Take Home Final

Nathan Rose

December 6, 2021

1 (30pts)

A High performance helicopter has a model shown in Figure 1. The goal is to control the pitch angle θ of the helicopter by adjusting the rotor thrust δ. The equations of motion of the helicopter are

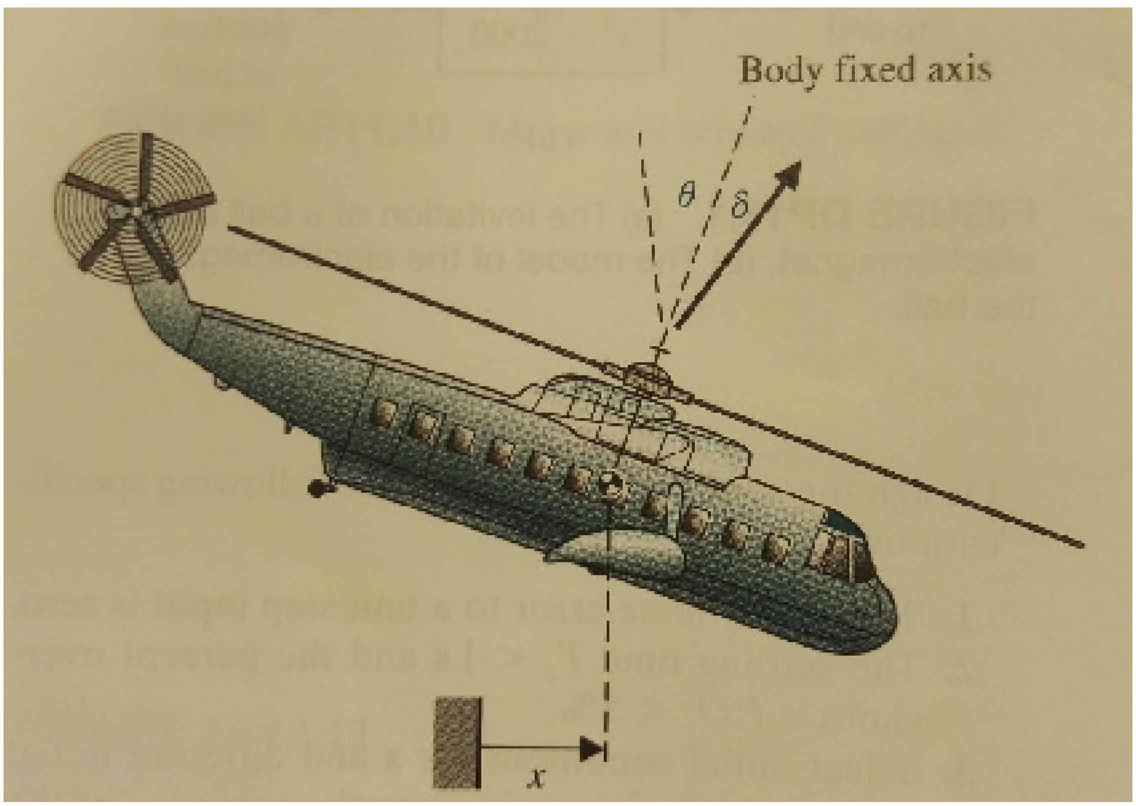

Figure 1:

$$\frac{d^2\theta}{dt^2} = -\sigma_1 \frac{d\theta}{dt} - \alpha_1 \frac{dx}{dt} + n\delta \tag{1}$$

$$\frac{d^2x}{dt^2} = g\theta - \alpha_2 \frac{d\theta}{dt} - \sigma_2 \frac{dx}{dt} + g\delta \tag{2}$$

Where x is the translation in the horizontal direction. For a miltary high-preformance helicopter we find: $\sigma_1 = 0.415$, $\sigma_2 = 0.0198$, $\alpha_1 = 0.0111$,$\alpha_2 = 1.43$, $n = 6.27$, $g = 9.8$ all in appropriate SI units. Find:

(a) A state variable representation of this system

(b) The transfer function representation for $\frac{\theta(s)}{\delta(s)}$

(c) Use state variable feedback to achieve adaquate performances for the controlled system. Desired specifications include:

   (1) A steady-state for an input step command for $\theta_d(s)$, the desired pitch angle, less than 20% of the input step magnitude

   (2) An overshoot for a step input command is less than 20%

   (3) a settling (with a 2% criterion) time for a step command of less than 1.5 seconds

(d) If the state variable is not available, design the observer and control law to meet the design specifications included in part

work:

(a) For the system with state variables:

$$x = \begin{bmatrix} x \\ \dot{x} \\ \theta \\ \dot{\theta} \end{bmatrix} \tag{3}$$

The state space representation is:

$$\dot{x} = \begin{bmatrix} 0 & 1 & 0 & 0 \\ 0 & -\sigma_2 & g & -\alpha_1 \\ 0 & 0 & 0 & 1 \\ 0 & -\sigma_1 & 0 & -\alpha_1 \end{bmatrix} x + \begin{bmatrix} 0 \\ g \\ 0 \\ n \end{bmatrix} u \tag{4}$$

$$y = \begin{bmatrix} 1 & 0 & 0 & 0 \\ 0 & 0 & 1 & 0 \end{bmatrix} x + \begin{bmatrix} 0 \\ 0 \end{bmatrix} u \tag{5}$$

(b) For the system described by:

$$\dot{x} = \begin{bmatrix} 0.0 & 1.0 & 0.0 & 0.0 \\ 0.0 & -0.02 & 9.8 & -1.43 \\ 0.0 & 0.0 & 0.0 & 1.0 \\ 0.0 & -0.415 & 0.0 & -0.011 \end{bmatrix} x + \begin{bmatrix} 0.0 \\ 9.8 \\ 0.0 \\ 6.27 \end{bmatrix} u \tag{6}$$

$$y = \begin{bmatrix} 1 & 0 & 0 & 0 \\ 0 & 0 & 1 & 0 \end{bmatrix} x + \begin{bmatrix} 0 \\ 0 \end{bmatrix} u \tag{7}$$

A realization was determined by taking the realization of the state space representation. The result of this is:

$$g(s) = \begin{bmatrix} \frac{9.8s^2-8.857s+61.446}{1.0s^4+0.031s^3-0.593s^2+4.067s} \\ \frac{6.27s^2-3.943s}{1.0s^4+0.031s^3-0.593s^2+4.067s} \end{bmatrix} \tag{8}$$

(c) part c

(d) If the state variable is not available, design the observer and control law to meet the design specifications included in part

Final Answer:

2 (30pts)

The open loop system

$$\dot{x} = \begin{bmatrix} 0 & 1 & 0 \\ 0 & 0 & 1 \\ -1 & 4 & -3 \end{bmatrix} x + \begin{bmatrix} 0 \\ 0 \\ 2 \end{bmatrix} u \tag{9}$$

$$y = \begin{bmatrix} 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix} x \tag{10}$$

$$x(0) = \begin{bmatrix} 1 \\ -1 \\ 1 \end{bmatrix} \tag{11}$$

1) Assume that x is availbale for state feedback. Design and LQR control law by letting $R = 1$ and choosing Q so that all the elements of the feedback gain K have absolute value less than 50.
   Requirement: $|y_1(t)| \leq 0.05$,$|y_2(t)| \leq 0.05$, for all $t > 5$. Plot $y_1(t)$ and $y_2(t)$ in the same figure for $t \in [0, 20]$

2) Assume that only the output y is available. Design an observer so that the poles of the observer are $-5 \pm j5, -10$. Choose the observer gain so that all the elments have absolute value less than 80. Form a closed loop system along with the LQR controller in step 1). Plot $y_1(t)$ and $y_2(t)$ in the same figure for $t \in [0, 20]$

work:

