Homework 3

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February 21, 2022

1. In the LTI system described for $\dot{\bar{x}}(t) = A\bar{x}$ with $A = \begin{bmatrix} 0 & 1 & 0 \\ 0 & 0 & 1 \\ -1 & -3 & -3 \end{bmatrix}$

   (a) Obtain all eigenvalues and eigenvectors of A

$$|\lambda I - A| = 0 \tag{1}$$

$$det(\begin{bmatrix} \lambda & -1 & 0 \\ 0 & \lambda & -1 \\ 1 & 3 & \lambda+3 \end{bmatrix}) = \lambda^3 + 3\lambda^2 + 3\lambda + 1 = 0 \tag{2}$$

$$\lambda = -1, -1, -1 \tag{3}$$

   (b) Use the eigenvectors in part 1a to obtain the m odal matrix V and Jordan Form J

   Solving for eigenvalue(s) of -1, with a multiplicity of 3

$$(A - \lambda i)^3 x_0 = 0 \tag{4}$$

$$\begin{bmatrix} 0 & 0 & 0 \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{bmatrix} \begin{bmatrix} x_0 \\ x_1 \\ x_2 \end{bmatrix} = 0 \tag{5}$$

$$x_0 = \begin{bmatrix} 1 \\ 0 \\ 0 \end{bmatrix} \tag{6}$$

$$(A - \lambda i)^2 \begin{bmatrix} 1 \\ 0 \\ 0 \end{bmatrix} = x_1 \tag{7}$$

$$\begin{bmatrix} 1 & 2 & 1 \\ -1 & -2 & -1 \\ 1 & 2 & 1 \end{bmatrix} \begin{bmatrix} 1 \\ 0 \\ 0 \end{bmatrix} = x_1 \tag{8}$$

$$x_1 = \begin{bmatrix} 1 \\ -1 \\ 1 \end{bmatrix} \tag{9}$$

$$(A - \lambda i)^1 \begin{bmatrix} 1 \\ -1 \\ 1 \end{bmatrix} = x_2 \tag{10}$$

$$\begin{bmatrix} 1 & 1 & 0 \\ 0 & 1 & 1 \\ -1 & -3 & -2 \end{bmatrix} \begin{bmatrix} 1 \\ -1 \\ 1 \end{bmatrix} = x_2 \tag{11}$$

$$x_2 = \begin{bmatrix} 0 \\ 0 \\ 0 \end{bmatrix} \tag{12}$$

$$V = \begin{bmatrix} 1 & 1 & 1 \\ -1 & 0 & 0 \\ 1 & -1 & 0 \end{bmatrix} \tag{13}$$

$$J = V^{-1}AV = \begin{bmatrix} 0 & -1 & 0 \\ 0 & -1 & -1 \\ 1 & 2 & 1 \end{bmatrix} \begin{bmatrix} 0 & 1 & 0 \\ 0 & 0 & 1 \\ -1 & -3 & -3 \end{bmatrix} \begin{bmatrix} 1 & 1 & 1 \\ -1 & 0 & 0 \\ 1 & -1 & 0 \end{bmatrix} \tag{14}$$

$$J = \begin{bmatrix} -1 & 1 & 0 \\ 0 & -1 & 1 \\ 0 & 0 & -1 \end{bmatrix} \tag{15}$$

2. In each case below discuss BIBS stability of the LTI system $\dot{\bar{x}}(t) = A\bar{x}(t)$:

   (a) $A = \begin{bmatrix} -1 & 0 \\ 0 & -2 \end{bmatrix}$
   This system is stable as both of its eigenvalues are less 0

   (b) $A = \begin{bmatrix} 0 & 0 & 0 \\ 0 & -1 & 0 \\ 0 & 0 & -3 \end{bmatrix}$
   This system is semi-stable as two of the three are less than 0, and one is 0

(c) $A = \begin{bmatrix} 1 & 0 & 0 \\ 0 & -1 & 0 \\ 0 & 0 & -3 \end{bmatrix}$
This system is not stable an eigenvalue is greater than 0

(d) $A = \begin{bmatrix} 0 & 1 & 0 \\ 0 & 0 & 0 \\ 0 & 0 & -3 \end{bmatrix}$
This system has two semi-simple eigenvalues that are 0, meaning this is not stable

