$Z_i \sim N(i, i^2)\ \forall i \in \mathbb{N} \Rightarrow E Z_i = i,\ Var(Z_i) = i^2 \Rightarrow EZ = E(Z_1 \dots Z_n^T) = (EZ_1 \dots EZ_n)^T = (1, \dots, n)^T$ c 1

$$\Rightarrow Var(Z) = cov(Z_i, Z_j) = \begin{cases} Var(Z_i) & : i = j \\ cov(Z_i, Z_j) & : i \neq j \end{cases}$$

$$= diag(1^2, 2^2 \dots n^2)$$

d

$$E(Z) = E(Z_1, Z_2 \dots Z_n)^T = E\left(Z_1, \tfrac{1}{2}\eta_1 + \eta_2, \tfrac{1}{2}\eta_2 + \eta_3 \dots \tfrac{1}{2}\eta_{n-1} + \eta_n\right)^T = \left(E(\eta_1), E(\tfrac{1}{2}\eta_1 + \eta_2), E(\tfrac{1}{2}\eta_2 + \eta_3), \dots\right)^T$$

$$= \left(0, \tfrac{1}{2}E(\eta_1) + E(\eta_2), \dots \tfrac{1}{2}E(\eta_{n-1}) + E(\eta_n)\right)^T = \bar{0}^T$$

$$cov(Z_i, Z_j) = E(Z_i Z_j) - \mu_{Z_i}\mu_{Z_j} = E(Z_i Z_j)$$

for $i = j$: $cov(Z_i, Z_j) = Var(Z_i)$

$$Var(Z_1) = 1$$
$$Var(Z_n) = Var\left(\tfrac{1}{2}\eta_{n-1} + \eta_n\right) = \left(\tfrac{1}{2}\right)^2 Var(\eta_{n-1}) + Var(\eta_n) = \tfrac{1}{4}\cdot 1 + 1 = 1.25 \quad \forall i \geq 2$$

for $j = i+1$: $E Z_i Z_j = E\left[\left(\tfrac{1}{2}\eta_{i-1} + \eta_i\right)\left(\tfrac{1}{2}\eta_i + \eta_{i+1}\right)\right] = \underbrace{E\left(\tfrac{1}{4}\eta_{i-1}\eta_i\right)}_{\tfrac{1}{4}E(\eta_{i-1}\eta_i) = 0} + \underbrace{E\left(\tfrac{1}{2}\eta_{i-1}\eta_{i+1}\right)}_{0} + \underbrace{E\left(\tfrac{1}{2}\eta_i^2\right)}_{\tfrac{1}{2}Var(\eta_i) = \tfrac{1}{2}} + \underbrace{E(\eta_i \eta_{i+1})}_{0} = \tfrac{1}{2}$

for $j > i+1$: $E Z_i Z_j = E\left[\left(\tfrac{1}{2}\eta_{i-1} + \eta_i\right)\left(\tfrac{1}{2}\eta_{j-1} + \eta_j\right)\right] = \underbrace{E\left(\tfrac{1}{4}\eta_{i-1}\eta_{j-1}\right)}_{0} + \underbrace{E\left(\tfrac{1}{2}\eta_{i-1}\eta_j\right)}_{0} + \underbrace{E\left(\tfrac{1}{2}\eta_{j-1}\eta_i\right)}_{0} + \underbrace{E(\eta_i \eta_j)}_{0} = 0$

$$\Rightarrow \Sigma = \begin{pmatrix} 1 & 0.5 & 0 & 0 & 0 & \dots \\ 0.5 & 1.25 & 0.5 & 0 & 0 & \\ 0 & 0.5 & 1.25 & 0.5 & 0 & \\ 0 & 0 & 0.5 & 1.25 & 0.5 & \\ 0 & 0 & 0 & 0.5 & \ddots & \\ \vdots & & & & & \ddots \end{pmatrix}$$

n trials for 3 categories, so $Z = (Z_1, Z_2, Z_3)^T \sim multinomial(n; p, q, 1-p-q)$ 2

the $E(Z_i) = \sum_{j=1}^{n} Pr(X_j = i) = np \Rightarrow E(Z) = \begin{pmatrix} np \\ nq \\ n(1-p-q) \end{pmatrix}$

$$cov(Z_i, Z_l) = \begin{cases} i = l \Rightarrow Var(Z_i) = \sum_{j=1}^{n} Var(\mathbb{1}\{X_j = i\}) = n(Pr(X_j = i))(1 - Pr(X_j = i)) \\ i \neq l \Rightarrow cov(Z_i, Z_l) = \sum_{j=1}^{n}\sum_{m=1}^{n} cov(\mathbb{1}\{X_j = i\}, \mathbb{1}\{X_m = l\}) = \end{cases}$$

$$E(Z_i Z_\ell) - E(Z_i)E(Z_\ell) = \left[\sum_{j=1}^{n}\sum_{m=1}^{n} E\left(1\!\!1\{x_j = i\} 1\!\!1\{x_m = \ell\}\right)\right] - \left[\sum_{j=1}^{n}\sum_{m=1}^{n} 1\!\!1\{x_j = i\} 1\!\!1\{x_m = \ell\}\right] = \left[n(n-1)p_i p_\ell\right] - np_i np_\ell$$

$= -np_i p_\ell$ where $p_1 = p$, $p_2 = q$, $p_3 = 1-p-q$

$$\Rightarrow \Sigma = n\begin{pmatrix} p(1-p) & -pq & -p(1-p-q) \\ -pq & q(1-q) & -q(1-p-q) \\ -p(1-p-q) & -q(1-p-q) & (1-p-q)(1-p+q) \end{pmatrix}$$

$2 \Rightarrow 1$ (2)

let B semi P, $\forall v \in \mathbb{R}^p$: $Var(v^T Z) - Var(v^T W) = v^T \Sigma_Z v - v^T \Sigma_W v = v^T B v \geq 0 \Rightarrow B \succeq 0$

$\Rightarrow Var(v^T Z) \geq Var(v^T W) \Rightarrow$ (1)

$1 \Rightarrow 2$

$v^T B v \Rightarrow v^T \Sigma_Z v - v^T \Sigma_W v \Rightarrow Var(v^T Z) - Var(v^T W) \geq 0 \quad \forall v \in \mathbb{R}^p \Rightarrow B \succeq 0 \Rightarrow$ (2)

$2 \Rightarrow 3$ B is semi pos. and symmetric, then by SVD: $B = U \Lambda U^T$ for U orthogonal and $\Lambda = Diag(\lambda_1 \dots \lambda_p)$ for $\lambda_i \geq 0 \; \forall i$

Def. $\sqrt{B} = U\, Diag(\sqrt{\lambda_1} \dots \sqrt{\lambda_p})\, U^T$ Then $B^{1/2}$ is symmetric, semi pos. and $\sqrt{B}\sqrt{B} = U \Lambda U^T = B$ so the root exist $\Rightarrow$ (3)

$3 \Rightarrow 2$

if $\exists R$ st $RR^T = B$, then $\forall v \in \mathbb{R}^p$: $v^T B v = v^T R R^T v = (R^T v)^T (R^T v) = \|R^T v\|^2 \geq 0 \Rightarrow B \succeq 0$

with B symmetric since RR^T is also

$\Sigma_x = E[(x-\mu_x)(x-\mu_x)^T] = E[xx^T - x\mu_x^T - \mu_x x^T + \mu_x\mu_x^T] = E(xx^T) - \mu_x\mu_x^T$ (3)

$\forall v \in \mathbb{R}^p \quad v^T\Sigma_x v = E[v^T(x-\mu_x)(x-\mu_x)^T v] = E[(v^T(x-\mu_x))^2] \geq 0 \rightarrow \Sigma_x \succeq 0$

$\mathrm{cov}(Ax+b) = E[(Ax+b-A\mu_x-b)(Ax+b-A\mu_x-b)^T] = E[A(x-\mu_x)(x-\mu_x)^T A^T] = A\Sigma_x A^T$

$\mathrm{cov}(Y,X)^T = (E[(Y-\mu_Y)(X-\mu_X)^T])^T = E[(Y-\mu_Y)^T(X-\mu_X)] = \mathrm{cov}(X,Y)$

$\mathrm{cov}(x_1,x_2,Y) = E[(x_1+x_2-\mu_{x_1}-\mu_{x_2})(Y-\mu_Y)^T] = E[(x_1-\mu_{x_1})(Y-\mu_Y)^T] - E[(x_2-\mu_{x_2})(Y-\mu_Y)^T] = \mathrm{cov}(x_1,Y) + \mathrm{cov}(x_2,Y)$

$\mathrm{cov}(Ax,By) = E[(Ax-A\mu_x)(By-B\mu_y)^T] = AE[(x-\mu_x)(y-\mu_y)]B^T = A\,\mathrm{cov}(x,y)B^T$

$\mathrm{cov}(\varepsilon_i,\varepsilon_j) = \rho\sigma^2$ is violated for persons i and j of the same family (4) (5)

$E(\hat{\beta}|X) = \beta + (X^TX)^{-1}X^T E(\varepsilon|X) = \beta$

$Var(\hat{\beta}|X) = (X^TX)^{-1}X^T(Var(\varepsilon|X))X(X^TX)^{-1}$

$E(\varepsilon_i | x_i) \neq 0$ (אנדוגניות) (2)