

Homework 3

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February 23, 2022

1. In the LTI system described for $\dot{\bar{x}}(t) = A\bar{x}$ with $A = \begin{bmatrix} 0 & 1 & 0 \\ 0 & 0 & 1 \\ -1 & -3 & -3 \end{bmatrix}$

- (a) Obtain all eigenvalues and eigenvectors of A

$$|\lambda I - A| = 0 \quad (1)$$

$$\det\left(\begin{bmatrix} \lambda & -1 & 0 \\ 0 & \lambda & -1 \\ 1 & 3 & \lambda + 3 \end{bmatrix}\right) = \lambda^3 + 3\lambda^2 + 3\lambda + 1 = 0 \quad (2)$$

$$\lambda = -1, -1, -1 \quad (3)$$

- (b) Use the eigenvectors in part 1a to obtain the modal matrix V and Jordan Form J

Solving for eigenvalue(s) of -1, with a multiplicity of 3

$$(A - \lambda i)^3 x_0 = 0 \quad (4)$$

$$\begin{bmatrix} 0 & 0 & 0 \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{bmatrix} \begin{bmatrix} x_0 \\ x_1 \\ x_2 \end{bmatrix} = 0 \quad (5)$$

$$x_0 = \begin{bmatrix} 1 \\ 0 \\ 0 \end{bmatrix} \quad (6)$$

$$(A - \lambda i)^2 \begin{bmatrix} 1 \\ 0 \\ 0 \end{bmatrix} = x_1 \quad (7)$$

$$\begin{bmatrix} 1 & 2 & 1 \\ -1 & -2 & -1 \\ 1 & 2 & 1 \end{bmatrix} \begin{bmatrix} 1 \\ 0 \\ 0 \end{bmatrix} = x_1 \quad (8)$$

$$x_1 = \begin{bmatrix} 1 \\ -1 \\ 1 \end{bmatrix} \quad (9)$$

$$(A - \lambda i)^1 \begin{bmatrix} 1 \\ -1 \\ 1 \end{bmatrix} = x_2 \quad (10)$$

$$\begin{bmatrix} 1 & 1 & 0 \\ 0 & 1 & 1 \\ -1 & -3 & -2 \end{bmatrix} \begin{bmatrix} 1 \\ -1 \\ 1 \end{bmatrix} = x_2 \quad (11)$$

$$x_2 = \begin{bmatrix} 0 \\ 0 \\ 0 \end{bmatrix} \quad (12)$$

$$V = \begin{bmatrix} 1 & 1 & 1 \\ -1 & 0 & 0 \\ 1 & -1 & 0 \end{bmatrix} \quad (13)$$

$$J = V^{-1}AV = \begin{bmatrix} 0 & -1 & 0 \\ 0 & -1 & -1 \\ 1 & 2 & 1 \end{bmatrix} \begin{bmatrix} 0 & 1 & 0 \\ 0 & 0 & 1 \\ -1 & -3 & -3 \end{bmatrix} \begin{bmatrix} 1 & 1 & 1 \\ -1 & 0 & 0 \\ 1 & -1 & 0 \end{bmatrix} \quad (14)$$

$$J = \begin{bmatrix} -1 & 1 & 0 \\ 0 & -1 & 1 \\ 0 & 0 & -1 \end{bmatrix} \quad (15)$$

2. In each case below discuss BIBS stability of the LTI system $\dot{\bar{x}}(t) = A\bar{x}(t)$:

(a) $A = \begin{bmatrix} -1 & 0 \\ 0 & -2 \end{bmatrix}$

This system is stable as both of its eigenvalues are less 0

(b) $A = \begin{bmatrix} 0 & 0 & 0 \\ 0 & -1 & 0 \\ 0 & 0 & -3 \end{bmatrix}$

This system is semi-stable as two of the three are less than 0, and one is 0

$$(c) \ A = \begin{bmatrix} 1 & 0 & 0 \\ 0 & -1 & 0 \\ 0 & 0 & -3 \end{bmatrix}$$

This system is not stable an eigenvalue is greater than 0

$$(d) \ A = \begin{bmatrix} 0 & 1 & 0 \\ 0 & 0 & 0 \\ 0 & 0 & -3 \end{bmatrix}$$

This system has two semi-simple eigenvalues that are 0, meaning this is not stable

$$(e) \ A = \begin{bmatrix} 0 & 0 & 0 \\ 0 & 0 & 0 \\ 0 & 0 & -3 \end{bmatrix}$$

this system has three eigenvalues that are all less than or equal to 0

3. The linearized equations of motion of a pendulum can be written in the form of

$$\dot{x}_1 = x_2 \quad (16)$$

$$\dot{x}_2 = -ax_1 - cx_2 \quad (17)$$

where $a > 0$ is a constant parameter of the system and $c > 0$ is the torsional friction coefficient

(a) Study BIBS stability of the system

$$\begin{bmatrix} \dot{x}_1 \\ \dot{x}_2 \end{bmatrix} = \begin{bmatrix} 0 & 1 \\ -a & -c \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \end{bmatrix} \quad (18)$$

This allows you to calculate the eigenvalues with:

$$\Delta = |\lambda I - A| = 0 \quad (19)$$

$$0 = \begin{vmatrix} \lambda & -1 \\ a & \lambda + c \end{vmatrix} = \lambda^2 + c\lambda + a \quad (20)$$

$$\lambda = \frac{-c \pm \sqrt{c^2 - 4a}}{2} \quad (21)$$

the system will be stable if the eigenvalues described by this equation are less than 0 (or equal to 0 if the eigenvalues produce linearly independent eigenvectors)

(b) Consider the quadratic Lyapunov function $V = \bar{x}^T P \bar{x}$ with $P = \begin{bmatrix} \frac{a}{2} & 0 \\ 0 & \frac{1}{2} \end{bmatrix}$. What can be said about the stability of the system based on this choice of the Lyapunov

function?

$$\begin{bmatrix} x_1 & x_2 \end{bmatrix} \begin{bmatrix} \frac{a}{2} & 0 \\ 0 & \frac{1}{2} \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \end{bmatrix} = \begin{bmatrix} x_1 & x_2 \end{bmatrix} \begin{bmatrix} \frac{ax_1}{2} \\ \frac{x_2}{2} \end{bmatrix} = \begin{bmatrix} \frac{a(x_1)^2}{2} \\ \frac{(x_2)^2}{2} \end{bmatrix} \quad (22)$$

V is positive definite if a is positive? **TODO: check**

- (c) What can be said about the stability of the system based on the analysis of b) and c)?

The system will be stable if a is positive

- (d) Study BIBS stability of the system when $c = 0$

$$\lambda = \pm \sqrt{a}j \quad (23)$$

if the eigenvectors are linearly independent, then the system is lyapunov stable

4. For the transfer function matrix

$$H(s) = \begin{bmatrix} \frac{s}{s-2} & 0 \\ \frac{2}{s-2} & 1 \end{bmatrix} \quad (24)$$

- (a) Obtain the controllable canonical form

$$\frac{1}{s+2} \begin{bmatrix} s & 0 \\ 2 & s+2 \end{bmatrix} \quad (25)$$

$$\dot{x} = \begin{bmatrix} -2 & 0 \\ 0 & -2 \end{bmatrix} x + \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix} u \quad (26)$$

$$N_0 = \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix} \quad (27)$$

$$N_1 = \begin{bmatrix} 0 & 0 \\ 2 & 2 \end{bmatrix} \quad (28)$$

$$y = \left(\begin{bmatrix} 0 & 0 \\ 2 & 2 \end{bmatrix} - \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix} 2 \right) x + \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix} u \quad (29)$$

$$y = \begin{bmatrix} -2 & 0 \\ 2 & 0 \end{bmatrix} x + \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix} u \quad (30)$$

TODO: check this

(b) obtain the observable canonical form

$$\frac{1}{s+2} \begin{bmatrix} s & 0 \\ 2 & s+2 \end{bmatrix} \quad (31)$$

$$N_0 = \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix} \quad (32)$$

$$N_1 = \begin{bmatrix} 0 & 0 \\ 2 & 2 \end{bmatrix} \quad (33)$$

$$\dot{x} = \begin{bmatrix} 2 & 0 \\ 0 & 2 \end{bmatrix} x + \left(\begin{bmatrix} 0 & 0 \\ 2 & 2 \end{bmatrix} - \begin{bmatrix} 2 & 0 \\ 0 & 2 \end{bmatrix} \right) u \quad (34)$$

$$\dot{x} = \begin{bmatrix} 2 & 0 \\ 0 & 2 \end{bmatrix} x + \begin{bmatrix} -2 & 0 \\ 2 & 0 \end{bmatrix} u \quad (35)$$

$$y = \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix} x + \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix} \quad (36)$$

(c) show that the realization in (a) and (b) are dual
as per

$$\begin{bmatrix} A & B \\ C & D \end{bmatrix}^T = \begin{bmatrix} A^T & B^T \\ C^T & D^T \end{bmatrix} \quad (37)$$

it can be seen that this is not the case for this example as $A_c \neq A_o^T$