Homework UPDATE ME

Nathan Rose

March 28, 2022

1. For each of the systems (a) and (b) do the following:
   (a) Obtain the eigenvalues

      i. A

$$0 = \Delta(\lambda) = |\lambda I - A| = \begin{vmatrix} \lambda+6 & 1 & 0 \\ 11 & \lambda & 1 \\ 6 & 0 & \lambda \end{vmatrix} \tag{1}$$

$$0 = (\lambda+6)(\lambda^2) + 11\lambda + 6 = \lambda^3 + 6\lambda^2 + 11\lambda + 6 = \tag{2}$$

$$0 = \lambda^3 + 6\lambda^2 + 11\lambda + 6 = (x+1)(x^2+5x+6) = (x+1)(x+2)(x+3) \tag{3}$$

$$\lambda = -1, -2, -3 \tag{4}$$

$$0 = |\lambda I - A| = \begin{bmatrix} \lambda+6 & 1 & 0 \\ 11 & \lambda & 1 \\ 6 & 0 & \lambda \end{bmatrix} v \tag{5}$$

$$0 = \begin{bmatrix} 5 & 1 & 0 \\ 11 & -1 & 1 \\ 6 & 0 & -1 \end{bmatrix} v_1 \tag{6}$$

$$v_1 = \begin{bmatrix} \frac{1}{2} \\ \frac{3}{2} \\ 1 \end{bmatrix} \tag{7}$$

$$0 = \begin{bmatrix} 4 & 1 & 0 \\ 11 & -2 & 1 \\ 6 & 0 & -2 \end{bmatrix} v_1 \tag{8}$$

$$v_2 = \begin{bmatrix} \frac{1}{3} \\ \frac{4}{3} \\ 1 \end{bmatrix} \tag{9}$$

$$0 = \begin{bmatrix} 3 & 1 & 0 \\ 11 & -3 & 1 \\ 6 & 0 & -3 \end{bmatrix} v_1 \tag{10}$$

$$v_3 = \begin{bmatrix} \frac{1}{6} \\ \frac{5}{6} \\ 1 \end{bmatrix} \tag{11}$$

$$V = \begin{bmatrix} \frac{1}{2} & \frac{1}{3} & \frac{1}{6} \frac{3}{2} & \frac{4}{3} & \frac{5}{6} \\ 1 & 1 & 1 \end{bmatrix} \tag{12}$$

ii. B

eigenvalues are: $\lambda = -1, -1, -1, -2, -3, -3$ from sympy.jordan_from, eigenvectors are:

$$V = \begin{bmatrix} 0 & 0 & 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & 0 & 1 & 0 \\ 0 & 0 & 0 & 0 & 1 & 0 \\ 1 & 0 & 0 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 & 0 & \frac{1}{2} \\ 0 & 0 & 0 & 0 & 0 & 1 \end{bmatrix} \tag{13}$$

(b) Use the similarity transformation $x(t) = Vz(t)$ to express the system in the modal form

i. A

$$\bar{A} = J = \begin{bmatrix} -3 & 0 & 0 \\ 0 & -2 & 0 \\ 0 & 0 & -1 \end{bmatrix} \tag{14}$$

$$\bar{B} = V^{-1}B = \begin{bmatrix} -4 \\ 9 \\ 0 \end{bmatrix} \tag{15}$$

$$\bar{C} = CV = \begin{bmatrix} \frac{1}{2} & \frac{1}{3} & \frac{1}{6} \end{bmatrix} \tag{16}$$

ii. B
jordan form:

$$\bar{A} = J = \begin{bmatrix} -3 & 0 & 0 & 0 & 0 & 0 \\ 0 & -3 & 0 & 0 & 0 & 0 \\ 0 & 0 & -2 & 0 & 0 & 0 \\ 0 & 0 & 0 & -1 & 0 & 0 \\ 0 & 0 & 0 & 0 & -1 & 0 \\ 0 & 0 & 0 & 0 & 0 & -1 \end{bmatrix} \tag{17}$$

$$\bar{B} = V^{-1}B = \begin{bmatrix} 0 & 1 \\ -\frac{1}{2} & 0 \\ 0 & -1 \\ 1 & 0 \\ 0 & 1 \\ 1 & 0 \end{bmatrix} \tag{18}$$

$$\bar{C} = CV = \begin{bmatrix} 0 & 0 & 2 & 1 & 2 & 0 \\ 1 & 1 & 0 & 0 & 0 & \frac{1}{2} \end{bmatrix} \tag{19}$$

(c) Use the modal form to study the controllability and observability of the system(slides 77, 85)

i. A
modal state variables 1 and 2 are controllable do to non-zero values but 3 is not. all state variables are observable due to non-zero values.

ii. B
state variables are 1-3 are controllable, 4-6 are not due to non-linear independence of bottom three rows of B. only the -2 eigenvalue state variable is observable because of its linear independence

(d) idicate teh controllability and onbservability of each mode

i. A
System is observable not controllable

ii. B
system is not controllobale or observable

(e) Study stabalizability and detectability of the system

i. A
system is stabalizable(and obviously) detectable

ii. B
system is stabalizable and detectable

(f) Plot the block diagram of the modal form

i. A
TODO: this

ii. B
TODO: this

2. For the systems of a and b do the following:

(a) Obtain the transfer function matrix

i. A

$$G(s) = C(sI - A)^{-1}B + D = \begin{bmatrix} 3 & 1 & 4 & 0 \\ 0 & 1 & 0 & 1 \end{bmatrix} \begin{bmatrix} s+6 & -1 & 0 & 0 \\ 0 & s+6 & 0 & 0 \\ 0 & 0 & s+6 & 0 \\ 0 & 0 & 0 & s-6 \end{bmatrix}^{-1} \begin{bmatrix} 0 & 0 & 0 \\ 1 & 1 & 1 \\ 2 & 0 & 2 \\ 0 & 1 & 0 \end{bmatrix} \tag{20}$$

$$G(s) = \frac{1}{(s+6)^2(s-6)} \begin{bmatrix} 3 & 1 & 4 & 0 \\ 0 & 1 & 0 & 1 \end{bmatrix} \begin{bmatrix} s^2-36 & s-6 & 0 & 0 \\ 0 & s^2-36 & 0 & 0 \\ 0 & 0 & s^2-36 & 0 \\ 0 & 0 & 0 & (s+6)^2 \end{bmatrix} \begin{bmatrix} 0 & 0 & 0 \\ 1 & 1 & 1 \\ 2 & 0 & 2 \\ 0 & 1 & 0 \end{bmatrix} \tag{21}$$

$$G(s) = \frac{1}{(s+6)^2(s-6)} \begin{bmatrix} 3 & 1 & 4 & 0 \\ 0 & 1 & 0 & 1 \end{bmatrix} \begin{bmatrix} s-6 & s-6 & s-6 \\ s^2-36 & s^2-36 & s^2-36 \\ 2s^2-72 & 0 & 2s^2-72 \\ 0 & (s+6)^2 & 0 \end{bmatrix} \tag{22}$$

$$G(s) = \frac{1}{(s+6)^2(s-6)} \begin{bmatrix} 9s^2+3s-342 & s^2+3s-54 & 9s^2+3s-342 \\ s^2-36 & 2s(s+6) & s^2-36 \end{bmatrix} \tag{23}$$

ii. B

from previous problem, only thing that changes is C

$$G(s) = \frac{1}{(s+6)^2(s-6)} \begin{bmatrix} 3 & 1 & 4 & 0 \\ 0 & 1 & 1 & 1 \end{bmatrix} \begin{bmatrix} s-6 & s-6 & s-6 \\ s^2-36 & s^2-36 & s^2-36 \\ 4s^2-144 & 0 & 4s^2-144 \\ 0 & (s+6)^2 & 0 \end{bmatrix} \tag{24}$$

$$G(s) = \frac{1}{(s+6)^2(s-6)} \begin{bmatrix} 9s^2+3s-342 & s^2+3s-54 & 9s^2+3s-342 \\ s^2-36 & 2s(s+6) & s^2-36 \end{bmatrix} \tag{25}$$

(b) Verify the system is a minimum realization, if the system is not minimal do steps 3 and 4

A minimum realization is both observable and controllable

i. A

Controllable because rows 2 and 3 of matrix B are LI, and row 4 is non-zero not observable because columns 1 and 3 of matrix C are not LI This is a not a min realization

ii. B

Controllable because rows 2 and 3 of matrix B are LI, and row 4 is non-zero Observable because columns 1 and 3 of matrix C are LI and column 4 is non-zero This is a min realization

(c) Use approach a discussed in slide 92 to obtain the minimal realization of the system.

i. A

Move to observable canonical form $q = 2, r = 3$

$$(s+6)^2(s-6) = (s^2+12s+36)(s-6) = s^3+6s^2-36s-216 \tag{26}$$

$$N_0 = \begin{bmatrix} 0 & 0 & 0 \\ 0 & 0 & 0 \end{bmatrix} \tag{27}$$

$$N_1 = \begin{bmatrix} 9 & 1 & 9 \\ 1 & 2 & 1 \end{bmatrix} \tag{28}$$

$$N_2 = \begin{bmatrix} 3 & 3 & 3 \\ 0 & 12 & 0 \end{bmatrix} \tag{29}$$

$$N_3 = \begin{bmatrix} -342 & -54 & -342 \\ -36 & 12 & -36 \end{bmatrix} \tag{30}$$

$$A = \begin{bmatrix} 0 & 0 & 0 & 0 & 216 & 0 \\ 0 & 0 & 0 & 0 & 0 & 216 \\ 1 & 0 & 0 & 0 & 36 & 0 \\ 0 & 1 & 0 & 0 & 0 & 36 \\ 0 & 0 & 1 & 0 & -6 & 0 \\ 0 & 0 & 0 & 1 & 0 & -6 \end{bmatrix} \tag{31}$$

$$B = \begin{bmatrix} -342 & -54 & -342 \\ -36 & 12 & -36 \\ 3 & 3 & 3 \\ 0 & 12 & 0 \\ 9 & 1 & 9 \\ 1 & 2 & 1 \end{bmatrix} \tag{32}$$

$$C = \begin{bmatrix} 0 & 0 & 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 0 & 0 & 1 \end{bmatrix} \tag{33}$$

$$D = \begin{bmatrix} 0 & 0 & 0 \\ 0 & 0 & 0 \end{bmatrix} \tag{34}$$

Now move it to jordan form:

$$P = \begin{bmatrix} -36 & -6 & 0 & 0 & 36 & 0 \\ 0 & 0 & -36 & -6 & 0 & 36 \\ 0 & 1 & 0 & 0 & 12 & 0 \\ 0 & 0 & 0 & 1 & 0 & 12 \\ 1 & 0 & 0 & 0 & 1 & 0 \\ 0 & 0 & 1 & 0 & 0 & 1 \end{bmatrix} \tag{35}$$

$$\bar{A} = J = \begin{bmatrix} -6 & 1 & 0 & 0 & 0 & 0 \\ 0 & -6 & 0 & 0 & 0 & 0 \\ 0 & 0 & -6 & 1 & 0 & 0 \\ 0 & 0 & 0 & -6 & 0 & 0 \\ 0 & 0 & 0 & 0 & 6 & 0 \\ 0 & 0 & 0 & 0 & 0 & 6 \end{bmatrix} \tag{36}$$

$$\bar{B} = P^{-1}B = \begin{bmatrix} -36 & -6 & 0 & 0 & 36 & 0 \\ 0 & 0 & -36 & -6 & 0 & 36 \\ 0 & 1 & 0 & 0 & 12 & 0 \\ 0 & 0 & 0 & 1 & 0 & 12 \\ 1 & 0 & 0 & 0 & 1 & 0 \\ 0 & 0 & 1 & 0 & 0 & 1 \end{bmatrix} \begin{bmatrix} -342 & -54 & -342 \\ -36 & 12 & -36 \\ 3 & 3 & 3 \\ 0 & 12 & 0 \\ 9 & 1 & 9 \\ 1 & 2 & 1 \end{bmatrix} = \begin{bmatrix} 9 & 1 & 9 \\ 3 & 3 & 3 \\ 1 & \frac{11}{12} & 1 \\ 0 & -1 & 0 \\ 0 & 0 & 0 \\ 0 & \frac{13}{12} & 0 \end{bmatrix} \tag{37}$$

$$\bar{C} = CP = \begin{bmatrix} 0 & 0 & 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 0 & 0 & 1 \end{bmatrix} \begin{bmatrix} -36 & -6 & 0 & 0 & 36 & 0 \\ 0 & 0 & -36 & -6 & 0 & 36 \\ 0 & 1 & 0 & 0 & 12 & 0 \\ 0 & 0 & 0 & 1 & 0 & 12 \\ 1 & 0 & 0 & 0 & 1 & 0 \\ 0 & 0 & 1 & 0 & 0 & 1 \end{bmatrix} = \begin{bmatrix} 1 & 0 & 0 & 0 & 1 & 0 \\ 0 & 0 & 1 & 0 & 0 & 1 \end{bmatrix} \tag{38}$$

Painfully remove a state variable

$$y(t) = \begin{bmatrix} 1 \\ 0 \end{bmatrix} \bar{x}_0 + \begin{bmatrix} 0 \\ 0 \end{bmatrix} \bar{x}_1 + \begin{bmatrix} 0 \\ 1 \end{bmatrix} \bar{x}_2 + \begin{bmatrix} 0 \\ 0 \end{bmatrix} \bar{x}_3 + \begin{bmatrix} 1 \\ 0 \end{bmatrix} \bar{x}_4 + \begin{bmatrix} 0 \\ 1 \end{bmatrix} \bar{x}_5 \tag{39}$$

Lets define the state variables as follows:

A. $q_o = \bar{x}_0 + \bar{x}_4$

B. $q_1 = \bar{x}_1$

C. $q_2 = \bar{x}_2$

D. $q_3 = \bar{x}_3$

E. $q_4 = \bar{x}_5$

Thus:

A. $\dot{q}_o = \dot{\bar{x}}_0 + \dot{\bar{x}}_4$

B. $\dot{q}_1 = \dot{\bar{x}}_1$

C. $\dot{q}_2 = \dot{\bar{x}}_2$

D. $\dot{q}_3 = \dot{\bar{x}}_3$

E. $\dot{q}_4 = \dot{\bar{x}}_5$

THe state update equation is the same as bar A but:

A. row 0 = row 0 + row 4

for the output equation you just delete column 4 resulting in the following stat equation same thing, but swap row for column for C

$$\dot{q} = \begin{bmatrix} -6 & 1 & 0 & 0 & 0 \\ 0 & -6 & 0 & 0 & 0 \\ 0 & 0 & -6 & 1 & 0 \\ 0 & 0 & 0 & -6 & 0 \\ 0 & 0 & 0 & 0 & 6 \end{bmatrix} q + \begin{bmatrix} 9 & 1 & 9 \\ 3 & 3 & 3 \\ 1 & \frac{11}{12} & 1 \\ 0 & -1 & 0 \\ 0 & \frac{13}{12} & 0 \end{bmatrix} u \tag{40}$$

$$y = \begin{bmatrix} 1 & 0 & 0 & 0 & 0 \\ 0 & 0 & 1 & 0 & 1 \end{bmatrix} q + \begin{bmatrix} 0 & 0 & 0 \\ 0 & 0 & 0 \end{bmatrix} u \tag{41}$$

ii. B
not needed

(d) Show that the transfer matrix of the minimal realization is the same as that in part i

i. A

$$G(s) = C(sI-A)^{-1}B+D = \begin{bmatrix} 1 & 0 & 0 & 0 & 0 \\ 0 & 0 & 1 & 0 & 1 \end{bmatrix} (\begin{bmatrix} s+6 & -1 & 0 & 0 & 0 \\ 0 & s+6 & 0 & 0 & 0 \\ 0 & 0 & s+6 & -1 & 0 \\ 0 & 0 & 0 & s+6 & 0 \\ 0 & 0 & 0 & 0 & s-6 \end{bmatrix})^{-1} \begin{bmatrix} 9 & 1 \\ 3 & 3 \\ 1 & \frac{11}{12} \\ 0 & -1 \\ 0 & \frac{13}{12} \end{bmatrix} \tag{42}$$

$$G(s) = \begin{bmatrix} 1 & 0 & 0 & 0 & 0 \\ 0 & 0 & 1 & 0 & 1 \end{bmatrix} \begin{bmatrix} \frac{1}{s+6} & \frac{1}{s^2+12s+36} & 0 & 0 & 0 \\ 0 & \frac{1}{s+6} & 0 & 0 & 0 \\ 0 & 0 & \frac{1}{s+6} & \frac{1}{s^2+12s+36} & 0 \\ 0 & 0 & 0 & \frac{1}{s+6} & 0 \\ 0 & 0 & 0 & 0 & \frac{1}{s-6} \end{bmatrix} \begin{bmatrix} 9 & 1 & 9 \\ 3 & 3 & 3 \\ 1 & \frac{11}{12} & 1 \\ 0 & -1 & 0 \\ 0 & \frac{13}{12} & 0 \end{bmatrix} +0 \tag{43}$$

$$G(s) = \begin{bmatrix} \frac{3(3s+19)}{s^2+12s+36} & \frac{s+9}{s^2+12s+36} & \frac{3(3s+19)}{s^2+12s+36} \\ \frac{1}{s+6} & \frac{2\left(s^2+6s+6\right)}{s^3+6s^2-36s-216} & \frac{1}{s+6} \end{bmatrix} \tag{44}$$

$$G(s) = \frac{1}{(s+6^2)(s-6)} \begin{bmatrix} 9s^2+3s-342 & s^2+3s-54 & 9s^2+3s-342 \\ s^2-36 & 2s^2+12s+12 & s^2-36 \end{bmatrix} \tag{45}$$

the same

ii. B

not needed

3. For the system with transfer function

$$H(s) = \begin{bmatrix} \frac{-s}{(s+1)^2} & \frac{1}{s+1} \\ \frac{2s+1}{s(s+1)} & \frac{1}{s+1} \end{bmatrix} \tag{46}$$

(a) Find a minimal realization using approach(b) Jordan canonical form method, discussed in Slides 94-96

Write as partial fractions

$$H(s) = \frac{1}{s(s+1)^2} \begin{bmatrix} -s^2 & s(s+1) \\ (s2+1)(s+1) & s(s+1) \end{bmatrix} \tag{47}$$

$$H(s) = \frac{N_0}{s} + \frac{N_{1-0}}{s+1} + \frac{N_{1-1}}{(s+1)^2} \tag{48}$$

$$H(s) = \frac{1}{s} \begin{bmatrix} 0 & 0 \\ 1 & 0 \end{bmatrix} + \frac{1}{s+1} \begin{bmatrix} -1 & 1 \\ 1 & 1 \end{bmatrix} + \frac{1}{(s+1)^2} \begin{bmatrix} 1 & 0 \\ 0 & 0 \end{bmatrix} \tag{49}$$

$$H(s) = \frac{1}{s} \begin{bmatrix} 0 \\ 1 \end{bmatrix} \begin{bmatrix} 1 & 0 \end{bmatrix} + \frac{1}{s+1} \begin{bmatrix} -1 & 1 \\ 1 & 1 \end{bmatrix} + \frac{1}{(s+1)^2} \begin{bmatrix} 1 \\ 0 \end{bmatrix} \begin{bmatrix} 1 & 0 \end{bmatrix} \tag{50}$$

rank = 4

TODO: Finish

(b) Show that the transfer function matrix of the minimal realization is the same as H(s) given above

TODO: this

1 Help Recieved

This section is a thank you for people who caught issues or otherwised helped(collaboration is allowed)

1. List people here

A jupyter notebook

For question two, I used a jupyter notebook to do calculations, and to check my work, the easiest way to see this is to open the following file in a jupyter notebook(or just look at it on github) `https://github.com/NathanRoseCE/ControlsClass/blob/master/HomeworkFour/Untitled.ipynb`

Disclaimer: This is just a few relevant fragments of the source code, as the entire code is a complicated system that takes these fragments and automatically renders them into the final pdf. However all of this is available online on github(its latex + python)