Homework 3

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1. In the LTI system described for $\dot{\bar{x}}(t) = A\bar{x}$ with $A = \begin{bmatrix} 0 & 1 & 0 \\ 0 & 0 & 1 \\ -1 & -3 & -3 \end{bmatrix}$

   (a) Obtain all eigenvalues and eigenvectors of A

   $$|\lambda I - A| = 0 \tag{1}$$

   $$det(\begin{bmatrix} \lambda & -1 & 0 \\ 0 & \lambda & -1 \\ 1 & 3 & \lambda+3 \end{bmatrix}) = \lambda^3 + 3\lambda^2 + 3\lambda + 1 = 0 \tag{2}$$

   $$\lambda = -1, -1, -1 \tag{3}$$

   (b) Use the eigenvectors in part 1a to obtain the m odal matrix V and Jordan Form J

   Solving for eigenvalue(s) of -1, with a multiplicity of 3

   $$(A - \lambda i)^3 x_0 = 0 \tag{4}$$

   $$\begin{bmatrix} 0 & 0 & 0 \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{bmatrix} \begin{bmatrix} x_0 \\ x_1 \\ x_2 \end{bmatrix} = 0 \tag{5}$$

   $$x_0 = \begin{bmatrix} 1 \\ 0 \\ 0 \end{bmatrix} \tag{6}$$

   $$(A - \lambda i)^2 \begin{bmatrix} 1 \\ 0 \\ 0 \end{bmatrix} = x_1 \tag{7}$$

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$$\begin{bmatrix} 1 & 2 & 1 \\ -1 & -2 & -1 \\ 1 & 2 & 1 \end{bmatrix} \begin{bmatrix} 1 \\ 0 \\ 0 \end{bmatrix} = x_1 \tag{8}$$

$$x_1 = \begin{bmatrix} 1 \\ -1 \\ 1 \end{bmatrix} \tag{9}$$

$$(A - \lambda i)^1 \begin{bmatrix} 1 \\ -1 \\ 1 \end{bmatrix} = x_2 \tag{10}$$

$$\begin{bmatrix} 1 & 1 & 0 \\ 0 & 1 & 1 \\ -1 & -3 & -2 \end{bmatrix} \begin{bmatrix} 1 \\ -1 \\ 1 \end{bmatrix} = x_2 \tag{11}$$

$$x_2 = \begin{bmatrix} 0 \\ 0 \\ 0 \end{bmatrix} \tag{12}$$

$$V = \begin{bmatrix} 1 & 1 & 1 \\ -1 & 0 & 0 \\ 1 & -1 & 0 \end{bmatrix} \tag{13}$$

$$J = V^{-1}AV = \begin{bmatrix} 0 & -1 & 0 \\ 0 & -1 & -1 \\ 1 & 2 & 1 \end{bmatrix} \begin{bmatrix} 0 & 1 & 0 \\ 0 & 0 & 1 \\ -1 & -3 & -3 \end{bmatrix} \begin{bmatrix} 1 & 1 & 1 \\ -1 & 0 & 0 \\ 1 & -1 & 0 \end{bmatrix} \tag{14}$$

$$J = \begin{bmatrix} -1 & 1 & 0 \\ 0 & -1 & 1 \\ 0 & 0 & -1 \end{bmatrix} \tag{15}$$

2. In each case below discuss BIBS stability of the LTI system $\dot{\bar{x}}(t) = A\bar{x}(t)$:

   (a) $A = \begin{bmatrix} -1 & 0 \\ 0 & -2 \end{bmatrix}$

   This system is stable as both of its eigenvalues are less 0

   (b) $A = \begin{bmatrix} 0 & 0 & 0 \\ 0 & -1 & 0 \\ 0 & 0 & -3 \end{bmatrix}$

   This system is semi-stable as two of the three are less than 0, and one is 0

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(c) $A = \begin{bmatrix} 1 & 0 & 0 \\ 0 & -1 & 0 \\ 0 & 0 & -3 \end{bmatrix}$

This system is not stable an eigenvalue is greater than 0

(d) $A = \begin{bmatrix} 0 & 1 & 0 \\ 0 & 0 & 0 \\ 0 & 0 & -3 \end{bmatrix}$

This system has two semi-simple eigenvalues that are 0, meaning this is not stable

(e) $A = \begin{bmatrix} 0 & 0 & 0 \\ 0 & 0 & 0 \\ 0 & 0 & -3 \end{bmatrix}$

this system has three eigenvaleus that are all less than or equal to 0

3. The linearized equatinos of motion of a pendulum can be written in the form of

$$\dot{x}_1 = x_2 \tag{16}$$

$$\dot{x}_2 = -ax_1 - cx_2 \tag{17}$$

where $a > 0$ is a constant parameter of the system and $c > 0$ is the torsional friction coefficient

(a) Study BIBS stability of the system

$$\begin{bmatrix} \dot{x}_1 \\ \dot{x}_2 \end{bmatrix} = \begin{bmatrix} 0 & 1 \\ -a & -c \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \end{bmatrix} \tag{18}$$

This allows you to calculate the eigenvalues with:

$$\Delta = |\lambda I - A| = 0 \tag{19}$$

$$0 = \begin{vmatrix} \lambda & -1 \\ a & \lambda + c \end{vmatrix} = \lambda^2 + c\lambda + a \tag{20}$$

$$\lambda = \frac{-c \pm \sqrt{c^2 - 4a}}{2} \tag{21}$$

the system will be stable if the eigenvalues described by this equation are less then 0(or equal to 0 if the eigenvalues produce linearly independent eigenvectors)

(b) Consider the quadratic Lyapunov functino $V = \bar{x}^T P \bar{x}$ with $P = \begin{bmatrix} \frac{a}{2} & 0 \\ 0 & \frac{1}{2} \end{bmatrix}$. What can be said about the stability of the system based on this choice of the Lyapunov

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function?

$$\begin{bmatrix} x_1 & x_2 \end{bmatrix} \begin{bmatrix} \frac{a}{2} & 0 \\ 0 & \frac{1}{2} \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \end{bmatrix} = \begin{bmatrix} x_1 & x_2 \end{bmatrix} \begin{bmatrix} \frac{ax_1}{2} \\ \frac{x_2}{2} \end{bmatrix} = \begin{bmatrix} \frac{a(x_1)^2}{2} \\ \frac{(x_2)^2}{2} \end{bmatrix} \tag{22}$$

(c) What can be said about the stability of the system based on the analysis of b) and c)?

(d) Study BIBS stability of the system when $c = 0$

$$\lambda = \pm\sqrt{a}j \tag{23}$$

if the eigenvectors are linearly independent, then the system is lyapunov stable

4. For the transfer function matrix

$$H(s) = \begin{bmatrix} \frac{s}{s-2} & 0 \\ \frac{2}{s-2} & 1 \end{bmatrix} \tag{24}$$

(a) Obtain the controllable canoncical form

$$\frac{1}{s+2} \begin{bmatrix} s & 0 \\ 2 & s+2 \end{bmatrix} \tag{25}$$

$$\dot{x} = \begin{bmatrix} -2 & 0 \\ 0 & -2 \end{bmatrix} x + \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix} u \tag{26}$$

$$N_0 = \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix} \tag{27}$$

$$N_1 = \begin{bmatrix} 0 & 0 \\ 2 & 2 \end{bmatrix} \tag{28}$$

$$y = \left(\begin{bmatrix} 0 & 0 \\ 2 & 2 \end{bmatrix} - \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix} 2 \right) x + \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix} u \tag{29}$$

$$y = \begin{bmatrix} -2 & 0 \\ 2 & 0 \end{bmatrix} x + \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix} u \tag{30}$$

TODO: check this

(b) obtain the observable canonical form

$$\frac{1}{s+2} \begin{bmatrix} s & 0 \\ 2 & s+2 \end{bmatrix} \tag{31}$$

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$$N_0 = \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix} \tag{32}$$

$$N_1 = \begin{bmatrix} 0 & 0 \\ 2 & 2 \end{bmatrix} \tag{33}$$

$$\dot{x} = \begin{bmatrix} 2 & 0 \\ 0 & 2 \end{bmatrix} x + (\begin{bmatrix} 0 & 0 \\ 2 & 2 \end{bmatrix} - \begin{bmatrix} 2 & 0 \\ 0 & 2 \end{bmatrix}) u \tag{34}$$

$$\dot{x} = \begin{bmatrix} 2 & 0 \\ 0 & 2 \end{bmatrix} x + \begin{bmatrix} -2 & 0 \\ 2 & 0 \end{bmatrix} u \tag{35}$$

$$y = \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix} x + \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix} \tag{36}$$

(c) show that the realization in (a) and (b) are dual
as per

$$\begin{bmatrix} A & B \\ C & D \end{bmatrix}^T = \begin{bmatrix} A^T & B^T \\ C^T & D^T \end{bmatrix} \tag{37}$$

it can be seen that this is the case for this example