1) Assume that x is availbale for state feedback. Design and LQR control law by letting $R = 1$ and choosing Q so that all the elements of the feedback gain K have absolute value less than 50.
   Requirement: $|y_1(t)| \leq 0.05$,$|y_2(t)| \leq 0.05$, for all $t > 5$. Plot $y_1(t)$ and $y_2(t)$ in the same figure for $t \in [0, 20]$
   For the system described by:

$$\dot{x} = \begin{bmatrix} 0 & 1 & 0 \\ 0 & 0 & 1 \\ -1 & 4 & 5 \end{bmatrix} x + \begin{bmatrix} 0 \\ 0 \\ 2 \end{bmatrix} u \tag{12}$$

$$y = \begin{bmatrix} 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix} x + \begin{bmatrix} 0 \\ 0 \end{bmatrix} u \tag{13}$$

   and $Q = \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix}$,$R = \begin{bmatrix} 1 \end{bmatrix}$

   The following is the outputs of the LQR system assuming the inputs are 0

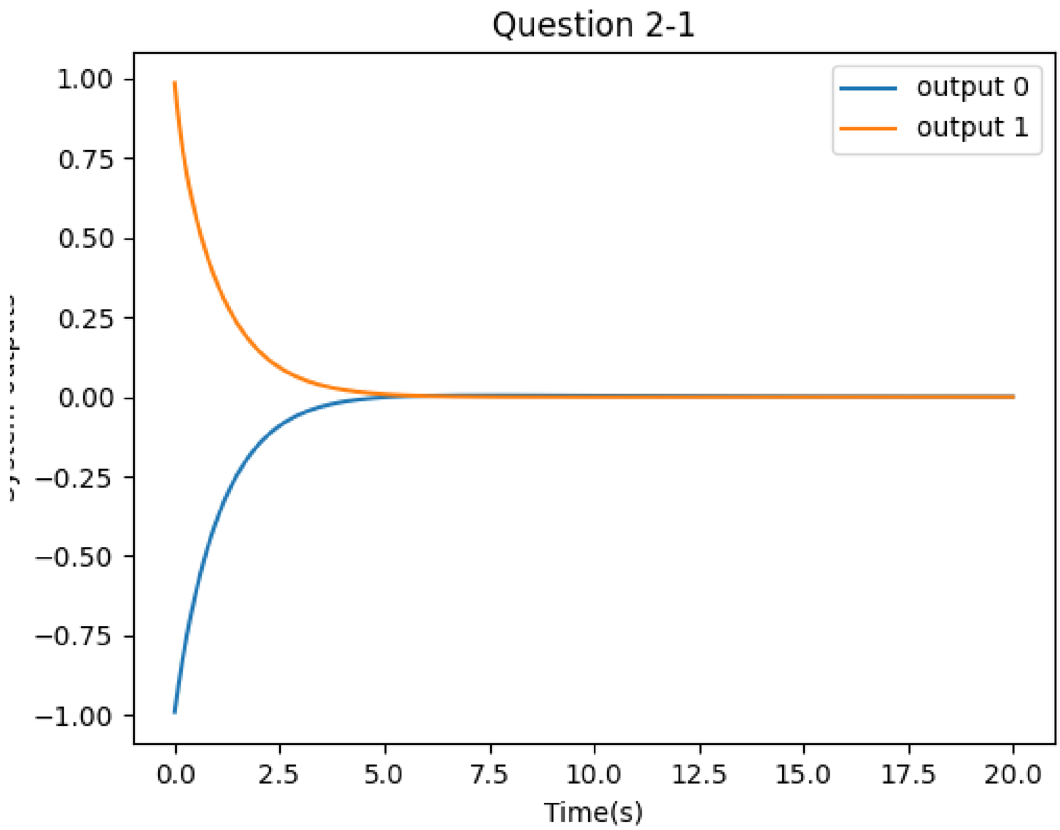

Figure 2: LQR system

2) Assume that only the output y is available. Design an observer so that the poles of the observer are $-5 \pm j5, -10$. Choose the observer gain so that all the elments have absolute value less than 80. Form a closed loop system along with the LQR controller in step 1). Plot $y_1(t)$ and $y_2(t)$ in the same figure for $t \in [0, 20]$ For the system described by:

$$\dot{x} = \begin{bmatrix} 0 & 1 & 0 \\ 0 & 0 & 1 \\ -1 & 4 & 5 \end{bmatrix} x + \begin{bmatrix} 0 \\ 0 \\ 2 \end{bmatrix} u \tag{14}$$

$$y = \begin{bmatrix} 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix} x + \begin{bmatrix} 0 \\ 0 \end{bmatrix} u \tag{15}$$

The following variables were chosen for the observer:

L_0	$\begin{bmatrix} 0.0 & 1.0 \\ 0.1 & 0.0 \\ 0.1 & 0.25 \end{bmatrix}$
F	$\begin{bmatrix} -5.0 & 5.0 & 0.0 \\ -5.0 & -5.0 & 0.0 \\ 0.0 & 0.0 & -10.0 \end{bmatrix}$

These were used to calculate:

$$
\begin{array}{c|l}
T & \begin{bmatrix} 0.003 & -0.002 & 0.078 \\ -0.012 & 0.061 & -0.045 \\ 0.002 & 0.003 & 0.016 \end{bmatrix} \\
L & \begin{bmatrix} 58.497 & 1.337 \\ 11.91 & 9.973 \\ -2.128 & 13.09 \end{bmatrix}
\end{array}
$$

The following is the outputs of the LQR system assuming the inputs are 0 and an initial estimate of state of $\begin{bmatrix} 1.0 \\ -1.0 \\ 1.0 \end{bmatrix}$

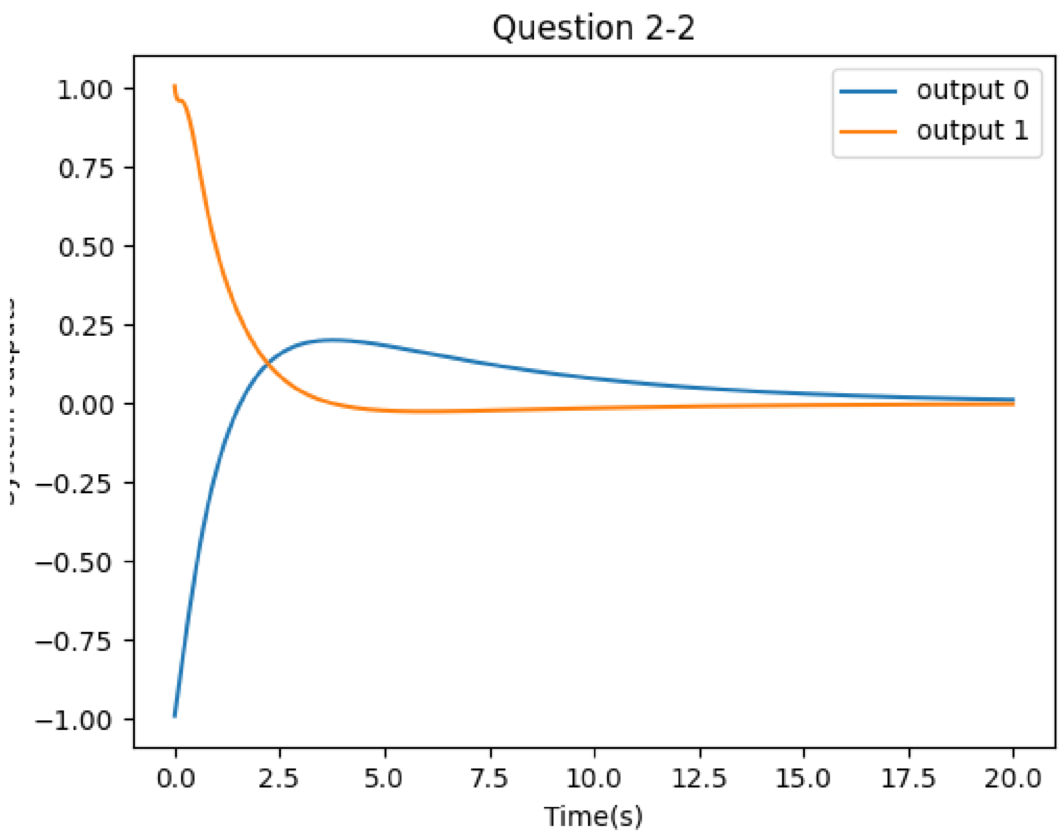

Figure 3: LQR system

Final Answer:

3 (40pts)

A cart with an inverted pendulum as seen in Figure 4

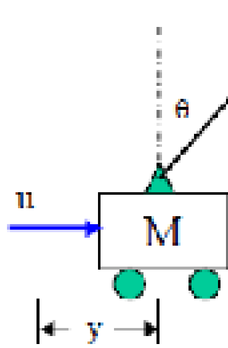

Figure 4:

n	control input(Newtons)
y	displacement of the cart(meters)
θ	angle of the pendulum(radians)

$$x = \begin{bmatrix} y \\ \dot{y} \\ \theta \\ \dot{\theta} \end{bmatrix} \tag{16}$$

The control problems are

1: Stabalization: Design a feedback law $u\ Fx$ such that $x(t) > 0$ for $x(0)$ close to the diagram

2: For $x(0) = (0, 0, -\pi, 0)$, apply an impulse force $u(t) = u_{max}$ for $t \in [0, 0.1]$ to bring 0 to a certain range and then switch to the linear controller so that $x(t) \to 0$.

Assume that there is no friction or damping. The nonlinear model is as follows.

$$\begin{bmatrix} M+m & ml\cos(\theta) \\ \cos(\theta) & l \end{bmatrix} \begin{bmatrix} \ddot{y} \\ \ddot{\theta} \end{bmatrix} - \begin{bmatrix} u + ml\dot{\theta}^2 \sin\theta \\ g\sin\theta \end{bmatrix} \tag{17}$$

with

$m = 1kg$	mass of the pendulum
$l = 0.2m$	length of the pendulum
$M = 5kG$	mass of the cart
$g = 9.8\frac{m}{s^2}$	mass of the cart

Linearize the system at $x = 0$

$$\begin{bmatrix} M+m & ml \\ 1 & l \end{bmatrix} \begin{bmatrix} \ddot{y} \\ \ddot{\theta} \end{bmatrix} - \begin{bmatrix} u \\ g\theta \end{bmatrix} \tag{18}$$

the state space description for the linearlized system.

$$\dot{x} = Ax + Bu \tag{19}$$

Problems:

1. Find matrices A, B for the state space equation.

2. Design a feedback law $u - F_1 x$ so that $A + BF_1$ has eignevalues as $-3 \pm j3, -6, -8$. Build a simulink model for the closed loop linear system. Plot the response under initial condition $x(0) = (-1.5, 0, 1, 3)$.

3. Build a simulink model for the original nonlinear system, verify that stabilization is achieved by $u = F_1 x$ when $x(0)$ is close to the origin. Find the maximal θ_0 so that nonlinear system can be stabalized from $x_0 = (0, 0, \theta_0, 0)$

4. For $x(0) = (0, 0, \frac{\pi}{5}, 0)$, compare the response $y(t)$ and $\theta(t)$ for the linearized system and the nonlinear system under the same feedback $u - F_1 x$

work:

1. For the system with state matrix:

$$x = \begin{bmatrix} y \\ \dot{y} \\ \theta \\ \dot{\theta} \end{bmatrix} \tag{20}$$

The following system is described:

$$\dot{x} = \begin{bmatrix} 0 & 1 & 0 & 0 \\ \frac{-mg}{M} & 0 & 0 & 0 \\ 0 & 0 & 0 & 1 \\ 0 & 0 & \frac{g}{l} + \frac{gm}{Ml} & 0 \end{bmatrix} x + \begin{bmatrix} 0 \\ \frac{1}{M} \\ 0 \\ -\frac{1}{m*l} \end{bmatrix} u \tag{21}$$

$$y = \begin{bmatrix} 1 & 0 & 0 & 0 \\ 0 & 0 & 1 & 0 \end{bmatrix} x + \begin{bmatrix} 0 \\ 0 \end{bmatrix} u \tag{22}$$

2. The input values were: $\begin{array}{c|c} K_0 & \begin{bmatrix} 10 & 20 & 30 & 40 \end{bmatrix} \\ F & \begin{bmatrix} -3.0 & -3.0 & 0.0 & 0.0 \\ 3.0 & -3.0 & 0.0 & 0.0 \\ 0.0 & 0.0 & -6.0 & 0.0 \\ 0.0 & 0.0 & 0.0 & -8.0 \end{bmatrix} \end{array}$ this produced a k of:

$$k = \begin{bmatrix} -47.222 & -41.211 & -216.284 & -28.242 \end{bmatrix} \tag{23}$$

With this feedback value the following outputs were made:

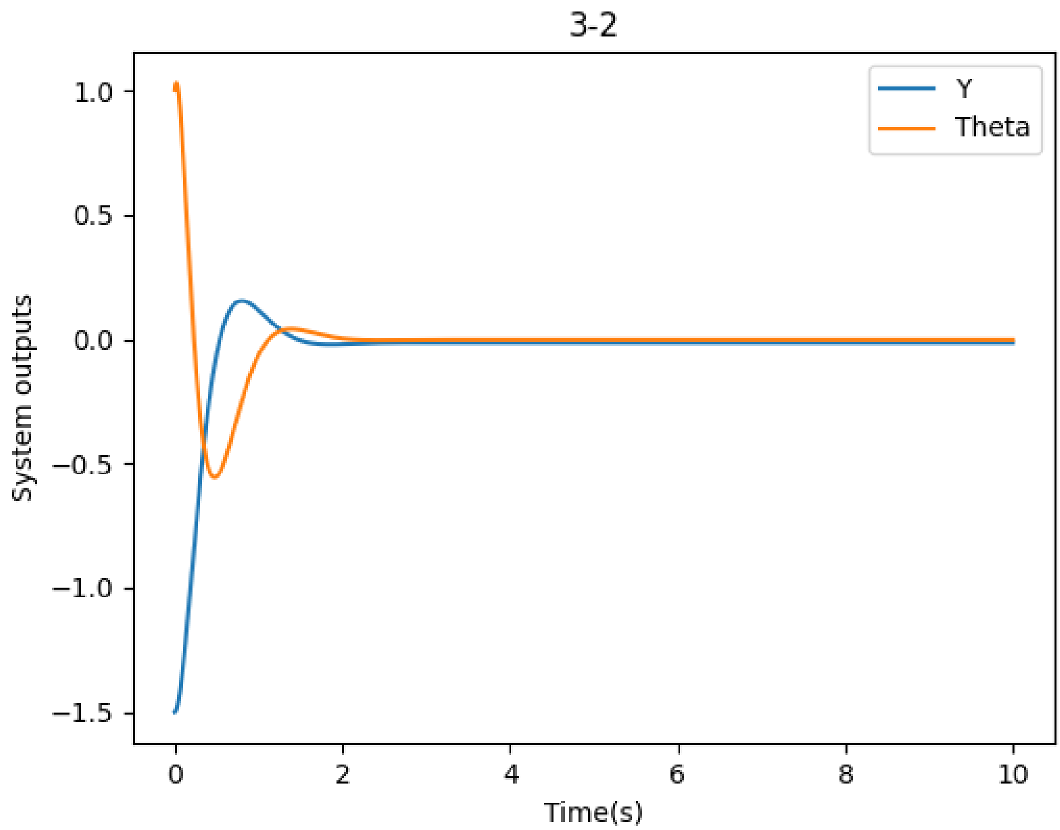

Figure 5: 3-2 system

3. The non-linear system was implimented with the following code

```
import numpy as np
from numpy.linalg import inv
from math import cos, sin, pi
import json


def nonLinearUpdate(x: np.matrix, r: np.matrix, k: np.matrix, config: json, dt: float)
    # a list of variables that make my life easier
    dot_y = x.item((1, 0))
    theta = x.item((2, 0))
    dot_theta = x.item((3, 0))
    u = r - (k*x)
    n = u.item((0, 0))
    M = float(config["M"])
    m = float(config["m"])
    L = float(config["l"])
    g = float(config["g"])

    # non-linear model
```

```python
    accels = inv(np.matrix([
        [M+m, m*L*cos(theta)],
        [cos(theta), L]
    ])) * np.matrix([
        [n + m*L*theta*theta*sin(theta)],
        [g*sin(theta)]
    ])
    ddot_y = accels.item((0, 0))
    ddot_theta = accels.item((1, 0))
    dx = np.matrix([
        [dot_y],
        [ddot_y],
        [dot_theta],
        [ddot_theta]
    ])
    next_x = x + (dx*dt)

    # rage quit if it falls over lol
    next_theta = next_x.item((2, 0))
    while (next_theta) > pi:
        raise AssertionError("collapsed")
        next_theta = next_theta - (2*pi)
    while (next_theta) < -pi:
        raise AssertionError("collapsed")
        next_theta = next_theta + (2*pi)
    next_x.itemset((2, 0), next_theta)

    return next_x


def nonLinearOutput(x: np.matrix, r: np.matrix, k: np.matrix, config: json, dt: float)
    theta = x.item((2, 0))
    return np.matrix([
        [theta]
    ])
```

A sample with starting inputs of: $\begin{bmatrix} 0 \\ 0 \\ 1 \\ 0 \end{bmatrix}$ produces the followung outputs

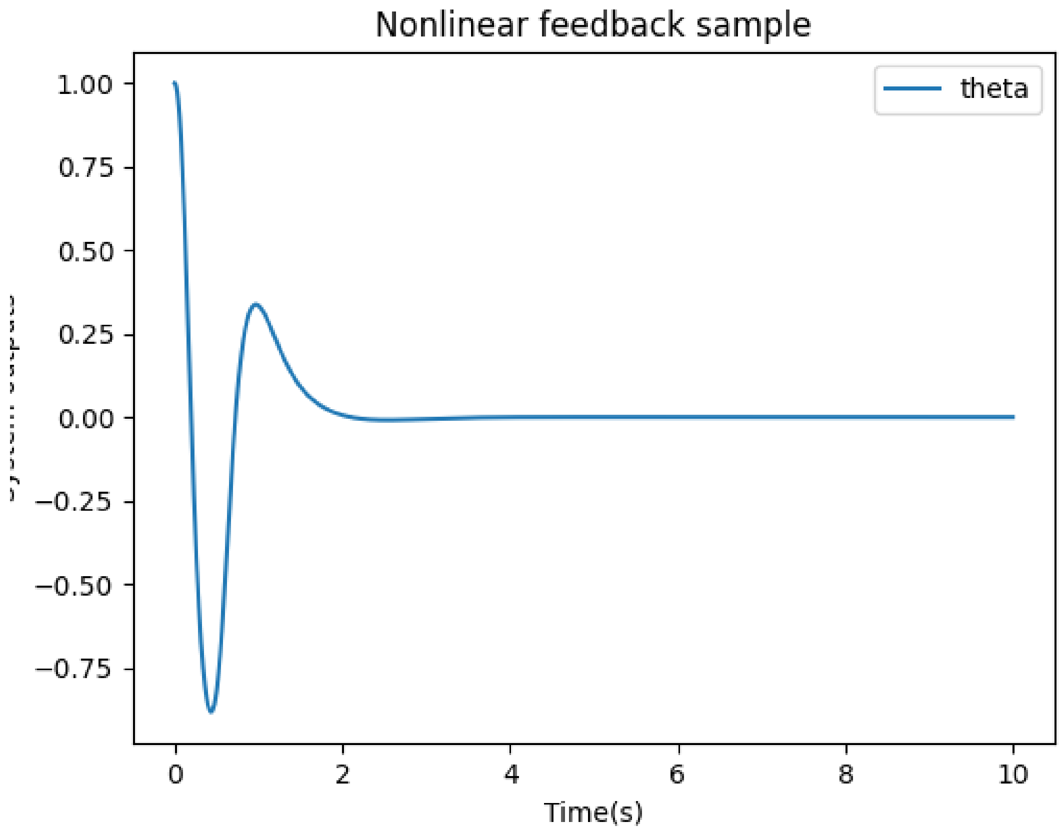

Figure 6: 3-3 system

For the limit, the system was considered stabalized if after 10 seconds the system state variable θ was within ± 0.01 of 0, and it had not fallen over it yet, the result of this is a theta limit of: 1.058 radians which is: 60.645 degrees

4. The comparison betweeen the linear and nonlinear system can be seen in the system below in Figure 7

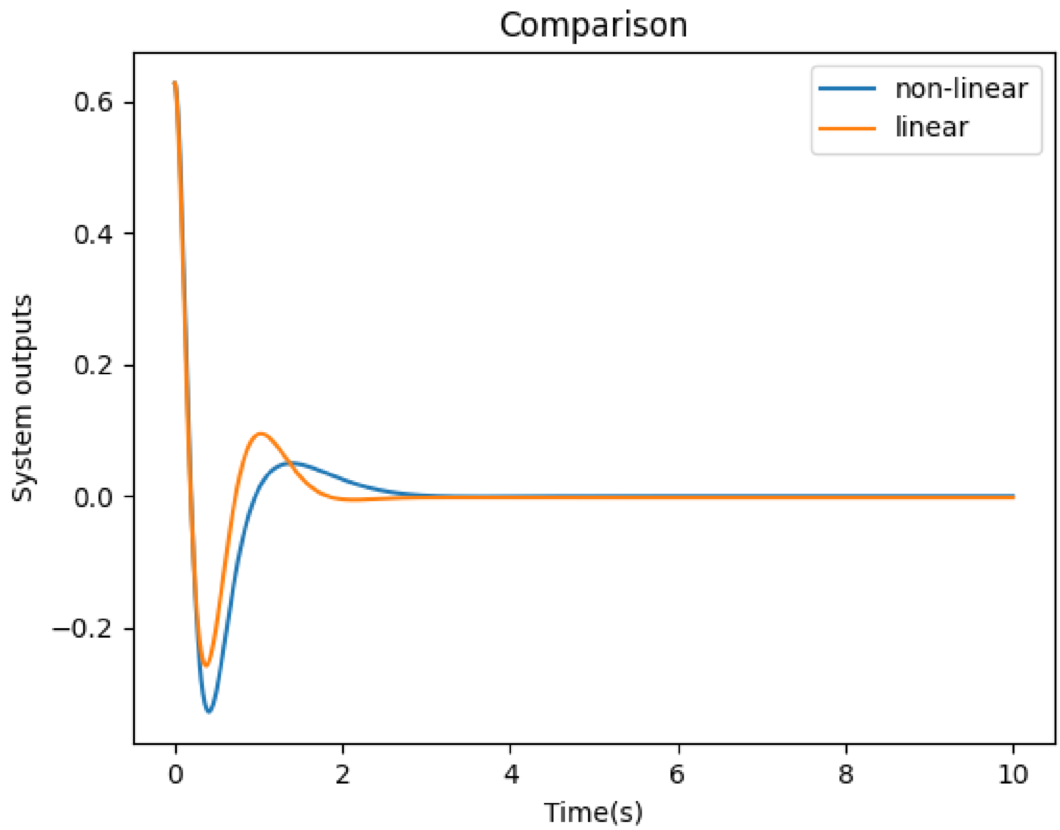

Figure 7: Comparison

Final Answer: