

Take Home Final

Nathan Rose

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1 (30pts)

A High performance helicopter has a model shown in Figure 1. The goal is to control the pitch angle θ of the helicopter by adjusting the rotor thrust δ . The equations of motion of the helicopter are

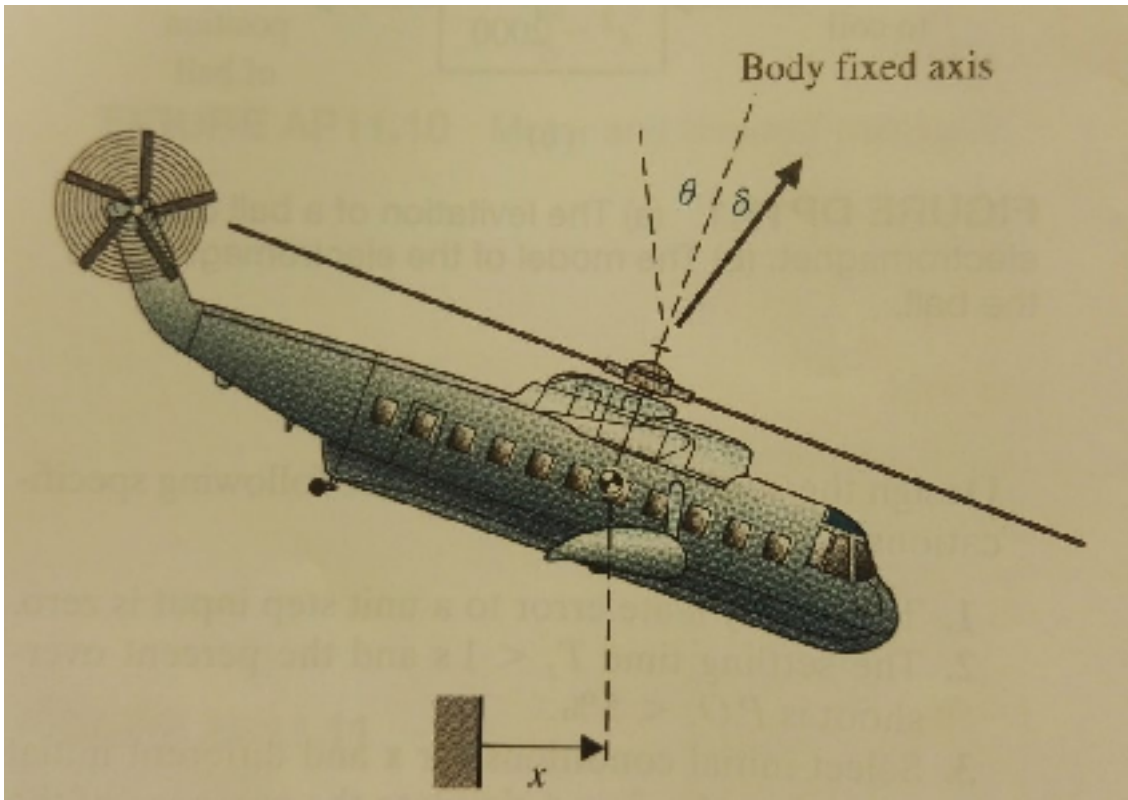


Figure 1:

$$\frac{d^2\theta}{dt^2} = -\sigma_1 \frac{d\theta}{dt} - \alpha_1 \frac{dx}{dt} + n\delta \quad (1)$$

$$\frac{d^2x}{dt^2} = g\theta - \alpha_2 \frac{d\theta}{dt} - \sigma_2 \frac{dx}{dt} + g\delta \quad (2)$$

Where x is the translation in the horizontal direction. For a military high-preformance helicopter we find: $\sigma_1 = 0.415$, $\sigma_2 = 0.0198$, $\alpha_1 = 0.0111$, $\alpha_2 = 1.43$, $n = 6.27$, $g = 9.8$ all in appropriate SI units. Find:

- (a) A state variable representation of this system
- (b) The transfer function representation for $\frac{\theta(s)}{\delta(s)}$
- (c) Use state variable feedback to achieve adequate performances for the controlled system. Desired specifications include:
 - (1) A steady-state for an input step command for $\theta_d(s)$, the desired pitch angle, less than 20% of the input step magnitude
 - (2) An overshoot for a step input command is less than 20%
 - (3) a settling (with a 2% criterion) time for a step command of less than 1.5 seconds
- (d) If the state variable is not available, design the observer and control law to meet the design specifications included in part (c)

work:

- (a) A state variable representation of this system
For the system with state variables:

$$x = \begin{bmatrix} x \\ \dot{x} \\ \theta \\ \dot{\theta} \end{bmatrix} \quad (3)$$

The state space representation is:

$$\dot{x} = \begin{bmatrix} 0 & 1 & 0 & 0 \\ 0 & -\sigma_2 & g & -\alpha_1 \\ 0 & 0 & 0 & 1 \\ 0 & -\sigma_1 & 0 & -\alpha_1 \end{bmatrix} x + \begin{bmatrix} 0 \\ g \\ 0 \\ n \end{bmatrix} u \quad (4)$$

$$y = \begin{bmatrix} 1 & 0 & 0 & 0 \\ 0 & 0 & 1 & 0 \end{bmatrix} x + \begin{bmatrix} 0 \\ 0 \end{bmatrix} u \quad (5)$$

- (b) The transfer function representation for $\frac{\theta(s)}{\delta(s)}$
For the system described by:

$$\dot{x} = \begin{bmatrix} 0.0 & 1.0 & 0.0 & 0.0 \\ 0.0 & -0.02 & 9.8 & -1.43 \\ 0.0 & 0.0 & 0.0 & 1.0 \\ 0.0 & -0.415 & 0.0 & -0.011 \end{bmatrix} x + \begin{bmatrix} 0.0 \\ 9.8 \\ 0.0 \\ 6.27 \end{bmatrix} u \quad (6)$$

$$y = \begin{bmatrix} 1 & 0 & 0 & 0 \\ 0 & 0 & 1 & 0 \end{bmatrix} x + \begin{bmatrix} 0 \\ 0 \end{bmatrix} u \quad (7)$$

A realization was determined by taking the realization of the state space representation. The result of this is:

$$g(s) = \left[\frac{9.8s^2 - 8.857s + 61.446}{1.0s^4 + 0.031s^3 - 0.593s^2 + 4.067s} \right] \quad (8)$$

- (c) Use state variable feedback to achieve adequate performances for the controlled system. Desired specifications include:

- (1) A steady-state for an input step command for $\theta_d(s)$, the desired pitch angle, less than 20% of the input step magnitude
- (2) An overshoot for a step input command is less than 20%
- (3) a settling (with a 2% criterion) time for a step command of less than 1.5 seconds
- (d) If the state variable is not available, design the observer and control law to meet the design specifications included in part (c)

Final Answer:

2 (30pts)

The open loop system

$$\dot{x} = \begin{bmatrix} 0 & 1 & 0 \\ 0 & 0 & 1 \\ -1 & 4 & -3 \end{bmatrix} x + \begin{bmatrix} 0 \\ 0 \\ 2 \end{bmatrix} u \quad (9)$$

$$y = \begin{bmatrix} 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix} x \quad (10)$$

$$x(0) = \begin{bmatrix} 1 \\ -1 \\ 1 \end{bmatrix} \quad (11)$$

- 1) Assume that x is available for state feedback. Design an LQR control law by letting $R = 1$ and choosing Q so that all the elements of the feedback gain K have absolute value less than 50.

Requirement: $|y_1(t)| \leq 0.05, |y_2(t)| \leq 0.05$, for all $t > 5$. Plot $y_1(t)$ and $y_2(t)$ in the same figure for $t \in [0, 20]$

- 2) Assume that only the output y is available. Design an observer so that the poles of the observer are $-5 \pm j5, -10$. Choose the observer gain so that all the elements have absolute value less than 80. Form a closed loop system along with the LQR controller in step 1). Plot $y_1(t)$ and $y_2(t)$ in the same figure for $t \in [0, 20]$

work:

- 1) Assume that x is available for state feedback. Design an LQR control law by letting $R = 1$ and choosing Q so that all the elements of the feedback gain K have absolute value less than 50.

Requirement: $|y_1(t)| \leq 0.05, |y_2(t)| \leq 0.05$, for all $t > 5$. Plot $y_1(t)$ and $y_2(t)$ in the same figure for $t \in [0, 20]$

For the system described by:

$$\dot{x} = \begin{bmatrix} 0 & 1 & 0 \\ 0 & 0 & 1 \\ -1 & 4 & 5 \end{bmatrix} x + \begin{bmatrix} 0 \\ 0 \\ 2 \end{bmatrix} u \quad (12)$$

$$y = \begin{bmatrix} 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix} x \quad (13)$$

and $Q = \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix}, R = [1]$

The following is the outputs of the LQR system assuming the inputs are 0

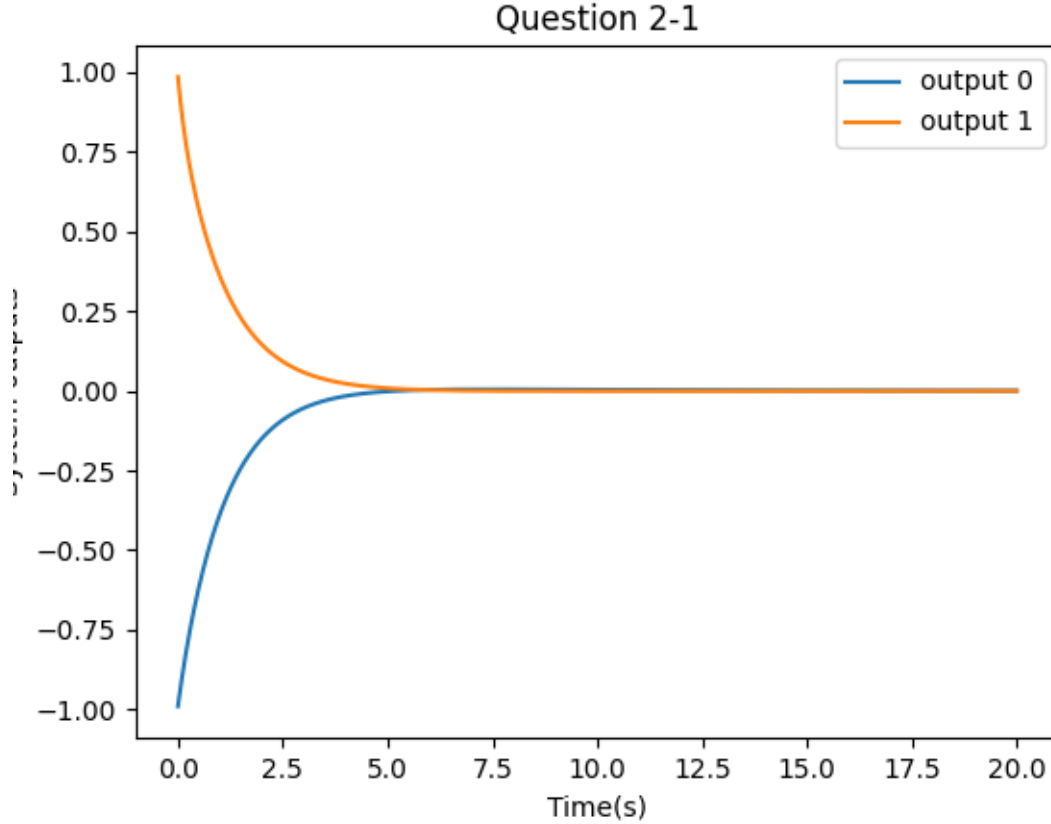


Figure 2: LQR system

- 2) Assume that only the output y is available. Design an observer so that the poles of the observer are $-5 \pm j5, -10$. Choose the observer gain so that all the elements have absolute value less than 80. Form a closed loop system along with the LQR controller in step 1). Plot $y_1(t)$ and $y_2(t)$ in the same figure for $t \in [0, 20]$ For the system described by:

$$\dot{x} = \begin{bmatrix} 0 & 1 & 0 \\ 0 & 0 & 1 \\ -1 & 4 & 5 \end{bmatrix} x + \begin{bmatrix} 0 \\ 0 \\ 2 \end{bmatrix} u \quad (14)$$

$$y = \begin{bmatrix} 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix} x + \begin{bmatrix} 0 \\ 0 \end{bmatrix} u \quad (15)$$

The following variables were chosen for the observer:

$$L_0 \begin{bmatrix} 0.0 & 1.0 \\ 0.5 & 0.0 \\ 0.1 & 0.25 \end{bmatrix}$$

$$F \begin{bmatrix} -5.0 & 5.0 & 0.0 \\ -5.0 & -5.0 & 0.0 \\ 0.0 & 0.0 & -5.0 \end{bmatrix}$$

These were used to calculate:

$$T \begin{bmatrix} 0.003 & 0.04 & 0.073 \\ -0.012 & 0.099 & -0.046 \\ 0.005 & -0.001 & 0.025 \end{bmatrix}$$

$$L \begin{bmatrix} 55.622 & -6.386 \\ 8.567 & 4.575 \\ -6.769 & 11.433 \end{bmatrix}$$

The following is the outputs of the LQR system assuming the inputs are 0 and an initial estimate of state of $\begin{bmatrix} 1.0 \\ -1.0 \\ 1.0 \end{bmatrix}$

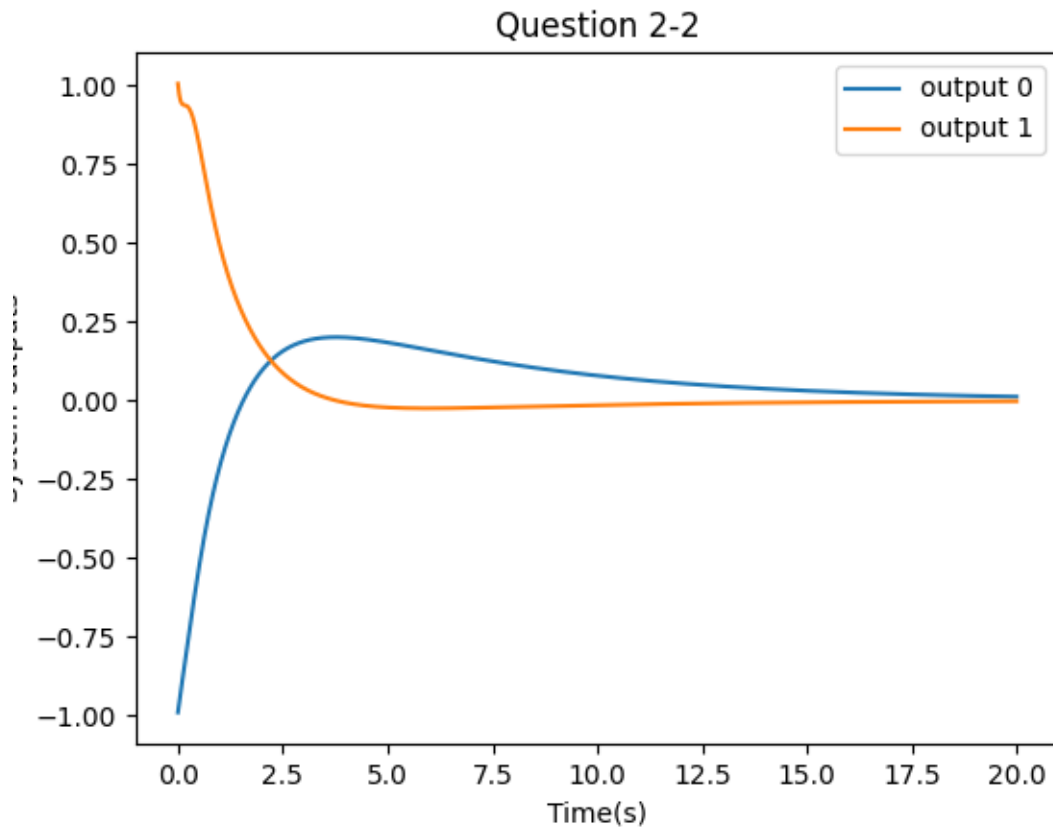


Figure 3: LQR system

Final Answer:

3 (40pts)

