Homework 5

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1. Consider the LTI SISO system with matrices:

$$A = \begin{bmatrix} 2 & 1 & 0 \\ 0 & 2 & 1 \\ 0 & 0 & 2 \end{bmatrix} \; B = \begin{bmatrix} 0 \\ 0 \\ 1 \end{bmatrix} \tag{1}$$

Assuming all state variabels can be measures, find a state feedback control law such that the eigenvalues of the closed loop system become: $\lambda_{d_1} = \lambda_{d_2} = \lambda_{d_3} = -1$

(a) Choose the desired eigenvalues and obtain the desired characteristic equation

$$\Delta_d(s) = (s-\lambda_1)(s-\lambda_2)(s-\lambda_3) = (s+1)(s+1)(s+1) = s^3+3s^2+3s+1 \tag{2}$$

$$\bar{\alpha} = \begin{bmatrix} 3 & 3 & 1 \end{bmatrix} \tag{3}$$

(b) Find the actual characterstic equation

$$\Delta(s) = |sI - A| = \begin{vmatrix} s-2 & -1 & 0 \\ 0 & s-2 & -1 \\ 0 & 0 & s-2 \end{vmatrix} \tag{4}$$

$$\Delta(s) = \begin{vmatrix} s-2 & -1 & 0 \\ 0 & s-2 & -1 \\ 0 & 0 & s-2 \end{vmatrix} = (s-2)^3 = s^3 - 6s^2 + 12s - 8 \tag{5}$$

$$\alpha = \begin{bmatrix} -6 & 12 & -8 \end{bmatrix} \tag{6}$$

(c) Calulate $\bar{k}$

$$\bar{k} = \bar{\alpha} - \alpha = \begin{bmatrix} 3 & 3 & 1 \end{bmatrix} - \begin{bmatrix} -6 & 12 & -8 \end{bmatrix} = \begin{bmatrix} 9 & -9 & 9 \end{bmatrix} \tag{7}$$

(d) Calculate P

$$Q = P^{-1} = \begin{bmatrix} b & AB & A^2B \end{bmatrix} \begin{bmatrix} 1 & \alpha_1 & \alpha_2 \\ 0 & 1 & \alpha_1 \\ 0 & 0 & 1 \end{bmatrix} = \begin{bmatrix} 0 & 0 & 1 \\ 0 & 1 & 4 \\ 1 & 2 & 4 \end{bmatrix} \begin{bmatrix} 1 & -6 & 12 \\ 0 & 1 & -6 \\ 0 & 0 & 1 \end{bmatrix} = \begin{bmatrix} 0 & 0 & 1 \\ 0 & 1 & -2 \\ 1 & -4 & 4 \end{bmatrix} \tag{8}$$

$$P = Q^{-1} = \begin{bmatrix} 0 & 0 & 1 \\ 0 & 1 & -2 \\ 1 & -4 & 4 \end{bmatrix}^{-1} = \begin{bmatrix} 4 & 4 & 1 \\ 2 & 1 & 0 \\ 1 & 0 & 0 \end{bmatrix} \tag{9}$$

(e) Calculate k

$$k = \bar{k}P = \begin{bmatrix} 9 & -9 & 9 \end{bmatrix} \begin{bmatrix} 4 & 4 & 1 \\ 2 & 1 & 0 \\ 1 & 0 & 0 \end{bmatrix} = \begin{bmatrix} 27 & 27 & 9 \end{bmatrix} \tag{10}$$

(f) Confirm eigenvalues of $A - bk$

$$A - bk = \begin{bmatrix} 2 & 1 & 0 \\ 0 & 2 & 1 \\ 0 & 0 & 2 \end{bmatrix} \begin{bmatrix} 0 \\ 0 \\ 1 \end{bmatrix} \begin{bmatrix} 27 & 27 & 9 \end{bmatrix} = \begin{bmatrix} 2 & 1 & 0 \\ 0 & 2 & 1 \\ -27 & -27 & -7 \end{bmatrix} \tag{11}$$

eigenvalues are: $-1, -1, -1$

2. Condider a LTI MIMO system with matrices:

$$A = \begin{bmatrix} -2 & -2 & 0 \\ 0 & 0 & 1 \\ 0 & -3 & -4 \end{bmatrix} \; B = \begin{bmatrix} 1 & 0 \\ 0 & 0 \\ 0 & 1 \end{bmatrix} \; C = \begin{bmatrix} 1 & 0 & 1 \\ 0 & 1 & 0 \end{bmatrix} \tag{12}$$

