Controls Project

Nathan Rose

April 6, 2022

1 Abstract

The project for Controls will involve controlling a doubly inverted pendulum with a linear "cart". The goals of this project are to

1. Linearize the system about the stability point and design a controller for it.
2. Determine the limits of the controller with respect to each state variable(in isolation)

In order to simplify the model, the mass of the pendulums will be assumed to be at the end of the rod, rather than in the middle as it would be for a constant density rod.

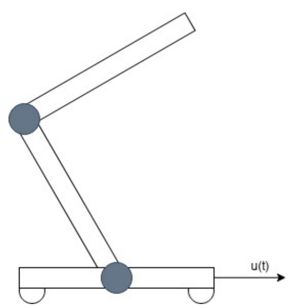

Figure 1: System to be modeled

2 Introduction

2.1 Defining the non-linear system

A doubly inverted pendulum is a pendulum on a pendulum. What makes this a classical control problem is the inverted nature of the system. That is instead of hanging the pendulum, the goal is to balence it upright.

The equations of motion were derived with assistance from **eq˙of˜motion**. TODO: fix cite with a filter

This will not be a derivation of the work, as the work follows closely from **eq˙of˜motion** which I used as a source to help derive the non-linear equations of motion.

$$M = \begin{bmatrix} m_1 + m_2 + m_c & l_1 \left(m_1 + m_2\right) \cos\left(\theta_1\right) & l_2 m_2 \cos\left(\theta_2\right) \\ l_1 \left(m_1 + m_2\right) \cos\left(\theta_1\right) & l_1^2 \left(m_1 + m_2\right) & l_1 l_2 m_2 \cos\left(\theta_1 - \theta_2\right) \\ l_2 m_2 \cos\left(\theta_2\right) & l_1 l_2 m_2 \cos\left(\theta_1 - \theta_2\right) & l_2^2 m_2 \end{bmatrix} \tag{1}$$

$$f = \begin{bmatrix} \dot{\theta}_1^2 l_1 \left(m_1 + m_2\right) \sin\left(\theta_1\right) + \dot{\theta}_2^2 l_2 m_2 \sin\left(\theta_2\right) - \dot{y} d_1 + u \\ -\dot{\theta}_1 d_2 - \dot{\theta}_2^2 l_1 l_2 m_2 \sin\left(\theta_1 - \theta_2\right) + g l_1 \left(m_1 + m_2\right) \sin\left(\theta_1\right) \\ \dot{\theta}_1^2 l_1 l_2 m_2 \sin\left(\theta_1 - \theta_2\right) - \dot{\theta}_2 d_3 + g l_2 m_2 \sin\left(\theta_2\right) \end{bmatrix} \tag{2}$$

$$F = \begin{bmatrix} \frac{-l_1 l_2\left(m_1 - m_2\cos^2\left(\theta_1-\theta_2\right)+m_2\right)\left(\dot{\theta}_1^2 l_1\left(m_1+m_2\right)\sin\left(\theta_1\right)+\dot{\theta}_2^2 l_2 m_2\sin\left(\theta_2\right)-\dot{y}d_1+u\right)-\frac{l_1\left(\dot{\theta}_1^2 l_1 l_2 m_2\sin\left(\theta_1-\theta_2\right)-\dot{\theta}_2 \cdots\right.}{\cdots}}{l_1 l_2\left(-m_1^2\sin^2\left(\theta_1\right)-m \cdots\right.} \\ \frac{l_1 l_2\left(m_1\cos\left(\theta_1\right)+m_2\cos\left(\theta_1\right)-m_2\cos\left(\theta_2\right)\cos\left(\theta_1-\theta_2\right)\right)\left(\dot{\theta}_1^2 l_1\left(m_1+m_2\right)\sin\left(\theta_1\right)+\dot{\theta}_2^2 l_2 m_2\sin\left(\theta_2\right)-\dot{y}d_1+u\right)+l_1\left(m \cdots\right.}{l_1^2 l_2\left(-m_1^2\sin^2\left(\theta_1\right)-m \cdots\right.} \\ \frac{-\frac{l_1 l_2 m_2\left(-m_1\cos\left(\theta_2\right)+m_1\cos\left(2\theta_1-\theta_2\right)-m_2\cos\left(\theta_2\right)+m_2\cos\left(2\theta_1-\theta_2\right)\right)\left(\dot{\theta}_1^2 l_1\left(m_1+m_2\right)\sin\left(\theta_1\right)+\dot{\theta}_2^2 l_2 m_2\sin\left(\theta_2\right)-\dot{y}d_1+u\right)}{2}-l_1\left(\dot{\theta}_1^2 l_1 l_2 m_2\sin\left(\theta_1-\cdots\right.\right.}{l_1 l_2^2 m_2\left(-m_1^2\sin^2\left(\theta_1\right)-\cdots\right.} \end{bmatrix} \tag{3}$$

$$\dot{x} = \frac{d}{dt}\begin{bmatrix} y \\ \dot{y} \end{bmatrix} = \begin{bmatrix} \cdots \\ \frac{-l_1 l_2\left(m_1 - m_2\cos^2\left(\theta_1-\theta_2\right)+m_2\right)\left(\dot{\theta}_1^2 l_1\left(m_1+m_2\right)\sin\left(\theta_1\right)+\dot{\theta}_2^2 l_2 m_2\sin\left(\theta_2\right)-\dot{y}d_1+u\right)-\frac{l_1\left(\dot{\theta}_1^2 l_1 l_2 m_2 \sin \cdots\right.}{\cdots}}{l_1 l_2\left(-m_1^2 \cdots\right.} \\ \frac{l_1 l_2\left(m_1\cos\left(\theta_1\right)+m_2\cos\left(\theta_1\right)-m_2\cos\left(\theta_2\right)\cos\left(\theta_1-\theta_2\right)\right)\left(\dot{\theta}_1^2 l_1\left(m_1+m_2\right)\sin\left(\theta_1\right)+\dot{\theta}_2^2 l_2 m_2\sin\left(\theta_2\right)-\dot{y} \cdots\right.}{l_1^2 l_2\left(-m_1^2 \cdots\right.} \\ \frac{-\frac{l_1 l_2 m_2\left(-m_1\cos\left(\theta_2\right)+m_1\cos\left(2\theta_1-\theta_2\right)-m_2\cos\left(\theta_2\right)+m_2\cos\left(2\theta_1-\theta_2\right)\right)\left(\dot{\theta}_1^2 l_1\left(m_1+m_2\right)\sin\left(\theta_1\right)+\dot{\theta}_2^2 l_2 m_2\sin\left(\theta_2\right)-\dot{y}d_1+u\right)}{2}-l_1\left(\dot{\theta}_1^2 l_1 l_2 \cdots\right.}{l_1 l_2^2 m_2\left(-m \cdots\right.} \end{bmatrix} \tag{4}$$

This is unreadable

For this, I assumed:

1. $m_c = 1$
2. $m_1 = 1$
3. $m_2 = 1$
4. $l_1 = 1$

5. $l_2 = 1$
6. $g = 1$
7. $d_1 = 1$
8. $d_2 = 1$
9. $d_3 = 1$

$$M = \begin{bmatrix} 2.0 & 2.0\cos(\theta_1) & 1.0\cos(\theta_2) \\ 2.0\cos(\theta_1) & 4.0 & 2.0\cos(\theta_1 - \theta_2) \\ 1.0\cos(\theta_2) & 2.0\cos(\theta_1 - \theta_2) & 2.0 \end{bmatrix} \tag{5}$$

$$f = \begin{bmatrix} 2.0\dot{\theta}_1^2\sin(\theta_1) + 1.0\dot{\theta}_2^2\sin(\theta_2) - 0.1\dot{y} + u \\ -0.1\dot{\theta}_1 - 2.0\dot{\theta}_2^2\sin(\theta_1 - \theta_2) + 19.62\sin(\theta_1) \\ 2.0\dot{\theta}_1^2\sin(\theta_1 - \theta_2) - 0.1\dot{\theta}_2 + 9.81\sin(\theta_2) \end{bmatrix} \tag{6}$$

$$F = \begin{bmatrix} \frac{-(64.0\cos(\theta_1)-32.0\cos(\theta_2)\cos(\theta_1-\theta_2))\big(0.1\dot{\theta}_1+2.0\dot{\theta}_2^2\sin(\theta_1-\theta_2)-19.62\sin(\theta_1)\big)+\big(64.0\cos^2(\theta_1-\theta_2)-128.0\big)\big(2.0\dot{\theta}_1^2\sin(\theta_1)+1.0\dot{\theta}}{-128.0\sin^2(\theta_1)+128.0\sin(\theta_1)\sin(\theta_2)\cos(\theta_1-\theta_2)-64.0\,\mathrm{s}} \\ \frac{-(16.0\cos(\theta_1)\cos(\theta_2)-32.0\cos(\theta_1-\theta_2))\big(2.0\dot{\theta}_1^2\sin(\theta_1-\theta_2)-0.1\dot{\theta}_2+9.81\sin(\theta_2)\big)+(32.0\cos(\theta_1)-16.0\cos(\theta_2)\cos(\theta_1-\theta_2))\big(2.0\dot{\theta}_1^2\sin(}{-64.0\sin^2(\theta_1)+64.0\sin(\theta_1)\sin(\theta_2)\cos(\theta_1-\theta_2)-32.0\,\mathrm{si}} \\ \frac{-1.0\dot{\theta}_1^2\sin(\theta_1-\theta_2)-0.0375\dot{\theta}_1\cos(\theta_1-\theta_2)+0.0125\dot{\theta}_1\cos(\theta_1+\theta_2)-0.25\dot{\theta}_2^2\sin(2\theta_1-2\theta_2)-0.025\dot{\theta}_2\cos(2\theta_1)+0.075\dot{\theta}_2-0.025\dot{y}\cos(\theta_2)+0.}{0.25\cos(2\theta_1)+0.25\cos(2\theta_1-2\theta_2)-1.0} \end{bmatrix} \tag{7}$$

$$\dot{x} = \frac{d}{dt}\begin{bmatrix} y \\ \dot{y} \end{bmatrix} = \begin{bmatrix} \dot{\theta}_2 \\ \dot{\theta}_1 \\ \dot{y} \\ \frac{-(64.0\cos(\theta_1)-32.0\cos(\theta_2)\cos(\theta_1-\theta_2))\big(0.1\dot{\theta}_1+2.0\dot{\theta}_2^2\sin(\theta_1-\theta_2)-19.62\sin(\theta_1)\big)+\big(64.0\cos^2(\theta_1-\theta_2)-128.0\big)\big(2.0\dot{\theta}_1^2\,\mathrm{s}}{-128.0\sin^2(\theta_1)+128.0\sin(\theta_1)\sin(\theta_2)\cos(\theta_1} \\ \frac{-(16.0\cos(\theta_1)\cos(\theta_2)-32.0\cos(\theta_1-\theta_2))\big(2.0\dot{\theta}_1^2\sin(\theta_1-\theta_2)-0.1\dot{\theta}_2+9.81\sin(\theta_2)\big)+(32.0\cos(\theta_1)-16.0\cos(\theta_2)\cos(\theta_1-\theta_2))}{-64.0\sin^2(\theta_1)+64.0\sin(\theta_1)\sin(\theta_2)\cos(\theta_1-} \\ \frac{-1.0\dot{\theta}_1^2\sin(\theta_1-\theta_2)-0.0375\dot{\theta}_1\cos(\theta_1-\theta_2)+0.0125\dot{\theta}_1\cos(\theta_1+\theta_2)-0.25\dot{\theta}_2^2\sin(2\theta_1-2\theta_2)-0.025\dot{\theta}_2\cos(2\theta_1)+0.075\dot{\theta}_2-0.025\dot{y}}{0.25\cos(2\theta_1)+0.25\cos(2\theta_1-} \end{bmatrix} \tag{8}$$

TODO: fix runnon equations...... somehow

2.2 Defining the linear system

The Linearization was simply a jacobian taken at 0,0,0 which is an equilibrium point that the system will be controlled to.

$$\dot{x} = \begin{bmatrix} 0 & 0 & 0 & 0 & 0 & 1.0 \\ 0 & 0 & 0 & 0 & 1.0 & 0 \\ 0 & 0 & 0 & 1.0 & 0 & 0 \\ 0 & -\frac{g(m_1+m_2)}{m_c} & 0 & -\frac{d_1}{m_c} & \frac{d_2}{l_1 m_c} & 0 \\ 0 & \frac{g(m_1+m_2)(m_1+m_c)}{l_1 m_1 m_c} & -\frac{g m_2}{l_1 m_1} & \frac{d_1}{l_1 m_c} & -\frac{d_2(m_1+m_c)}{l_1^2 m_1 m_c} & \frac{d_3}{l_1 l_2 m_1} \\ 0 & -\frac{g(m_1+m_2)}{l_2 m_1} & \frac{g(m_1 m_c + m_2 m_c)}{l_2 m_1 m_c} & 0 & \frac{d_2}{l_1 l_2 m_1} & -\frac{d_3(m_1 m_c + m_2 m_c)}{l_2^2 m_1 m_2 m_c} \end{bmatrix} \begin{bmatrix} y \\ \theta_1 \\ \theta_2 \\ \dot{y} \\ \dot{\theta}_1 \\ \dot{\theta}_2 \end{bmatrix} + \begin{bmatrix} 0 \\ 0 \\ 0 \\ \frac{1}{m_c} \\ -\frac{1}{l_1 m_c} \\ 0 \end{bmatrix} u \tag{9}$$

$$\dot{x} = \begin{bmatrix} 0 & 0 & 0 & 0 & 0 & 1.0 \\ 0 & 0 & 0 & 0 & 1.0 & 0 \\ 0 & 0 & 0 & 1.0 & 0 & 0 \\ 0 & -9.81 & 0 & -0.1 & 0.05 & 0 \\ 0 & 14.715 & -4.905 & 0.05 & -0.075 & 0.05 \\ 0 & -9.81 & 9.81 & -1.0 \cdot 10^{-141} & 0.05 & -0.1 \end{bmatrix} \begin{bmatrix} y \\ \theta_1 \\ \theta_2 \\ \dot{y} \\ \dot{\theta}_1 \\ \dot{\theta}_2 \end{bmatrix} + \begin{bmatrix} 0 \\ 0 \\ 0 \\ 1.0 \\ -0.5 \\ -2.0 \cdot 10^{-140} \end{bmatrix} u \tag{10}$$

3 Help Recieved

This section is a thank you for people who caught issues or otherwised helped(collaboration is allowed)

1. People will be listed here