Homework UPDATE ME

Nathan Rose

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1. Consider the LTI SISO system with matrices:

$$A = \begin{bmatrix} 2 & 1 & 0 \\ 0 & 2 & 1 \\ 0 & 0 & 2 \end{bmatrix} B = \begin{bmatrix} 0 \\ 0 \\ 1 \end{bmatrix} \tag{1}$$

Assuming all state variabels can be measures, find a state feedback control law such that the eigenvalues of the closed loop system become: $\lambda_{d_1} = \lambda_{d_2} = \lambda_{d_3} = -1$

(a) Choose the desired eigenvalues and obtain the desired characteristic equation

$$\Delta_d(s) = (s - \lambda_1)(s - \lambda_2)(s - \lambda_3) = (s+1)(s+1)(s+1) = s^3 + 3s^2 + 3s + 1 \tag{2}$$

$$\bar{\alpha} = \begin{bmatrix} 3 & 3 & 1 \end{bmatrix} \tag{3}$$

(b) Find the actual characterstic equation

$$\Delta(s) = |sI - A| = \begin{vmatrix} s-2 & -1 & 0 \\ 0 & s-2 & -1 \\ 0 & 0 & s-2 \end{vmatrix} \tag{4}$$

$$\Delta(s) = \begin{vmatrix} s-2 & -1 & 0 \\ 0 & s-2 & -1 \\ 0 & 0 & s-2 \end{vmatrix} = (s-2)^3 = s^3 - 6s^2 + 12s - 8 \tag{5}$$

$$\alpha = \begin{bmatrix} -6 & 12 & -8 \end{bmatrix} \tag{6}$$

(c) Calulate $\bar{k}$

$$\bar{k} = \bar{\alpha} - \alpha = \begin{bmatrix} 3 & 3 & 1 \end{bmatrix} - \begin{bmatrix} -6 & 12 & -8 \end{bmatrix} = \begin{bmatrix} 9 & -9 & 9 \end{bmatrix} \tag{7}$$

(d) Calculate P

$$Q = P^{-1} = \begin{bmatrix} b & AB & A^2B \end{bmatrix} \begin{bmatrix} 1 & \alpha_1 & \alpha_2 \\ 0 & 1 & \alpha_1 \\ 0 & 0 & 1 \end{bmatrix} = \begin{bmatrix} 0 & 0 & 1 \\ 0 & 1 & 4 \\ 1 & 2 & 4 \end{bmatrix} \begin{bmatrix} 1 & -6 & 12 \\ 0 & 1 & -6 \\ 0 & 0 & 1 \end{bmatrix} = \begin{bmatrix} 0 & 0 & 1 \\ 0 & 1 & -2 \\ 1 & -4 & 4 \end{bmatrix} \tag{8}$$

$$P = Q^{-1} = \begin{bmatrix} 0 & 0 & 1 \\ 0 & 1 & -2 \\ 1 & -4 & 4 \end{bmatrix}^{-1} = \begin{bmatrix} 4 & 4 & 1 \\ 2 & 1 & 0 \\ 1 & 0 & 0 \end{bmatrix} \tag{9}$$

(e) Calculate k

$$k = \bar{k}P = \begin{bmatrix} 9 & -9 & 9 \end{bmatrix} \begin{bmatrix} 4 & 4 & 1 \\ 2 & 1 & 0 \\ 1 & 0 & 0 \end{bmatrix} = \begin{bmatrix} 27 & 27 & 9 \end{bmatrix} \tag{10}$$

(f) Confirm eigenvalues of $A - bk$

$$A - bk = \begin{bmatrix} 2 & 1 & 0 \\ 0 & 2 & 1 \\ 0 & 0 & 2 \end{bmatrix} \begin{bmatrix} 0 \\ 0 \\ 1 \end{bmatrix} \begin{bmatrix} 27 & 27 & 9 \end{bmatrix} = \begin{bmatrix} 2 & 1 & 0 \\ 0 & 2 & 1 \\ -27 & -27 & -7 \end{bmatrix} \tag{11}$$

eigenvalues are: $-1, -1, -1$

2. Condider a LTI MIMO system with matrices:

$$A = \begin{bmatrix} -2 & -2 & 0 \\ 0 & 0 & 1 \\ 0 & -3 & -4 \end{bmatrix} \quad B = \begin{bmatrix} 1 & 0 \\ 0 & 0 \\ 0 & 1 \end{bmatrix} \tag{12}$$

Assuming all stat variabes can be measured find a state feedback control law such that the eigenvalues of the closed loop system become: $\lambda_{d_1} = \lambda_{d_2} = -3$ and $\lambda_{d_3} = -4$. For this purpose use:

(a) Method 1 in slide 106 and verify the eigenvalues of the closed loop system

i. For each deisred eigenvalue λ_{d_i}: solve:

$$\begin{bmatrix} \lambda_{d_i} I - A & B \end{bmatrix} \bar{\phi}_i = \bar{0} \tag{13}$$

A. 1, 2

$$\begin{bmatrix} -1 & 2 & 0 & 1 & 0 \\ 0 & -3 & -1 & 0 & 0 \\ 0 & 3 & 1 & 0 & 1 \end{bmatrix} \phi_i = \bar{0} \tag{14}$$

$$\phi_1 = \begin{bmatrix}1\\0\\0\\1\\0\end{bmatrix} \phi_2 = \begin{bmatrix}2\\1\\-3\\0\\0\end{bmatrix} \tag{15}$$

B. 3

$$\begin{bmatrix}-2 & 2 & 0 & 1 & 0\\ 0 & -4 & -1 & 0 & 0\\ 0 & 3 & 0 & 0 & 1\end{bmatrix} \phi_3 = \bar{0} \tag{16}$$

$$phi_3 = \begin{bmatrix}1\\1\\-4\\0\\-3\end{bmatrix} \tag{17}$$

C. split the matrix

$$\begin{bmatrix}\bar{\psi}_1 & \bar{\psi}_2 & \bar{\psi}_3\end{bmatrix} = \begin{bmatrix}1 & 2 & 1\\ 0 & 1 & 1\\ 0 & -3 & -4\end{bmatrix} \tag{18}$$

$$\begin{bmatrix}K\bar{\psi}_1 & K\bar{\psi}_2 & K\bar{\psi}_3\end{bmatrix} = \begin{bmatrix}1 & 0 & 0\\ 0 & 0 & -3\end{bmatrix} \tag{19}$$

D. Solve for K

$$K = \begin{bmatrix}K\bar{\psi}_1 & K\bar{\psi}_2 & K\bar{\psi}_3\end{bmatrix} \begin{bmatrix}\bar{\psi}_1 & \bar{\psi}_2 & \bar{\psi}_3\end{bmatrix}^{-1} = \begin{bmatrix}0 & 2 & 2\\ -\frac{3}{2} & -\frac{3}{2} & 3\end{bmatrix} \begin{bmatrix}1 & 2 & 0\\ 1 & 1 & -1\\ -\frac{3}{2} & -\frac{3}{2} & 2\end{bmatrix}^{-1} \tag{20}$$

$$K = \begin{bmatrix}0 & 2 & 2\\ -\frac{3}{2} & -\frac{3}{2} & 3\end{bmatrix} \begin{bmatrix}1 & 2 & 0\\ 1 & 1 & -1\\ -\frac{3}{2} & -\frac{3}{2} & 2\end{bmatrix}^{-1} = \begin{bmatrix}1 & -5 & -1\\ 0 & 9 & 3\end{bmatrix} \tag{21}$$

E. evalute eigenvalues of feedback matrix

$$A - BK = \begin{bmatrix}-2 & -2 & 0\\ 0 & 0 & 1\\ 0 & -3 & -4\end{bmatrix} - \begin{bmatrix}1 & 0\\ 0 & 0\\ 0 & 1\end{bmatrix} \begin{bmatrix}1 & -5 & -1\\ 0 & 9 & 3\end{bmatrix} = \begin{bmatrix}-3 & 3 & 1\\ 0 & 0 & 1\\ 0 & -12 & -7\end{bmatrix} \tag{22}$$

eigenvalues of this are: $-3, -3, -4$

(b) Method 2 in slide 107 and verify the eigenvalues of the closed loop system

i. get a k_V where you have distict eigenvalues
already done $k_v = 0$

ii. select a $\bar{v}$ such that $(A, b\bar{v})$ is controllable:

$$\bar{v} = \begin{bmatrix} 0 \\ 1 \end{bmatrix} \tag{23}$$

$$Bv = \begin{bmatrix} 0 \\ 0 \\ 1 \end{bmatrix} \tag{24}$$

$$\begin{bmatrix} A^2Bv & ABv & Bv \end{bmatrix} = \begin{bmatrix} -2 & 0 & 0 \\ -4 & 1 & 0 \\ -13 & -4 & 1 \end{bmatrix} \tag{25}$$

full row rank $\implies$ controllable

$$A = \begin{bmatrix} -2 & -2 & 0 \\ 0 & 0 & 1 \\ 0 & -3 & -4 \end{bmatrix} B = \begin{bmatrix} 0 \\ 0 \\ 1 \end{bmatrix} \tag{26}$$

$$\Delta_d(s) = (s+3)(s+3)(s+4) = (s^2+6s+9)(s+4) \tag{27}$$

$$\Delta_d(s) = s^3 + 10s^2 + 33s + 36 \tag{28}$$

$$\bar{\alpha} = \begin{bmatrix} 10 & 33 & 36 \end{bmatrix} \tag{29}$$

$$\Delta(s) = |sI - A| = \begin{vmatrix} s+2 & 2 & 0 \\ 0 & s & -1 \\ 0 & 3 & s+4 \end{vmatrix} = (s+2)((s)(s+4)+3) \tag{30}$$

$$\Delta(s) = (s+2)(s^s + 4s + 3) = s^3 + 6s^2 + 11s + 6 \tag{31}$$

$$\alpha = \begin{bmatrix} 6 & 11 & 6 \end{bmatrix} \tag{32}$$

$$\bar{k} = \begin{bmatrix} 4 & 22 & 30 \end{bmatrix} \tag{33}$$

$$Q = P^{-1} = \begin{bmatrix} 0 & 0 & -2 \\ 0 & 1 & -4 \\ 1 & -4 & 13 \end{bmatrix} \begin{bmatrix} 1 & 6 & 11 \\ 0 & 1 & 6 \\ 0 & 0 & 1 \end{bmatrix} = \begin{bmatrix} 0 & 0 & -2 \\ 0 & 1 & 2 \\ 1 & 2 & 0 \end{bmatrix} \tag{34}$$

$$P = Q^{-1} = \begin{bmatrix} 0 & 0 & -2 \\ 0 & 1 & 2 \\ 1 & 2 & 0 \end{bmatrix} = \begin{bmatrix} -2 & -2 & 1 \\ 1 & 1 & 0 \\ -\frac{1}{2} & 0 & 0 \end{bmatrix} \tag{35}$$

$$k = \bar{k}P = \begin{bmatrix} 4 & 22 & 30 \end{bmatrix} \begin{bmatrix} -2 & -2 & 1 \\ 1 & 1 & 0 \\ -\frac{1}{2} & 0 & 0 \end{bmatrix} = \begin{bmatrix} -1 & 14 & 4 \end{bmatrix} \tag{36}$$

$$K = K_v + \bar{v}k = 0 + \begin{bmatrix} 0 \\ 1 \end{bmatrix} \begin{bmatrix} -1 & 14 & 4 \end{bmatrix} = \begin{bmatrix} 0 & 0 & 0 \\ -1 & 14 & 4 \end{bmatrix} \tag{37}$$

3. For the system in problem 2, design an ouput feedback control law such that two of the eigenvalues of the closed loop system become -3 and -4. Verify the eigenvalues of the closed loop system.

4. Consider the nonlinear control system in which it is desired to stabalize the origin

$$\dot{x}_1 = -2x_1 + x_2 + \sin(x_2) \tag{38}$$

$$\dot{x}_2 = -x_2 \cos(x_1) + \cos(2x_1)u \tag{39}$$

   (a) Veirfy that $(x_1, x_2, u) = (0, 0, 0)$ is an equilibrium.

   (b) linearize the system about the origin

   (c) Study the controllability of the linearized system

   (d) Assuming bost state variables can be measured, find a state feedback control law such that both eigenvalues of the linearized closed-loop system -4

   (e) Apply the state feedback control law obtained in pard d to the non=linear system above. Simulate and plot the state response and closed-loop system using the folowing initial conditions and briefly interpret the results:

      i. $\bar{x}_0 = \begin{bmatrix} 1 \\ 0 \end{bmatrix}$

      ii. $\bar{x}_0 = \begin{bmatrix} 1 \\ 1 \end{bmatrix}$

      iii. $\bar{x}_0 = \begin{bmatrix} -1 \\ 1 \end{bmatrix}$

1 Help Recieved

This section is a thank you for people who caught issues or otherwised helped(collaboration is allowed)

1. List people here