of masses of the component rings, and $r_0 = R_0/R$ is the dimensionless minor radius of the torus (a geometrical parameter). Then, the potential of a homogeneous circular torus takes the form

$$\varphi_{torus}(\rho, \zeta) = \frac{GM}{\pi^2 R r_0^2}\int_{-r_0}^{r_0}\int_{-\sqrt{r_0^2-\eta'^2}}^{\sqrt{r_0^2-\eta'^2}} \phi_r(\rho, \zeta; \eta', \zeta')\,\mathrm{d}\eta'\,\mathrm{d}\zeta'. \tag{8}$$

This integral expression for the torus potential is valid for both the inner and outer points. The validity of expression (8) is confirmed by calculation of the potential made by direct integration over the torus volume. Hereafter, in analysing approximate expressions, we will use the term ‘exact’ for the values of potential obtained from the integral formula (8). In Fig. 2, dependences of the torus potential on the radial coordinate are presented; these were obtained numerically from formula (8) for tori with different values of the geometrical

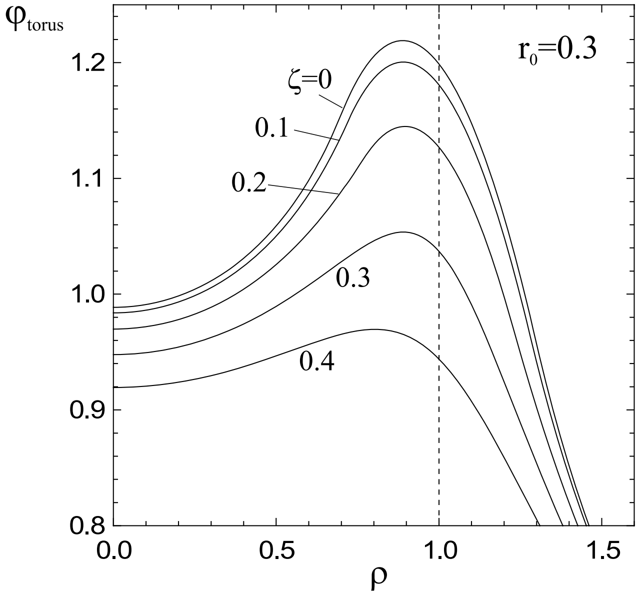

Figure 3. Potential of the torus with $r_0 = 0.3$ as a function of radial coordinate for various values of $\zeta = 0, 0.1, 0.2, 0.3, 0.4$.

parameter r_0. The potential curves for all values of r_0 are seen to be inscribed into the potential curve of an infinitely thin ring of the same mass and radius, located in the torus symmetry plane. The potential curve to the right of the torus surface ($\rho > 1 + r_0$) almost coincides with the potential curve of the ring, while to the left ($\rho < 1 + r_0$) it passes lower and differs by a quantity that depends on r_0 (see Section 3). In Fig. 3, the dependences of the torus potential on the radial coordinate are presented; these were calculated from expression (8) for different values of ζ.

Note that in contrast to the work by Kondratyev (2007, p. 196, expression 7.26), where the torus potential is expressed only through a single integration of the elliptical integrals of all three kinds, the torus potential (8) in our work is expressed by double integration of the elliptical integral of the first kind. However, further analysis of this expression for the torus potential (8) allows us to obtain approximations that are physically understandable and enable the solution of practical astrophysical tasks that need multiple calculations of the gravitational potential of the torus.

For further analysis of the torus potential, we define the inner region as the volume bounded by the torus surface (inside the torus body) and the outer region as the region outside this surface.

3 TORUS POTENTIAL IN THE OUTER REGION

It is seen from Fig. 2 that the outer potential of the torus can be approximately represented by the potential of an infinitely thin ring of the same mass up to the torus surface. For $\rho \to 0$, the values of the torus potential and potential of an infinitely thin ring differ by a quantity that depends on a geometric parameter r_0, which is especially evident for a thick torus ($r_0 > 0.5$). We find a relationship between the outer potential of the torus and the potential of a ring of the same mass, i.e. derive an approximate expression for the torus potential in the outer region, where a condition $(\rho - 1)^2 + \zeta^2 \geq r_0^2$ holds. Within this region, the integrand $\phi_r(\rho, \zeta; \eta', \zeta')$ in (8) does not have singularities for all η', ζ'; therefore it can be expanded as a Maclaurin series in powers of η', ζ' in the vicinity of a point $\eta' = \zeta' = 0$. Since the integrals in symmetrical limits from the series terms that contain cross-derivatives and derivatives of odd orders are equal to zero, only summands with even orders remain in the expansion. With the quadratic terms of the series being restricted, the potential of the component ring is

$$\phi_r(\rho, \zeta; \eta', \zeta') \approx \phi_c(\rho, \zeta) + \frac{1}{2}\left.\frac{\partial^2 \phi_r}{\partial \eta'^2}\right|_{\substack{\eta'=0\\ \zeta'=0}} \eta'^2 + \frac{1}{2}\left.\frac{\partial^2 \phi_r}{\partial \zeta'^2}\right|_{\substack{\eta'=0\\ \zeta'=0}} \zeta'^2. \tag{9}$$

Substituting (9) into (8), we will have after integration

$$\varphi_{\rm torus} \approx \frac{GM}{\pi R}\phi_c\left(1 + \frac{r_0^2}{8\phi_c}\left[\left.\frac{\partial^2 \phi_r}{\partial \eta'^2}\right|_{\substack{\eta'=0\\ \zeta'=0}} + \left.\frac{\partial^2 \phi_r}{\partial \zeta'^2}\right|_{\substack{\eta'=0\\ \zeta'=0}}\right]\right) \tag{10}$$

