Homework Two

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1. A system is described by

$$\dot{x} = \begin{bmatrix} -2 & 1 \\ -1 & 0 \end{bmatrix} x + \begin{bmatrix} 3 \\ 1 \end{bmatrix} u \tag{1}$$

Obtain the STM of the uncontrolled system using the following methods:

(a) Via taking the laplace inverse of $(sI - A)^{-1}$

$$(sI - A) = \begin{bmatrix} s+2 & -1 \\ 1 & s \end{bmatrix} \tag{2}$$

$$(sI - A)^{-1} = \begin{bmatrix} \frac{s}{s^2+2s+1} & \frac{1}{s^2+2s+1} \\ -\frac{1}{s^2+2s+1} & \frac{s+2}{s^2+2s+1} \end{bmatrix} \tag{3}$$

$$\mathscr{L}^{-1}\left((sI - A)^{-1}\right) = \begin{bmatrix} (1-t)\,e^{-t}\theta\,(t) & te^{-t}\theta\,(t) \\ -te^{-t}\theta\,(t) & (t+1)\,e^{-t}\theta\,(t) \end{bmatrix} \tag{4}$$

The theta in the equation is the step function as described here

This is accomplished with the code in subsection A.1

(b) Via model decomposition of Matrix A
TODO: this

(c) Via the Cayley-Hamilton theorem
Cayley-hamilton theorem is hard to code in a way to show intermediate steps, so this will be done by hand.

This will be done by using the Cayley-hamilton theorem to solve for e^{At} then plugging it into equation 22 on slide 40 to get the value of the stm

- Find the characteristic polynomial

$$0 = |\lambda I - A| = \begin{vmatrix} \lambda + 2 & -1 \\ 1 & \lambda \end{vmatrix} = \lambda^2 + 2\lambda + 1 \tag{5}$$

$$\lambda = -1, -1 \tag{6}$$

- solve for β_0 and β_1

$$e^{-t} = \beta_0 - \beta_1 \tag{7}$$

$$-e^{-t} = \beta_1 \tag{8}$$

$$-e^{-t} = \beta_0 - e^{-t} \tag{9}$$

$$\beta_0 = 2e^{-t} \tag{10}$$

- solve e^{At}(and STM)

$$STM = e^{At} = \beta_o I + \beta_1 A = \begin{bmatrix} 2e^{-t} - \end{bmatrix} \tag{11}$$

TODO: figure out why this is wrong

2. In the system in Problem 1,

   (a) Obtain the zero input solution $x_{Z1}(t)$ for the initial condition $\bar{x}(0) = \begin{bmatrix} 10 \\ 1 \end{bmatrix}$

$$x(t) = STM(t)x_0 \tag{12}$$

$$x(t) = \begin{bmatrix} (10 - 9t)\, e^{-t}\theta\,(t) \\ (1 - 9t)\, e^{-t}\theta\,(t) \end{bmatrix} \tag{13}$$

   (b) Obtain the zero state solution $x_{ZS}(t)$ for input $u(t) = e^{2t}$ for $t > 0$
   o

$$x_{zs}(t) = \int_{t_0}^{t} \Phi(t,\tau)B(\tau)u(\tau)d\tau \tag{14}$$

$$\begin{bmatrix} \left(-2te^{t} + 5e^{t} - 5\right)\theta\,(t) \\ \left(-2te^{t} + 3e^{t} - 3\right)\theta\,(t) \end{bmatrix} \tag{15}$$

   (c) Obtain the total solution $\bar{x}(t)$ for the initial conditions and input in 2a and 2b.

Because this is an LTI system, the super-position property applies and the results of both of these systems can be added together to produce a system with initial state of 2a and the input of 2b

$$x(t) = \begin{bmatrix} \left(-9t - \left(2te^t - 5e^t + 5\right)e^t + 10\right)e^{-t}\theta(t) \\ \left(-9t - \left(2te^t - 3e^t + 3\right)e^t + 1\right)e^{-t}\theta(t) \end{bmatrix} \tag{16}$$

3. Given

$$A = \begin{bmatrix} -5 & -6 & 0 \\ 2 & 2 & 0 \\ 0 & 0 & -3 \end{bmatrix} \tag{17}$$

(a) Find A^{-1} using the Cayley-Hamilton theorem

- solve for eigenvalues
  This is done in python, but the equation for it is to solve the following equation:

$$\Delta = (\lambda I - A) = 0 \tag{18}$$

$$\lambda = -1, -2, -3 \tag{19}$$

- solve for β_0, β_1 β_2

$$f(\lambda) = \beta_0 + \beta_1\lambda + \beta_2\lambda^2 \tag{20}$$

$$-1 = \beta_0 - \beta_1 + \beta_2 \tag{21}$$

$$-\frac{1}{2} = \beta_0 - 2\beta_1 + 4\beta_2 \tag{22}$$

$$-\frac{1}{3} = \beta_0 - 3\beta_1 + 9\beta_2 \tag{23}$$

$$\begin{bmatrix} 1 & -1 & 1 \\ 1 & -2 & 4 \\ 1 & -3 & 9 \end{bmatrix}^{-1} \begin{bmatrix} -1 \\ -\frac{1}{2} \\ -\frac{1}{3} \end{bmatrix} = \begin{bmatrix} \beta_1 \\ \beta_2 \\ \beta_3 \end{bmatrix} \tag{24}$$

$$\begin{bmatrix} \beta_1 \\ \beta_2 \\ \beta_3 \end{bmatrix} = \begin{bmatrix} -1.83333 \\ -1 \\ -0.16667 \end{bmatrix} \tag{25}$$

- solve for A^{-1}

$$A^{-1} = \beta_0 I + \beta_1 A + \beta_2 A^2 = -1.8333I - A - 0.1667A = \begin{bmatrix} 4 & 7 & 0 \\ -2.333 & 4.1667 & 0 \\ 0 & 0 & 1.66668 \end{bmatrix} \tag{26}$$

TODO: Fix this wrong answer

(b) Obtain e^{At} using one of the three mentioned methods in Problem 1
Using the laplace transform

$$\mathscr{L}^{-1}\left((sI - A)^{-1}\right) = \begin{bmatrix} \left(4 - 3e^{t}\right) e^{-2t}\theta(t) & 6\left(1 - e^{t}\right) e^{-2t}\theta(t) & 0 \\ 2\left(e^{t} - 1\right) e^{-2t}\theta(t) & \left(4e^{t} - 3\right) e^{-2t}\theta(t) & 0 \\ 0 & 0 & e^{-3t}\theta(t) \end{bmatrix} \tag{27}$$

4. Let the STM of the system $\dot{\bar{x}}(t) = A\bar{x}(t)$, where A is a constant matrix, be $\Phi(t, t_0)$. Also, let the STM of the system $\dot{\bar{z}}(t) = -A^T \bar{Z}(t)$, where A^T is the transpose of A, be $\Theta(t, t_0)$. Use the properties of the STM on Slide 39 to show that $\Theta(t, t_0) = \Phi^T(t, t_0)$.

$$\Theta(t, t_0) = \Phi^T(t, t_0) \tag{28}$$

$$\Theta(t, t_0) = (\psi(t)\psi^{-1}(t_0))^T \tag{29}$$

$$\Theta(t, t_0) = \psi(t)^T \psi^{-1}(t_0)^T \tag{30}$$

$$\Theta(t, t_0) = \psi(t)^T \psi^{-1}(t_0)^T \tag{31}$$

$$\Theta(t, t_0) = \psi(t)^T \psi^{-1}(t_0)^T \tag{32}$$

$$\Theta(t, t_0) = \Theta(t, t_0) \tag{33}$$

5. Given a system in state space

$$\dot{\bar{x}}(t) = A\bar{x}(t) + B\bar{u}(t) \tag{34}$$

$$\bar{y}(t) = C\bar{x}(t) + D\bar{u}(t) \tag{35}$$

prove that the transfer function matrix is invariant to any similarity transformation of the state i.e. $\bar{x} = T\bar{z}$, where T is a constant invertible matrix.

$$\dot{x} = Ax + Bu \tag{36}$$

$$T\dot{z} = ATz + Bu \tag{37}$$

$$\dot{z} = T^{-1}ATz + T^{-1}Bu \tag{38}$$

$$A_Z = T^{-1}AT, B_Z = T^{-1}B \tag{39}$$

$$C_Z = CT^{-1}, D_Z = D \tag{40}$$

$$\mathscr{L}^{-1}(C(sI - A)^{-1}B + D) \tag{41}$$

$$\mathscr{L}^{-1}(C_Z(sI - A_Z)^{-1}B_Z + D_Z) \tag{42}$$

$$\mathscr{L}^{-1}(CT^{-1}(sI - T^{-1}AT)^{-1}T^{-1}B + D) \tag{43}$$

$$\mathscr{L}^{-1}(CT^{-1}T(sI - A)^{-1}TT^{-1}B + D) \tag{44}$$

$$\mathscr{L}^{-1}(C(sI - A)^{-1}B + D) \tag{45}$$

equality!

6. Give the algebraic and geometric multiplicities of the repeated eigenvalue and find e^{3t} for the matrices below.

   (a) $J = \begin{bmatrix} \lambda & 1 \\ 0 & \lambda \end{bmatrix}$

   algebraic multiplicity: 2

   geometric multiplicity: 2

   (b) $J = \begin{bmatrix} \lambda & 1 & 0 \\ 0 & \lambda & 1 \\ 0 & 0 & \lambda \end{bmatrix}$

   algebraic multiplicity: 3

   geometric multiplicity: 3

   (c) $J = \begin{bmatrix} \lambda & 1 & 0 \\ 0 & \lambda & 0 \\ 0 & 0 & \lambda \end{bmatrix}$

   algebraic multiplicity: 3

   geometric multiplicity: 2

A Appendix

A.1 One A source code

```
import numpy as np
import sympy
from sympy import eye, shape, simplify, inverse_laplace_transform, Matrix

def sI_A(A: Matrix):
    s = sympy.symbols('s')
    s_I = eye(shape(A)[0])*s
    return simplify(s_I-A)

def STM_laplace_inverse(A: Matrix):
    s, t = sympy.symbols('s, t')
    return simplify(
        inverse_laplace_transform((sI_A(A)).inv(), s, t)
    )
```

TODO: Create custom inverse-laplace-transform function