Controls Project

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1 Introduction

The project for Controls will involve controlling a doubly inverted pendulum with a linear "cart". The goals of this project are to

1. Linearize the system about the stability point and design a controller for it.
2. Determine the limits of the controller with respect to each state variable(in isolation)

In order to simplify the model, the mass of the pendulums will be assumed to be at the end of the rod, rather than in the middle as it would be for a constant density rod.

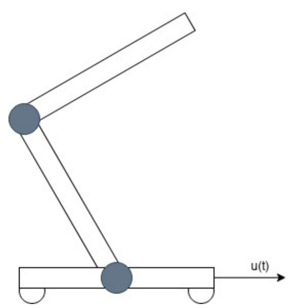

Figure 1: System to be modeled

2 Controller and Observer Design

2.1 State and input response

This system is really complicated and I am unable to find a mathmatical solution to e^AT or $\mathscr{L}^{-1}((sI - A)^{-1})$. I even tried to have my computer solve it to no luck as a result, I am going to plot the response by modeling the system with code required later. As the system is non-stable it quickly explodes Zero State Response

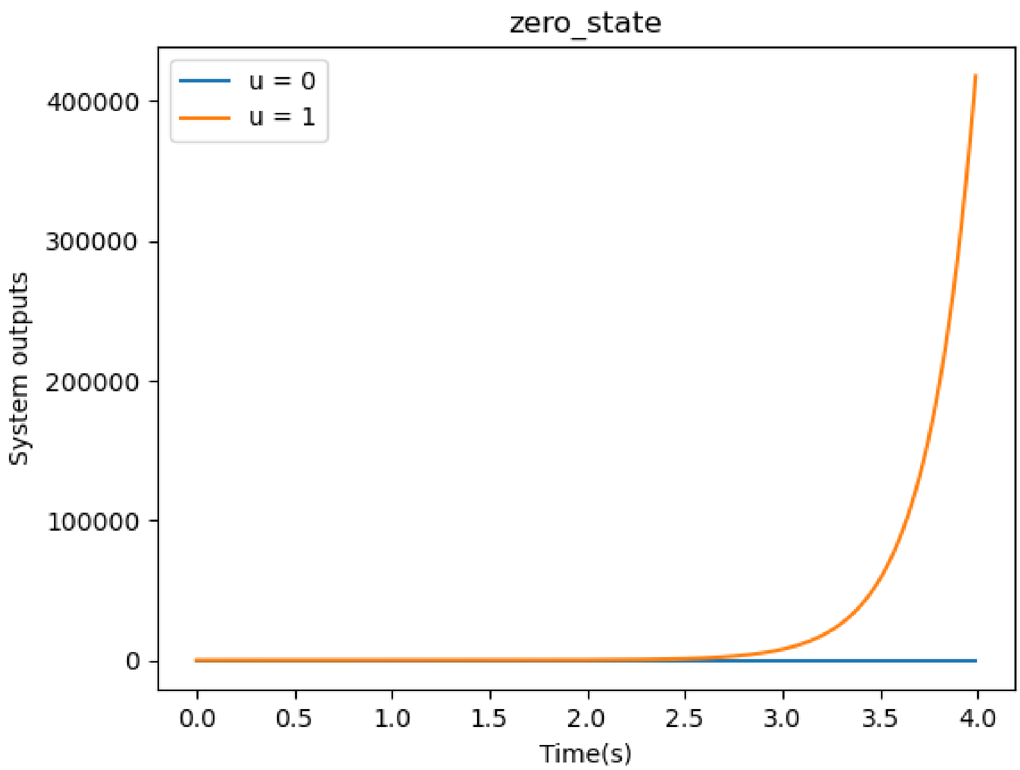

Figure 2: zero state

Zero Input Response

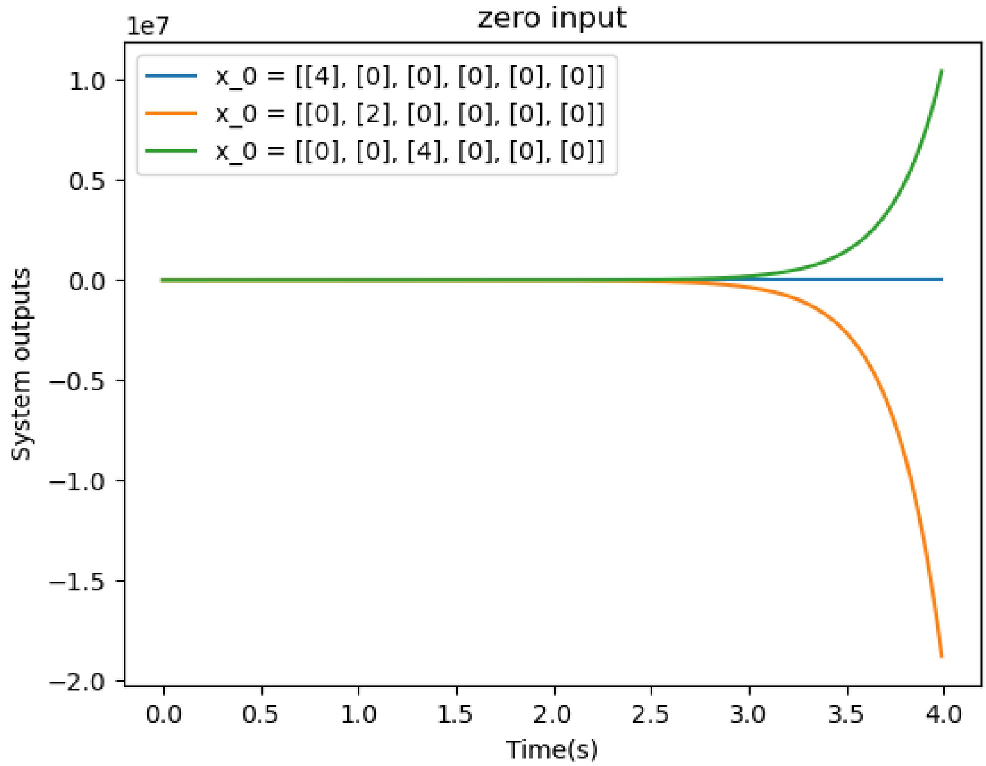

Figure 3: Zero input Response

As this system is massively unstable, I will only be doing simulation runs with zero input(shown again below) however, the total state response for a less unstable system is achieved by adding the zero state response with the zero state response for a given input and state.

Total Response

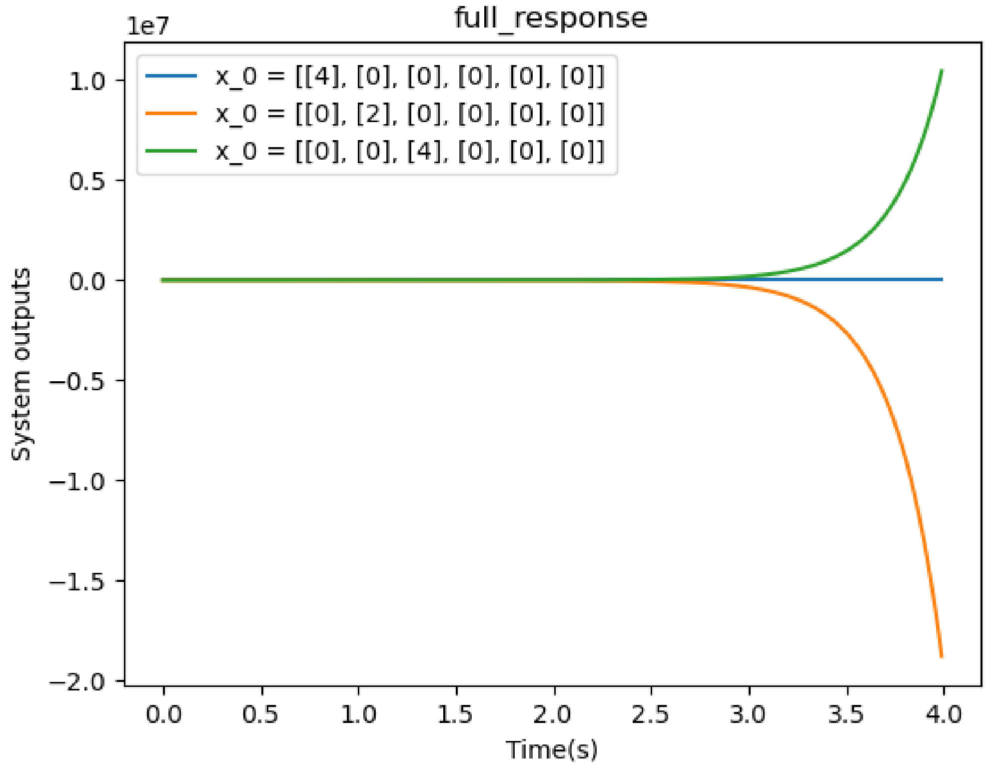

Figure 4: Full Response

3 Introduction

3.1 Defining the non-linear system

A doubly inverted pendulum is a pendulum on a pendulum. What makes this a classical control problem is the inverted nature of the system. That is instead of hanging the pendulum, the goal is to balence it upright.