Ultimately, the approximate expression for the torus potential in the outer region $(\rho - 1)^2 + \zeta^2 \geq r_0^2$ has the form

$$\varphi_{\rm torus}(\rho, \zeta; r_0) \approx \frac{GM}{\pi R}\phi_c\left[1 - \frac{r_0^2}{16} + \frac{r_0^2}{16}S(\rho, \zeta)\right], \tag{11}$$

where $\phi_c = \sqrt{m/\rho}\,K(m)$ is the dimensionless potential of central ring (2) and

$$S(\rho, \zeta) = \frac{\rho^2 + \zeta^2 - 1}{(\rho + 1)^2 + \zeta^2}\frac{E(m)}{K(m)}. \tag{12}$$

$E(m) = \int_0^{\pi/2} \mathrm{d}\beta \sqrt{1 - m \sin^2\beta}$ is the complete elliptical integral of the second kind. We may conveniently proceed to a new variable $\eta = \rho - 1$ that allows expression (12) to be represented as

$$S(\eta, \zeta) = \frac{\eta^2 + \zeta^2 + 2\eta}{(\eta + 2)^2 + \zeta^2}\frac{E(m)}{K(m)}, \tag{13}$$

where

$$m = 4\frac{\eta + 1}{(\eta + 2)^2 + \zeta^2}.$$

Expression (11) for the torus potential (we will further call it the S-approximation), with (12) or (13) taken into account, represents the torus potential accurately enough in the outer region $\eta^2 + \zeta^2 \geq r_0^2$ (Fig. 2). Since $|S| \leq 1$, the second multiplier in (11) is a slowly varying function of ρ and ζ. Let us simplify the expression (11), replacing the second multiplier by its asymptotic approximations.

In the first case, $\rho \to 0$ corresponding to $\eta \to -1$, the parameter $m \to 0$ and $E(m)/K(m) \to 1$, therefore, $S \to (\zeta^2 - 1)/(\zeta^2 + 1)$. The expression for the torus potential in this case is

$$\varphi_{\rm torus}(\rho, \zeta; r_0) \approx \frac{GM}{\pi R}\phi_c(\rho, \zeta)\left(1 - \frac{r_0^2}{16} + \frac{r_0^2}{16}\frac{\zeta^2 - 1}{\zeta^2 + 1}\right). \tag{14}$$

Since the dimensionless potential of the ring (2) at the symmetry axis is $\phi_c = \pi/\sqrt{1 + \zeta^2}$, we obtain for the torus

$$\varphi_{\rm torus}(0, \zeta; r_0) \approx \frac{GM}{R}\frac{1}{\sqrt{1 + \zeta^2}}\left(1 - \frac{r_0^2}{16} + \frac{r_0^2}{16}\frac{\zeta^2 - 1}{\zeta^2 + 1}\right) \tag{15}$$

and for $\zeta = 0$

$$\varphi_{\rm torus}(0, 0; r_0) \approx \frac{GM}{R}\left(1 - \frac{r_0^2}{8}\right). \tag{16}$$

The second summand $(GM/R \times r_0^2/8)$ in (16) describes the displacement of the torus potential at the symmetry axis compared with the potential of an infinitely thin ring (Fig. 2).

In the second case, at large η, the parameter $m \to 0$ and $S \to 1$ in (11), therefore

$$\varphi_{\rm torus}(\rho, \zeta; r_0) \approx \frac{GM}{\pi R}\phi_c(\rho, \zeta), \tag{17}$$

i.e. the torus potential is equal to the potential of an infinitely thin ring with the same mass M and radius R in this case. It is seen from

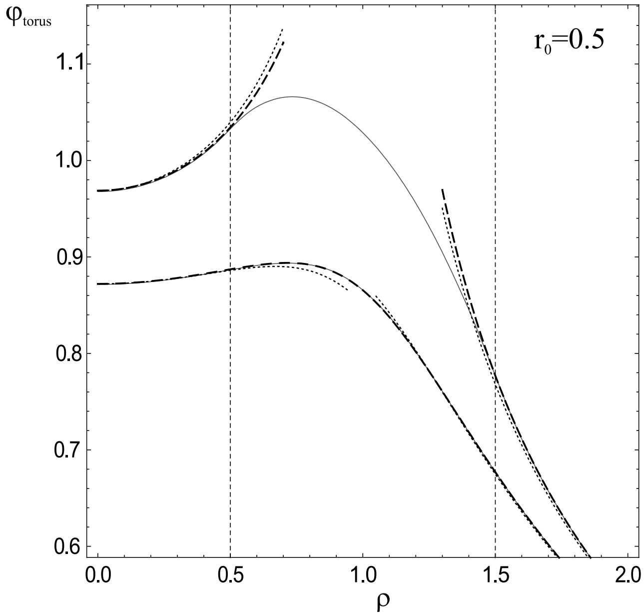

Figure 4. Dependence of the torus potential with $r_0 = 0.5$ on ρ for $\zeta = 0$ (upper curves) and $\zeta = 0.5$ (lower curves). Solid lines show the potential calculated with the exact formula (8). The S-approximation (11) of the potential is shown by dashed lines, and dotted lines represent the limiting cases of the S-approximation: the potential curve for an infinitely thin ring (17) is to the right of the torus cross-section and the curve representing a 'shifted' potential of an infinitely thin ring (14) is to the left. The boundaries of the torus cross-section are dotted vertical lines.

Fig. 4 that *the S-approximation for the torus outer potential (11) is applicable up to the torus surface* (upper curves). Indeed, in the region $\rho \le 1 - r_0$, the difference between the potential obtained from the integral expression (8) and its value taken from the S-approximation reaches a maximum near the torus surface and does not exceed 0.2 per cent for $r_0 = 0.5$. The difference remains small even for a thick torus: it does not exceed 1.5 per cent for $r_0 = 0.9$. For $\zeta = r_0$ all the points are outer and the curves for the exact potential and S-approximation virtually coincide (the deviation is less than 0.1 per cent).

Note that the asymptotics of the S-approximation for the outer potential (15) and (17) also describe the torus potential well enough (dotted line in Fig. 4). Thus, for $|\zeta| < r_0$, the approximation (15) can be used to estimate the potential inside the region bounded by a cylinder with radius $\rho - r_0$, while the approximation (17) is applicable outside the region bounded by a cylinder with radius $\rho + r_0$. At $|\zeta| \gg 1$, expression (15) tends to (17) and the expression for the potential of an infinitely thin ring (2) can be used within the whole outer region to evaluate the torus potential approximately.

Therefore, *the outer potential of the torus can be represented with good accuracy by the potential of an infinitely thin ring of the same mass. The dependence of the geometrical parameter r_0 appears only in the torus hole; it is taken into account in the 'shifted' potential of the infinitely thin ring (14). These approximations are valid up to the surface of the torus.*[3]

[3]There is some analogy with a known result: the outer potential of a solid sphere of mass M is the same as that generated by a point mass M located at the sphere's centre. Note, however, that a torus has another system of equigravitating elements (Kondratyev 2007).

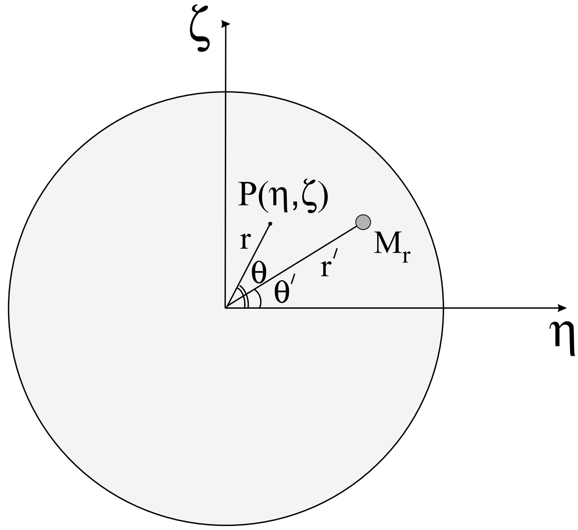

Figure 5. Scheme for the torus cross-section.

4 TORUS POTENTIAL IN THE INNER REGION

To analyse the inner potential of the torus, it is convenient to select a coordinate system with origin in the centre of the torus cross-section (Fig. 5). Then, the dimensionless potential of the central ring takes the form

$$\phi_c(\eta, \zeta) = \sqrt{\frac{m}{1+\eta}}\, K(m), \tag{18}$$

where

$$m = \frac{4(1+\eta)}{(2+\eta)^2 + \zeta^2}.$$

Consider the potential of the central ring (18) in the vicinity of $\eta \to 0, \zeta \to 0$, which corresponds to $m \to 1$. In this case, the elliptical integral in (18) can be expanded in terms of a small parameter $m_1 = 1 - m$. With the series clipped by two terms, we will have

$$K(m_1) \approx \ln\frac{4}{\sqrt{m_1}} + \frac{1}{4} m_1 \ln\frac{4}{e\sqrt{m_1}}, \tag{19}$$

where

$$m_1 = \frac{r^2}{r^2 + 4(1+\eta)},$$

with $r^2 = \eta^2 + \zeta^2$. The approximate formula for the ring potential expressed through the parameter m_1 is then

$$\phi_c(\eta, \zeta) \approx \frac{2\sqrt{m_1}}{r}\left(\ln\frac{4}{\sqrt{m_1}} + \frac{1}{4} m_1 \ln\frac{4}{e\sqrt{m_1}}\right). \tag{20}$$

Passage to the potential of an arbitrary component ring is fulfilled by substitutions $1 + \eta \to (1 + \eta)/(1 + \eta')$ and $\zeta \to (\zeta - \zeta')/(1 + \eta')$, which results in an expression

$$\phi_r(\eta, \zeta; \eta', \zeta') \approx \frac{2(1+\eta')}{\sqrt{q}}\left[\ln\frac{4}{\sqrt{m'_1}}\left(1 + \frac{m'_1}{4}\right) - \frac{m'_1}{4}\right], \tag{21}$$

where $m'_1 = (\boldsymbol{r} - \boldsymbol{r}')^2/q, q = (\boldsymbol{r} - \boldsymbol{r}')^2 + 4(1 + \eta)(1 + \eta')$. The summand $(\boldsymbol{r} - \boldsymbol{r}')^2 = (\eta - \eta')^2 + (\zeta - \zeta')^2$ is the square of the distance between the component ring and a point P (Fig. 5). Expansion (21) is valid for $m'_1 \to 0$, therefore $(\boldsymbol{r} - \boldsymbol{r}')^2 \ll 1$.

We confine ourselves to the case of a thin torus ($r_0 \ll 1$). Then $(\boldsymbol{r}-\boldsymbol{r}')^2 \ll 4(1+\eta)(1+\eta')$ and $(1+\eta)(1+\eta') \approx 1$, and the first multiplier in (21) can be written to second order as

$$f_1 \equiv \frac{2(1+\eta')}{\sqrt{q}} \approx \sqrt{\frac{1+\eta'}{1+\eta}}\left[1-\frac{1}{8}(\boldsymbol{r}-\boldsymbol{r}')^2\right]. \quad (22)$$

After expanding the square root in (22) in powers of η and η', we obtain

$$f_1 \approx \left(1+\frac{\eta'}{2}-\frac{\eta}{2}-\frac{\eta\eta'}{4}-\frac{\eta'^2}{8}+\frac{3}{8}\eta^2\right)\left[1-\frac{1}{8}(\boldsymbol{r}-\boldsymbol{r}')^2\right]. \quad (23)$$

Similarly, the second multiplier (in square brackets) in expression (21) can be written, to terms quadratic in coordinates, as

$$f_2 \equiv \ln\frac{4}{\sqrt{m_1'}}\left(1+\frac{m_1'}{4}\right)-\frac{m_1'}{4} \approx \frac{1}{2}(\eta+\eta')-\frac{1}{4}(\eta^2+\eta'^2)$$
$$+\ln\frac{8}{|\,\boldsymbol{r}-\boldsymbol{r}'\,|}+\frac{(\boldsymbol{r}-\boldsymbol{r}')^2}{16}\ln\frac{8e}{|\,\boldsymbol{r}-\boldsymbol{r}'\,|}. \quad (24)$$

Thus, we obtain the following expression for the potential of the component ring:

$$\phi_{\rm r}(\eta,\zeta;\eta',\zeta') \approx f_1\, f_2. \quad (25)$$

In consideration of the inner potential of the torus, we rewrite expression (8) in polar coordinates (Fig. 5):

$$\varphi_{\rm torus}(r,\theta;r_0) = \frac{GM}{\pi^2 R r_0^2}\int_0^{r_0}\int_0^{2\pi}\phi_{\rm r}(r,\theta;r',\theta')r'\,{\rm d}r'\,{\rm d}\theta', \quad (26)$$

where the coordinates of the component ring are $\eta' = r'\cos\theta'$, $\zeta' = r'\sin\theta'$ and the coordinates of a point P are $\eta = r\cos\theta$, $\zeta = r\sin\theta$. We substitute (23) and (24) into (25) and, after multiplying, restrict ourselves to terms quadratic in η, ζ and η', ζ'. Then, after integration of (26), we obtain an approximate expression for the inner potential of the torus:

$$\varphi_{\rm torus}(\eta,\zeta;r_0) \approx \frac{GM}{2\pi R}\left[c+\tilde{a}_1\eta+\tilde{a}_2\eta^2+\tilde{b}_2\zeta^2\right], \quad (27)$$

where

$$k \equiv \frac{r_0}{8}, \quad c = 1+2k^2-2\ln k+8k^2\ln k, \quad \tilde{a}_1 = 1+\ln k,$$

$$\tilde{a}_2 = -\frac{1}{(8k)^2}-4k^2(11+10\ln k),$$

$$\tilde{b}_2 = -\frac{1}{(8k)^2}+4k^2(3+2\ln k).$$

The first summand in (27) is the value of the torus potential in the centre of the torus cross-section: $c = \phi_{\rm torus}(0, 0;r_0)$. To analyse the inner potential further, it is convenient to transfer to a coordinate system normalized to the geometrical parameter of the torus r_0. Then the series coefficients will transform to the form

$$a_1 = 8k(1+\ln k), \qquad a_2 = -1-4k^2(11+10\ln k),$$

$$b_2 = -1+4k^2(3+2\ln k).$$

The expression for the torus potential (27) can be written as

$$\varphi_{\rm torus}(\eta,\zeta;r_0) \approx \frac{GM}{2\pi R}\left[c+a_1\frac{\eta}{r_0}+a_2\left(\frac{\eta}{r_0}\right)^2+b_2\left(\frac{\zeta}{r_0}\right)^2\right]. \quad (28)$$

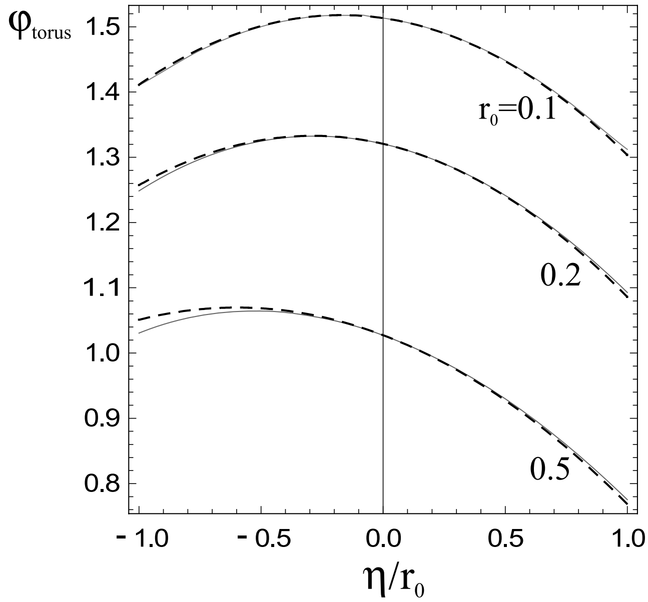

Figure 6. Dependence of the inner potential on the normalized coordinate η/r_0 at $\zeta = 0$ for various values of the geometric parameter: $r_0 = 0.1$, 0.2, 0.5. Solid curves represent dependences of the potential on the distance to the centre of the torus cross-section, which were calculated with the exact formula (8). Dashed lines are the potential curves taken with the approximate expression for the inner potential of the torus (28).

It follows from (28) that the maximal value of the potential is reached at a point $\eta_{\max} = -(a_1 r_0)/(2a_2)$, $\zeta = 0$, while equipotential lines are ellipses with their centres displaced by an amount $\eta_{\max}$ with respect to the centre of the torus cross-section, and the ratio of semi-axes of the ellipses is $\sqrt{b_2/a_2}$. Note that the location of the potential maximum $\eta = \eta_{\max}$, $\zeta = 0$ corresponds to the weightlessness point, where the resultant of all forces affecting a particle inside the torus equals zero. In such an approximation, components of the force inside the torus depend on the coordinates linearly. In Fig. 6, the curves of the inner potential in the coordinate system normalized to r_0 are presented for three values of r_0.

Although we confined ourselves to the case of a thin torus, the curves of potential taken from expression (28) are consistent with the curves for the exact potential (8) up to $r_0 = 0.5$, where the deviation is maximal near the torus surface and is of the order of 2 per cent (Fig. 6). The value of potential in the centre of the torus cross-section [a constant $c = \phi_{\rm torus}(0, 0;r_0)$ in (27)] also coincides with its exact value.

It is of interest to investigate the solutions obtained for limiting cases. Indeed, the case $r_0 = R_0/R \to 0$ corresponds to two limiting passages, to an infinitely thin ring (when R is fixed while $R_0 \to 0$) and to a cylinder (when R_0 is fixed and $R \to \infty$). We dwell on the limiting passage to the cylinder potential. At $R_0 \to 0$, the coefficients $c \to 1-2\ln k$, $a_1 \to 0$, and coefficients a_2, $b_2 \to -1$; expression (28) takes the following form in this case:

$$\varphi_{\rm torus}(\eta,\zeta;r_0\to 0) \approx \frac{GM}{2\pi R}\left[1-2\ln\frac{r_0}{8}-\left(\frac{r}{r_0}\right)^2\right], \quad (29)$$

where $r^2 = \eta^2+\zeta^2$. It is known that the inner potential of a circular cylinder with length $2H$, much larger than the radius R_0 of its cross-section (Kondratyev 2007), has the form

$$\varphi_{\rm cyl} = \frac{GM}{2H}\left[2\ln\frac{2\sqrt{e}H}{R_0} - \left(\frac{r}{r_0}\right)^2\right]. \tag{30}$$

After a formal substitution $2H = 2\pi R$ in (30) (the cylinder length is equal to the length of the central ring), we get an expression

$$\varphi_{\rm cyl} = \frac{GM}{2\pi R}\left[1 - 2\ln\left(\frac{r_0}{2\pi}\right) - \left(\frac{r}{r_0}\right)^2\right]. \tag{31}$$

Expression (31) coincides with (29) save for a constant.[4]

The quadratic dependence on r of the inner potential of a thin torus can be also derived in the case in which the minor radius $R_0 \to 0$. The outer potential of the torus was shown in Section 3 to be approximately equal to the potential of an infinitely thin ring of the same mass and radius. In this case, the smaller the torus geometrical parameter r_0, the more accurate this approximation is. Therefore, at $\eta^2 + \zeta^2 \ge r_0 \to 0$ the outer potential of the torus tends to the potential of an infinitely thin ring. In this case, $\eta, \zeta \to 0$ and thus the elliptical integral in the expression for an infinitely thin ring (2) can be expanded in the vicinity of $m \to 1$. If we confine ourselves to the first term of the expansion, we get an approximate expression for the potential of a central infinitely thin ring:

$$\varphi_{\rm c}(\eta, \zeta) \approx \frac{GM}{2\pi R}\left[-\ln(\eta^2+\zeta^2) + 2\ln 8\right], \tag{32}$$

which remains valid for the outer potential of the thin torus as well. It should be noted that there is no dependence on r_0, because in such an approximation all thin tori with the same masses and major radii are equigravitating for the outer potential. The derivatives of the outer potential of the thin torus in η, ζ are then

$$\frac{\partial\varphi_{\rm c}}{\partial\eta} \approx -\frac{GM}{\pi R}\frac{\eta}{r^2}, \qquad \frac{\partial\varphi_{\rm c}}{\partial\zeta} \approx -\frac{GM}{\pi R}\frac{\zeta}{r^2},$$

and take the following forms at the torus surface ($\eta^2 + \zeta^2 = r_0^2$):

$$\left.\frac{\partial\varphi_{\rm c}}{\partial\eta}\right|_{r=r_0} \approx -\frac{GM}{\pi R r_0}\cos\theta, \qquad \left.\frac{\partial\varphi_{\rm c}}{\partial\zeta}\right|_{r=r_0} \approx -\frac{GM}{\pi R r_0}\sin\theta.$$

It is the linear dependence of the force on coordinates η, ζ that satisfies such boundary conditions. Thus, the inner potential of the thin torus can be represented in the form

$$\varphi_{\rm torus}(\eta, \zeta; r_0) \approx \frac{GM}{2\pi R}\left[-\frac{\eta^2+\zeta^2}{r_0^2} + c(r_0)\right], \tag{33}$$

where $c(r_0)$ is the integration constant. Equating (32) with (33) at the torus surface, we obtain the expression for the constant $c(r_0) = -2\ln(r_0/8) + 1$ that coincides with expression (27) obtained above at $r_0 \ll 1$.

It becomes evident from analysis of the inner potential for the two limiting cases ($R_0 \to 0$ and $R \to \infty$) that the first summand in coefficients a_2, b_2 of the power series (28) represents the properties of the inner potential of a cylinder. With the cylinder potential separated, the inner potential of the torus (28) can be written as

$$\varphi_{\rm torus}(\eta, \zeta) = \varphi_{\rm cyl}(r) + \varphi_{\rm curv}(\eta, \zeta), \tag{34}$$

where

$$\varphi_{\rm curv} \approx \frac{GM}{2\pi R}\left[c_{\rm curv} + a_1\left(\frac{\eta}{r_0}\right) + c_a\left(\frac{\eta}{r_0}\right)^2 + c_b\left(\frac{\zeta}{r_0}\right)^2\right], \tag{35}$$

$$c_{\rm curv} = 2\ln\frac{8}{2\pi} + 2k^2(1 + 4\ln k),$$

$$c_a = 1 + a_2, \qquad c_b = 1 + b_2.$$

The second summand $\varphi_{\rm curv}(\eta, \zeta)$, which we will call a potential of curvature, implies the curvature of the torus surface. Indeed, all the coefficients of series (35) tend to zero in the limiting passage to a cylinder ($r_0 \to 0$), and $\varphi_{\rm curv} \to 0$. Therefore, *the inner potential of the torus can be represented as a sum of the cylinder potential and a term comprising a geometrical curvature of the torus surface*.

5 SEWING TOGETHER THE INNER AND OUTER POTENTIALS AT THE TORUS SURFACE

In the previous sections, we derived approximate expressions for the torus potential in the outer ($\eta^2 + \zeta^2 \ge r_0^2$) and inner ($\eta^2 + \zeta^2 \le r_0^2$) regions. It has also been shown that the inner potential of the torus can be represented by a series in powers of η/r_0 and ζ/r_0, and the constant, linear and quadratic terms of the series were determined analytically. To find a larger number of series terms sufficient to represent the inner potential accurately enough, and to obtain a continuous approximate solution for the potential and its derivatives in the whole region that will satisfy the boundary conditions at the surface, we will act in the following way. We represent the inner potential of the torus as a power series[5]:

$$\begin{aligned}\phi(\eta, \zeta; r_0) = \frac{1}{2\pi}\Bigg[&c(r_0) + \sum_{i=1} a_i(r_0)\left(\frac{\eta}{r_0}\right)^i \\ &+ \sum_{i=1}\sum_{j=1} t_{ij}(r_0)\left(\frac{\eta}{r_0}\right)^i\left(\frac{\zeta}{r_0}\right)^j \\ &+ \sum_{j=1} b_j(r_0)\left(\frac{\zeta}{r_0}\right)^j\Bigg],\end{aligned} \tag{36}$$

where $c(r_0), a_i(r_0), b_j(r_0), t_{ij}(r_0)$ are unknown coefficients. Note that the series (36) contains only the terms with even powers of ζ, because the torus potential is symmetric in ζ.

Suppose that we have an analytical expression for the torus potential $\Psi(\eta, \zeta; r_0)$ at its surface ($\eta^2 + \zeta^2 = r_0^2$). Also, we write the inner potential of the torus (36) at its surface ($\eta = r_0\cos\theta$ and $\zeta = r_0\sin\theta$) as

$$\begin{aligned}\phi(\theta, r_0) = \frac{1}{2\pi}\Bigg(&c + \sum_{i=1} a_i\cos^i\theta \\ &+ \sum_{i=1}\sum_{j=1} t_{ij}\cos^i\theta\sin^j\theta + \sum_{j=1} b_j\sin^j\theta\Bigg).\end{aligned} \tag{37}$$

From conditions of equality of the inner and outer potential and its derivatives in coordinates at the torus surface for several angles θ_k,

[4]The difference in the constant may be caused by the curvature of the torus surface.

[5]We consider a dimensionless potential here. To pass to the dimensional case, (36) must be multiplied by *GM/R*.

we obtain a system of $3k$ linear equations to determine coefficients c, a_i, b_j, t_{ij}:

$$c + \sum_{i=1} a_i \cos^i\theta_k + \sum_{i=1}\sum_{j=1} t_{ij}\cos^i\theta_k \sin^j\theta_k + \sum_{j=1} b_j \sin^j\theta_k$$
$$= 2\pi\,\Psi(\theta_k, r_0),$$
$$\sum_{i=1} i\, a_i \cos^{i-1}\theta_k + \sum_{i=1}\sum_{j=1} i\, t_{ij}\cos^{i-1}\theta_k \sin^j\theta_k$$
$$= 2\pi r_0 \frac{\partial}{\partial\eta}\Psi(\theta_k, r_0),$$
$$\sum_{j=1} j\, b_j \sin^{j-1}\theta_k + \sum_{i=1}\sum_{j=1} j\, t_{ij}\cos^{i}\theta_k \sin^{j-1}\theta_k$$
$$= 2\pi r_0 \frac{\partial}{\partial\zeta}\Psi(\theta_k, r_0). \quad (38)$$

Thus, if we had the analytic solution for the outer potential of the torus, we could obtain an exact expression for the inner potential as an infinite series in powers of $\cos\theta_k$, $\sin\theta_k$, using the boundary conditions and solving the system of equations (38). Since there is no analytic expression for the outer potential, we can use the above approximate expression (11) for the torus potential in the outer region (the S-approximation) and introduce designations

$$\Phi \;= \sum_k \left[\phi_{\rm in}(\theta_k, r_0) - \phi_{\rm out}(\theta_k, r_0)\right]^2,$$
$$\Phi_1 = \sum_k \left(\frac{\partial}{\partial\eta}\left[\phi_{\rm in}(\theta_k, r_0) - \phi_{\rm out}(\theta_k, r_0)\right]\right)^2,$$
$$\Phi_2 = \sum_k \left(\frac{\partial}{\partial\zeta}\left[\phi_{\rm in}(\theta_k, r_0) - \phi_{\rm out}(\theta_k, r_0)\right]\right)^2,$$

where $\phi_{\rm in}, \phi_{\rm out}$ are solutions for the inner (37) and outer (11) potentials at the torus boundary, respectively. The unknown coefficients of the series can be then determined from a condition of the minimal value of a functional:

$$F = \Phi + \Phi_1 + \Phi_2 \rightarrow \min. \quad (39)$$

The functional (39) was minimized with the least-squares method, and coefficients of the series (37) were determined up to fourth power. The coefficients of the series are presented in Appendix A (Table A1). In Fig. 7 the dependence of the potential on the radial coordinate from the exact expression (8) is presented for the entire region, as well as its approximate solution obtained by sewing together the S-approximation (11) and the inner potential (37). Though the approximate solutions were obtained assuming that the torus is thin, $r_0 \ll 1$, we see that the exact (8) and approximate solutions are consistent even for a torus with $r_0 = 0.5$. In Fig. 8, the equipotential curves on the plane of the torus cross-section are shown, where a good agreement for all values of ρ, ζ is seen as well.

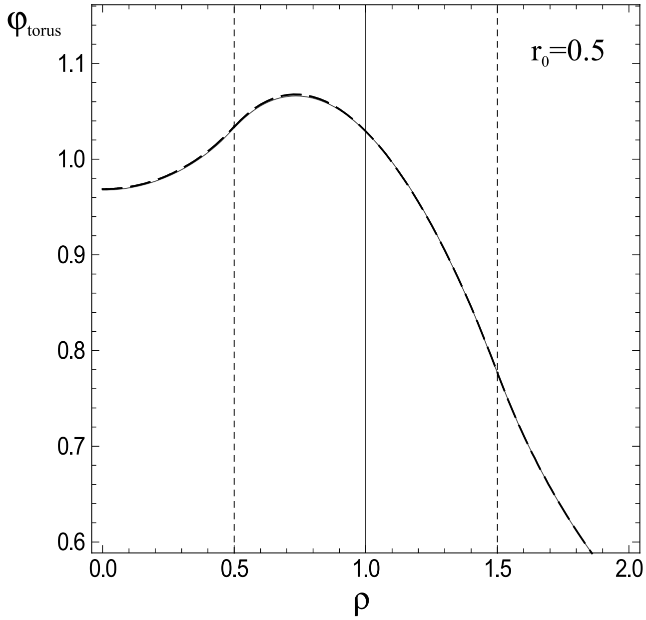

Figure 7. Dependence of the torus potential on ρ for $r_0 = 0.5$ ($\zeta = 0$) in the whole region: the potential curve calculated from the exact expression (8) is shown by a solid line, while the dashed line demonstrates the result of sewing together the S-approximation in the outer region (11) and the inner potential represented by the power series (37) with the coefficients found from the sewing condition (39).

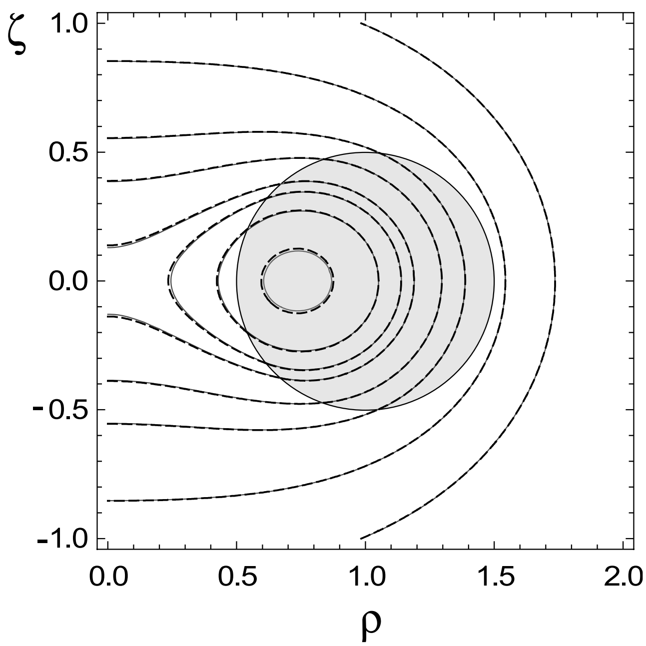

Figure 8. Equipotential curves for the torus with $r_0 = 0.5$. Solid lines are those calculated from the exact expression for the potential; dashed lines are those for the approximate formulae for the outer and inner potentials (see Fig. 7 caption). The torus body cross-section is indicated as a grey circle.

6 CONCLUSIONS

In the present work, the gravitational potential of a homogeneous circular torus is investigated in detail. An integral expression for the potential that is valid for an arbitrary point is obtained by composing the torus from infinitely thin rings. This approach has made it possible to find an approximate expression for the outer potential of the torus (S-approximation) that has a sufficiently simple form. It is shown that the outer potential of the torus can be represented with good accuracy by the potential of an infinitely thin ring of the same mass. The dependence of the geometrical parameter r_0 appears only in the torus hole; it is taken into account in the 'shifted' potential of the infinitely thin ring. These approximations are valid up to the surface of the torus.

For the inner potential, an approximate expression is found in the form of a power series up to second-order terms, where the coefficients depend only on the geometric parameter r_0. Expressions for the potential in the centre of the torus cross-section and for coordinates of the potential maximum are obtained, and the limiting passage to a cylinder potential is considered. It is shown that the inner potential of the torus can be represented as the sum of the cylinder potential and a term comprising the geometrical curvature of the torus surface. A method for determining the torus potential over the whole region is proposed that implies a 'sewing together' at the surface of the outer potential (S-approximation) and the inner potential represented by the power series. This method provided a continuous approximate solution for the potential and its derivatives, working throughout the region.

Surely, matter distribution within a torus is inhomogeneous in actual astrophysical objects and a torus cross-section may differ from a circular one. Therefore, it is of further interest to account for inhomogeneity of matter distribution inside a torus, difference of the torus cross-section from a circular form, etc.

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REFERENCES

Antonucci R., 1993, ARA&A, 31, 473
Block D. L. et al., 2006, Nat, 443, 832
de Boer W., Sander C., Zhukov V., Gladyshev A. V., Kazakov D. I., 2005, A&A, 444, 51
Gerber R. A., Lamb S. A., Balsara D. S., 1992, ApJ, 399, 51
Ibata R. A., Irwin M. J., Lewis G. L., Ferguson A. M. N., Tanvir N., 2003, MNRAS, 340, 21
Jaffe W. et al., 2004, Nat, 429, 47
Jee M. J. et al., 2007, ApJ, 661, 728
Kondratyev B. P., 2003, The potential theory and equilibrium figures. Regular and Chaotic Dynamic Press, Moscow-Izhevsk (in Russian)
Kondratyev B. P., 2007, The potential theory. New methods and problems with solutions. Mir, Moscow, 512 pp (in Russian)
Kondratyev B. P., Trubitsina N. G., 2010, Tech. Phys., 55, 22
Kondratyev B. P., Dubrovskii A. S., Trubitsina N. G., Mukhametshina E. Sh., 2009, Tech. Phys., 54, 176
Riemann B., 1948, inGoncharov V.L., ed., Collected papers. OGIZ, Moscow (Russian translation)
Smythe W. R., 1950, Static and Dynamic Electricity. McGraw-Hill, New York
Urry C. M., Padovani P., 1995, PASP, 107, 803

APPENDIX A: COEFFICIENTS OF THE POWER SERIES FOR THE INNER POTENTIAL OF A TORUS

In Table A1, the coefficients of the power series (up to fourth power) for the inner potential of a torus are presented, calculated from the sewing condition for the torus with various values of the geometrical parameter r_0. The analytic expression (28) was used to determine the zeroth coefficient c of the series. In Fig. A1, the linear (a_1) and quadratic (a_2, b_2) coefficients of the power series as functions of r_0 obtained analytically from (27) are shown by solid lines; the dots are the proper values of these coefficients obtained from the condition of sewing (see Table A1). The values of the analytic coefficients are seen to coincide up to $r_0 = 0.5$ with values obtained independently with the method of sewing from (39).

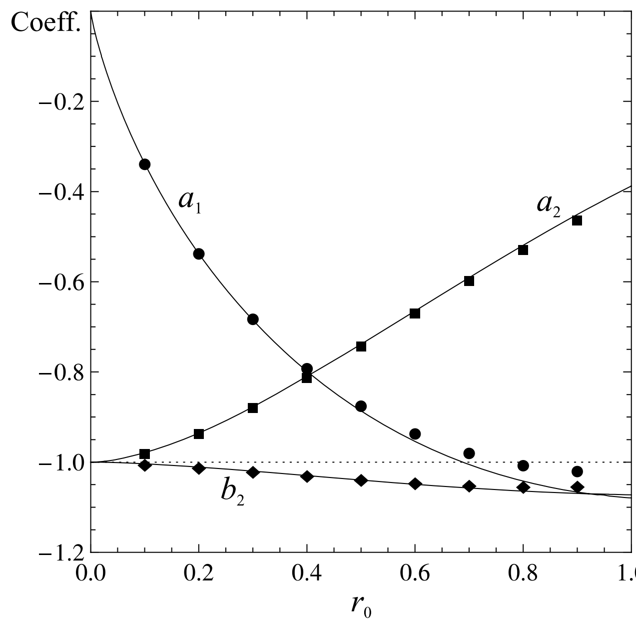

Figure A1. Dependence of the first coefficients of the power series on r_0 for the inner potential obtained with the sewing method.

Table A1. Coefficients of the power series for the inner potential of the torus for various values of r_0, obtained with the method of sewing.

Coeff.	Geometrical parameter r_0								
	0.1	0.2	0.3	0.4	0.5	0.6	0.7	0.8	0.9
a_1	−0.33798	−0.53651	−0.68154	−0.79129	−0.87439	−0.93587	−0.97906	−1.00628	−1.01928
a_2	−0.98002	−0.93543	−0.87773	−0.81171	−0.74107	−0.66865	−0.59677	−0.52739	−0.46224
b_2	−1.00411	−1.01086	−1.01970	−1.02892	−1.03759	−1.04495	−1.05030	−1.05299	−1.05237
a_3	0.02392	0.04364	0.05781	0.06608	0.06853	0.06550	0.05753	0.04525	0.02938
t_{12}	0.02550	0.05329	0.08404	0.11791	0.15454	0.19323	0.23295	0.27238	0.30991
a_4	−0.00182	−0.00785	−0.01610	−0.02576	−0.03580	−0.04535	−0.05371	−0.06036	−0.06495
b_4	0.00061	0.00131	0.00308	0.00570	0.00922	0.01362	0.01880	0.02453	0.03045
t_{22}	−0.00122	−0.00812	−0.01948	−0.03681	−0.06076	−0.09157	−0.12900	−0.17213	−0.21931

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