Assuming all stat variabes can be measured find a state feedback control law such that the eigenvalues of the closed loop system become: $\lambda_{d_1} = \lambda_{d_2} = -3$ and $\lambda_{d_3} = -4$. For this purpose use:

(a) Method 1 in slide 106 and verify the eigenvalues of the closed loop system

i. For each deisred eigenvalue λ_{d_i}: solve:

$$\begin{bmatrix} \lambda_{d_i} I - A & B \end{bmatrix} \bar{\phi}_i = \bar{0} \tag{13}$$

A. 1, 2

$$\begin{bmatrix} -1 & 2 & 0 & 1 & 0 \\ 0 & -3 & -1 & 0 & 0 \\ 0 & 3 & 1 & 0 & 1 \end{bmatrix} \phi_i = \bar{0} \tag{14}$$

$$\phi_1 = \begin{bmatrix} 1 \\ 0 \\ 0 \\ 1 \\ 0 \end{bmatrix} \phi_2 = \begin{bmatrix} 2 \\ 1 \\ -3 \\ 0 \\ 0 \end{bmatrix} \tag{15}$$

B. 3

$$\begin{bmatrix} -2 & 2 & 0 & 1 & 0 \\ 0 & -4 & -1 & 0 & 0 \\ 0 & 3 & 0 & 0 & 1 \end{bmatrix} \phi_3 = \bar{0} \tag{16}$$

$$phi_3 = \begin{bmatrix} 1 \\ 1 \\ -4 \\ 0 \\ -3 \end{bmatrix} \tag{17}$$

C. split the matrix

$$\begin{bmatrix} \bar{\psi}_1 & \bar{\psi}_2 & \bar{\psi}_3 \end{bmatrix} = \begin{bmatrix} 1 & 2 & 1 \\ 0 & 1 & 1 \\ 0 & -3 & -4 \end{bmatrix} \tag{18}$$

$$\begin{bmatrix} K\bar{\psi}_1 & K\bar{\psi}_2 & K\bar{\psi}_3 \end{bmatrix} = \begin{bmatrix} 1 & 0 & 0 \\ 0 & 0 & -3 \end{bmatrix} \tag{19}$$

D. Solve for K

$$K = \begin{bmatrix} K\bar{\psi}_1 & K\bar{\psi}_2 & K\bar{\psi}_3 \end{bmatrix} \begin{bmatrix} \bar{\psi}_1 & \bar{\psi}_2 & \bar{\psi}_3 \end{bmatrix}^{-1} = \begin{bmatrix} 0 & 2 & 2 \\ -\frac{3}{2} & -\frac{3}{2} & 3 \end{bmatrix} \begin{bmatrix} 1 & 2 & 0 \\ 1 & 1 & -1 \\ -\frac{3}{2} & -\frac{3}{2} & 2 \end{bmatrix}^{-1} \tag{20}$$

$$K = \begin{bmatrix} 0 & 2 & 2 \\ -\frac{3}{2} & -\frac{3}{2} & 3 \end{bmatrix} \begin{bmatrix} 1 & 2 & 0 \\ 1 & 1 & -1 \\ -\frac{3}{2} & -\frac{3}{2} & 2 \end{bmatrix}^{-1} = \begin{bmatrix} 1 & -5 & -1 \\ 0 & 9 & 3 \end{bmatrix} \tag{21}$$

