# Take Home Final

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## 1 (30pts)

A High performance helicopter has a model shown in Figure 1. The goal is to control the pitch angle $\theta$ of the helicopter by adjusting the rotor thrust $\delta$. The equations of motion of the helicopter are

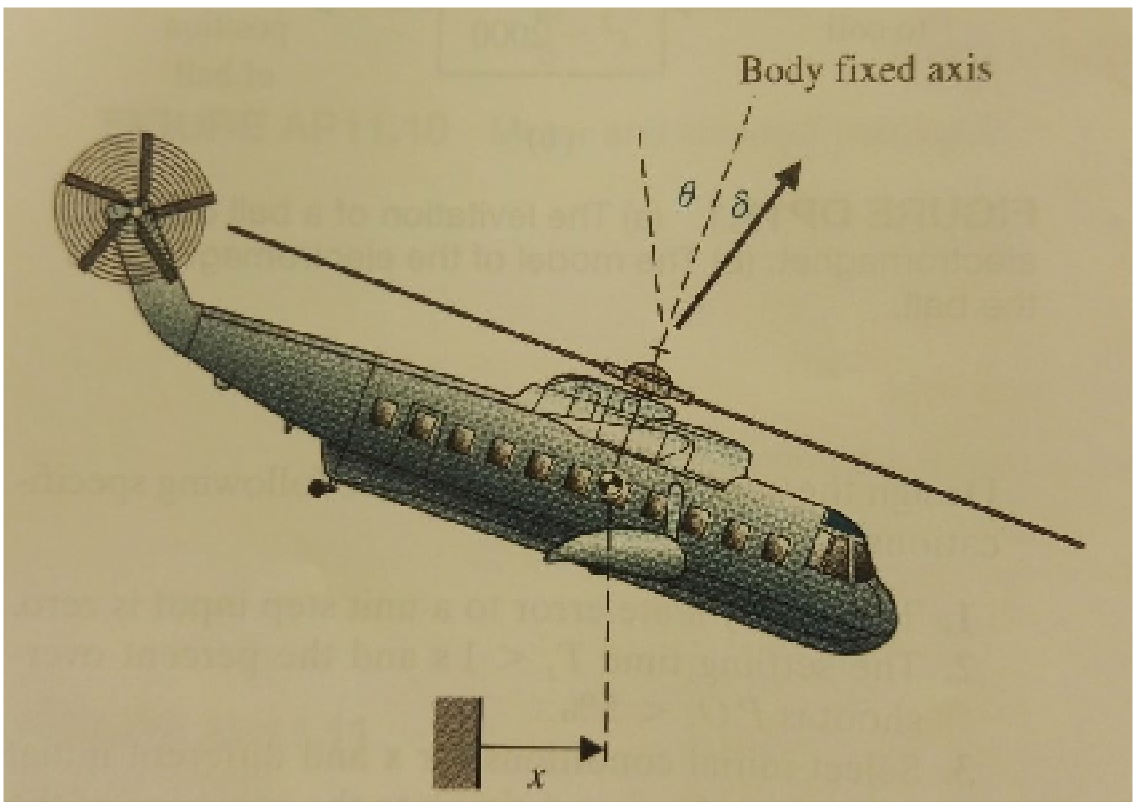


Figure 1:

$$\frac{d^2\theta}{dt^2} = -\sigma_1 \frac{d\theta}{dt} - \alpha \frac{dx}{dt} + n\delta \tag{1}$$

$$\frac{d^2x}{dt^2} = g\theta - \alpha_2 \frac{d\theta}{dx} - \sigma_2 \frac{dx}{dt} + g\delta \tag{2}$$

Where x is teh translation in the horizontal direction. FOr a miltary high-preformance helicopter we find: $\sigma_1 = 0.415$, $\sigma_2 = 0.0198$, $\alpha_1 = 0.0111$,$\alpha_2 = 1.43$, $n = 6.27$, $g = 9.8$ all in appropriate SI units. Find:

(a) A state variable representation of this system

(b) The transfer function representation for $\frac{\theta(s)}{\delta(s)}$

(c) Use state variable feedback to achieve adaquate performances for the controlled system. Desired specifications include:

  (1) A steady-state for an input step command for $\theta_d(s)$, the desired pitch angle, less than 20% of the input step magnitude

  (2) An overshoot for a step input command is less than 20%

  (3) a settling (with a 2% criterion) time for a step command of less than 1.5 seconds

(d) If the state variable is not available, design the observer and control law to meet the design specifications included in part (c)

work:

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(d) If the state variable is not available, design the observer and control law to meet the design specifications included in part (c)

Final Answer:

## 2 (30pts)

The open loop system

$$\dot{x} = \begin{bmatrix} 0 & 1 & 0 \\ 0 & 0 & 1 \\ -1 & 4 & -3 \end{bmatrix} x + \begin{bmatrix} 0 \\ 0 \\ 2 \end{bmatrix} u \tag{3}$$

$$y = \begin{bmatrix} 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix} x \tag{4}$$

$$x(0)0 = \begin{bmatrix} 1 \\ -1 \\ 1 \end{bmatrix} \tag{5}$$

1) Assume that x is availbale for state feedback. Design and LQR control law by letting $R = 1$ and choosing $Q$ so that all the elements of the feedback gain $K$ have absolute value less than 50.
   Requirement: $|y_1(t)| \leq 0.05$,$|y_2(t)| \leq 0.05$, for all $t > 5$. Plot $y_1(t)$ and $y_2(t)$ in the same figure for $t \in [0, 20]$

2) Assume that only the output $y$ is available. Design an observer so that the poles of the observer are $-5 \pm j5, -10$. Choose the observer gain so that all the elments have absolute value less than 80. Form a closed loop system along with the LQR controller in step 1). Plot $y_1(t)$ and $y_2(t)$ in the same figure for $t \in [0, 20]$

work:

1) Assume that x is availbale for state feedback. Design and LQR control law by letting $R = 1$ and choosing $Q$ so that all the elements of the feedback gain $K$ have absolute value less than 50.

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   For the system described by:

$$\dot{x} = \begin{bmatrix} 0 & 1 & 0 \\ 0 & 0 & 1 \\ -1 & 4 & 5 \end{bmatrix} x + \begin{bmatrix} 0 \\ 0 \\ 2 \end{bmatrix} u \tag{6}$$

$$\dot{x} = \begin{bmatrix} 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix} x + \begin{bmatrix} 0 \\ 0 \end{bmatrix} u \tag{7}$$

and $Q = \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix}$,$R = [1]$

The following is the outputs of the LQR system assuming the inputs are 0

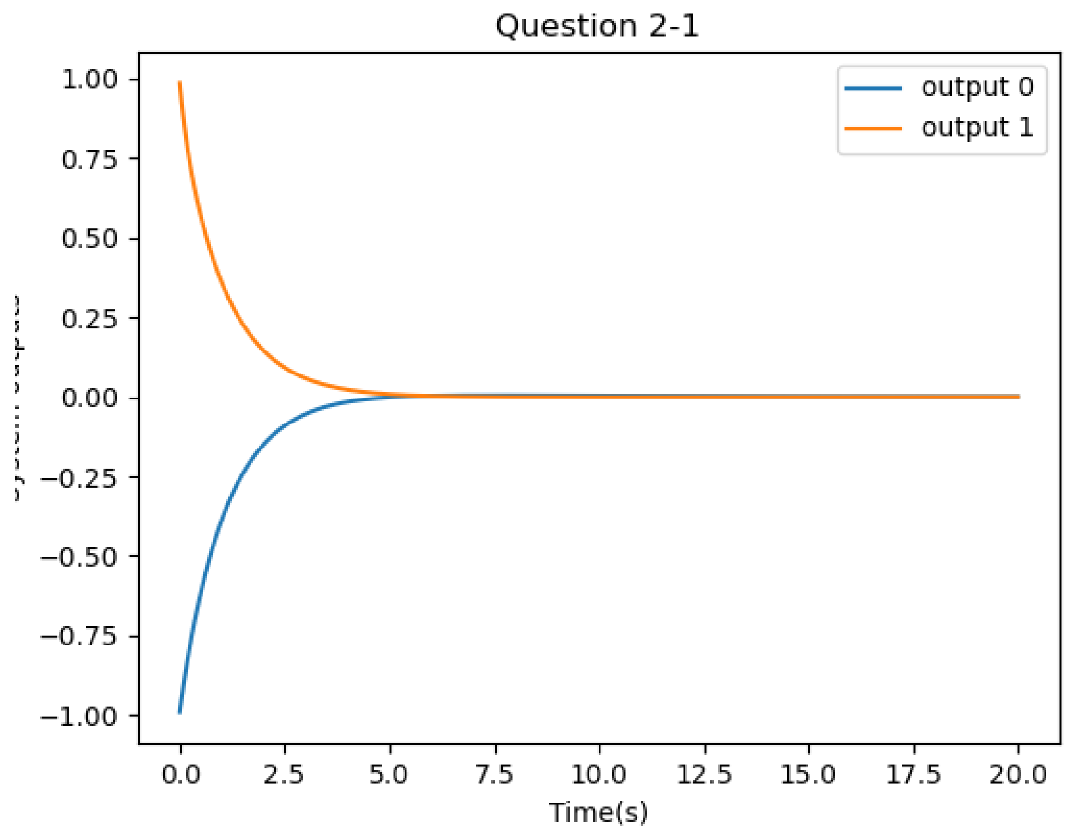


Figure 2: LQR system

2) Assume that only the output $y$ is available. Design an observer so that the poles of the observer are $-5 \pm j5, -10$. Choose the observer gain so that all the elments have absolute value less than 80. Form a closed loop system along with the LQR controller in step 1). Plot $y_1(t)$ and $y_2(t)$ in the same figure for $t \in [0, 20]$ For the system described by:

$$\dot{x} = \begin{bmatrix} 0 & 1 & 0 \\ 0 & 0 & 1 \\ -1 & 4 & 5 \end{bmatrix} x + \begin{bmatrix} 0 \\ 0 \\ 2 \end{bmatrix} u \tag{8}$$

$$\dot{x} = \begin{bmatrix} 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix} x + \begin{bmatrix} 0 \\ 0 \end{bmatrix} u \tag{9}$$

The following variables were chosen for the observer:

| | |
|---|---|
| $L_0$ | $\begin{bmatrix} 0. & 1. \\ 0.5 & 0. \\ 0.1 & 0.25 \end{bmatrix}$ |
| $F$ | $\begin{bmatrix} -5. & 5. & 0. \\ -5. & -5. & 0. \\ 0. & 0. & -5. \end{bmatrix}$ |

These were used to calculate:

$$
\begin{array}{c|l}
T & \begin{bmatrix} 0.00264941 & 0.04032866 & 0.07280946 \\ -0.01191248 & 0.09910611 & -0.04631534 \\ 0.00502183 & -0.0010917 & 0.02510917 \end{bmatrix} \\
L & \begin{bmatrix} 55.62246651 & -6.38612161 \\ 8.56733081 & 4.57523188 \\ -6.76939196 & 11.43266919 \end{bmatrix}
\end{array}
$$

The following is the outputs of the LQR system assuming the inputs are 0 and an initial estimate of state of $\begin{bmatrix} 1. \\ -1. \\ 1. \end{bmatrix}$

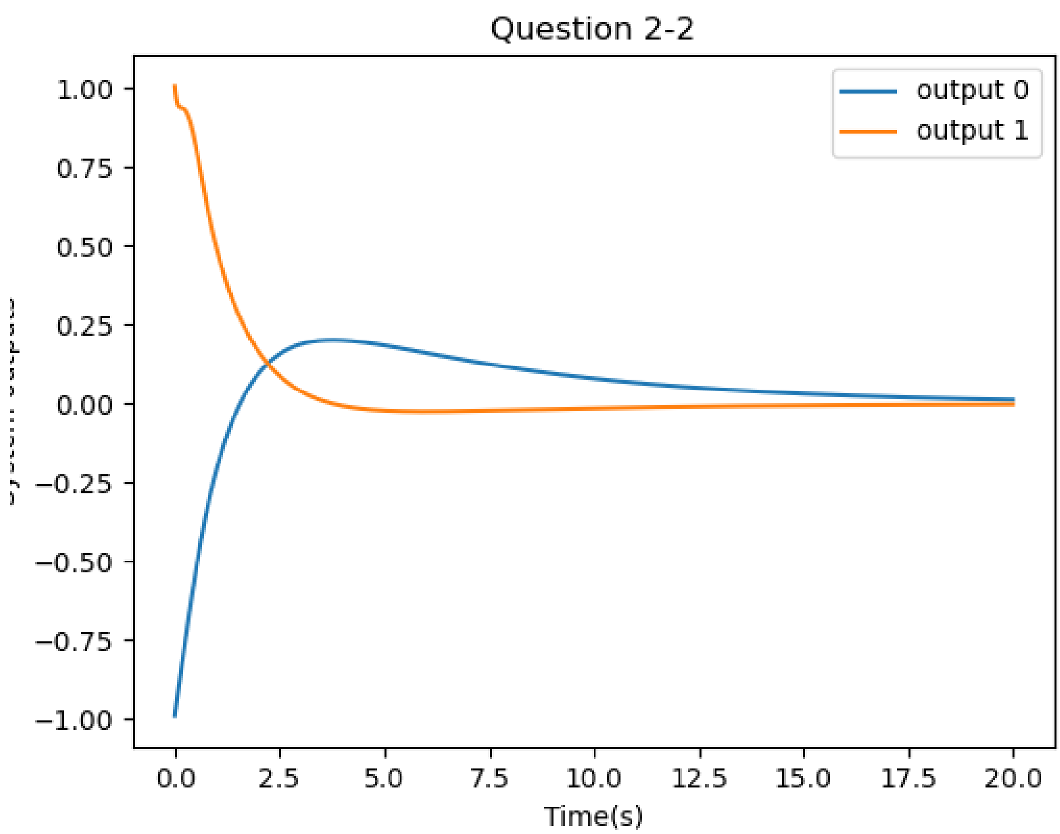


Figure 3: LQR system

Final Answer:

## 3 (40pts)

A cart with an inverted pendulum as seen in Figure 4

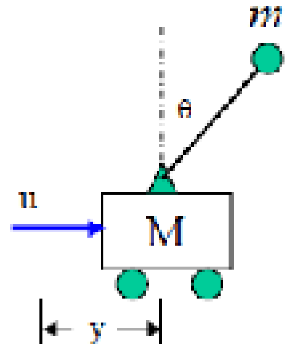


Figure 4:

| | |
|---|---|
| n | control input(Newtons) |
| y | displacement of the cart(meters) |
| $\theta$ | angle of the pendulum(radians) |

$$x = \begin{bmatrix} y \\ \dot{y} \\ \theta \\ \dot{\theta} \end{bmatrix} \tag{10}$$

The control problems are

1: Stabalization: Design a feedback law $u\ Fx$ such that $x(t) > 0$ for $x(0)$ close to the diagram

2: For $x(0) = (0, 0, -\pi, 0)$, apply an impulse force $u(t) = u_{max}$ for $t \in [0, 0.1]$ to bring 0 to a certain range and then switch to the linear controller so that $x(t) \to 0$.

Assume that there is no friction or damping. The nonlinear model is as follows.

$$\begin{bmatrix} M+m & ml\cos(\theta) \\ \cos(\theta) & l \end{bmatrix} \begin{bmatrix} \ddot{y} \\ \ddot{\theta} \end{bmatrix} - \begin{bmatrix} u + ml\dot{\theta}^2 \sin\theta \\ g\sin\theta \end{bmatrix} \tag{11}$$

with

| | |
|---|---|
| $m = 1kg$ | mass of the pendulum |
| $l = 0.2m$ | length of the pendulum |
| $M = 5kG$ | mass of the cart |
| $g = 9.8\frac{m}{s^2}$ | mass of the cart |

Linearize the system at $x = 0$

$$\begin{bmatrix} M+m & ml\cos(\theta) \\ \cos(\theta) & l \end{bmatrix} \begin{bmatrix} \ddot{y} \\ \ddot{\theta} \end{bmatrix} - \begin{bmatrix} u \\ g\theta \end{bmatrix} \tag{12}$$

the state space description for the linearlized system.

$$\dot{x} = Ax + Bu \tag{13}$$

Problems:

1. Find matrices $A$, $B$ for the state space equation.

2. Design a feedback law $u - F_1 x$ so that $A + BF_1$ has eignevalues as $-3 \pm j3, -6, -8$. Build a simulink model for the closed loop linear system. Plot the response under initial condition $x(0) = (-1.5, 0, 1, 3)$.

3. Build a simulink model for the original nonlinear system, verify that stabilization is achieved by $u = F_1 x$ when $x(0)$ is close to the origin. Find the maximal $\theta_0$ so that nonlinear system can be stabalized from $x_0 = (0, 0, \theta_0, 0)$

4. For $x(0) = (0, 0, \frac{\pi}{5}, 0)$, compare the response $y(t)$ and $\theta(t)$ for the linearized system and the nonlinear system under the same feedback $u - F_1 x$

work:

1. Find matrices $A$, $B$ for the state space equation.

$$\dot{x} = \begin{bmatrix} \dot{y} \\ \ddot{y} \\ \dot{\theta} \\ \ddot{\theta} \end{bmatrix} = \begin{bmatrix} 0 & 1 & 0 & 0 \\ 0 & 0 & 0 & 0 \end{bmatrix} \begin{bmatrix} y \\ \dot{y} \\ \theta \\ \dot{\theta} \end{bmatrix} \tag{14}$$

2. Design a feedback law $u - F_1 x$ so that $A + BF_1$ has eignevalues as $-3 \pm j3, -6, -8$. Build a simulink model for the closed loop linear system. Plot the response under initial condition $x(0) = \begin{bmatrix} -1.5 \\ 0 \\ 1 \\ 3 \end{bmatrix}$.

3. Build a simulink model for the original nonlinear system, verify that stabilization is achieved by $u = F_1 x$ when $x(0)$ is close to the origin. Find the maximal $\theta_0$ so that nonlinear system can be stabalized from $x_0 = \begin{bmatrix} 0 \\ 0 \\ \theta_0 \\ 0 \end{bmatrix}$

4. For $x(0) = \begin{bmatrix} 0 \\ 0 \\ \frac{\pi}{5} \\ 0 \end{bmatrix}$, compare the response $y(t)$ and $\theta(t)$ for the linearized system and the nonlinear system under the same feedback $u - F_1 x$

Final Answer: