POLYTECH
NICE-SOPHIA

Algorithms & Data Structures

Lesson 2: Math Review, Algorithm Analysis

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Logarithms and Exponents

- Definition: $x = 2^y$ if $\log_2 x = y$
  - $8 = 2^3$, so $\log_2 8 = 3$
  - $65536 = 2^{16}$, so $\log_2 65536 = 16$

- The exponent of a number says how many times to use the number in a multiplication. e.g. $2^3 = 2 \times 2 \times 2 = 8$

  (2 is used 3 times in a multiplication to get 8)

- A logarithm says how many of one number to multiply to get another number. It asks "what exponent produced this?”

  e.g. $\log_2 8 = 3$ *(2 makes 8 when used 3 times in a multiplication)*

Logarithms and Exponents

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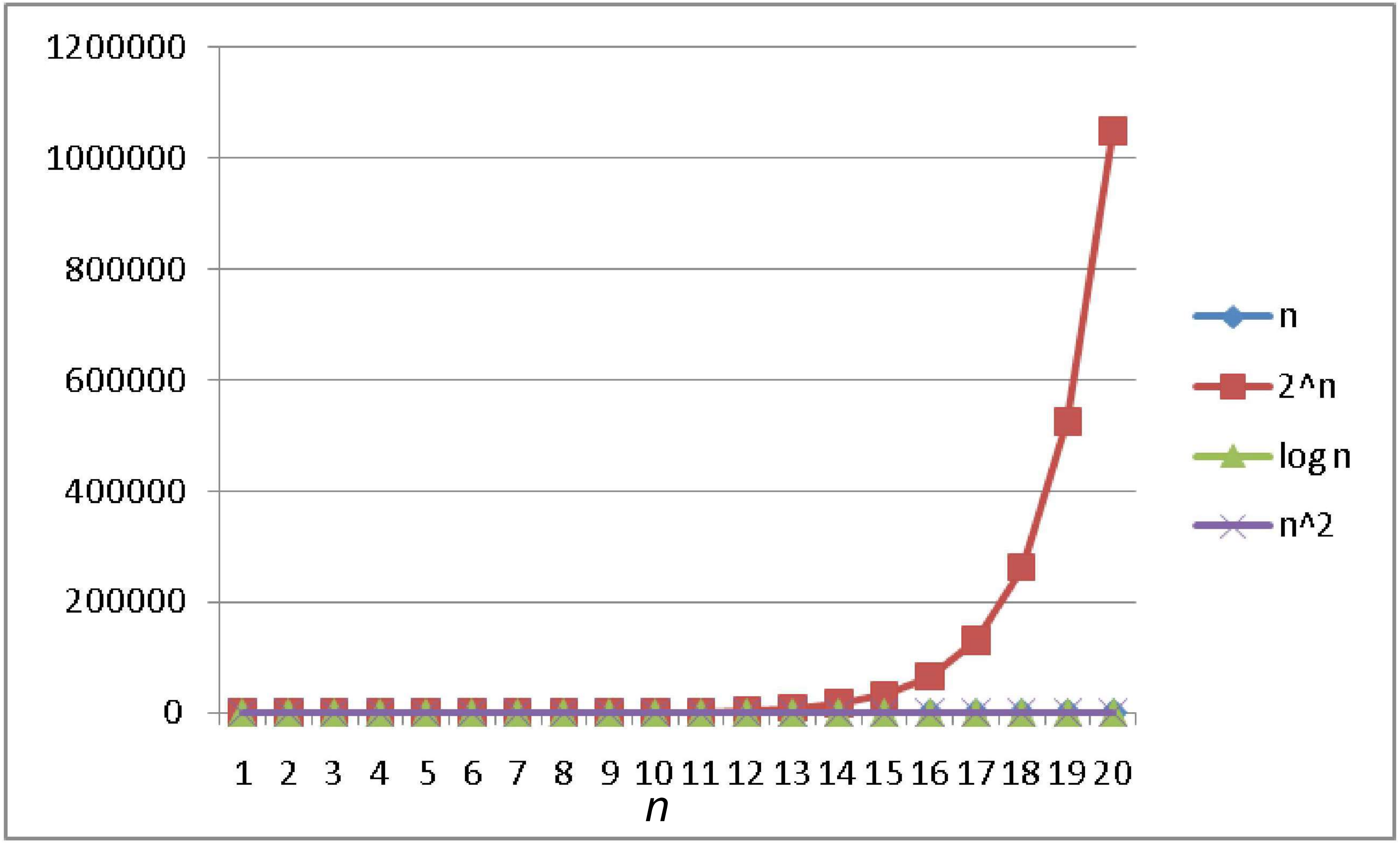
- Since so much is binary in CS, **log** almost always means $\log_2$
- $\log_2 n$ tells you how many bits needed to represent n combinations.

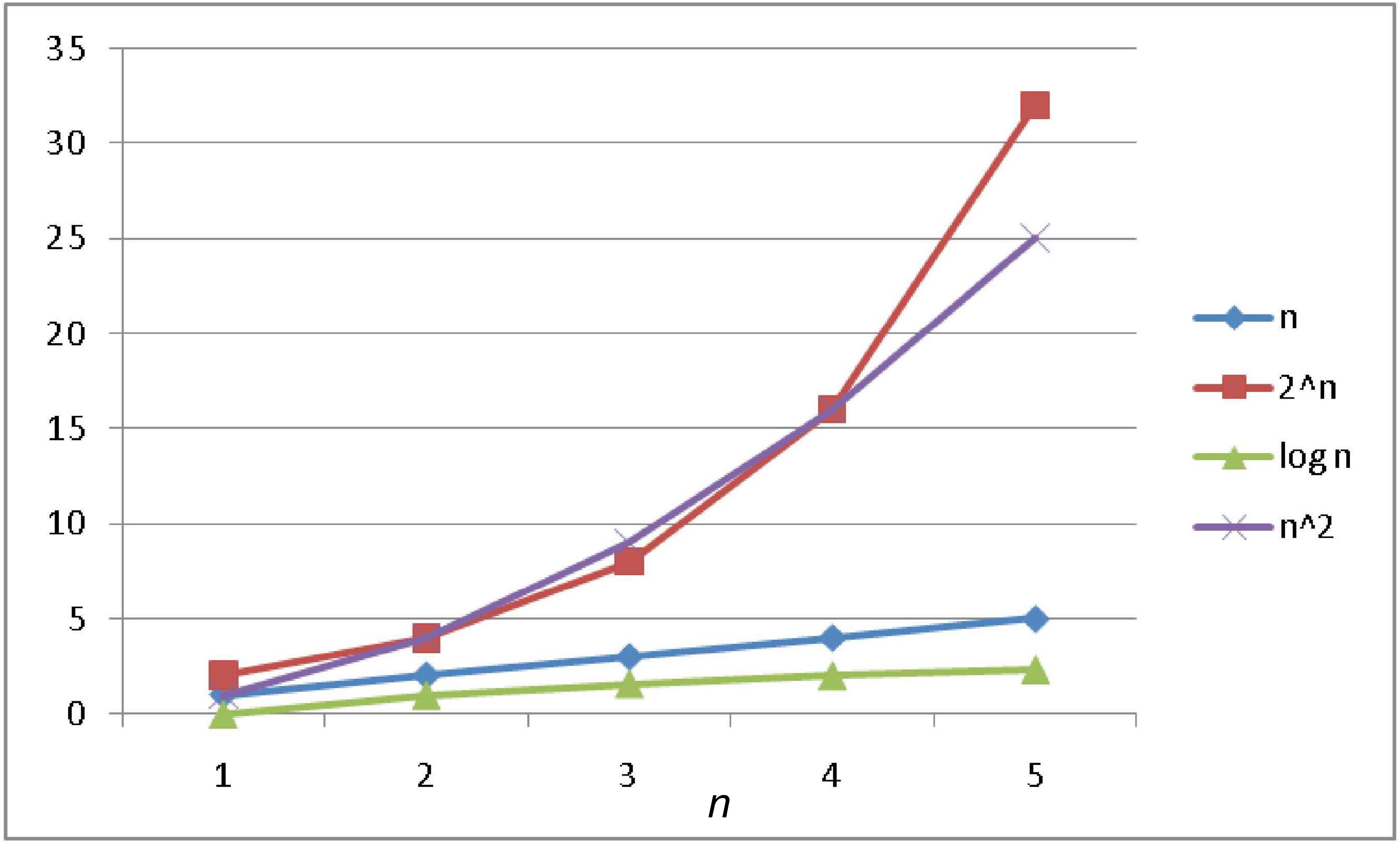
- So, $\log_2 1{,}000{,}000$ = "a little under 20"

- Logarithms and exponents are inverse functions. Just as exponents grow *very* quickly, logarithms grow *very* slowly.

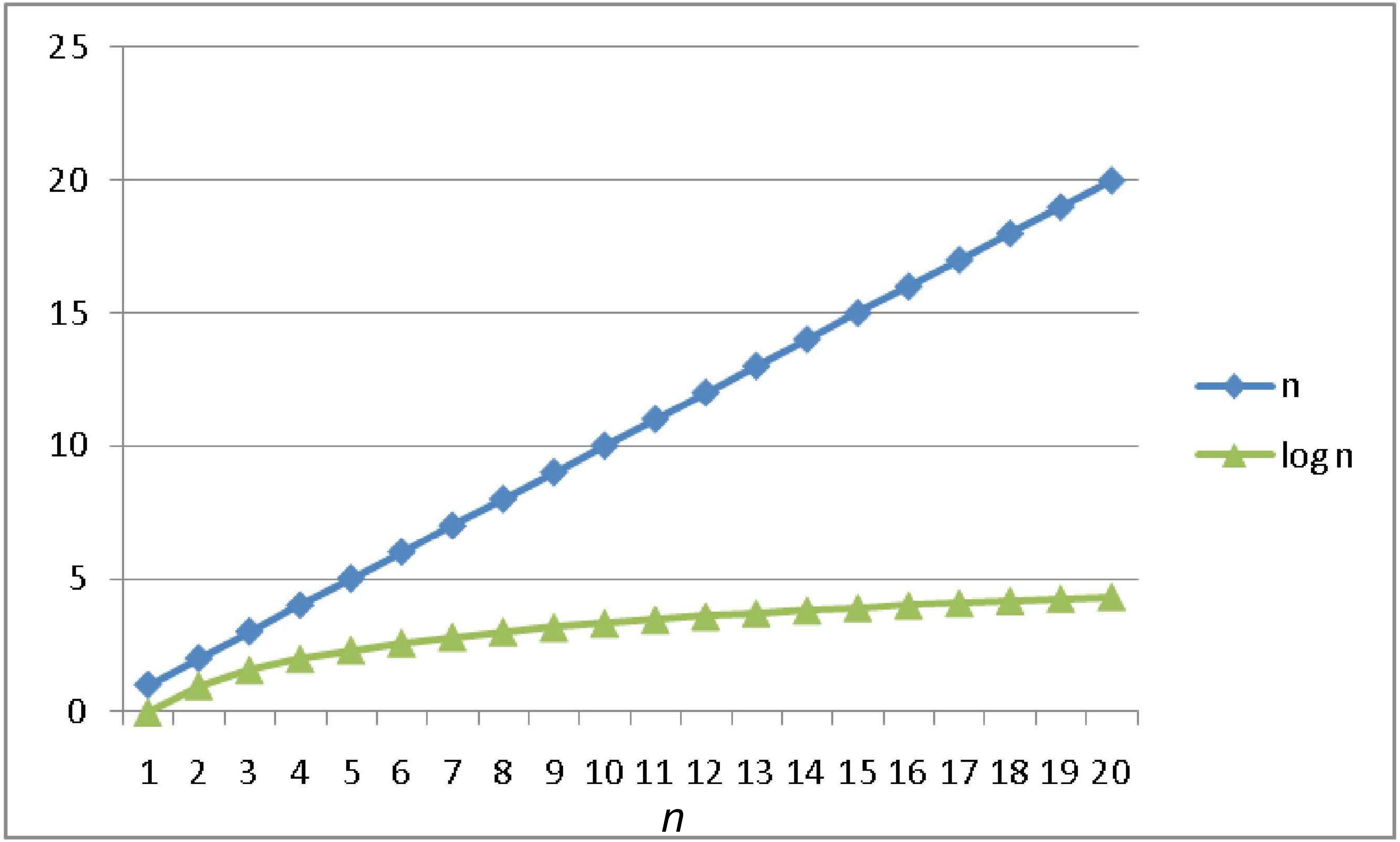
Logarithms and Exponents



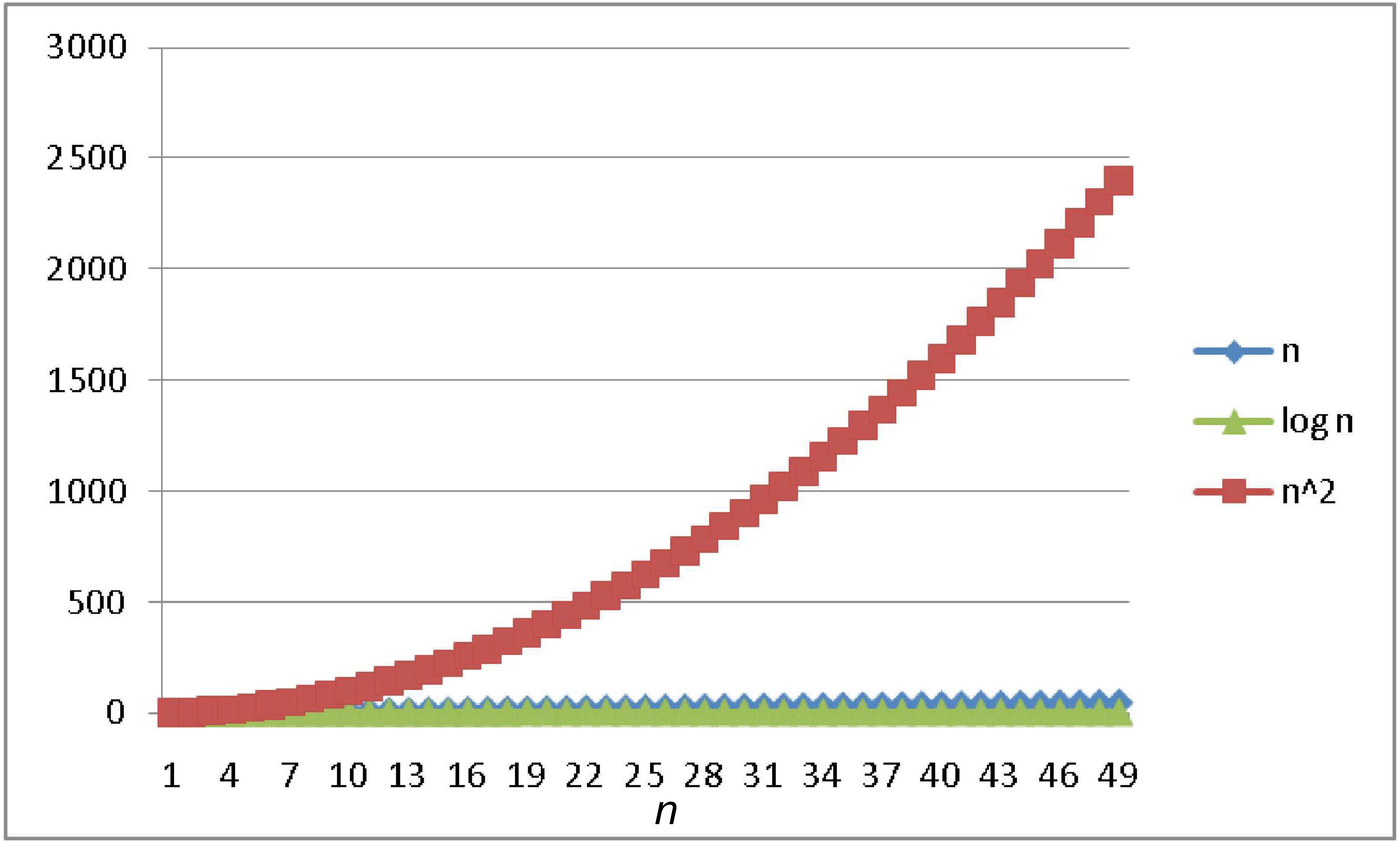
Logarithms and Exponents



Logarithms and Exponents



Logarithms and Exponents



Properties of logarithms

- `log(A*B) = log A + log B`
- `log(N`^{k}`)= k log N`
- `log(A/B) = log A – log B`
- `log(log x)` is written `log log x`
  - Grows as slowly as 2^{2^y} grows quickly
- `(log x)(log x)` is written $\log^2 x$
  - It is greater than `log x` for all `x > 2`
  - It is not the same as `log log x`

Algorithm Analysis

As the "size" of an algorithm's input grows (integer, length of array, size of queue, etc.), we want to know

- How much longer does the algorithm take to run? (time)
- How much more memory does the algorithm need? (space)

Because the curves we saw are so different, often care about only "which curve we are like"

Separate issue: Algorithm *correctness* – does it produce the right answer for all inputs

- Usually more important, naturally

Algorithm Analysis: A first example

- Consider the following program segment:

```
x:= 0;
for i = 1 to n do
    for j = 1 to i do
    x := x + 1;
```

- What is the value of x at the end?

i	**j**	**x**
1	1 to 1	1
2	1 to 2	3
3	1 to 3	6
4	1 to 4	10
…		
n	1 to n	?

Number of times x gets incremented by 1 is

$= 1 + 2 + 3 + \ldots + (n-1) + n$

$= n*(n+1)/2$

Analyzing the loop

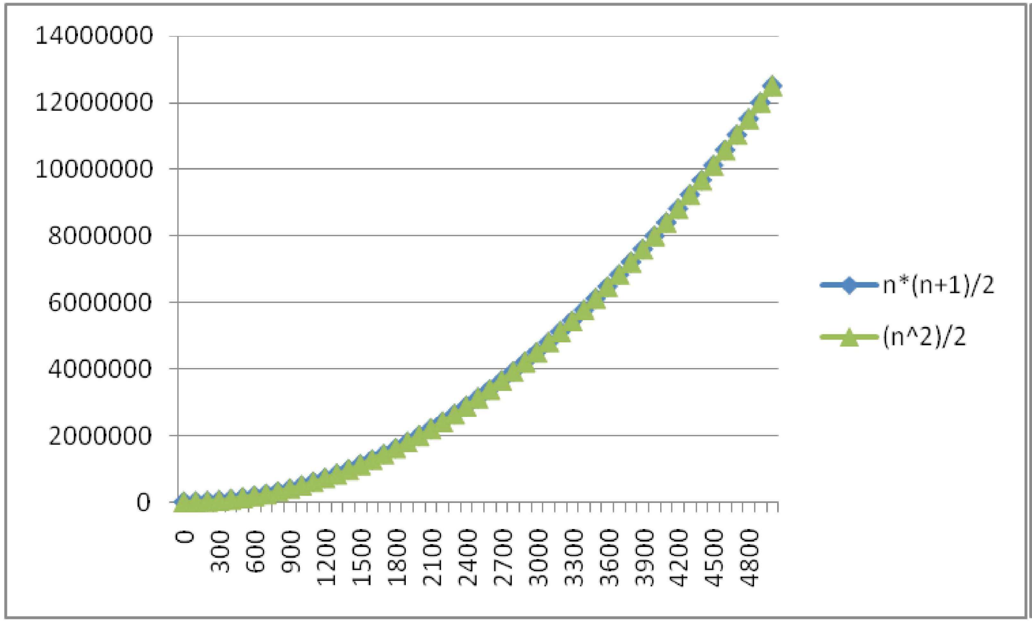
- Consider the following program segment:

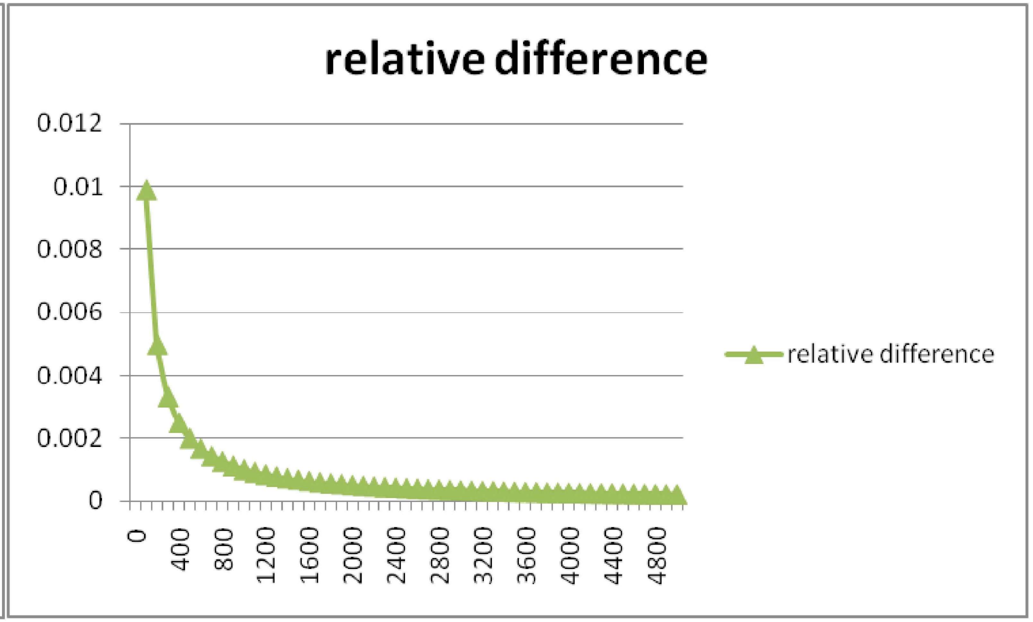
```
x := 0;
for i = 1 to n do
    for j = 1 to i do
    x := x + 1;
```

- The total number of loop iterations is n*(n+1)/2
  - This is a very common loop structure, worth memorizing
  - This is *proportional to* n^2 , and we say $O(n^2)$, “big-Oh of”
    - $n*(n+1)/2 = (n^2+ n)/2$
    - For large enough n, the lower order and constant terms are irrelevant, as are the assignment statements
    - See plot… $(n^2+ n)/2$ vs. just $n^2/2$

Lower-order terms don't matter

$n*(n+1)/2$ vs. just $n^2/2$





We just say $O(n^2)$

Big-O: Common Names

$O(1)$	constant (same as $O(k)$ for constant k)
$O(\mathtt{log}\ n)$	logarithmic
$O(n)$	linear
$O(n\ \mathtt{log}\ n)$	"n $\mathtt{log}$ n"
$O(n^2)$	quadratic
$O(n^3)$	cubic
$O(n^k)$	polynomial (where is k is any constant)
$O(k^n)$	exponential (where k is any constant > 1)
$O(n!)$	factorial

Note: "exponential" does not mean "grows really fast", it means "grows at rate proportional to k^n for some k>1"

Big-O running times

- For a processor capable of one million instructions per second

	n	$n \log_2 n$	n^2	n^3	1.5^n	2^n	$n!$
$n = 10$	< 1 sec	< 1 sec	< 1 sec	< 1 sec	< 1 sec	< 1 sec	4 sec
$n = 30$	< 1 sec	< 1 sec	< 1 sec	< 1 sec	< 1 sec	18 min	10^{25} years
$n = 50$	< 1 sec	< 1 sec	< 1 sec	< 1 sec	11 min	36 years	very long
$n = 100$	< 1 sec	< 1 sec	< 1 sec	1 sec	12,892 years	10^{17} years	very long
$n = 1{,}000$	< 1 sec	< 1 sec	1 sec	18 min	very long	very long	very long
$n = 10{,}000$	< 1 sec	< 1 sec	2 min	12 days	very long	very long	very long
$n = 100{,}000$	< 1 sec	2 sec	3 hours	32 years	very long	very long	very long
$n = 1{,}000{,}000$	1 sec	20 sec	12 days	31,710 years	very long	very long	very long

Analyzing code

Basic operations take "some amount of" constant time

- Arithmetic (fixed-width)
- Assignment
- Access one Java field **or array index**
- Etc.

(This is an *approximation of reality*: a very useful "lie".)

Consecutive statements	Sum of times
Conditionals	Time of test plus slower branch
Loops	Sum of iterations
Calls	Time of call's body
Recursion	Solve *recurrence equation* *(next lecture)*

Analyzing code

1. Add up time for all parts of the algorithm
   e.g. number of iterations = $(n^2 + n)/2$
2. Eliminate low-order terms i.e. eliminate n: $(n^2)/2$
3. Eliminate coefficients i.e. eliminate 1/2: (n^2)

Examples:

– $4n + 5$ $= O(n)$

– $0.5n \log n + 2n + 7$ $= O(n \log n)$

– $n^3 + 2^n + 3n$ $= O(2^n)$

– $n \log (10n^2) =$

$n\log(10) + n \log(n^2) =$

$n\log(10) + 2n \log(n) = O(n \log n)$