E. evalute eigenvalues of feedback matrix

$$A - BK = \begin{bmatrix} -2 & -2 & 0 \\ 0 & 0 & 1 \\ 0 & -3 & -4 \end{bmatrix} - \begin{bmatrix} 1 & 0 \\ 0 & 0 \\ 0 & 1 \end{bmatrix} \begin{bmatrix} 1 & -5 & -1 \\ 0 & 9 & 3 \end{bmatrix} = \begin{bmatrix} -3 & 3 & 1 \\ 0 & 0 & 1 \\ 0 & -12 & -7 \end{bmatrix} \tag{22}$$

eigenvalues of this are: $-3, -3, -4$

(b) Method 2 in slide 107 and verify the eigenvalues of the closed loop system

i. get a k_V where you have distict eigenvalues
already done $k_v = 0$

ii. select a $\bar{v}$ such that $(A, b\bar{v})$ is controllable:

$$\bar{v} = \begin{bmatrix} 0 \\ 1 \end{bmatrix} \tag{23}$$

$$Bv = \begin{bmatrix} 0 \\ 0 \\ 1 \end{bmatrix} \tag{24}$$

$$\begin{bmatrix} A^2Bv & ABv & Bv \end{bmatrix} = \begin{bmatrix} -2 & 0 & 0 \\ -4 & 1 & 0 \\ -13 & -4 & 1 \end{bmatrix} \tag{25}$$

full row rank $\implies$ controllable

$$A = \begin{bmatrix} -2 & -2 & 0 \\ 0 & 0 & 1 \\ 0 & -3 & -4 \end{bmatrix} B = \begin{bmatrix} 0 \\ 0 \\ 1 \end{bmatrix} \tag{26}$$

$$\Delta_d(s) = (s+3)(s+3)(s+4) = (s^2 + 6s + 9)(s+4) \tag{27}$$

$$\Delta_d(s) = s^3 + 10s^2 + 33s + 36 \tag{28}$$

$$\bar{\alpha} = \begin{bmatrix} 10 & 33 & 36 \end{bmatrix} \tag{29}$$

$$\Delta(s) = |sI - A| = \begin{vmatrix} s+2 & 2 & 0 \\ 0 & s & -1 \\ 0 & 3 & s+4 \end{vmatrix} = (s+2)((s)(s+4) + 3) \tag{30}$$

$$\Delta(s) = (s+2)(s^s + 4s + 3) = s^3 + 6s^2 + 11s + 6 \tag{31}$$

$$\alpha = \begin{bmatrix} 6 & 11 & 6 \end{bmatrix} \tag{32}$$

$$\bar{k} = \begin{bmatrix} 4 & 22 & 30 \end{bmatrix} \tag{33}$$

$$Q = P^{-1} = \begin{bmatrix} 0 & 0 & -2 \\ 0 & 1 & -4 \\ 1 & -4 & 13 \end{bmatrix} \begin{bmatrix} 1 & 6 & 11 \\ 0 & 1 & 6 \\ 0 & 0 & 1 \end{bmatrix} = \begin{bmatrix} 0 & 0 & -2 \\ 0 & 1 & 2 \\ 1 & 2 & 0 \end{bmatrix} \tag{34}$$

$$P = Q^{-1} = \begin{bmatrix} 0 & 0 & -2 \\ 0 & 1 & 2 \\ 1 & 2 & 0 \end{bmatrix} = \begin{bmatrix} -2 & -2 & 1 \\ 1 & 1 & 0 \\ -\frac{1}{2} & 0 & 0 \end{bmatrix} \tag{35}$$

$$k = \bar{k}P = \begin{bmatrix} 4 & 22 & 30 \end{bmatrix} \begin{bmatrix} -2 & -2 & 1 \\ 1 & 1 & 0 \\ -\frac{1}{2} & 0 & 0 \end{bmatrix} = \begin{bmatrix} -1 & 14 & 4 \end{bmatrix} \tag{36}$$

$$K = K_v + \bar{v}k = 0 + \begin{bmatrix} 0 \\ 1 \end{bmatrix} \begin{bmatrix} -1 & 14 & 4 \end{bmatrix} = \begin{bmatrix} 0 & 0 & 0 \\ -1 & 14 & 4 \end{bmatrix} \tag{37}$$

$$A - Bk = \begin{bmatrix} 0 & 0 & 0 \\ -1 & 14 & 4 \\ 4 & -56 & -16 \end{bmatrix} \tag{38}$$

eigenvalues are $-3, -3, -4$

3. For the system in problem 2, design an ouput feedback control law such that two of the eigenvalues of the closed loop system become -3 and -4. Verify the eigenvalues of the closed loop system.
   Assuming this question is asking to design an observer shooting for $-4, -3, -4$

$$\begin{bmatrix} \mu_{d_i} I - A^T & C^T \end{bmatrix} \Phi_i = 0 \tag{39}$$

$$\begin{bmatrix} \mu_{d_i} + 2 & 0 & 0 & 1 & 0 \\ 2 & \mu_{d_i} & 3 & 0 & 1 \\ 0 & -1 & \mu_{d_i} + 4 & 1 & 0 \end{bmatrix} \Phi_i = 0 \tag{40}$$

$$\begin{bmatrix} -2 & 0 & 0 & 1 & 0 \\ 2 & -4 & 3 & 0 & 1 \\ 0 & -1 & 0 & 1 & 0 \end{bmatrix} \Phi_{1,2} = 0 \tag{41}$$

$$\Phi_{1,2} = \begin{bmatrix} 1 & 1 \\ 2 & 2 \\ 2 & 0 \\ 2 & 2 \\ 0 & 6 \end{bmatrix} \tag{42}$$

$$\begin{bmatrix} -1 & 0 & 0 & 1 & 0 \\ 2 & -3 & 3 & 0 & 1 \\ 0 & -1 & 1 & 1 & 0 \end{bmatrix} \Phi_{1,2} = 0 \tag{43}$$

$$\Phi_3 = \begin{bmatrix} 0 \\ 1 \\ 1 \\ 0 \\ 0 \end{bmatrix} \tag{44}$$

$$\bar{\Phi} = \begin{bmatrix} 1 & 1 & 0 \\ 2 & 2 & 1 \\ 2 & 0 & 1 \end{bmatrix} \tag{45}$$

$$L^T \bar{\Phi} = \begin{bmatrix} 2 & 2 & 0 \\ 0 & 6 & 0 \end{bmatrix} \tag{46}$$

$$\begin{bmatrix} 2 & 2 & 0 \\ 0 & 6 & 0 \end{bmatrix} \bar{\Phi}^{-1} = L = \begin{bmatrix} 2 & 0 \\ 0 & 3 \\ 0 & -3 \end{bmatrix} \tag{47}$$

checking with python values are correct

4. Consider the nonlinear control system in which it is desired to stabalize the origin

$$\dot{x}_1 = -2x_1 + x_2 + \sin(x_2) \tag{48}$$

$$\dot{x}_2 = -x_2 \cos(x_1) + \cos(2x_1)u \tag{49}$$

(a) verify that $(x_1, x_2, u) = (0, 0, 0)$ is an equilibrium.

$$\dot{x}_1 = 0 + 0 + 0 = 0 \tag{50}$$

$$\dot{x}_2 = 0(1) + 1(0) = 0 \tag{51}$$

yes

(b) linearize the system about the origin

$$\dot{x}_1 = -2x_1 + x_2 = -2x_1 + 2x_2 \tag{52}$$

$$\dot{x}_1 = -x_2 + u \tag{53}$$

$$\dot{x} = \begin{bmatrix} -2 & 1 \\ 0 & -1 \end{bmatrix} + \begin{bmatrix} 0 \\ 1 \end{bmatrix} \tag{54}$$

(c) Study the controllability of the linearized system

$$\begin{bmatrix} B & AB \end{bmatrix} = \begin{bmatrix} 0 & 1 \\ 1 & -1 \end{bmatrix} \tag{55}$$

full rank, controllable

(d) Assuming bost state variables can be measured, find a state feedback control law such that both eigenvalues of the linearized closed-loop system -4

$$\Delta_d(s) = (s+4)^2 = s^2 + 8s + 16 \tag{56}$$

$$\bar{\alpha} = \begin{bmatrix} 8 & 16 \end{bmatrix} \tag{57}$$

$$\Delta(s) = |sI - A| = \begin{bmatrix} s+2 & -1 \\ 0 & s+1 \end{bmatrix} = s^2 + 3s + 2 \tag{58}$$

$$\alpha = \begin{bmatrix} 3 & 2 \end{bmatrix} \tag{59}$$

$$\bar{k} = \bar{\alpha} - \alpha = \begin{bmatrix} 5 & 14 \end{bmatrix} \tag{60}$$

$$Q = P^{-1} = \begin{bmatrix} 0 & 1 \\ 1 & -1 \end{bmatrix} \begin{bmatrix} 1 & 3 \\ 0 & 1 \end{bmatrix} = \begin{bmatrix} 0 & 1 \\ 1 & 2 \end{bmatrix} \tag{61}$$

$$P = Q^{-1} = \begin{bmatrix} 0 & 1 \\ 1 & 2 \end{bmatrix}^{-1} = \begin{bmatrix} -2 & 1 \\ 1 & 0 \end{bmatrix} \tag{62}$$

$$k = \bar{k}P = \begin{bmatrix} 5 & 14 \end{bmatrix} \begin{bmatrix} -2 & 1 \\ 1 & 0 \end{bmatrix} = \begin{bmatrix} 4 & 5 \end{bmatrix} \tag{63}$$

confirming:

$$A - bk = \begin{bmatrix} -2 & 1 \\ 0 & -1 \end{bmatrix} - \begin{bmatrix} 0 \\ 1 \end{bmatrix} \begin{bmatrix} 4 & 5 \end{bmatrix} = \begin{bmatrix} -2 & 1 \\ 0 & -1 \end{bmatrix} - \begin{bmatrix} 0 & 0 \\ 4 & 5 \end{bmatrix} = \begin{bmatrix} -2 & 1 \\ -4 & -6 \end{bmatrix} \tag{64}$$

eigenvalues are: $\lambda = -4, -4$

(e) Apply the state feedback control law obtained in pard d to the non=linear system above. Simulate and plot the state response and closed-loop system using the folowing initial conditions and briefly interpret the results:

i. $\bar{x}_0 = \begin{bmatrix} 1 \\ 0 \end{bmatrix}$

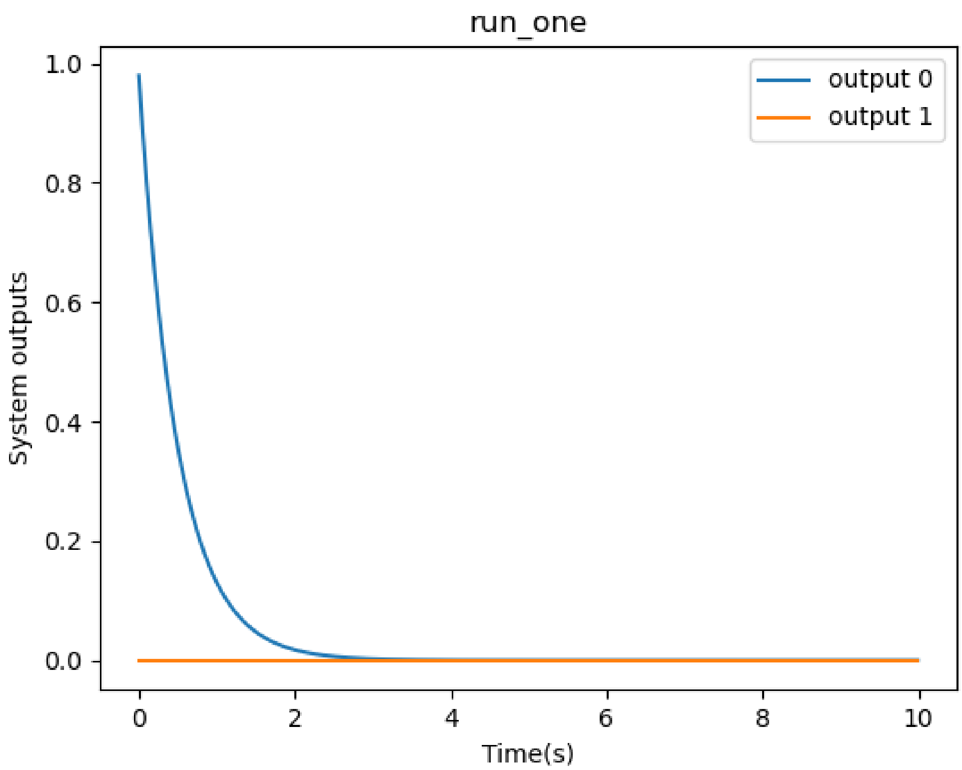

Figure 1: Run One

ii. $\bar{x}_0 = \begin{bmatrix} 1 \\ 1 \end{bmatrix}$

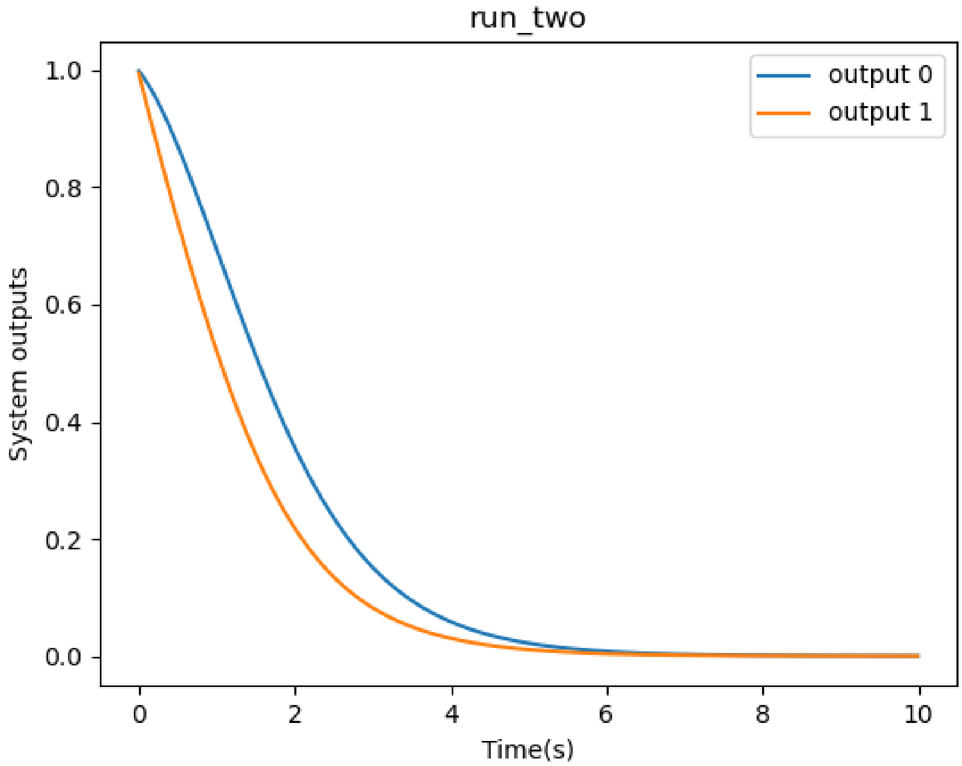

Figure 2: Run Two

iii. $\bar{x}_0 = \begin{bmatrix} -1 \\ 1 \end{bmatrix}$

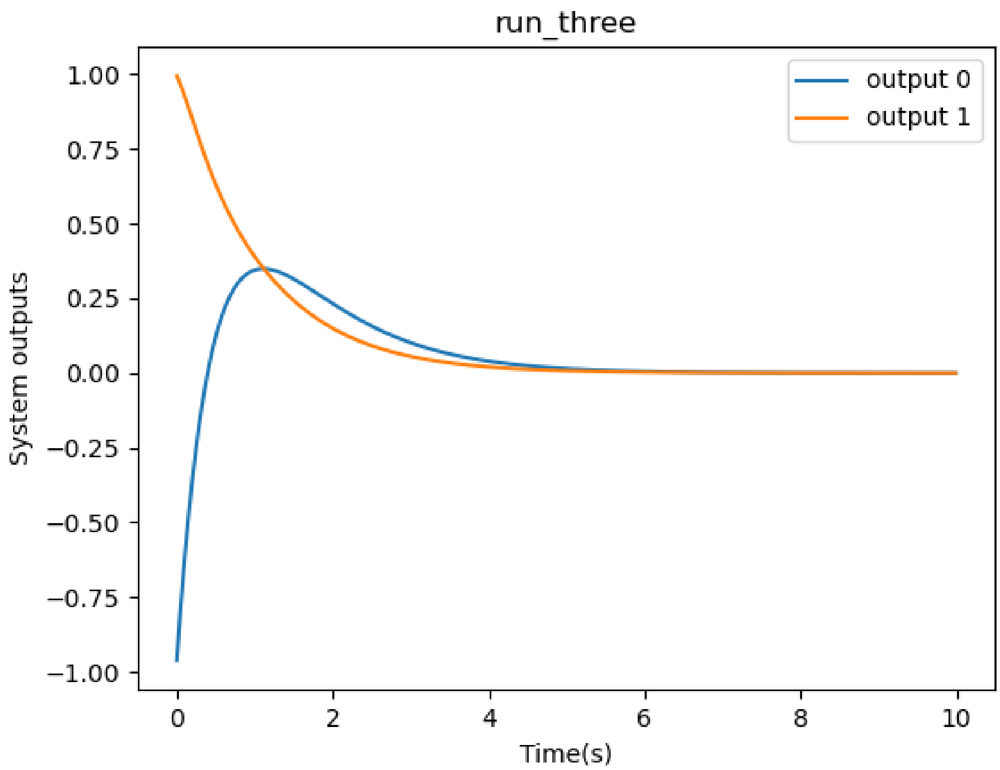

Figure 3: Run Three

as can be seen from Figure 1, Figure 2, and Figure 3. The system takes longer to converge and has a larger overshoot the further it is from point where the linearization was taken.

Just for fun, I wanted to see how the system would perform at a point much further out, and the result can be seen below in Figure 4. As can be seen.... it can't really control the system and bring the outputs to 0, thus showing the limits of linearizing a non-linear system.

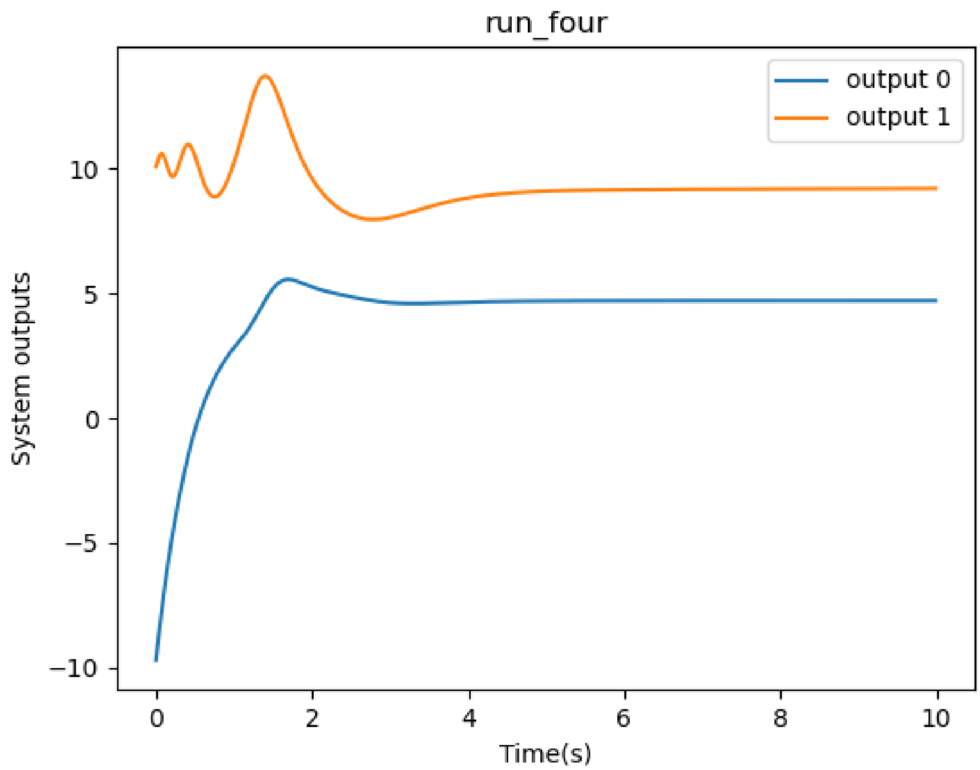

Figure 4: Run Four

1 Help Recieved

This section is a thank you for people who caught issues or otherwised helped(collaboration is allowed)

1. List people here

A System Modeling Code

```python
import sympy
from sympy import cos, sin
from typing import Callable, Iterable, Tuple
from copy import deepcopy

def ModelSystem(update: Callable[[sympy.Matrix, sympy.Matrix], sympy.Matrix],
                output: Callable[[sympy.Matrix, sympy.Matrix], sympy.Matrix],
                x_0: sympy.Matrix,
                inputs: Iterable[sympy.Matrix],
                *args,
                **dargs) -> Iterable[Tuple]:
    """
    A generic model for any system can work with linear and nonlinear
    systems. update functions returns the next state, and output outputs
    the output matrix. Both of these functions take in state, system input,
    then *args.
    This function will return list of outputs

    This is an underlying function see other models for examples
    """
    outputs = []
    x = deepcopy(x_0)
    inputs = deepcopy(inputs)
    for r in inputs:
        x = update(x, r, *args, **dargs)
        outputs.append(output(x, r, *args, **dargs))
    return outputs

def non_linear_update(x: sympy.Matrix,
                      r: sympy.Matrix,
                      k: sympy.Matrix,
                      dt: float,
                      *args, **dargs) -> sympy.Matrix:
    """
    Non linear update equation to handle question 4
    @arg x is the previous state sys
    @arg r input before any feedback
    @arg k feedback matrix
    @returns next state
```

```
    """
    x_1 = float(x[0, 0])
    x_2 = float(x[1, 0])
    u = float( ((k*x) * r)[0,0] )
    dot_x_1 = -2*x_1 + x_2 + sin(x_2)
    dot_x_2 = -x_2*cos(x_1) + cos(2*x_1)*u
    new_x = sympy.Matrix([
        [x_1 + (dt*dot_x_1)],
        [x_2 + (dt*dot_x_2)]
    ])
    return new_x

def linear_update(x: sympy.Matrix,
                  r: sympy.Matrix,
                  A: sympy.Matrix,
                  B: sympy.Matrix,
                  dt: float,
                  k: sympy.Matrix=None,
                  *args, **dargs) -> sympy.Matrix:
    """
    Linear output equation
    """
    if k is not None:
        u = (k*x) * r
    else:
        u = r
    dx = A*x + B*r
    return x + (dx*dt)
def linear_output(x: sympy.Matrix,
                  r: sympy.Matrix,
                  C: sympy.Matrix,
                  D: sympy.Matrix,
                  *args, **dargs) -> sympy.Matrix:
    """
    Linear output equation
    """
    return C*x + D*r
```

There is also some work on github(link below) in a jupyter notebook

Disclaimer: This is just a few relevant fragments of the source code, as the entire code is a complicated system that takes these fragments and automatically renders them into the final pdf. However all of this is available online on github(its latex + python)