Algorithms & Data Structures

Lesson 12:Topological Sort / Graph Traversals

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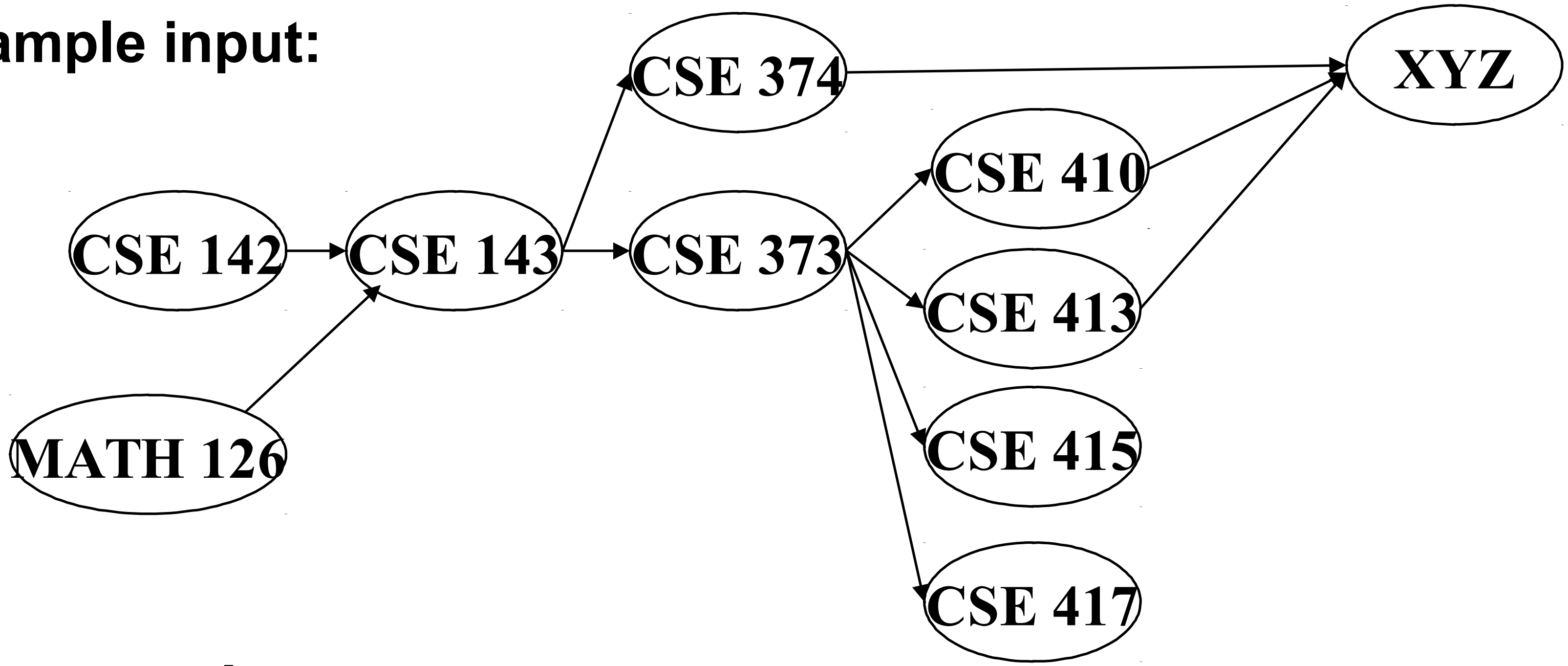
Edition 2017-2018

Topological Sort

Problem: Given a DAG `G=(V,E)`, output all vertices in an order such that no vertex appears before another vertex that has an edge to it

Example input:



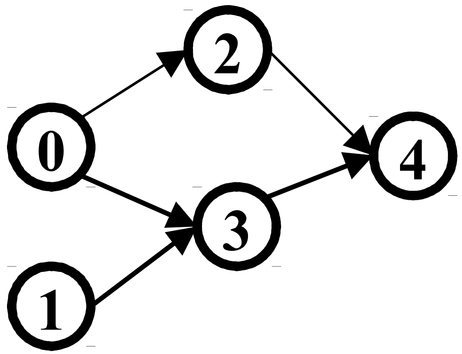
One example output:

126, 142, 143, 374, 373, 417, 410, 413, XYZ, 415

Questions and comments

- **Why do we perform topological sorts only on DAGs?**
  - Because a cycle means there is no correct answer

- **Is there always a unique answer?**
  - No, there can be 1 or more answers; depends on the graph
  - Graph with 5 topological orders:



- **Do some DAGs have exactly 1 answer?**
  - Yes, including all lists

- Terminology: A DAG represents a partial order and a topological sort produces a total order that is consistent with it

Uses

- Figuring out how to graduate
- Computing an order in which to recompute cells in a spreadsheet
- Determining an order to compile files using a Makefile
- In general, taking a dependency graph and finding an order of execution
- …

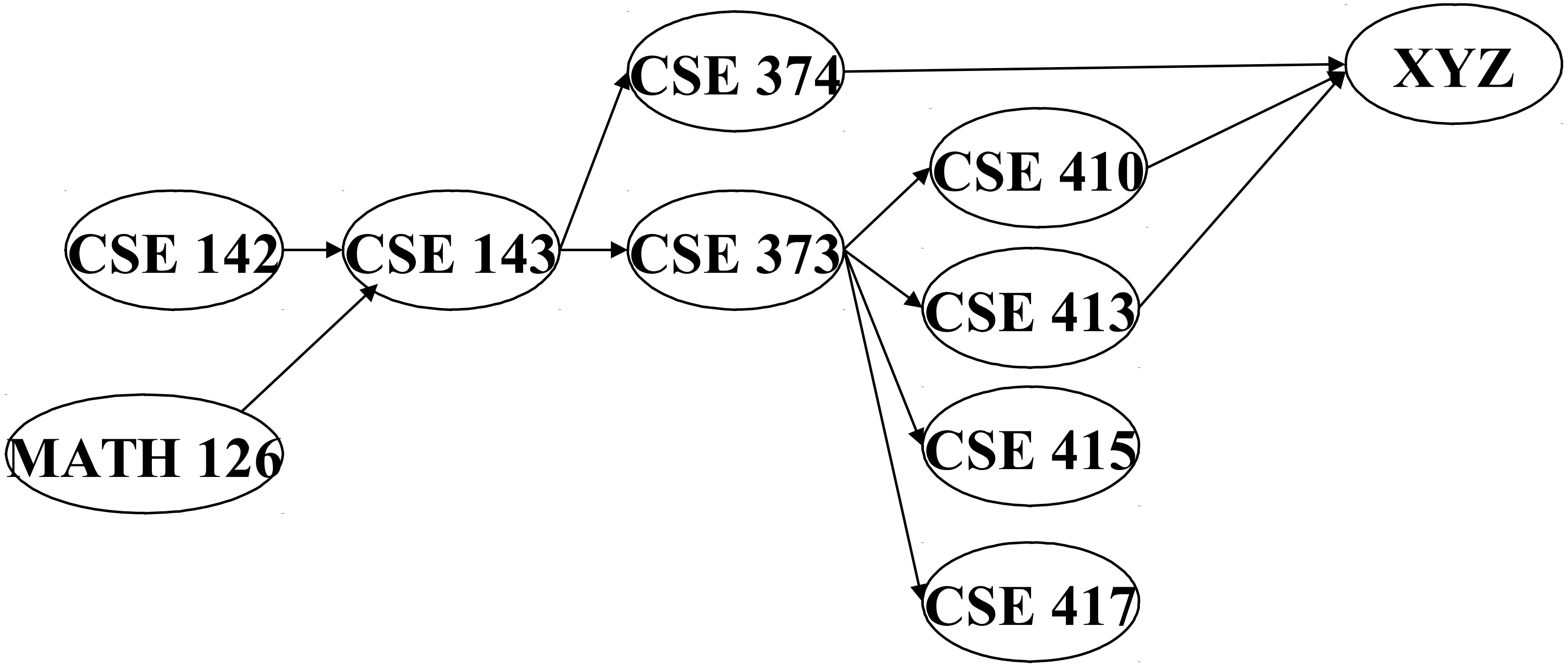
A First Algorithm for Topological Sort

1. Label (“mark”) each vertex with its in-degree
   – Think “write in a field in the vertex”
   – Could also do this via a data structure (e.g., array) on the side

2. While there are vertices not yet output:
   a) Choose a vertex **v** with labeled with in-degree of 0
   b) Output **v** and *conceptually* remove it from the graph
   c) For each vertex **u** adjacent to **v** (i.e. **u** such that (**v**,**u**) in `E`), decrement the in-degree of **u**

Example

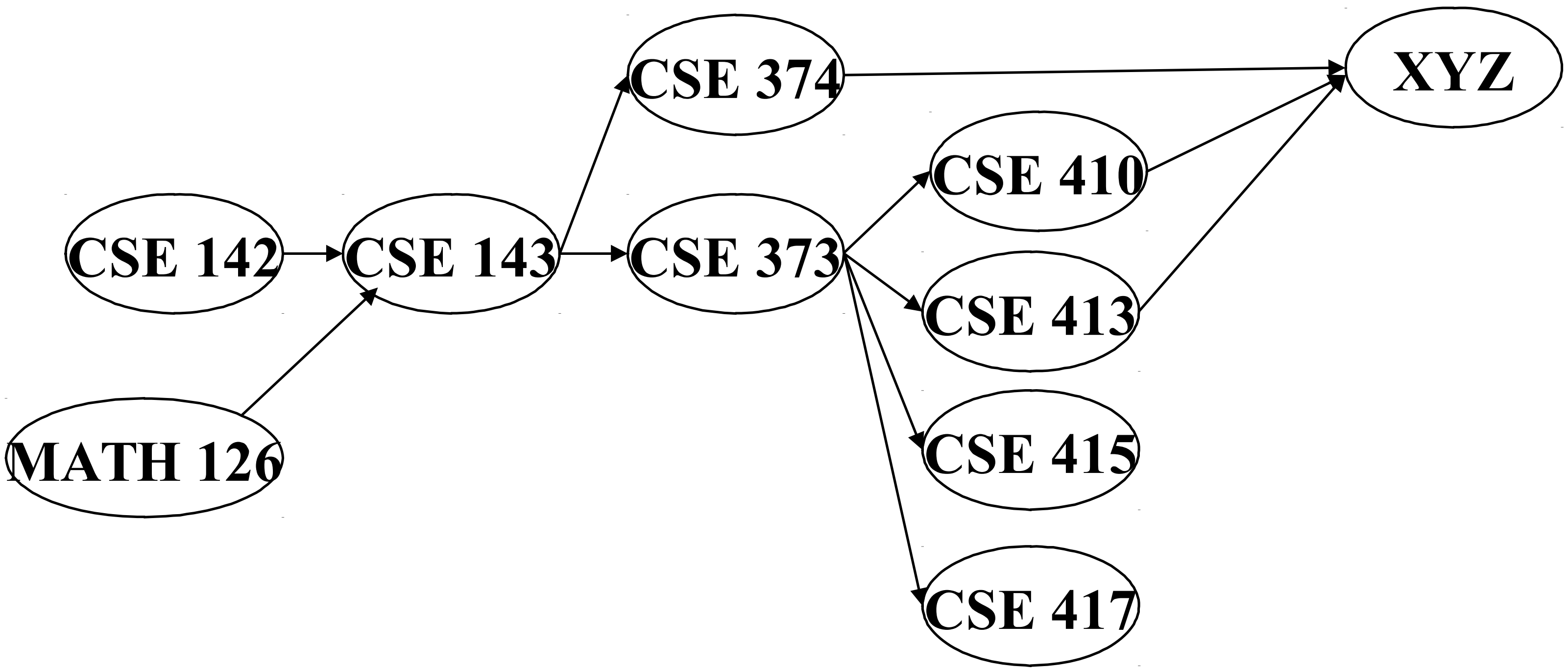
Output:



Node:	126	142	143	374	373	410	413	415	417	XYZ
Removed?										
In-degree:	0	0	2	1	1	1	1	1	1	3

