Homework 1

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1. Consider the following system consisting of a cart of mass m attached to a rigid wall by a spring and a damper. The spring stiffness $k(w)$ is a nonlinear function of displacement w such that the spring pull(force) applied to the mass is in the form of $f_s = k_1 w - k_3 w^3$, where k_1 and k_3 are constant scalars. Assume that the cart rolls freely without friction. It can be shown that the motion is $m\ddot{w} + c\dot{w} + k_1 w - k_3 w^3 = u$

   (a) The cart is instrumented with an accelerometer, which provides a measurement equation of the form $y = \ddot{w}$. Express the nonlinear equations of motion, including the output equation in state space form
   This system has two state variables: position and velocity. In order to keep it consistent with the notation in question 1c. the following will be used.

$$x = \begin{bmatrix} position \\ velocity \end{bmatrix} = \begin{bmatrix} w \\ \dot{w} \end{bmatrix} \tag{1}$$

$$k_1 w = u + k_3 w^3 - c\dot{w} - m\ddot{w} \tag{2}$$

$$w = \frac{u + k_3 w^3 - c\dot{w} - m\ddot{w}}{k_1} \tag{3}$$

$$c\dot{w} = u + k_3 w^3 - k_1 w - m\ddot{w} \tag{4}$$

$$\dot{w} = \frac{u + k_3 w^3 - k_1 w - m\ddot{w}}{c} \tag{5}$$

   (b) Assume $m = 1$, $k_1 = 4$, $k_3 = 1$, $c = 1$. Determine all the equilibrium points of the system assuming $u = 0$
   TODO: Figure out how to find equilibrium points

(c) Using the numerical assumptions above, linearize the equations of motion about the equilibrium point

$$\begin{bmatrix} 2 \\ 0 \end{bmatrix} \tag{6}$$

That is provide the lienar state space model about the point in the form of:

$$\begin{aligned} \delta\dot{\bar{x}} &= A\delta\bar{x}(t) + B\delta\bar{u}(t) \\ \delta y &= C\delta\bar{x}(t) + D\delta\bar{u}(t) \\ \delta\bar{x} &= C\delta\bar{x}(t) \end{aligned}$$

For the system that is described by:

$$w = \frac{0 + 1w^3 - 1\dot{w} - 1\ddot{w}}{4} \tag{7}$$

$$\dot{w} = \frac{0 + 1w^3 - 4w - 1\ddot{w}}{1} \tag{8}$$

It can be linearized into:

$$w = 1w + 2\dot{w} + 5u \tag{9}$$

$$w = 3w + 4\dot{w} + 6u \tag{10}$$

which in state space can be represented as:

$$\dot{x} = \begin{bmatrix} 1 & 2 \\ 3 & 4 \end{bmatrix} x + \begin{bmatrix} 5 \\ 6 \end{bmatrix} u \tag{11}$$

$$y = \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix} x + \begin{bmatrix} 0 \\ 0 \end{bmatrix} u \tag{12}$$

2. For the mass-spring-dampener system given in figure 2. Where k_1, k_2, c_1, and c_2 are constant scalars

   (a) obtain the differential equations of motion of the system

   (b) Write the equations of motion in the state space form

$$\begin{aligned} \dot{\bar{x}} &= A\bar{x} + B\bar{u} \\ \bar{y} &= C\bar{x} + D\bar{u} \end{aligned}$$

such that the A matrix is in the form of:

$$A = \begin{bmatrix} 0_{2\times 2} & I_2 \\ A_{21} & A_{22} \end{bmatrix} \tag{13}$$

3. Find the state space representation for the system described as

$$\dot{w}_1 + 3(w_1 + w_2) = u_1 \tag{14}$$

$$\ddot{w}_2 + 4\dot{w}_2 + 3w_2 = u_2 \tag{15}$$

Show the A and b matrix of your solution

Work:

For this problem there needs to be 3 state variables, 1 for w_1 and two for w_2

$$\dot{w}_1 = u_1 - 3w_1 - 3w_2) \tag{16}$$

$$\ddot{w}_2 = u_2 - 4\dot{w}_2 - 3w_2 \tag{17}$$

$$\dot{w}_2 = \frac{u_2 - \ddot{w}_2 - 3w_2}{4} \tag{18}$$

$$\dot{w}_2 = \frac{u_2 - (u_2 - 4\dot{w}_2 - 3w_2) - 3w_2}{4} \tag{19}$$

Final: TODO: Figure out second state variable

$$\begin{bmatrix} \dot{w}_1 \\ \dot{w}_2 \\ \ddot{w}_2 \end{bmatrix} = \begin{bmatrix} -3 & -3 & 0 \\ idk & & \\ 0 & -3 & -4 \end{bmatrix} \begin{bmatrix} w_1 \\ w_2 \\ \dot{w}_2 \end{bmatrix} + \begin{bmatrix} 1 & 0 \\ idk & \\ 0 & 1 \end{bmatrix} \begin{bmatrix} u_1 \\ u_2 \end{bmatrix} \tag{20}$$

4. Find the transfer function matrix for the system described as

$$\dot{x} = \begin{bmatrix} -1 & -2 & 6 \\ 2 & -3 & -2 \\ -2 & -2 & 1 \end{bmatrix} x + \begin{bmatrix} 1 & 1 \\ 1 & -1 \\ 1 & 0 \end{bmatrix} u \tag{21}$$

$$y = \begin{bmatrix} -1 & -1 & 2 \\ 1 & 1 & -1 \end{bmatrix} x + \begin{bmatrix} 0 & 0 \\ 0 & 0 \end{bmatrix} u \tag{22}$$

$$F(s) = C(sI - A)B + D \tag{23}$$

That is the equation for how to do that and I could grind through this process but honestly that sounds terrible and ill probably make a bunch of minor math mistakes. So I am going to put this in python using thier control library and spit out an answer.

$$\begin{bmatrix} \frac{6.0s-30.0}{1.0s^3+3.0s^2+11.0s+57.0} & \frac{6.0s-18.0}{1.0s^3+3.0s^2+11.0s+57.0} \\ \frac{1.0s^2+6.0s+29.0}{1.0s^3+3.0s^2+11.0s+57.0} & \frac{6.0s+6.0}{1.0s^3+3.0s^2+11.0s+57.0} \end{bmatrix} \tag{24}$$

5. The model of a control system with input $u(t)$ and output $y(t)$ is given as

$$\ddot{y}(t) + 3\dot{y}(t) + 7y(t) = \dot{u}(t) + 2u(t) \tag{25}$$

All initial conditions are zero and teh system is subjected to a unit step time input.

(a) Find the transfer function $\frac{Y(s)}{U(s)}$

$$\mathscr{L}(\ddot{y}(t) + 3\dot{y}(t) + 7y(t)) = \mathscr{L}(\dot{u}(t) + 2u(t)) \tag{26}$$

$$(s^2Y(s) - sf(0) - f'(0)) + 3(sY(s) - f(0)) + 7Y(s) = sU(s) + u(0) + 2U(s) \tag{27}$$

$$s^2Y(s) + 3sY(s) + 7Y(s) = sU(s) + 2U(s) \tag{28}$$

$$\frac{Y(s)}{U(s)}(s^2 + 3s + 7) = s + 2 \tag{29}$$

$$\frac{Y(s)}{U(s)} = \frac{s+2}{s^2 + 3s + 7} \tag{30}$$

(b) Find the values of all poles and zeros poles are calculated by getting the zeros of the denominator and zeros are calculated by getting the zeros of the denominator

zeros: -2.0

poles: -1.5+2.179, -1.5–2.179

(c) Find the steady state output response using the final value theorem

Approximate Anser for part 5c $y_{ss} \approx 0.3$ (Obtain the exact value accurate to the fourth decimal place