Homework 1

Nathan Rose

January 20, 2022

1. Consider the following system consisting of a cart of mass m attached to a rigid wall by a spring and a damper. The spring stiffness $k(w)$ is a nonlinear function of displacement w such that the spring pull(force) applied to the mass is in the form of $f_s = k_1 w - k_3 w^3$, where k_1 and k_3 are constant scalars. Assume that the cart rolls freely without friction. It can be shown that the motion is $m\ddot{w} + c\dot{w} + k_1 w - k_3 w^3 = u$

   (a) The cart is instrumented with an accelerometer, which provides a measurement equation of the form $y = \ddot{w}$. Express the nonlinear equations of motion, including the output equation in state space form
   This system has two state variables: position and velocity. In order to keep it consistent with the notation in question 1c. the following will be used.

$$x = \begin{bmatrix} position \\ velocity \end{bmatrix} = \begin{bmatrix} w \\ \dot{w} \end{bmatrix} \tag{1}$$

$$\ddot{w} = \frac{u - c\dot{w} - k_1 w + k_3 w^3}{m} \tag{2}$$

$$c\dot{w} = u + k_3 w^3 - k_1 w - m\ddot{w} \tag{3}$$

$$\dot{w} = \frac{u + k_3 w^3 - k_1 w - m\ddot{w}}{c} \tag{4}$$

   (b) Assume $m = 1$, $k_1 = 4$, $k_3 = 1$, $c = 1$. Determine all the equilibrium points of the system assuming $u = 0$

$$\begin{bmatrix} \frac{u + k_3 w^3 - k_1 w - m\ddot{w}}{c} \\ \frac{u - c\dot{w} - k_1 w + k_3 w^3}{m} \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \end{bmatrix} \tag{5}$$

$$\begin{bmatrix} \frac{1w^3 - 4w - 1\ddot{w}}{c} \\ \frac{-c\dot{w} - 4w + 1w^3}{1} \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \end{bmatrix} \tag{6}$$

TODO: not sure how to deal with ddot w as a non-state variable

(c) Using the numerical assumptions above, linearize the equations of motion about the equilibrium point

$$\begin{bmatrix} 2 \\ 0 \end{bmatrix} \tag{7}$$

That is provide the lienar state space model about the point in the form of:
TODO: input linearlization

$$\begin{aligned} \delta\dot{\bar{x}} &= A\delta\bar{x}(t) + B\delta\bar{u}(t) \\ \delta y &= C\delta\bar{x}(t) + D\delta\bar{u}(t) \\ \delta\bar{x} &= C\delta\bar{x}(t) \end{aligned}$$

For the system that is described by:

$$w = \frac{0 + 1w^3 - 1\dot{w} - 1\ddot{w}}{4} \tag{8}$$

$$\dot{w} = \frac{0 + 1w^3 - 4w - 1\ddot{w}}{1} \tag{9}$$

It can be linearized into:

$$w = 1w + 2\dot{w} + 5u \tag{10}$$

$$w = 3w + 4\dot{w} + 6u \tag{11}$$

which in state space can be represented as:

$$\dot{x} = \begin{bmatrix} 1 & 2 \\ 3 & 4 \end{bmatrix} x + \begin{bmatrix} 5 \\ 6 \end{bmatrix} u \tag{12}$$

$$y = \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix} x + \begin{bmatrix} 0 \\ 0 \end{bmatrix} u \tag{13}$$

2. For the mass-spring-dampener system given in figure 2. Where k_1, k_2, c_1, and c_2 are constant scalars

(a) obtain the differential equations of motion of the system
for this problem, positive is to the right

$$F_{n1} = m_1\ddot{w}_1 = -k_1w_1 - c_1\dot{w}_1 - c_2(\dot{w}_2 - \dot{w}_1) + k_2(w_2 - w_1) \tag{14}$$

$$F_{n2} = m_2\ddot{w}_2 = c_2(\dot{w}_2 - \dot{w}_1) - k_2(w_2 - w_1) + u \tag{15}$$

(b) Write the equations of motion in the state space form

$$\dot{\bar{x}} = A\bar{x} + B\bar{u}$$
$$\bar{y} = C\bar{x} + D\bar{u}$$

such that the A matrix is in the form of:

$$A = \begin{bmatrix} 0_{2\times 2} & I_2 \\ A_{21} & A_{22} \end{bmatrix} \tag{16}$$

$$\ddot{w}_1 = \frac{-k_1 w_1 - c_1\dot{w}_1 - c_2(\dot{w}_2 - \dot{w}_1) + k_2(w_2 - w_1)}{m_1} \tag{17}$$

$$\ddot{w}_1 = \frac{-k_1 w_1 - c_1\dot{w}_1 - c_2\dot{w}_2 - c_2\dot{w}_1 + k_2 w_2 - k_2 w_1}{m_1} \tag{18}$$

$$\ddot{w}_1 = \frac{(-k_1 - k_2)w_1 - (c_1 + c_2)\dot{w}_1 - c_2\dot{w}_2 + k_2 w_2}{m_1} \tag{19}$$

$$\ddot{w}_2 = \frac{c_2(\dot{w}_2 - \dot{w}_1) - k_2(w_2 - w_1) + u}{m_2} \tag{20}$$

$$\ddot{w}_2 = \frac{c_2\dot{w}_2 - c_2\dot{w}_1 - k_2 w_2 + k_2 w_1 + u}{m_2} \tag{21}$$

$$\begin{bmatrix} \dot{w}_1 \\ \dot{w}_2 \\ \ddot{w}_1 \\ \ddot{w}_2 \end{bmatrix} = \begin{bmatrix} 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \\ -\frac{(k_1+k_2)w_1}{m_1} & \frac{k_2}{m_1} & -\frac{c_1+c_2}{m_1} & -\frac{c_2}{m_2} \\ k_2 & -k_2 & -c_2 & c_2 \end{bmatrix} \begin{bmatrix} w_1 \\ w_2 \\ \dot{w}_1 \\ \dot{w}_2 \end{bmatrix} + \begin{bmatrix} 0 \\ 0 \\ 0 \\ \frac{1}{m} \end{bmatrix} u \tag{22}$$

$$\begin{bmatrix} y \end{bmatrix} = \begin{bmatrix} 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \end{bmatrix} \begin{bmatrix} w_1 \\ w_2 \\ \dot{w}_1 \\ \dot{w}_2 \end{bmatrix} \tag{23}$$

3. Find the state space representation for the system described as

$$\dot{w}_1 + 3(w_1 + w_2) = u_1 \tag{24}$$

$$\ddot{w}_2 + 4\dot{w}_2 + 3w_2 = u_2 \tag{25}$$

Show the A and b matrix of your solution

Work:

For this problem there needs to be 3 state variables, 1 for w_1 and two for w_2

$$\dot{w}_1 = u_1 - 3w_1 - 3w_2) \tag{26}$$

$$\ddot{w}_2 = u_2 - 4\dot{w}_2 - 3w_2 \tag{27}$$

$$\dot{w}_2 = \frac{u_2 - \ddot{w}_2 - 3w_2}{4} \tag{28}$$

$$\dot{w}_2 = \frac{u_2 - (u_2 - 4\dot{w}_2 - 3w_2) - 3w_2}{4} \tag{29}$$

Final:

$$\begin{bmatrix} \dot{w}_1 \\ \dot{w}_2 \\ \ddot{w}_2 \end{bmatrix} = \begin{bmatrix} -3 & -3 & 0 \\ 0 & 0 & 1 \\ 0 & -3 & -4 \end{bmatrix} \begin{bmatrix} w_1 \\ w_2 \\ \dot{w}_2 \end{bmatrix} + \begin{bmatrix} 1 & 0 \\ 0 & 0 \\ 0 & 1 \end{bmatrix} \begin{bmatrix} u_1 \\ u_2 \end{bmatrix} \tag{30}$$

4. Find the transfer function matrix for the system described as

$$\dot{x} = \begin{bmatrix} -1 & -2 & 6 \\ 2 & -3 & -2 \\ -2 & -2 & 1 \end{bmatrix} x + \begin{bmatrix} 1 & 1 \\ 1 & -1 \\ 1 & 0 \end{bmatrix} u \tag{31}$$

$$y = \begin{bmatrix} -1 & -1 & 2 \\ 1 & 1 & -1 \end{bmatrix} x + \begin{bmatrix} 0 & 0 \\ 0 & 0 \end{bmatrix} u \tag{32}$$

$$F(s) = C(sI - A)B + D \tag{33}$$

That is the equation for how to do that and I could grind through this process but honestly that sounds terrible and ill probably make a bunch of minor math mistakes. So I am going to put this in python using thier control library and spit out an answer.

$$\begin{bmatrix} \frac{6.0s-30.0}{1.0s^3+3.0s^2+11.0s+57.0} & \frac{6.0s-18.0}{1.0s^3+3.0s^2+11.0s+57.0} \\ \frac{1.0s^2+6.0s+29.0}{1.0s^3+3.0s^2+11.0s+57.0} & \frac{6.0s+6.0}{1.0s^3+3.0s^2+11.0s+57.0} \end{bmatrix} \tag{34}$$

5. The model of a control system with input $u(t)$ and output $y(t)$ is given as

$$\ddot{y}(t) + 3\dot{y}(t) + 7y(t) = \dot{u}(t) + 2u(t) \tag{35}$$

All initial conditions are zero and teh system is subjected to a unit step time input.

(a) Find the transfer function $\frac{Y(s)}{U(s)}$

$$\mathscr{L}(\ddot{y}(t) + 3\dot{y}(t) + 7y(t)) = \mathscr{L}(\dot{u}(t) + 2u(t)) \tag{36}$$

$$(s^2Y(s) - sf(0) - f'(0)) + 3(sY(s) - f(0)) + 7Y(s) = sU(s) + u(0) + 2U(s) \tag{37}$$

$$s^2Y(s) + 3sY(s) + 7Y(s) = sU(s) + 2U(s) \tag{38}$$

$$\frac{Y(s)}{U(s)}(s^2 + 3s + 7) = s + 2 \tag{39}$$

$$\frac{Y(s)}{U(s)} = \frac{s+2}{s^2 + 3s + 7} \tag{40}$$

(b) Find the values of all poles and zeros

poles are calculated by getting the zeros of the denominator and zeros are calculated by getting the zeros of the denominator. (I used python to calculate these values)

zeros: -2.0

poles: $-1.5 \pm 2.179j$

(c) Find the steady state output response using the final value theorem

$$\lim_{s \to 0} F(s) = \lim_{s \to 0} \frac{s+2}{s^2 + 3s + 7} = \frac{2}{7} = 0.2857 \tag{41}$$

Approximate Anser for part 5c $y_{ss} \approx 0.3$ (Obtain the exact value accurate to the fourth decimal place