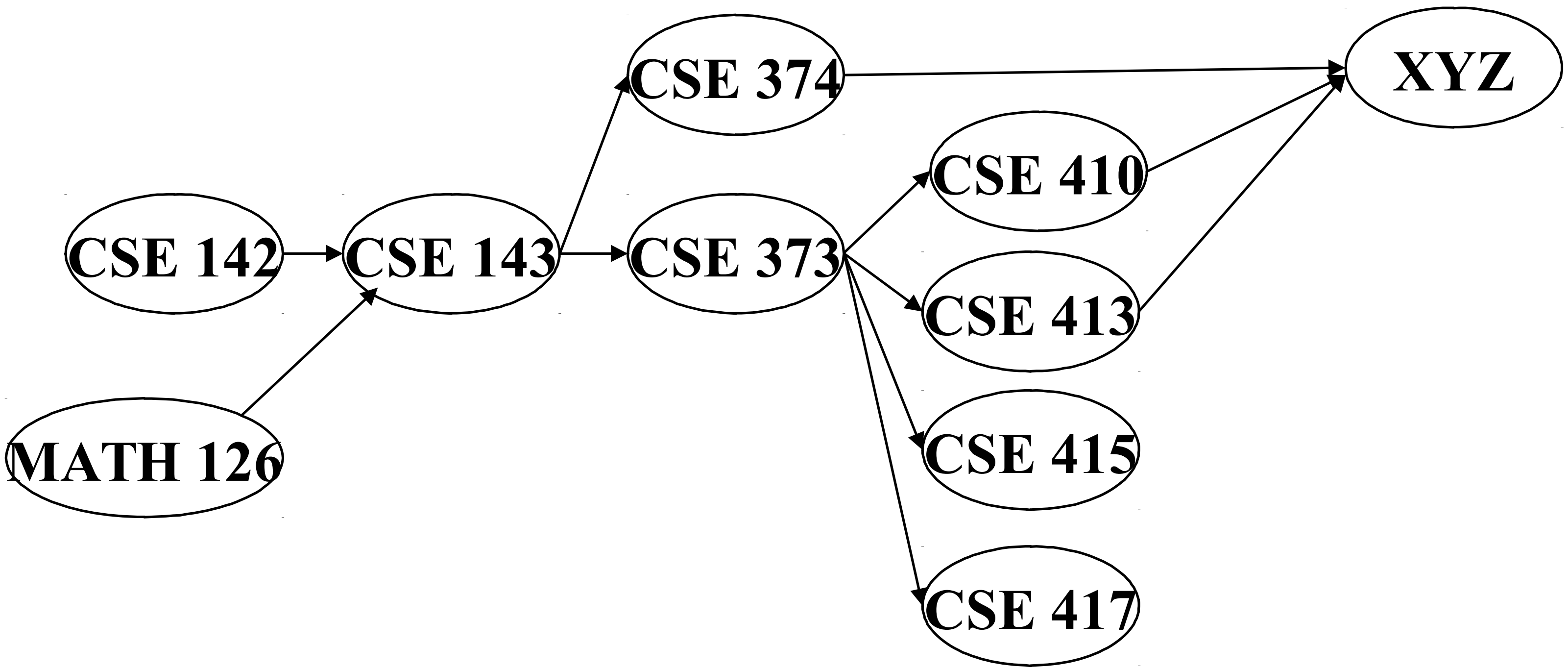
Example

Output:
126



Node:	126	142	143	374	373	410	413	415	417	XYZ
Removed?	x									
In-degree:	0	0	~~2~~ 1	1	1	1	1	1	1	3

Example



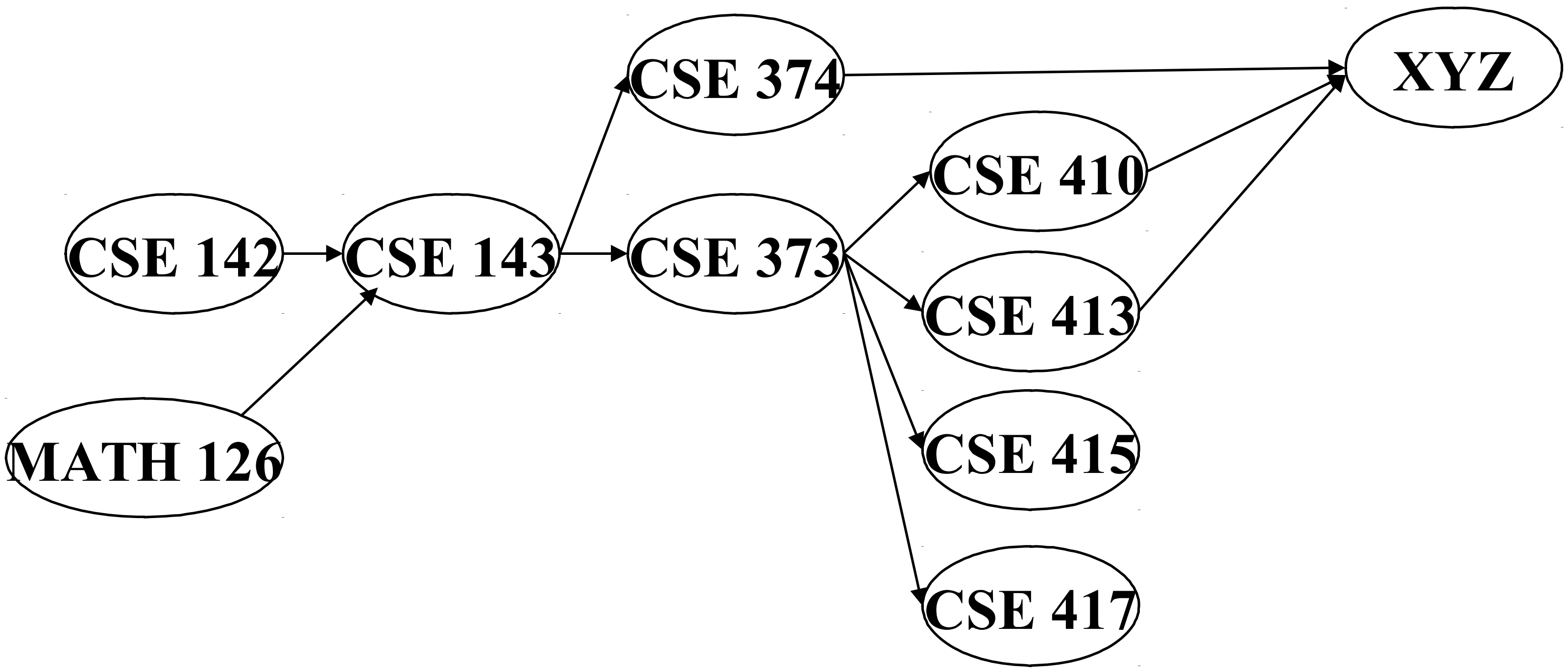
Output:

126

142

Node:	126	142	143	374	373	410	413	415	417	XYZ
Removed?	x	x								
In-degree:	0	0	~~2~~ ~~1~~ 0	1	1	1	1	1	1	3

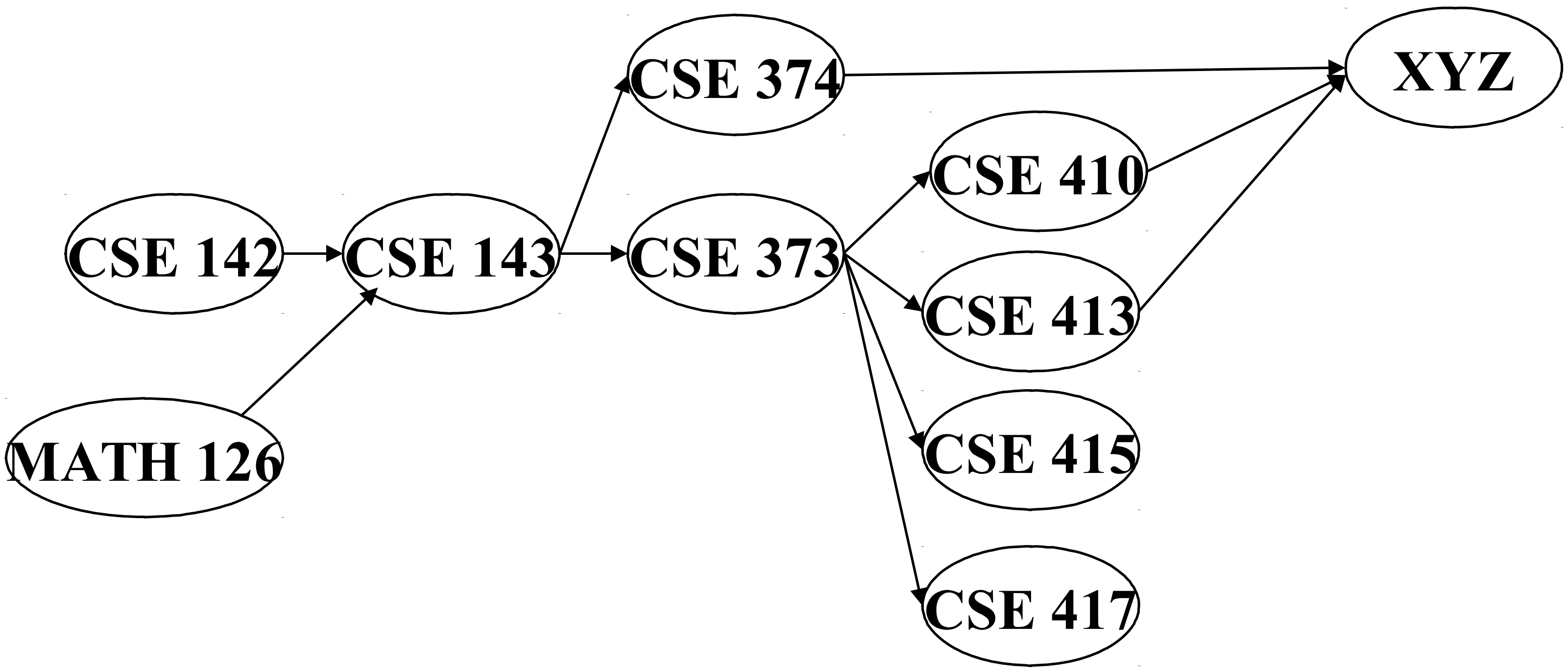
Example



Output:
126
142
143

Node:	126	142	143	374	373	410	413	415	417	XYZ
Removed?	x	x	x							
In-degree:	0	0	~~2~~	~~1~~	~~1~~	1	1	1	1	3
			~~1~~	0	0					
			0							

Example



Output:
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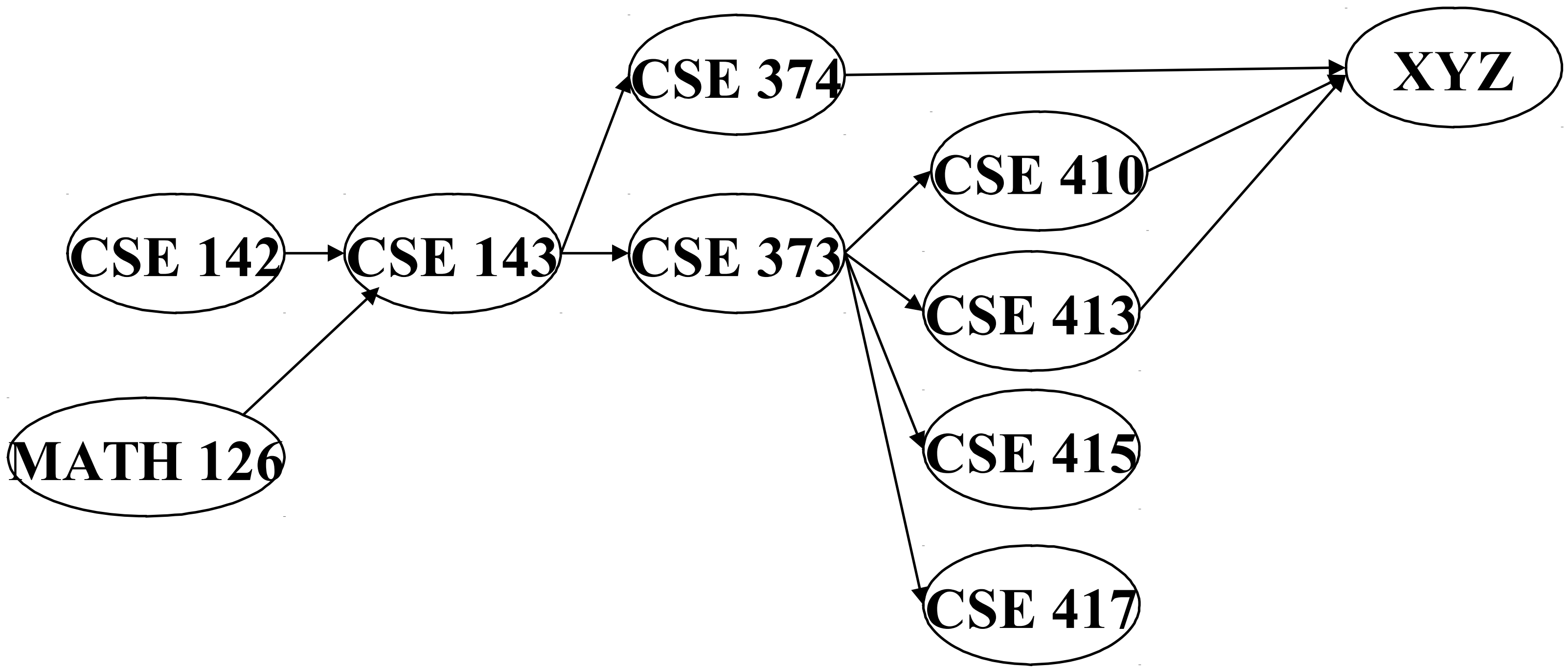
374

Node:	126	142	143	374	373	410	413	415	417	XYZ
Removed?	x	x	x	x						
In-degree:	0	0	~~2~~ ~~1~~ 0	~~1~~ 0	~~1~~ 0	1	1	1	1	~~3~~ 2

Example



Output:
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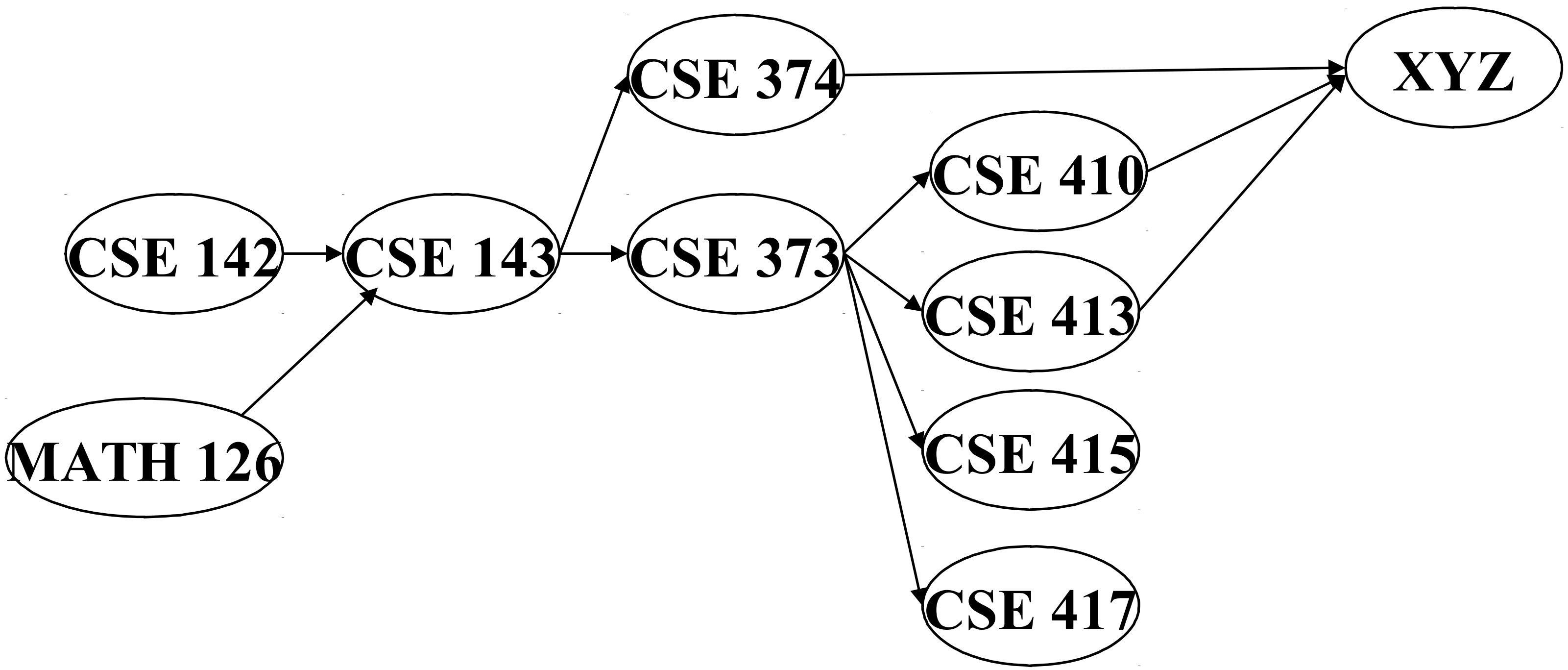
Node:	126	142	143	374	373	410	413	415	417	XYZ
Removed?	x	x	x	x	x					
In-degree:	0	0	~~2~~ ~~1~~ 0	~~1~~ 0	~~1~~ 0	~~1~~ 0	~~1~~ 0	~~1~~ 0	~~1~~ 0	~~3~~ 2

Example



Output:
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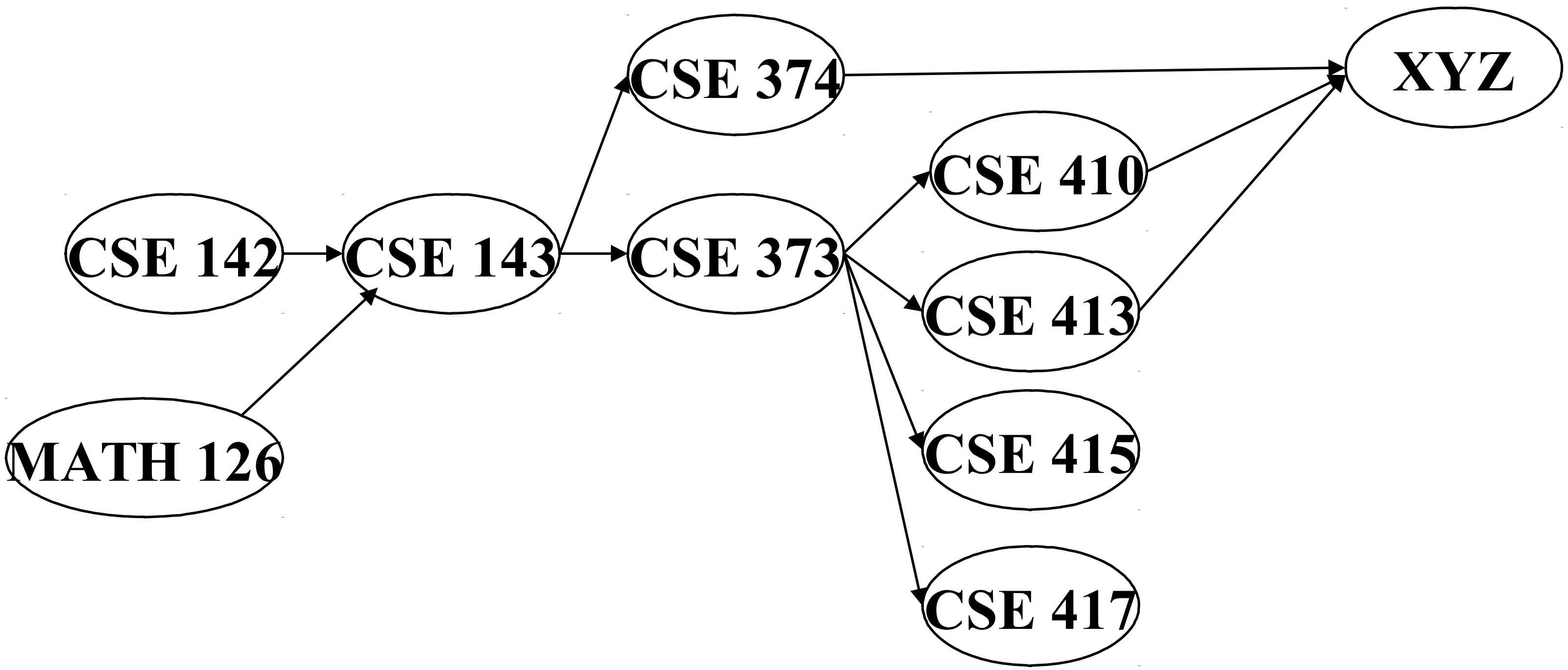
Node:	126	142	143	374	373	410	413	415	417	XYZ
Removed?	x	x	x	x	x				x	
In-degree:	0	0	~~2~~	~~1~~	~~1~~	~~1~~	~~1~~	~~1~~	~~1~~	~~3~~
			~~1~~	0	0	0	0	0	0	2
			0							

Example



Output:
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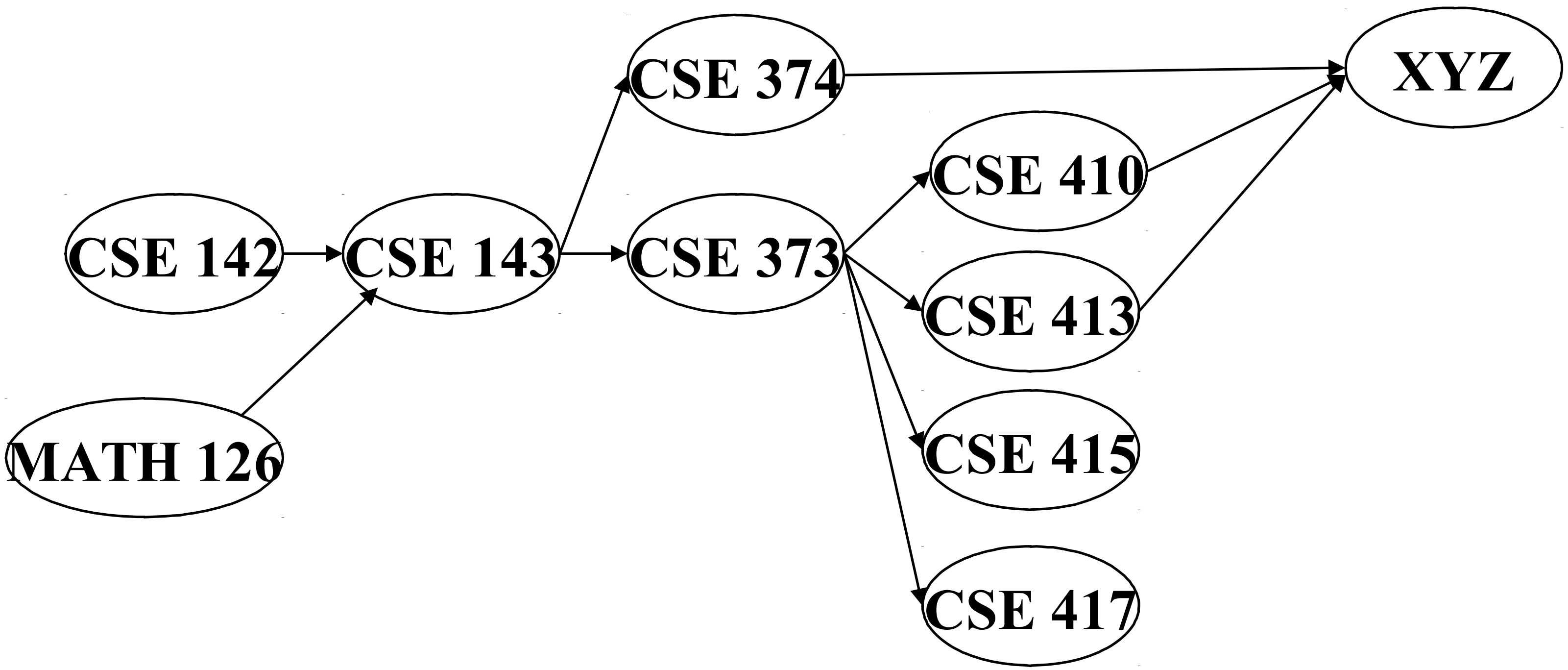
Node:	126	142	143	374	373	410	413	415	417	XYZ
Removed?	x	x	x	x	x	x			x	
In-degree:	0	0	~~2~~ ~~1~~ 0	~~1~~ 0	~~1~~ 0	~~1~~ 0	~~1~~ 0	~~1~~ 0	~~1~~ 0	~~3~~ ~~2~~ 1

Example



Output:
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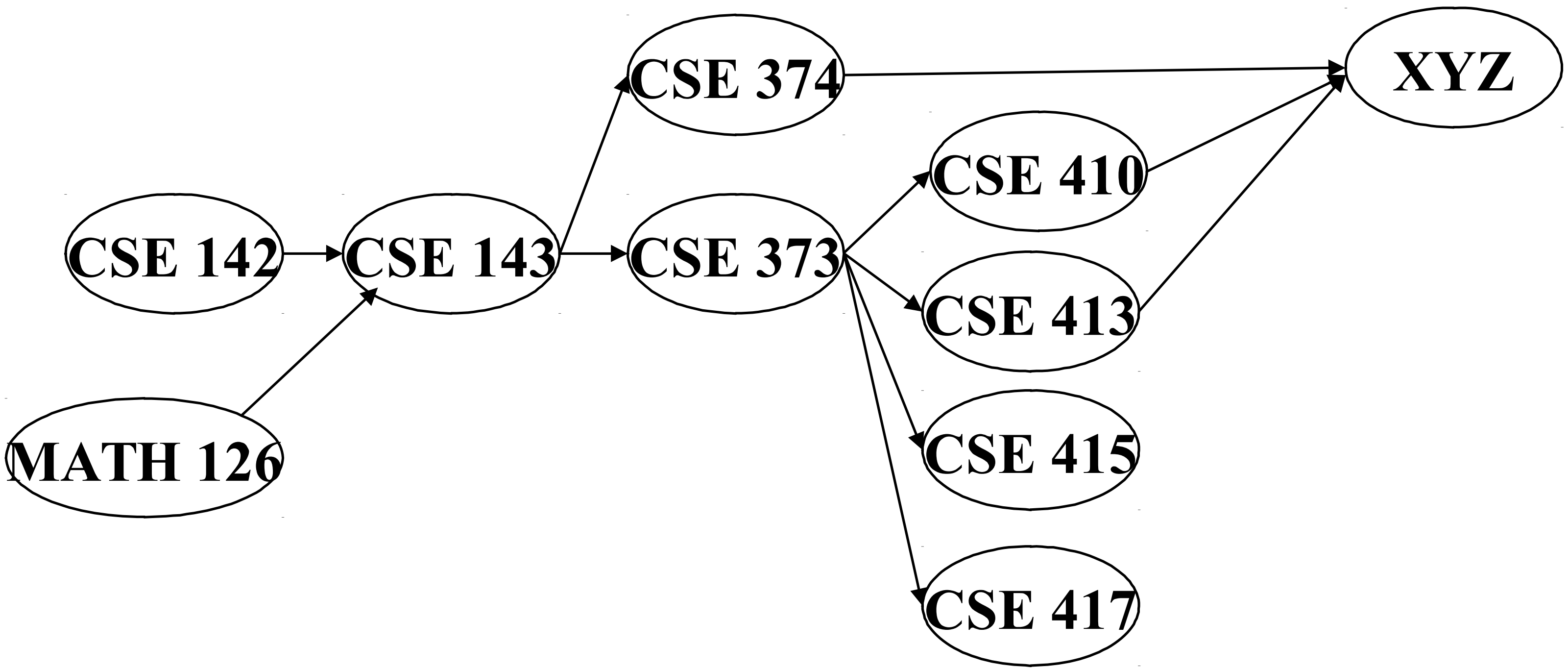
Node:	126	142	143	374	373	410	413	415	417	XYZ
Removed?	x	x	x	x	x	x	x		x	
In-degree:	0	0	~~2~~ ~~1~~ 0	~~1~~ 0	~~1~~ 0	~~1~~ 0	~~1~~ 0	~~1~~ 0	~~1~~ 0	~~3~~ ~~2~~ ~~1~~ 0

Example



Output:
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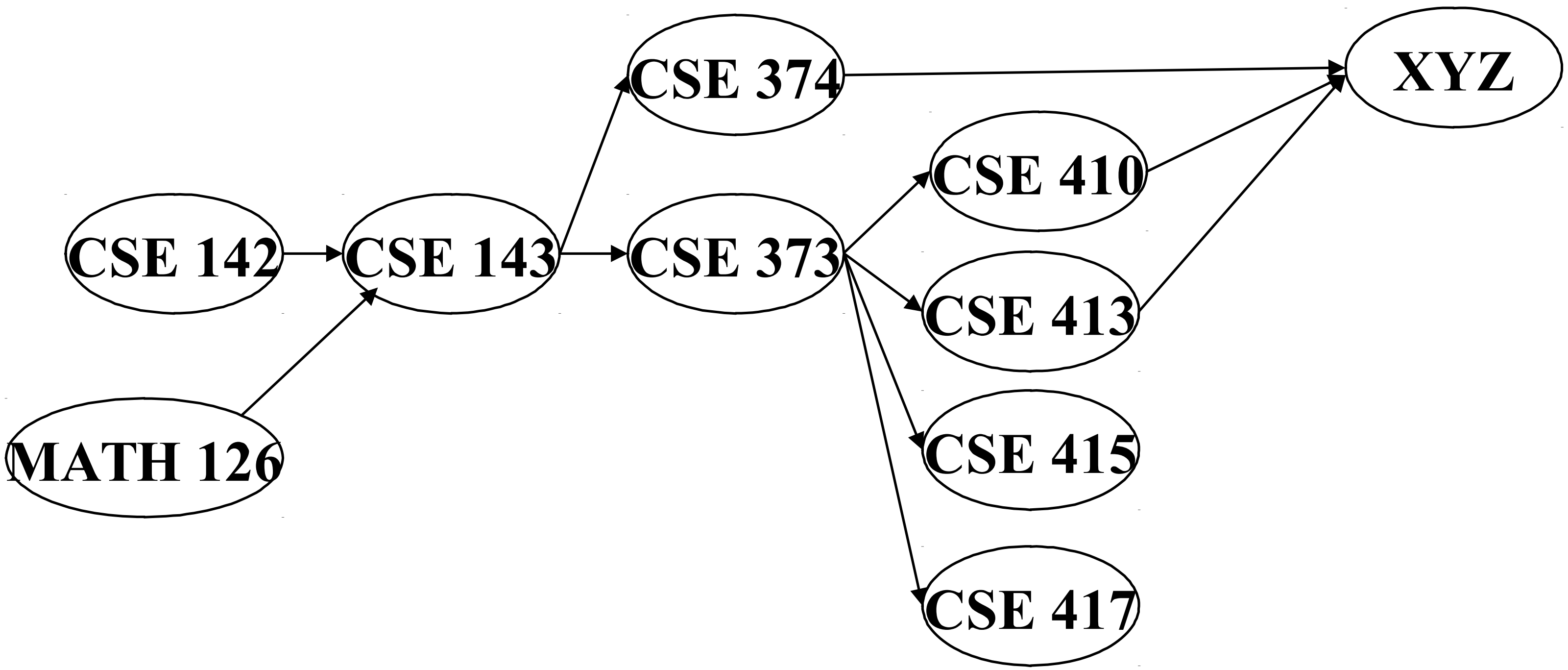
Node:	126	142	143	374	373	410	413	415	417	XYZ
Removed?	x	x	x	x	x	x	x		x	x
In-degree:	0	0	~~2~~ ~~1~~ 0	~~1~~ 0	~~1~~ 0	~~1~~ 0	~~1~~ 0	~~1~~ 0	~~1~~ 0	~~3~~ ~~2~~ ~~1~~ 0

Example



Output:
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Node:	126	142	143	374	373	410	413	415	417	XYZ
Removed?	x	x	x	x	x	x	x	x	x	x
In-degree:	0	0	~~2~~ ~~1~~ 0	~~1~~ 0	~~1~~ 0	~~1~~ 0	~~1~~ 0	~~1~~ 0	~~1~~ 0	~~3~~ ~~2~~ ~~1~~ 0

Notice

- Needed a vertex with in-degree 0 to start
  - Will always have at least 1 because no cycles

- Ties among vertices with in-degrees of 0 can be broken arbitrarily
  - Can be more than one correct answer, by definition, depending on the graph

Running time?

```
labelEachVertexWithItsInDegree();
for(ctr=0; ctr < numVertices; ctr++){
    v = findNewVertexOfDegreeZero();
    put v next in output
     for each w adjacent to v
        w.indegree--;
}
```

Running time?

```
labelEachVertexWithItsInDegree();
for(ctr=0; ctr < numVertices; ctr++){
    v = findNewVertexOfDegreeZero();
    put v next in output
    for each w adjacent to v
       w.indegree--;
}
```

- What is the worst-case running time?
  - Initialization $O(|V|+|E|)$ (assuming adjacency list)
  - Sum of all find-new-vertex $O(|V|^2)$ (because each $O(|V|)$)
  - Sum of all decrements $O(|E|)$ (assuming adjacency list)
  - So total is $O(|V|^2)$ – not good for a sparse graph!

Doing better

The trick is to avoid searching for a zero-degree node every time!

- Keep the "pending" zero-degree nodes in a list, stack, queue, bag, table, or something
- Order we process them affects output but not correctness or efficiency provided add/remove are both $O(1)$

Using a queue:

1. Label each vertex with its in-degree, enqueue 0-degree nodes
2. While queue is not empty
   a) **v** = dequeue()
   b) Output **v** and remove it from the graph
   c) For each vertex **u** adjacent to **v** (i.e. **u** such that (**v**,**u**) in **E**), decrement the in-degree of **u**, if new degree is 0, enqueue it

Running time?

```
labelAllAndEnqueueZeros();
while queue not empty {
   v = dequeue();
   put v next in output
   for each w adjacent to v {
      w.indegree--;
      if (w.indegree==0)
         enqueue(v);
   }
}
```

Running time?

```
labelAllAndEnqueueZeros();
while queue not empty {
   v = dequeue();
   put v next in output
   for each w adjacent to v {
      w.indegree--;
      if (w.indegree==0)
         enqueue(v);
   }
}
```

- What is the worst-case running time?
  - Initialization: $O(|V|+|E|)$ (assuming adjacenty list)
  - Sum of all enqueues and dequeues: $O(|V|)$
  - Sum of all decrements: $O(|E|)$ (assuming adjacency list)
  - So total is $O(|E| + |V|)$ – much better for sparse graph!

Graph Traversals

Next problem: For an arbitrary graph and a starting node **v**, find all nodes *reachable* from **v** (i.e., there exists a path from **v**)

- Possibly "do something" for each node
- Examples: print to output, set a field, etc.

- **Subsumed problem**: Is an undirected graph connected?
- **Related but different problem**: Is a directed graph strongly connected?
  - Need cycles back to starting node

Basic idea:

- Keep following nodes
- But "mark" nodes after visiting them, so the traversal terminates and processes each reachable node exactly once

Abstract Idea

```
void traverseGraph(Node start) {
     Set pending = emptySet()
     pending.add(start)
     mark start as visited
     while(pending is not empty) {
       next = pending.remove()
       for each node u adjacent to next
          if (u is not marked) {
             mark u
             pending.add(u)
          }
     }
  }
```

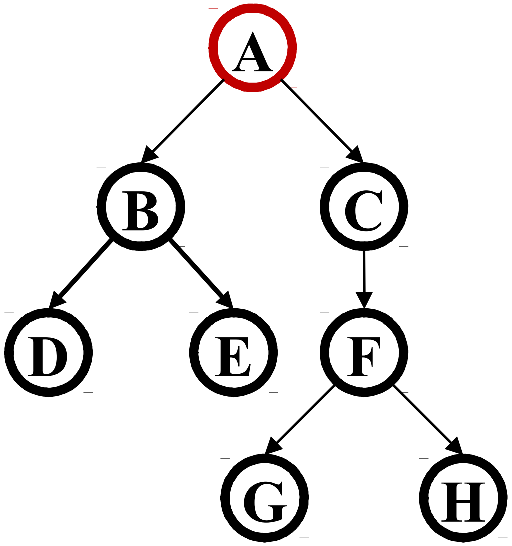
Running Time and Options

- Assuming **`add`** and **`remove`** are $O(1)$, entire traversal is $O(|E|)$
  - Use an adjacency list representation
- The order we traverse depends entirely on **`add`** and **`remove`**
  - Popular choice: a stack "depth-first graph search" DFS
  - Popular choice: a queue "breadth-first graph search" BFS
- DFS and BFS are "big ideas" in computer science
  - Depth: recursively explore one part before going back to the other parts not yet explored
  - Breadth: explore areas closer to the start node first

Cool visualization: http://visualgo.net/dfsbfs.html

Example: trees

- A tree is a graph and DFS and BFS are particularly easy to "see"

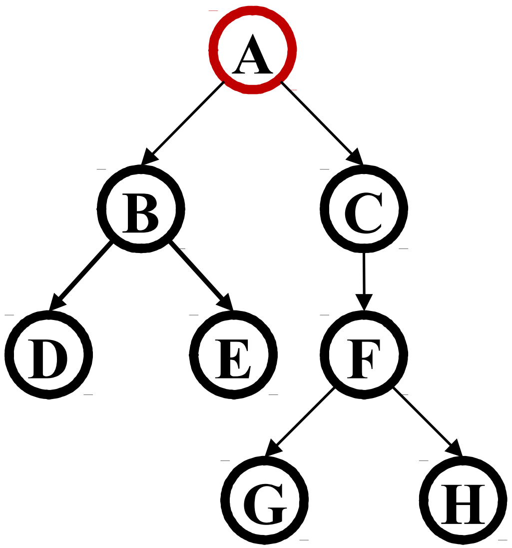



```
DFS(Node start) {
  mark and process start
  for each node u adjacent to start
    if u is not marked
      DFS(u)
}
```

- A, B, D, E, C, F, G, H
- Exactly what we called a "pre-order traversal" for trees
  – The marking is because we support arbitrary graphs and we want to process each node exactly once

Example: trees

- A tree is a graph and DFS and BFS are particularly easy to "see"

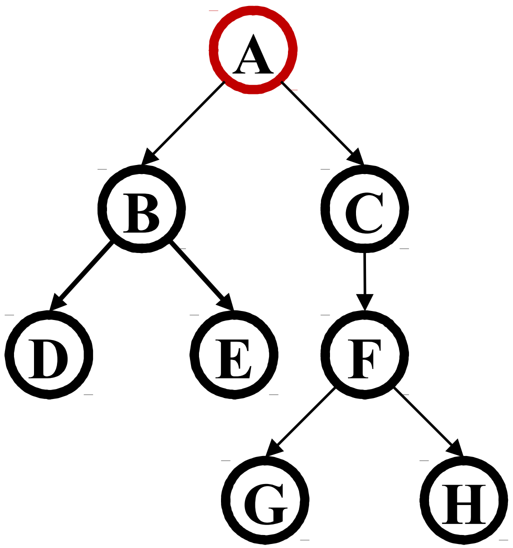



```
DFS2(Node start) {
  initialize stack s to hold start
  mark start as visited
  while(s is not empty) {
    next = s.pop() // and "process"
    for each node u adjacent to next
      if(u is not marked)
        mark u and push onto s
  }
}
```

- A, C, F, H, G, B, E, D
- A different but perfectly fine traversal

Example: trees

- A tree is a graph and DFS and BFS are particularly easy to “see”




```
BFS(Node start) {
  initialize queue q to hold start
  mark start as visited
  while(q is not empty) {
    next = q.dequeue() // and "process"
    for each node u adjacent to next
     if(u is not marked)
       mark u and enqueue onto q
  }
}
```

- A, B, C, D, E, F, G, H
- A “level-order” traversal

Comparison

- Breadth-first always finds shortest paths, i.e., “optimal solutions”
  - Better for “what is the shortest path from **`x`** to **`y`**”

- But depth-first can use less space in finding a path
  - If *longest path* in the graph is **`p`** and highest out-degree is **`d`** then DFS stack never has more than **`d*p`** elements
  - But a queue for BFS may hold *O*(|V|) nodes

- A third approach:
  - *Iterative deepening (IDFS)*:
    - Try DFS but disallow recursion more than **`K`** levels deep
    - If that fails, increment **`K`** and start the entire search over
  - Like BFS, finds shortest paths. Like DFS, less space.

Saving the Path

- Our graph traversals can answer the reachability question:
  - "Is there a path from node x to node y?"

- But what if we want to actually output the path?
  - Like getting driving directions rather than just knowing it's possible to get there!

- How to do it:
  - Instead of just "marking" a node, store the previous node along the path (when processing **`u`** causes us to add **`v`** to the search, set **`v.path`** field to be **`u`**)
  - When you reach the goal, follow **`path`** fields back to where you started (and then reverse the answer)
  - If just wanted path *length*, could put the integer distance at each node instead

Example using BFS

What is a path from Seattle to Tyler

- Remember marked nodes are not re-enqueued
- Note shortest paths may not be unique