A cart with an inverted pendulum as seen in Figure 4

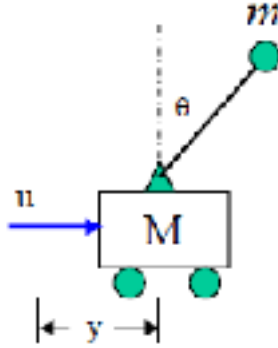


Figure 4:

u	control input(Newtons)
y	displacement of the cart(meters)
θ	angle of the pendulum(radians)

$$x = \begin{bmatrix} y \\ \dot{y} \\ \theta \\ \dot{\theta} \end{bmatrix} \quad (16)$$

The control problems are

- 1: Stabilization: Design a feedback law $u = Fx$ such that $x(t) \rightarrow 0$ for $x(0)$ close to the diagram
- 2: For $x(0) = (0, 0, -\pi, 0)$, apply an impulse force $u(t) = u_{max}$ for $t \in [0, 0.1]$ to bring θ to a certain range and then switch to the linear controller so that $x(t) \rightarrow 0$.

Assume that there is no friction or damping. The nonlinear model is as follows.

$$\begin{bmatrix} M + m & ml \cos(\theta) \\ \cos(\theta) & l \end{bmatrix} \begin{bmatrix} \ddot{y} \\ \ddot{\theta} \end{bmatrix} = \begin{bmatrix} u + ml\dot{\theta}^2 \sin \theta \\ g \sin \theta \end{bmatrix} \quad (17)$$

with

$m = 1kg$	mass of the pendulum
$l = 0.2m$	length of the pendulum
$M = 5kg$	mass of the cart
$g = 9.8 \frac{m}{s^2}$	mass of the cart

Linearize the system at $x = 0$

$$\begin{bmatrix} M + m & ml \cos(\theta) \\ \cos(\theta) & l \end{bmatrix} \begin{bmatrix} \ddot{y} \\ \ddot{\theta} \end{bmatrix} = \begin{bmatrix} u \\ g\theta \end{bmatrix} \quad (18)$$

the state space description for the linearized system.

$$\dot{x} = Ax + Bu \quad (19)$$

Problems:

1. Find matrices A , B for the state space equation.
2. Design a feedback law $u - F_1x$ so that $A + BF_1$ has eigenvalues as $-3 \pm j3, -6, -8$. Build a simulink model for the closed loop linear system. Plot the response under initial condition $x(0) = (-1.5, 0, 1, 3)$.
3. Build a simulink model for the original nonlinear system, verify that stabilization is achieved by $u = F_1x$ when $x(0)$ is close to the origin. Find the maximal θ_0 so that nonlinear system can be stabilized from $x_0 = (0, 0, \theta_0, 0)$
4. For $x(0) = (0, 0, \frac{\pi}{5}, 0)$, compare the response $y(t)$ and $\theta(t)$ for the linearized system and the nonlinear system under the same feedback $u - F_1x$

work:

1. Find matrices A , B for the state space equation.

$$\dot{x} = \begin{bmatrix} \dot{y} \\ \ddot{y} \\ \dot{\theta} \\ \ddot{\theta} \end{bmatrix} = \begin{bmatrix} 0 & 1 & 0 & 0 \\ 0 & 0 & 0 & 0 \end{bmatrix} \begin{bmatrix} y \\ \dot{y} \\ \theta \\ \dot{\theta} \end{bmatrix} \quad (20)$$

2. Design a feedback law $u - F_1x$ so that $A + BF_1$ has eigenvalues as $-3 \pm j3, -6, -8$. Build a simulink model for the closed loop linear system. Plot the response under initial condition

$$x(0) = \begin{bmatrix} -1.5 \\ 0 \\ 1 \\ 3 \end{bmatrix}.$$

3. Build a simulink model for the original nonlinear system, verify that stabilization is achieved by $u = F_1x$ when $x(0)$ is close to the origin. Find the maximal θ_0 so that nonlinear system

$$\text{can be stabilized from } x_0 = \begin{bmatrix} 0 \\ 0 \\ \theta_0 \\ 0 \end{bmatrix}$$

4. For $x(0) = \begin{bmatrix} 0 \\ 0 \\ \frac{\pi}{5} \\ 0 \end{bmatrix}$, compare the response $y(t)$ and $\theta(t)$ for the linearized system and the nonlinear system under the same feedback $u - F_1x$

Final Answer: