Algorithms & Data Structures

Lesson 14: Disjoint Sets & Union-Find

Marc Gaetano

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The plan

- What are *disjoint sets*
  - And how are they "the same thing" as *equivalence relations*

- The union-find ADT for disjoint sets

- Applications of union-find

- Basic implementation of the ADT with "up trees"

- Optimizations that make the implementation much faster

Disjoint sets

- A set is a collection of elements (no-repeats)

- Two sets are disjoint if they have no elements in common
  - $\mathbf{S_1 \cap S_2 = \varnothing}$

- Example: {a, e, c} and {d, b} are disjoint

- Example: {x, y, z} and {t, u, x} are not disjoint

Partitions

A partition P of a set S is a set of sets $\{S_1,S_2,\ldots,S_n\}$ such that every element of S is in **exactly one** S_i

Put another way:

- $S_1 \cup S_2 \cup \ldots \cup S_k = S$
- For all i and j, $i \neq j$ implies $S_i \cap S_j = \emptyset$

Example:

- Let S be {a,b,c,d,e}
- One partition: {a}, {d,e}, {b,c}
- Another partition: {a,b,c}, $\emptyset$, {d}, {e}
- A third: {a,b,c,d,e}
- Not a partition: {a,b,d}, {c,d,e}
- Not a partition of S: {a,b}, {e,c}

Binary relations

- *S x S* is the set of all pairs of elements of *S*
  - Example: If *S* = {a,b,c}
    then *S x S* = {(a,a),(a,b),(a,c),(b,a),(b,b),(b,c),(c,a),(c,b),(c,c)}
- A binary relation *R* on a set *S* is any subset of *S x S*
  - Write *R*(x,y) to mean (x,y) is "in the relation"
  - (Unary, ternary, quaternary, … relations defined similarly)
- Examples of binary relations for *S* = people-in-this-room
  - Sitting-next-to-each-other relation
  - First-sitting-right-of-second relation
  - Went-to-same-high-school relation
  - Same-gender-relation
  - First-is-younger-than-second relation

Properties of binary relations

- A binary relation *R* over set *S* is reflexive if:

  R*(a,a) for *all* a in *S

- A binary relation *R* over set *S* is symmetric if:

  R*(a,b) if and only if *R*(b,a) for *all* a,b in *S

- A binary relation *R* over set *S* is transitive if:

  If *R*(a,b) and *R*(b,c) then *R*(a,c) for *all* a,b,c in *S*

- Examples for *S* = people-in-this-room
  - Sitting-next-to-each-other relation
  - First-sitting-right-of-second relation
  - Went-to-same-high-school relation
  - Same-gender-relation
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Equivalence relations

- A binary relation *R* is an equivalence relation if *R* is reflexive, symmetric, and transitive

- Examples
  - Same gender
  - Connected roads in the world
  - *Graduated* from same high school?
  - …

Punch-line

- Every partition induces an *equivalence relation*
- Every equivalence relation *induces* a partition

- Suppose $P=\{S_1,S_2,\ldots,S_n\}$ be a partition
  – Define $R(x,y)$ to mean x and y are in the same S_i
    - R is an equivalence relation

- Suppose R is an equivalence relation over S
  – Consider a set of sets $S_1,S_2,\ldots,S_n$ where

    (1) x and y are in the same S_i if and only if $R(x,y)$

    (2) Every x is in some S_i

    - This set of sets is a partition

Example

- Let S be {a,b,c,d,e}
- One partition: {a,b,c}, {d}, {e}
- The corresponding equivalence relation:

  (a,a), (b,b), (c,c), (a,b), (b,a), (a,c), (c,a), (b,c), (c,b), (d,d), (e,e)

The operations

- Given an unchanging set *S*, `create` an initial partition of a set
  - Typically each item in its own subset: {a}, {b}, {c}, …
  - Give each subset a “name” by choosing a *representative element*

- Operation `find` takes an element of *S* and returns the representative element of the subset it is in

- Operation `union` takes two subsets and (permanently) makes one larger subset
  - A different partition with one fewer set
  - Affects result of subsequent `find` operations
  - Choice of representative element up to implementation

Example

- Let S = {1,2,3,4,5,6,7,8,9}
- Let initial partition be (will highlight representative elements red)

  {1}, {2}, {3}, {4}, {5}, {6}, {7}, {8}, {9}
- **`union`**(2,5):

  {1}, {2, 5}, {3}, {4}, {6}, {7}, {8}, {9}
- **`find`**(4) = 4, **`find`**(2) = 2, **`find`**(5) = 2
- **`union`**(4,6), **`union`**(2,7)

  {1}, {2, 5, 7}, {3}, {4, 6}, {8}, {9}
- **`find`**(4) = 6, **`find`**(2) = 2, **`find`**(5) = 2
- **`union`**(2,6)

  {1}, {2, 4, 5, 6, 7}, {3}, {8}, {9}

No other operations

- All that can "happen" is sets get unioned
  - No "un-union" or "create new set" or …

- As always: trade-offs – implementations will exploit this small ADT

- Surprisingly useful ADT: list of applications after one example
  - But not as common as dictionaries or priority queues

Applications

- Maze-building is:
  - A surprising use of the union-find ADT
- Many other uses:
  - Road/network/graph connectivity (will see this again)
    - "connected components" e.g., in social network
  - Partition an image by connected-pixels-of-similar-color
  - Type inference in programming languages
- Not as common as dictionaries, queues, and stacks, but valuable because implementations are very fast, so when applicable can provide big improvements

Basic implementation

- Start with an initial partition of *n* subsets
  - Often 1-element sets, e.g., {1}, {2}, {3}, …, {*n*}

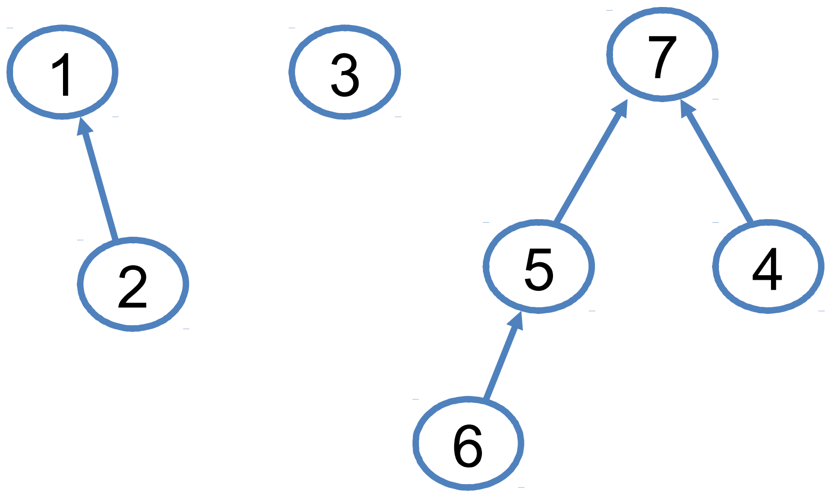
- May have *m* **`find`** operations and up to *n*-1 **`union`** operations in any order
  - After *n*-1 **`union`** operations, every **`find`** returns same 1 set

- If total for all these operations is $O(m+n)$, then amortized $O(1)$
  - We will get very, very close to this
  - $O(1)$ worst-case is impossible for **`find`** *and* **`union`**
    - Trivial for one *or* the other

Up-tree data structure

- Tree with:
  - No limit on branching factor
  - References from children to parent
- Start with *forest* of 1-node trees

- Possible forest after several unions:
  - Will use roots for set names

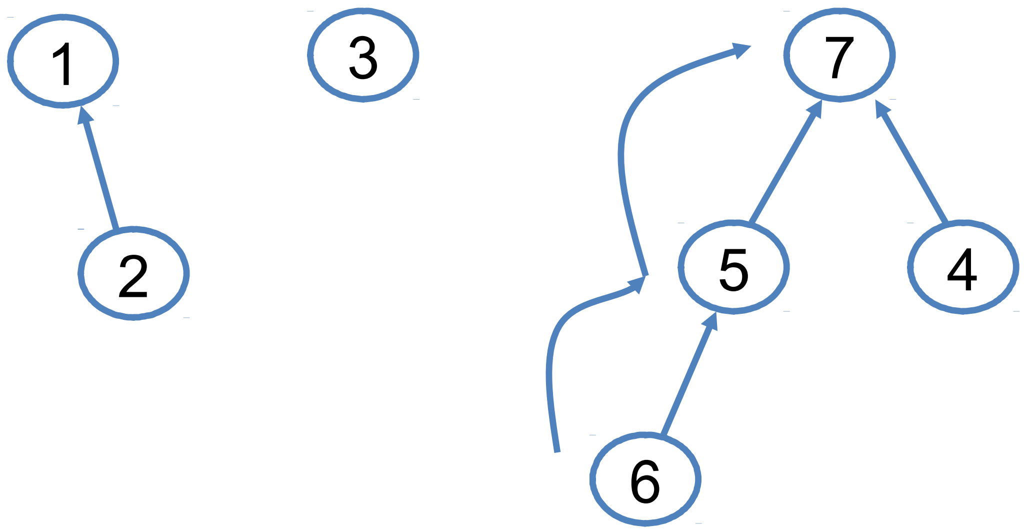


Find

`find`(**`x`**):

- *Assume* we have *O*(1) access to each node
  - Will use an array where index **`i`** holds node **`i`**
- Start at **`x`** and follow parent pointers to root
- Return the root

`find`(6) = 7

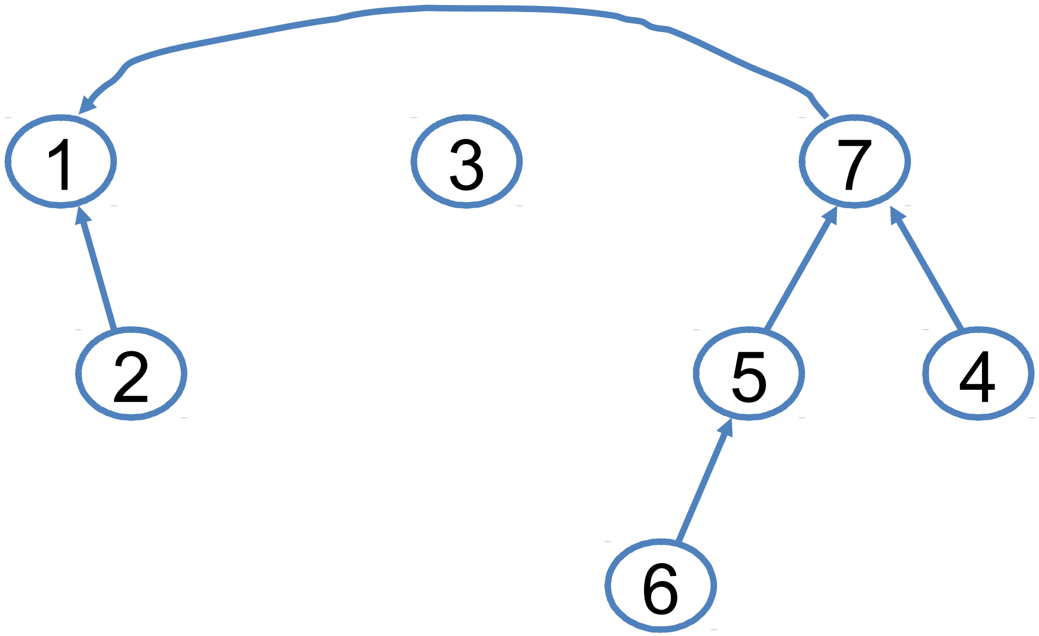


Union

`union(x,y)`:

- Assume **`x`** and **`y`** are roots
  - If they are not, just find the roots of their trees
- Assume distinct trees (else do nothing)
- Change root of one to have parent be the root of the other
  - Notice no limit on branching factor

`union`(1,7)

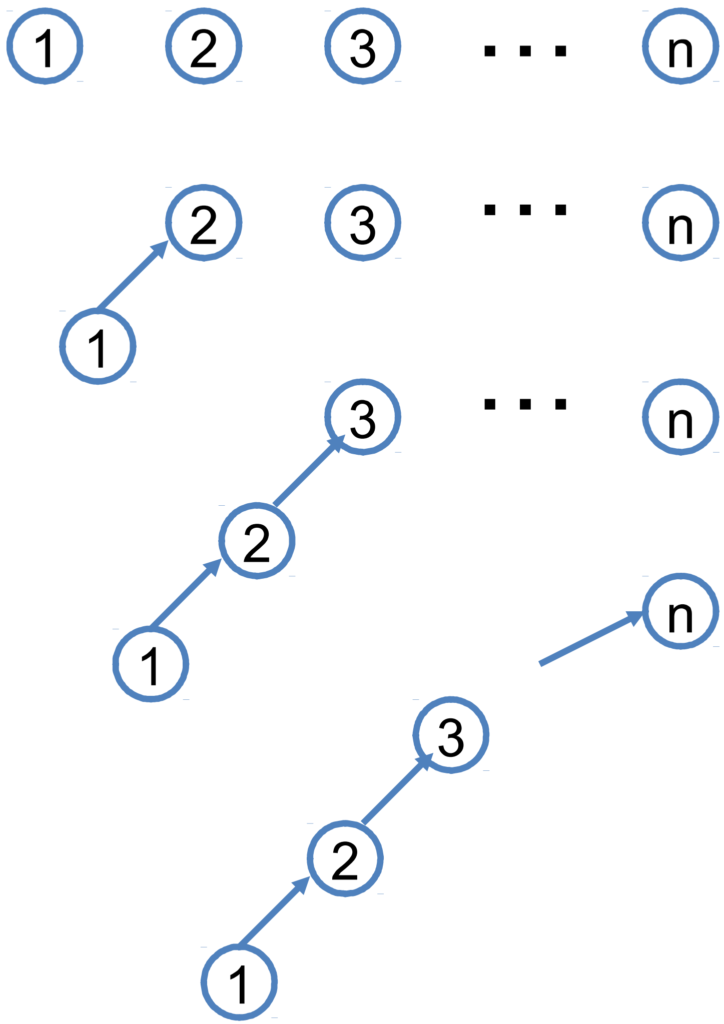


Two key optimizations

1. Improve **`union`** so it stays *O(1)* but makes **`find`** $O(\texttt{log}\ n)$
   - So m **`finds`** and n-1 **`unions`** is $O(m\ \texttt{log}\ n + n)$
   - *Union-by-size:* connect smaller tree to larger tree

2. Improve **`find`** so it becomes even faster
   - Make m **`finds`** and n-1 **`unions`** ***almost*** $O(m + n)$
   - *Path-compression:* connect directly to root during finds

The bad case to avoid



`union`(2,1)

`union`(3,2)

:

.

`union`(n,n-1)

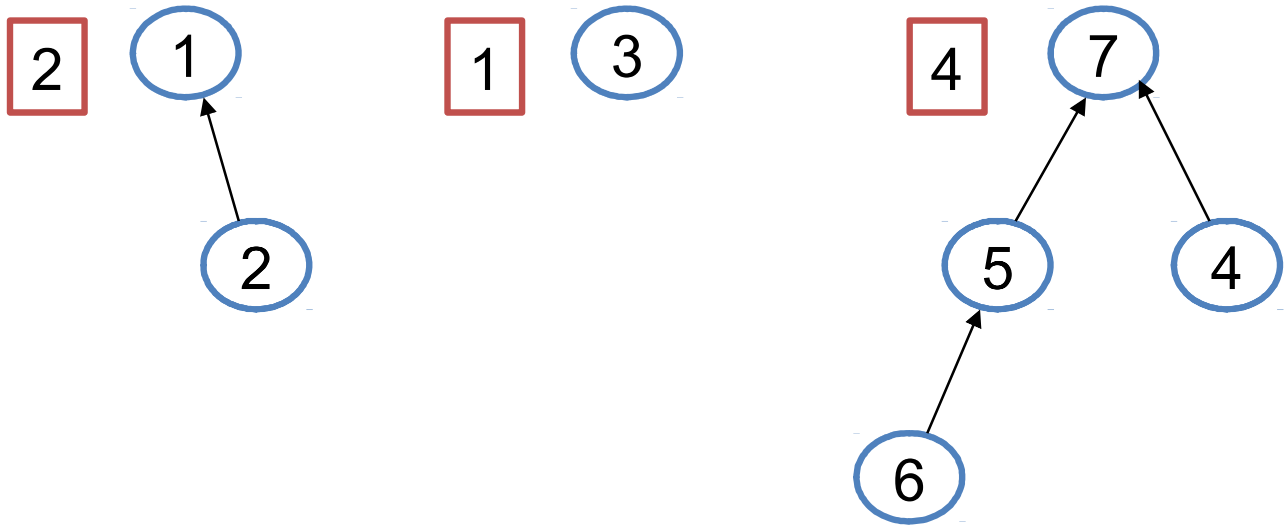
`find`(1)   *n* steps!!

Weighted union

Weighted union:

– Always point the *smaller* (total # of nodes) tree to the root of the larger tree

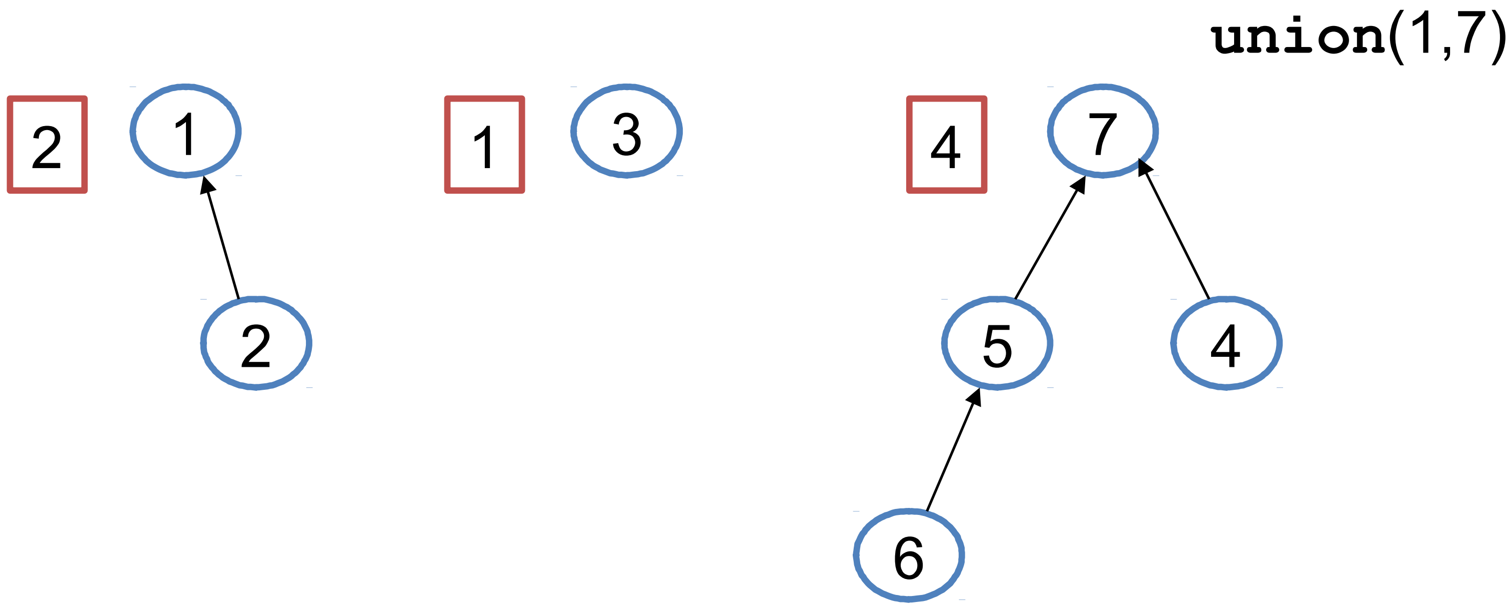
`union`(1,7)



Weighted union

Weighted union:

- Always point the *smaller* (total # of nodes) tree to the root of the larger tree

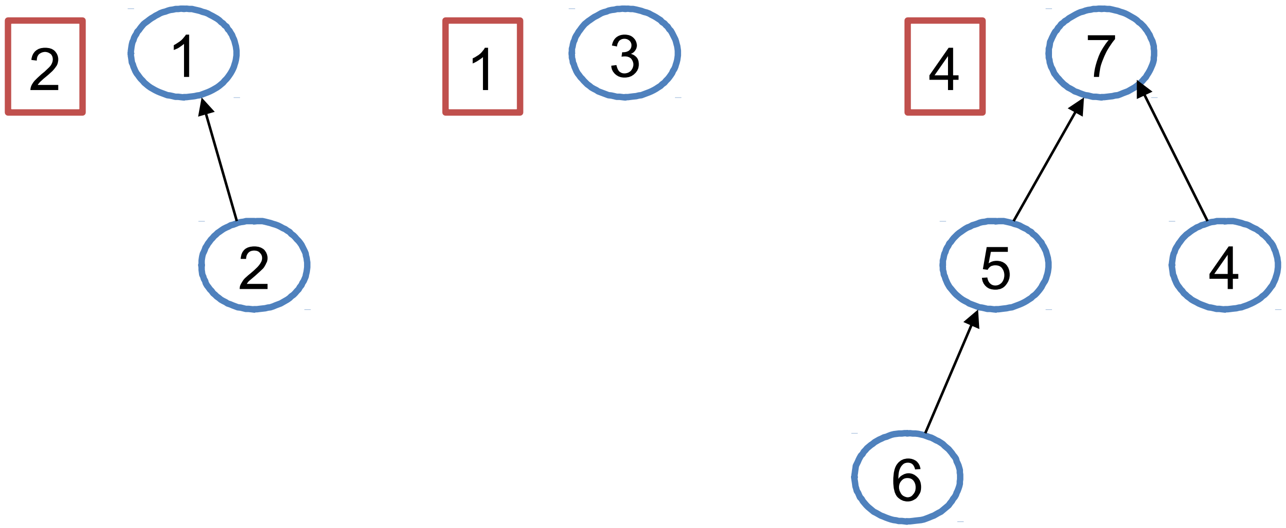


Weighted union

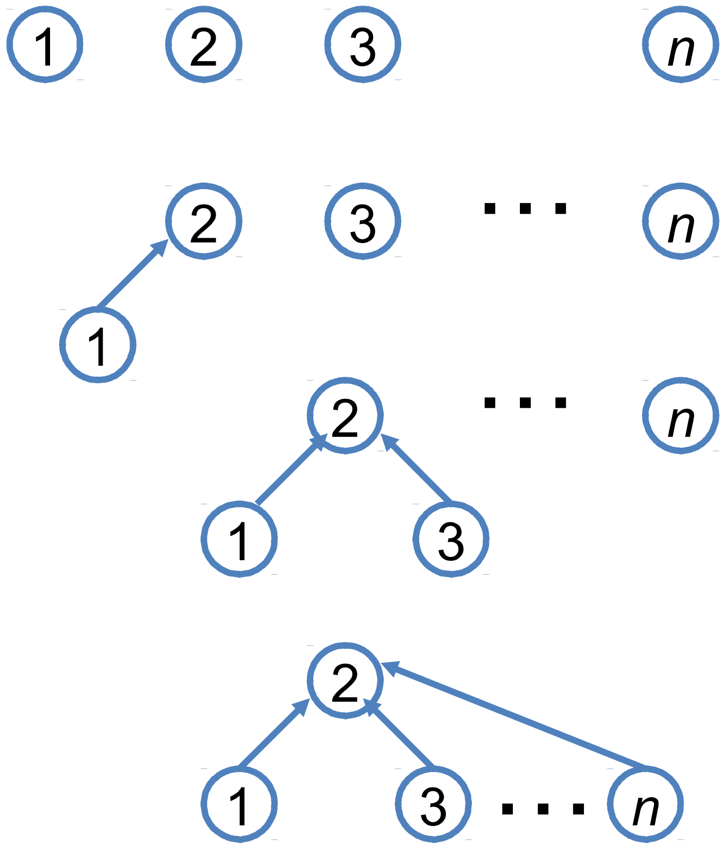
Weighted union:

- Always point the *smaller* (total # of nodes) tree to the root of the larger tree

union(1,7)



Bad example? Great example…



`union`(2,1)

`union`(3,2)

:

`union`(n,n-1)

`find`(1) *constant here*

General analysis

- Showing that one worst-case example is now good is *not* a proof that the worst-case has improved

- So let's prove:
  - **`union`** is still $O(1)$ – this is fairly easy to show
  - **`find`** is now $O(\texttt{log}\ n)$

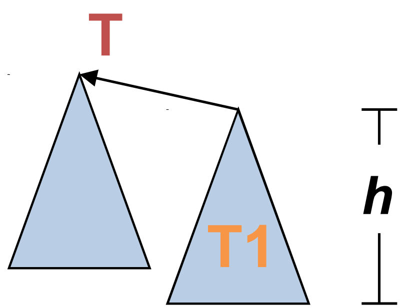
- Claim: If we use weighted-union, an up-tree of height h has at least 2^h nodes
  - Proof by induction on h…

Exponential number of nodes

P(h)= With weighted-union, up-tree of height h has at least 2^h nodes

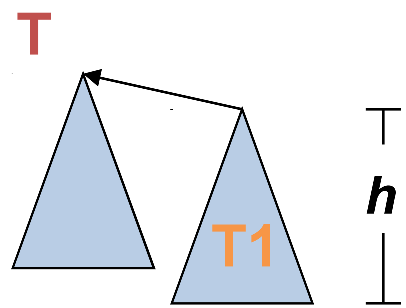
Proof by induction on h…

- Base case: h = 0: The up-tree has 1 node and 2^0= 1
- Inductive case: Assume P(h) and show P(h+1)
  - A height h+1 tree T has at least one height h child T1
  - T1 has at least 2^h nodes by induction
  - And T has *at least* as many nodes not in T1 than in T1
    - Else weighted-union would have had T point to T1, not T1 point to T (!!)
  - So total number of nodes is *at least* $2^h + 2^h = 2^{h+1}$



The key idea

Intuition behind the proof: No one child can have more than half the nodes

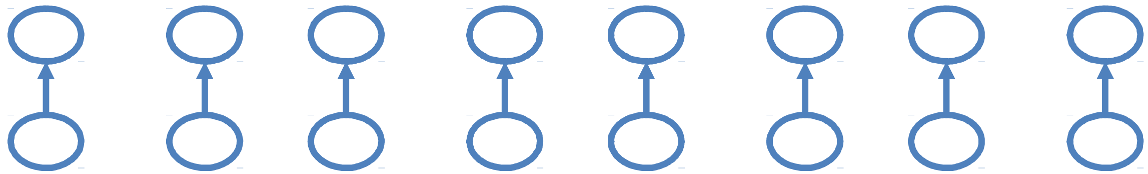


So, as usual, if number of nodes is exponential in height,
then height is logarithmic in number of nodes

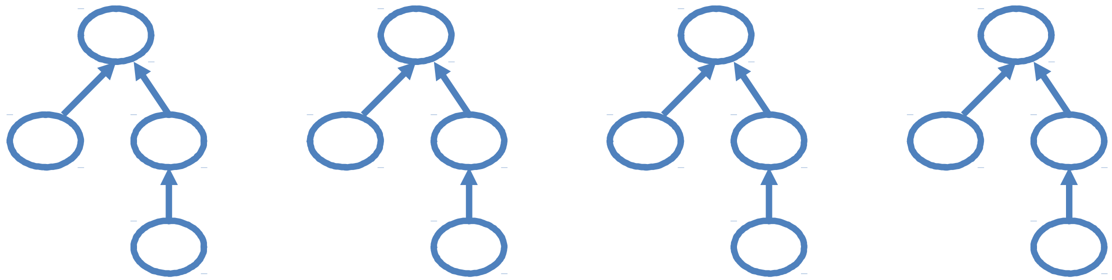
So **`find`** is $O($**`log`** $n)$

The new worst case

n/2 Weighted Unions

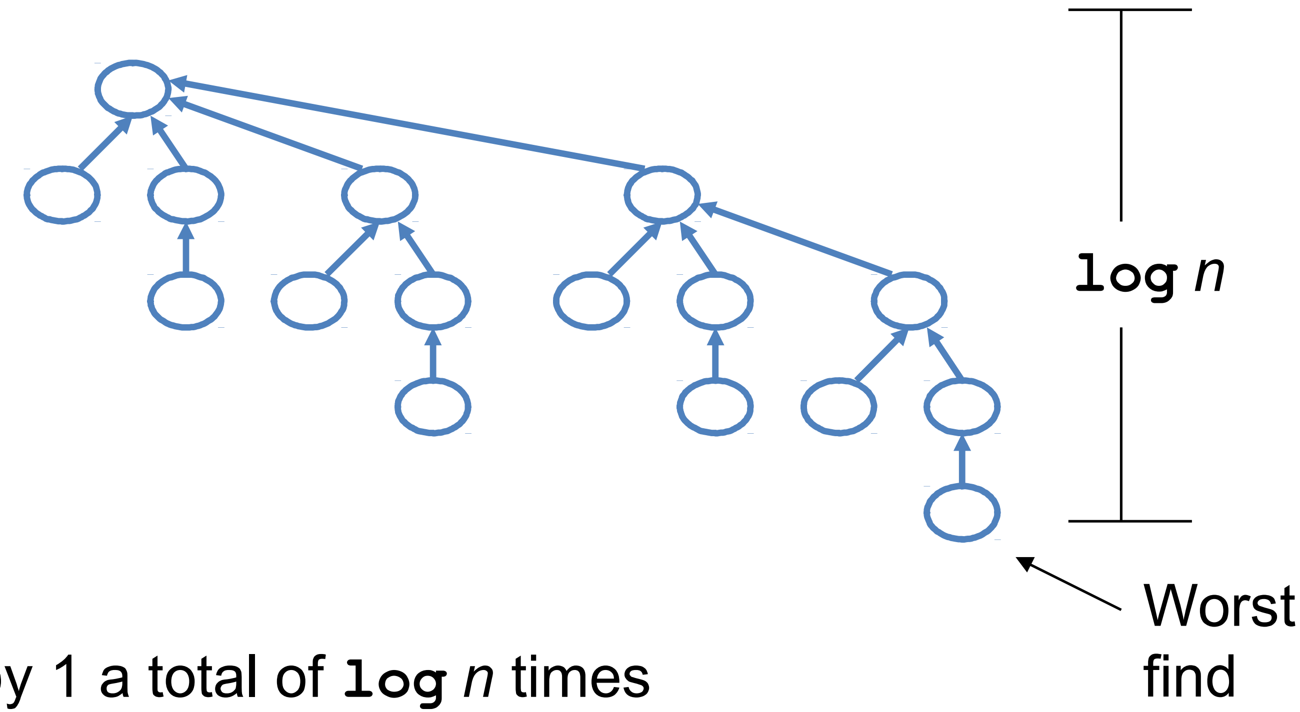


n/4 Weighted Unions



The new worst case (continued)

After n/2 + n/4 + …+ 1 Weighted Unions:



Height grows by 1 a total of `log` *n* times

Two key optimizations

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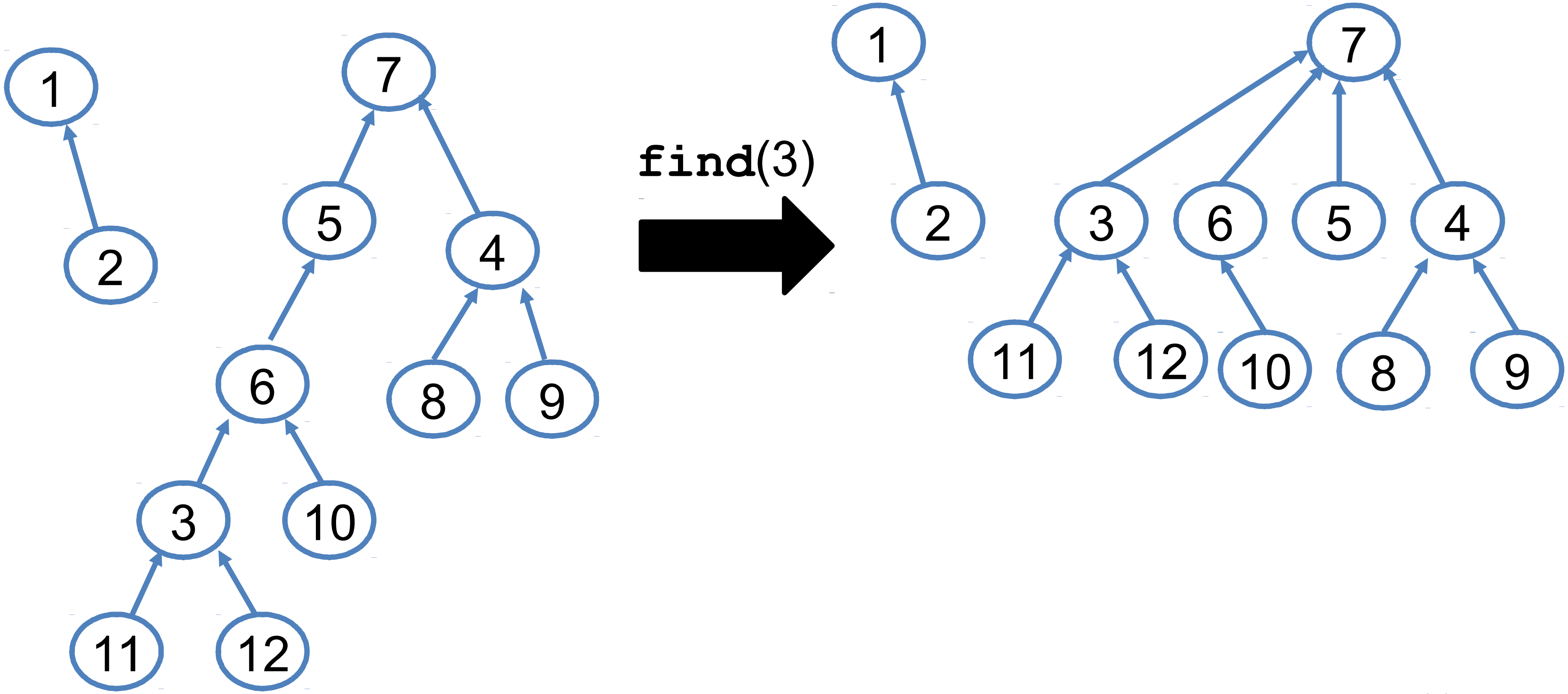
   – So *m* **`finds`** and *n*-1 **`unions`** is $O(m\ \texttt{log}\ n + n)$
   – *Union-by-size:* connect smaller tree to larger tree

2. Improve **`find`** so it becomes even faster
   – Make *m* **`finds`** and *n*-1 **`unions`** ***almost*** $O(m + n)$
   – *Path-compression:* connect directly to root during finds

Path compression

- Simple idea: As part of a **`find`**, change each encountered node's parent to point directly to root
  - Faster future **`find`**s for everything on the path (and their descendants)



So, how fast is it?

A single worst-case **`find`** could be $O(\texttt{log}\ n)$

- But only if we did a lot of worst-case unions beforehand
- And path compression will make future finds faster

Turns out the amortized worst-case bound is much better than $O(\texttt{log}\ n)$

- We won't *prove* it – see text if curious
- Result is that it is *almost* $O(1)$ because total for m **`find`**s and n-1 **`union`**s is *almost* $O(m+n)$