(e) $A = \begin{bmatrix} 0 & 0 & 0 \\ 0 & 0 & 0 \\ 0 & 0 & -3 \end{bmatrix}$
this system has three eigenvaleus that are all less than or equal to 0

3. The linearized equatinos of motion of a pendulum can be written in the form of

$$\dot{x}_1 = x_2 \tag{16}$$

$$\dot{x}_2 = -ax_1 - cx_2 \tag{17}$$

where $a > 0$ is a constant parameter of the system and $c > 0$ is the torsional friction coefficient

(a) Study BIBS stability of the system

$$\begin{bmatrix} \dot{x}_1 \\ \dot{x}_2 \end{bmatrix} = \begin{bmatrix} 0 & 1 \\ -a & -c \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \end{bmatrix} \tag{18}$$

This allows you to calculate the eigenvalues with:

$$\Delta = |\lambda I - A| = 0 \tag{19}$$

$$0 = \begin{vmatrix} \lambda & -1 \\ a & \lambda + c \end{vmatrix} = \lambda^2 + c\lambda + a \tag{20}$$

$$\lambda = \frac{-c \pm \sqrt{c^2 - 4a}}{2} \tag{21}$$

the system will be stable if the eigenvalues described by this equation are less then 0(or equal to 0 if the eigenvalues produce linearly independent eigenvectors)

(b) Consider the quadratic Lyapunov functino $V = \bar{x}^T P \bar{x}$ with $P = \begin{bmatrix} \frac{a}{2} & 0 \\ 0 & \frac{1}{2} \end{bmatrix}$. What can be said about the stability of the system based on this choice of the Lyapunov function?

(c) What can be said about the stability of the system based on the analysis of b) and c)?

(d) Study BIBS stability of the system when $c = 0$

$$\lambda = \pm\sqrt{a}j \tag{22}$$

if the eigenvectors are linearly independent, then the system is lyapunov stable

4. For the transfer function matrix

$$H(s) = \begin{bmatrix} \frac{s}{s-2} & 0 \\ \frac{2}{s-2} & 1 \end{bmatrix} \tag{23}$$

(a) Obtain the controllable canoncical form

$$\frac{1}{s+2}\begin{bmatrix} s & 0 \\ 2 & s+2 \end{bmatrix} \tag{24}$$

$$\dot{x} = \begin{bmatrix} -2 & 0 \\ 0 & -2 \end{bmatrix} x + \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix} u \tag{25}$$

$$N_0 = \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix} \tag{26}$$

$$N_1 = \begin{bmatrix} 0 & 0 \\ 2 & 2 \end{bmatrix} \tag{27}$$

$$y = (\begin{bmatrix} 0 & 0 \\ 2 & 2 \end{bmatrix} - \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix} 2)x + \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix} u \tag{28}$$

$$y = \begin{bmatrix} -2 & 0 \\ 2 & 0 \end{bmatrix} x + \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix} u \tag{29}$$

TODO: check this

(b) obtain the observable canonical form

$$\frac{1}{s+2}\begin{bmatrix} s & 0 \\ 2 & s+2 \end{bmatrix} \tag{30}$$

$$N_0 = \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix} \tag{31}$$

$$N_1 = \begin{bmatrix} 0 & 0 \\ 2 & 2 \end{bmatrix} \tag{32}$$

$$\dot{x} = \begin{bmatrix} 2 & 0 \\ 0 & 2 \end{bmatrix} x + \left(\begin{bmatrix} 0 & 0 \\ 2 & 2 \end{bmatrix} - \begin{bmatrix} 2 & 0 \\ 0 & 2 \end{bmatrix} \right) u \tag{33}$$

$$\dot{x} = \begin{bmatrix} 2 & 0 \\ 0 & 2 \end{bmatrix} x + \begin{bmatrix} -2 & 0 \\ 2 & 0 \end{bmatrix} u \tag{34}$$

$$y = \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix} x + \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix} \tag{35}$$

(c) show that the realization in (a) and (b) are dual as per

$$\begin{bmatrix} A & B \\ C & D \end{bmatrix}^T = \begin{bmatrix} A^T & B^T \\ C^T & D^T \end{bmatrix} \tag{36}$$

it can be seen that this is the case for this example