The equations of motion were derived with assistance from **eq˙of˜motion**.

This will not be a derivation of the work, as the work follows closely from **eq˙of˜motion** which I used as a source to help derive the non-linear equations of motion.

3.2 Defining the linear system

The Linearization was simply a jacobian taken at 0,0,0 which is an equilibrium point that the system will be controlled to.

$$\dot{x} = \begin{bmatrix} 0 & 0 & 0 & 0 & 0 & 1.0 \\ 0 & 0 & 0 & 0 & 1.0 & 0 \\ 0 & 0 & 0 & 1.0 & 0 & 0 \\ 0 & -9.81 & 0 & -0.1 & 0.05 & 0 \\ 0 & 14.715 & -4.905 & 0.05 & -0.075 & 0.05 \\ 0 & -9.81 & 9.81 & -1.0 \cdot 10^{-141} & 0.05 & -0.1 \end{bmatrix} \begin{bmatrix} y \\ \theta_1 \\ \theta_2 \\ \dot{y} \\ \dot{\theta}_1 \\ \dot{\theta}_2 \end{bmatrix} + \begin{bmatrix} 0 \\ 0 \\ 0 \\ 1.0 \\ -0.5 \\ -2.0 \cdot 10^{-140} \end{bmatrix} u \tag{1}$$

3.3 Transfer Function

The transfer function was calculated as follows:

$$G(s) = C(sI - A)^{-1}B + D \tag{2}$$

$$G(s) = \begin{bmatrix} \frac{1\cdots}{4630.693s^7+1\cdots} \\ \frac{1.0\left(22281.994s^8-2175279.664s^7-1145628.722s^6+31941104.707s^5+6948321.196s^4+105500215.\cdots\right.}{445639.88s^{11}+289665.922s^{10}-13038030.264s^9-6656077.248s^8+52259381.006s^7+17016437.957s^6+633240645.131s^5+178\cdots} \\ -\frac{0.5\left(109293.181s^8-21391407.767s^7-11784263.962s^6+313100152.783s^5+104940973.11s^4+1044421391.697s^3+\cdots\right.}{2185863.611s^{11}+1420811.347s^{10}-63951538.446s^9-32648058.9s^8+256332263.833s^7+83465628.18s^6+3106045364.367s^5+87774\cdots} \end{bmatrix} \tag{3}$$

3.4 Controllability and Observability of the system

Controllability of the system:

$$\begin{bmatrix} B & AB & A^2B & A^3B & A^4B & A^5B \end{bmatrix} \tag{4}$$

$$\begin{bmatrix} 0 & 0 & -0.025 & 14.722 & -4.171 & 169.324 \\ 0 & -0.5 & 0.088 & -12.277 & 3.804 & -205.383 \\ 0 & 1.0 & -0.125 & 4.922 & -1.964 & 120.82 \\ 1.0 & -0.125 & 4.922 & -1.964 & 120.82 & -59.665 \\ -0.5 & 0.088 & -12.277 & 3.804 & -205.383 & 95.517 \\ 0 & -0.025 & 14.722 & -4.171 & 169.324 & -83.786 \end{bmatrix} \tag{5}$$

The determinate is non-zero thus the system is controllable

Observability of the system:

$$\begin{bmatrix} C \\ CA \\ CA^2 \\ CA^3 \\ CA^4 \\ CA^5 \end{bmatrix} \tag{6}$$

$$\begin{bmatrix} 1 & 0 & 0 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 & 0 & 0 \\ 0 & 0 & 1 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 1.0 \\ 0 & 0 & 0 & 0 & 1.0 & 0 \\ 0 & 0 & 0 & 1.0 & 0 & 0 \end{bmatrix} \tag{7}$$

The determinate is non-zero thus the system is observable

3.5 The Controller

To begin with, we will develop a controller that will be used in order to apply feedback and make the system controllable.

TODO: re-enable this

First the goal is to see if the system behaves well in a non-linear system from various distances from the origin. In order to make sure none of the state space variable explode but also wanting to make sure that multiple starting states can be overlaid on the same graph the following C matrix was used in a linear model

$$C = \begin{bmatrix} 1 & 1 & 1 & 1 & 1 & 1 \end{bmatrix} \tag{8}$$

from these graphs, it can be seen that even non-sensically high values can be controlled down to zero, thanks to the linear approximation and the controller being able to compensate for it. Howver this does show that the controller works in at least this limited case.

Now to see the usefulness of the controller, let us re-apply the system to the non-linear system. The inital states are adjusted by a factor of 100, which was necessary in order to make sure that all of the states converged. THe same C matrix was used for a linear output

3.6 The Observer

The previous section assumed that the state variables can be modeled, which is not necessaryily the case. To make it interesting I am going to assume that the system's velocities can be measured(ie only relative encoders are available for the system).

This is going to create a c matrix of:

$$\begin{bmatrix} 1 & 0 & 0 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 & 0 & 0 \\ 0 & 0 & 1 & 0 & 0 & 0 \end{bmatrix} \tag{9}$$

The desired eigenvalues are 3 times the controller eigenvalues in order to allow the observer to converge faster than the controller, this resulted in:

1. $(-9 + 9j)$
2. $(-9 - 9j)$
3. $(-10.5 + 10.5j)$
4. $(-10.5 - 10.5j)$
5. $(-12 + 12j)$
6. $(-12 - 12j)$

The first goal will be to reduce the multi-output model to a single output model the desired eigenvalues for the observer are:

$$\lambda = (-9 + 9j), (-9 - 9j), (-10.5 + 10.5j), (-10.5 - 10.5j), (-12 + 12j), (-12 - 12j) \tag{10}$$

Desired Characteristic equation

$$\Delta_d(\lambda) = \lambda^6 + 63.0\lambda^5 + 1984.5\lambda^4 + 36855.0\lambda^3 + 431649.0\lambda^2 + 2980152.0\lambda + 10287648.0 \tag{11}$$

Desired Characteristic equation

$$\bar{\alpha} = \begin{bmatrix} 10287648.0 \\ 2980152.0 \\ 431649.0 \\ 36855.0 \\ 1984.5 \\ 63.0 \end{bmatrix} \tag{12}$$

Desired Characteristic equation

$$\Delta_d(\lambda) = \mathrm{PurePoly}\left(1.0\lambda^6 + 0.2750244140625\lambda^5 - 14.6950073242188\lambda^4 - 1.71600341796875\lambda^3 - 48.166992187 \tag{13}$$

Desired Characteristic equation

$$\alpha = \begin{bmatrix} 0 \\ 0 \\ -48.167 \\ -1.716 \\ -14.695 \\ 0.275 \end{bmatrix} \tag{14}$$

$\bar{l}$

$$\bar{l} = \bar{\alpha} - \alpha = \tag{15}$$

$$\bar{l} = \begin{bmatrix} 10287648.0 \\ 2980152.0 \\ 431649.0 \\ 36855.0 \\ 1984.5 \\ 63.0 \end{bmatrix} - \begin{bmatrix} 0 \\ 0 \\ -48.167 \\ -1.716 \\ -14.695 \\ 0.275 \end{bmatrix} = \tag{16}$$

$$\bar{l} = \begin{bmatrix} 10287648.0 \\ 2980152.0 \\ 431697.167 \\ 36856.716 \\ 1999.195 \\ 62.725 \end{bmatrix} \tag{17}$$

Calculate P

$$Q = P.T = \begin{bmatrix} (A.T)^5 C.T & (A.T)^4 C.T & (A.T)^3 C.T & (A.T)^2 C.T & (A.T)C.T & C.T \end{bmatrix} \begin{bmatrix} 1 & 0 & 0 & 0 & 0 & 0 \\ \alpha_1 & 1 & 0 & 0 & 0 & 0 \\ \alpha_2 & \alpha_1 & 1 & 0 & 0 & 0 \\ \alpha_3 & \alpha_2 & \alpha_1 & 1 & 0 & 0 \\ \alpha_4 & \alpha_3 & \alpha_2 & \alpha_1 & 1 & 0 \\ \alpha_5 & \alpha_4 & \alpha_3 & \alpha_2 & \alpha_1 & 1 \end{bmatrix} \tag{18}$$

$$Q = \begin{bmatrix} 0 & 0 & 0 & 0 & 0 & 1.84 \\ 119.414 & -354.53 & 3.34 & -15.827 & 0 & 1.163 \\ -33.035 & 77.876 & -1.63 & 12.344 & 0 & 1.518 \\ 78.499 & -3.658 & 12.358 & -0.094 & 1.518 & 0 \\ -355.139 & 5.147 & -15.844 & 0.081 & 1.163 & 0 \\ 0.337 & -0.794 & 0.017 & -0.126 & 1.84 & 0 \end{bmatrix} \begin{bmatrix} 1 & 0 & 0 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 & 0 & 0 \\ 0 & 0 & 1 & 0 & 0 & 0 \\ -48.167 & 0 & 0 & 1 & 0 & 0 \\ -1.716 & -48.167 & 0 & 0 & 1 & 0 \\ -14.695 & -1.716 & -48.167 & 0 & 0 & 1 \end{bmatrix} \tag{19}$$

$$Q = \begin{bmatrix} -27.038 & -3.158 & -88.625 & 0 & 0 & 1.84 \\ 864.638 & -356.527 & -52.689 & -15.827 & 0 & 1.163 \\ -649.936 & 75.27 & -74.757 & 12.344 & 0 & 1.518 \\ 80.404 & -76.785 & 12.358 & -0.094 & 1.518 & 0 \\ -361.021 & -50.882 & -15.844 & 0.081 & 1.163 & 0 \\ 3.24 & -89.419 & 0.017 & -0.126 & 1.84 & 0 \end{bmatrix} \tag{20}$$

$$P = Q.T = \begin{bmatrix} -27.038 & 864.638 & -649.936 & 80.404 & -361.021 & 3.24 \\ -3.158 & -356.527 & 75.27 & -76.785 & -50.882 & -89.419 \\ -88.625 & -52.689 & -74.757 & 12.358 & -15.844 & 0.017 \\ 0 & -15.827 & 12.344 & -0.094 & 0.081 & -0.126 \\ 0 & 0 & 0 & 1.518 & 1.163 & 1.84 \\ 1.84 & 1.163 & 1.518 & 0 & 0 & 0 \end{bmatrix} \tag{21}$$

$$l = P^{-1}\bar{l} = \begin{bmatrix} 16931.654 \\ -11720.663 \\ -11498.595 \\ -14541.918 \\ -40028.382 \\ 38391.429 \end{bmatrix} \tag{22}$$

$$L = lv = \begin{bmatrix} 31153.499 & 19695.321 & 25705.401 \\ -21565.505 & -13633.767 & -17794.148 \\ -21156.909 & -13375.452 & -17457.007 \\ -26756.489 & -16915.52 & -22077.336 \\ -73650.464 & -46562.011 & -60770.532 \\ 70638.542 & 44657.866 & 58285.332 \end{bmatrix} \tag{23}$$

The final values of this ended up as follows:

$$|A - LC| = \begin{bmatrix} -31153.499 & -19695.321 & -25705.401 & 0 & 0 & 1.0 \\ 21565.505 & 13633.767 & 17794.148 & 0 & 1.0 & 0 \\ 21156.909 & 13375.452 & 17457.007 & 1.0 & 0 & 0 \\ 26756.489 & 16905.71 & 22077.336 & -0.1 & 0.05 & 0 \\ 73650.464 & 46576.726 & 60765.627 & 0.05 & -0.075 & 0.05 \\ -70638.542 & -44667.676 & -58275.522 & 0 & 0.05 & -0.1 \end{bmatrix} \tag{24}$$

1. $-15.833 - 15.011i$
2. $-7.404 - 13.269i$
3. -5.044

4. -11.483
5. $-7.404 + 13.269i$
6. $-15.833 + 15.011i$

Ya that isn't the best, but considering this is a 6 variablesystem, im calling it close enough

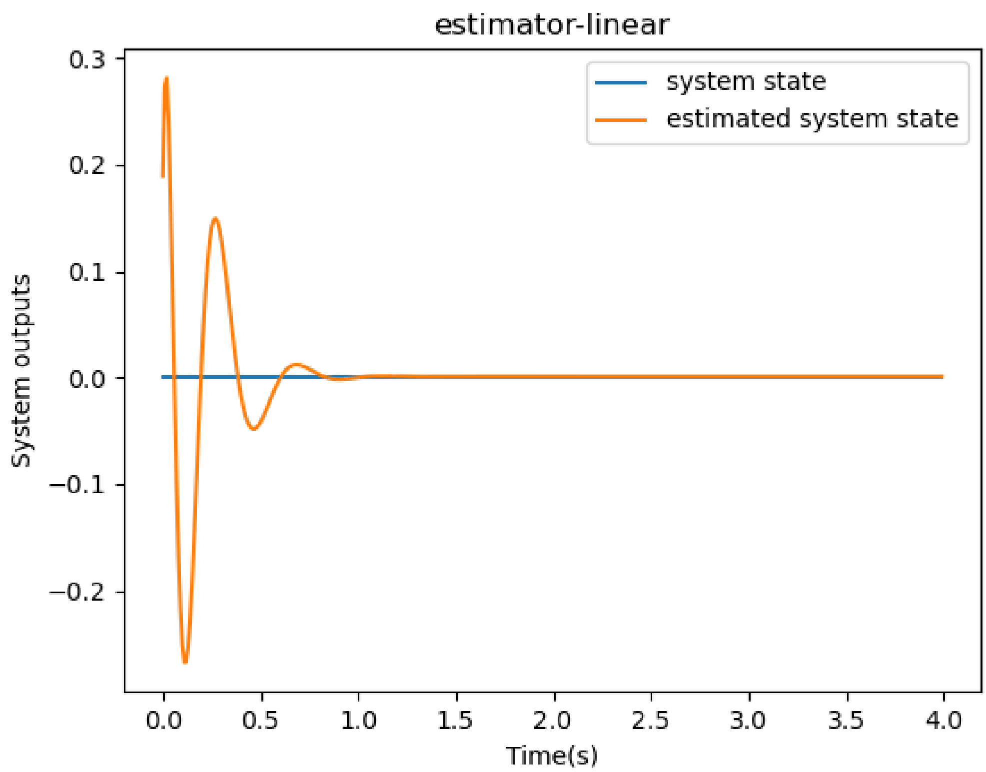

Figure 5: Estimator in Linear performance

For the next two plots, I was not able to get the observer to converge fast enough, with little overshoot in order to cause the system to be stable. I know parrallel axis theorem is a thing, and I cannot explain it, so for the next two plots, the feedback is given the true state rather than the estimated state.

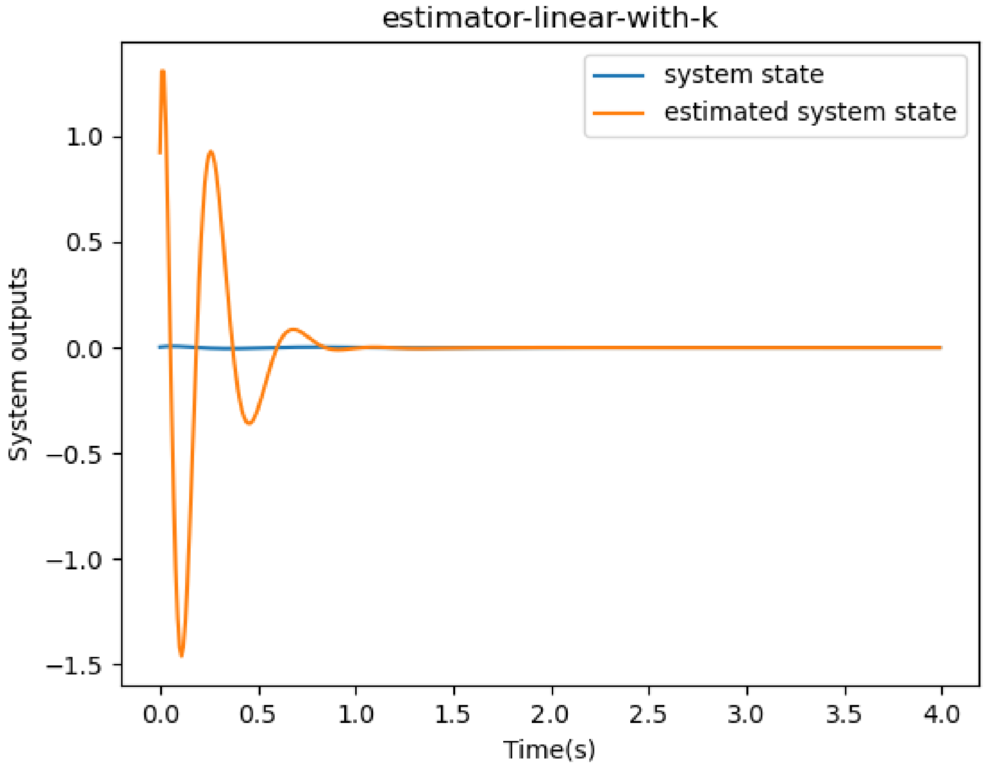

Figure 6: Estimator in Linear performance

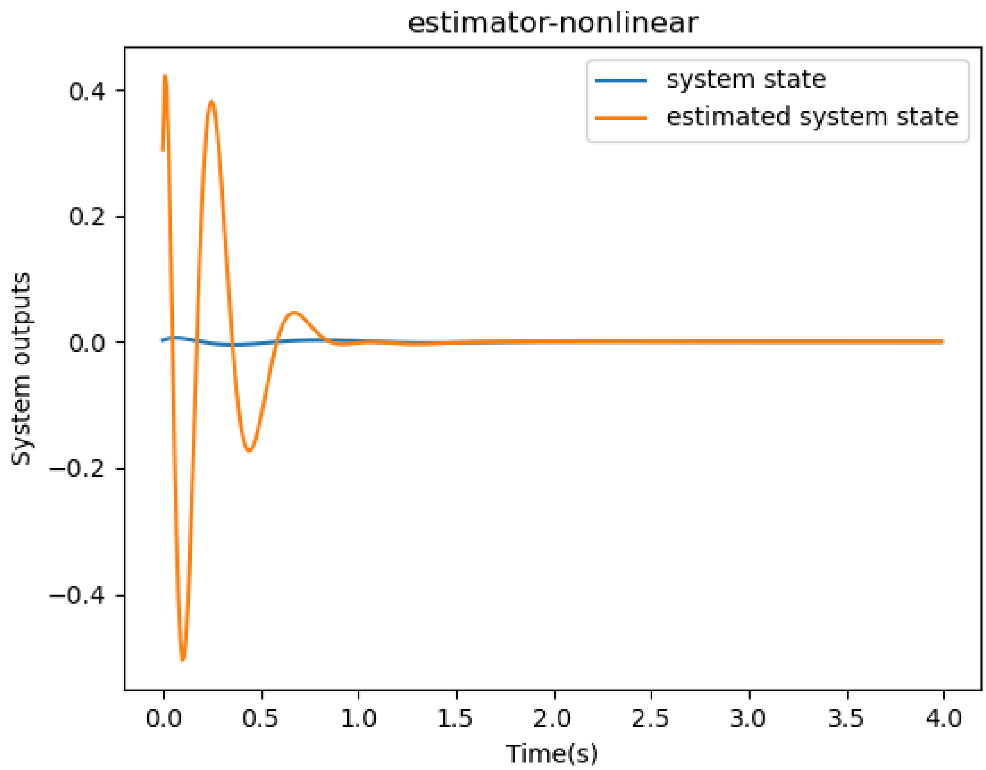

Figure 7: Estimator in NonLinear performance

4 Conclusion

In conclusion, this was a very hard project, the system being analyzed, that being a doubly-inverted pendulum being controlled only by a moving pad is incredibly hard and I now have a newfound respect for the videos online I have seen of people being able to control this system in real life. This difficulty was exasperated by the fact that as a non-Mechanical engineer, I did not have the requisite background to really be able to take on a problem of this caliber by myself.

As you are likely to wonder why I did not ask for help during this project, I find that I learn best when I just sit in my room and just struggle to solve the problem, even if it means that I spend significantly more time on the project than I would if I asked for help. I have spent an ordinate ammount of time on this project of constantly running into problems, figuring out my mis-understandings and then solving the issue, this is a strategy that has worked well for me in the past however for this project, I do feel that it backfired on me a bit on this project as there are several areas that I fell short on that could have been fixed early on if I asked for help specifically around equations of motion/state variable selection/linearization. As that could have resulted in a less complex system that is likely more in line with the expected difficulty of this project, as I do feel that the complexity was higher than expected.

Disclaimer: This is just a few relevant fragments of the source code, as the entire code is a complicated system that takes these fragments and automatically renders them into the final pdf. However all of this is available online on github(its latex + python)

A Help Recieved

This section is a thank you for people who caught issues or otherwised helped(collaboration is allowed)

1. People will be listed here