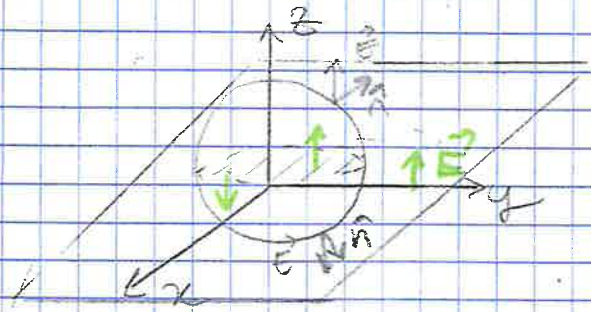
2.6 Distribution planaire de charge



① Gauss : Forme intégrale

$$\oint \vec{E}(\vec{r})\cdot\hat{n}\, dS = \frac{Q_{int}}{\varepsilon_0}$$

avec $Q_{int} = \rho_S \pi R^2$

Donc $\oint \vec{E}\cdot\hat{n}\, dS = \frac{\rho_S \pi R^2}{\varepsilon_0}$

② Calcul de l'intégrale

On choisit le système de coordonnées sphérique

S_1 est la surface de la demi-sphère sup
S_2 ———— inf

$$\oint_S \vec{E}(\vec{r})\,\hat{n}\, dS = \int_{S_1} \vec{E}(\vec{r})\,\hat{n}_1\, dS + \int_{S_2} \vec{E}(\vec{r})\,\hat{n}_2\, dS$$

$$= 2\int_{S_1} \frac{\rho_S}{2\varepsilon_0}\,\hat{e}_z\cdot\hat{e}_r\, dS - \cancel{\int_{S_2} \frac{\rho_S}{2\varepsilon_0}\,\hat{e}_z\cdot\hat{e}_r\, dS}$$

$$= 2\int_{S_1} \frac{\rho_S}{2\varepsilon_0}\cos\theta\, dS_1 - \cancel{\int_{S_2} \frac{\rho_S}{2\varepsilon_0}\cos\theta\, dS_2}$$

$$= \frac{2\rho_S}{2\varepsilon_0}\left(\int_{S_1}\cos\theta\, dS_1 - \cancel{\int_{S_2}\cos\theta\, dS_2}\right)$$

$dS = R^2 \sin\theta\, d\theta\, d\varphi$

$$\oint_S \vec{E}(\vec{r})\,\hat{n}\, dS = \frac{\rho_S 2}{\cancel{2}\varepsilon_0}\left(\int_0^{2\pi}\int_0^{\frac{\pi}{2}} R^2 \underbrace{\sin\theta\cos\theta}_{\frac{\sin 2\theta}{2}}\, d\theta\, d\varphi\right.$$

$$\left.\cancel{\int_0^{2\pi}\int_{\frac{\pi}{2}}^{\frac{3}{2}\pi} R^2 \underbrace{\sin\theta\cos\theta}_{\frac{\sin 2\theta}{2}}\, d\theta\, d\varphi}\right)$$

$$= \frac{2\rho_S\, 2\pi R^2}{2\varepsilon_0\, 2}\left(\left[-\frac{\cos 2\theta}{2}\right]_0^{\pi/2}\right)$$

$$= \frac{\rho_S \pi R^2}{\varepsilon_0}\left(+\frac{1}{2}+\frac{1}{2}\right)$$

$$= \frac{\rho_S \pi R^2}{\varepsilon_0}$$

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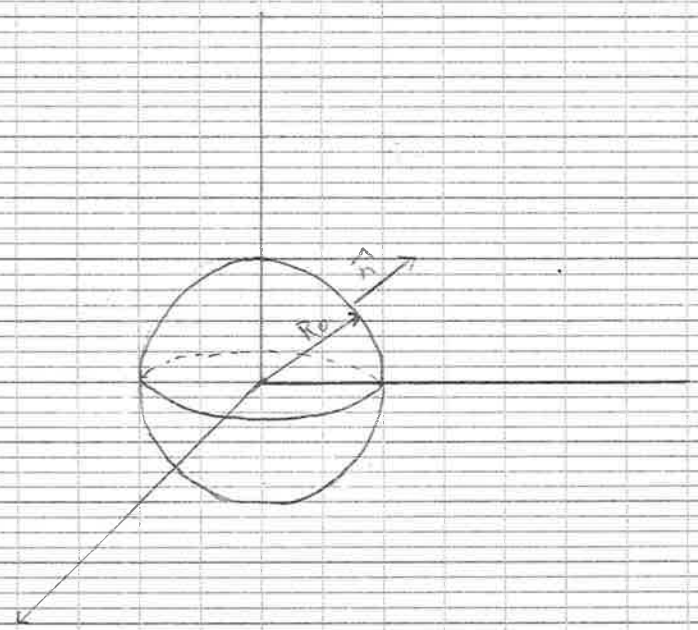
Electromagnetisme

TD4

4.1 Une sphere chargée



$$\vec{E}(\vec{r}) \begin{cases} \frac{1}{4\pi\varepsilon_0} Q_0 \frac{r}{R_0^3} \hat{e}_r & r \leq R_0 \\ \frac{1}{4\pi\varepsilon_0} \frac{Q_0}{r^2} \hat{e}_r & r > R_0 \end{cases}$$

P à l'infini

Soit M un point de l'espace

- $V(\vec{r_P}) = 0$ $\qquad V(\vec{r_M}) - V(\vec{r_P}) = -\int_{\vec{r_P} \to \vec{r_M}} \vec{E} \cdot \hat{t}\, dl$

<u>Cas $r > R_0$:</u> (Exterieur)

$$V(\vec{r_M}) = -\int_{\vec{r_P} \to \vec{r_M}} \vec{E} \cdot \hat{t}\, dl$$

couleur irrégularité

11,25	16,08	20	17,33	13,25 ↓ 14,55 - 14,85
12,50	9,52	11,50	17,04	7,52 ↓ 8,82 - 10,16

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$$V(\vec{r}_M) = -\int \vec{E}\,\vec{t}\,dl = \int_{r}^{+\infty} \vec{E}$$

or $\vec{E} = \dfrac{1}{4\pi\varepsilon_0} \dfrac{Q_0}{r^2} \vec{e}_r$ donc $V(\vec{r}_M) = \displaystyle\int_{r_M}^{+\infty} \frac{1}{4\pi\varepsilon_0} \frac{Q_0}{r^2} \cdot dr$

$dl = dr$

$$V(\vec{r}_M) = \frac{1\,Q_0}{4\pi\varepsilon_0} \int_{r_M}^{+\infty} \frac{1}{r^2}\,dr$$

$$\int_{r_M}^{+\infty} \frac{1}{r^2}\,dr = \lim_{x \to +\infty} \int_{r_M}^{x} \frac{1}{r^2} = \lim_{x \to +\infty} \left[\frac{-1}{r}\right]_{r_M}^{x}$$

$$= \lim_{x \to +\infty} \left(\frac{-1}{x} + \frac{1}{r_M}\right) = \frac{1}{r_M}$$

Donc $V(\vec{r}_M) = \dfrac{1}{4\pi\varepsilon_0} Q_0 \dfrac{1}{r_M}$

Cas r $\leq R_0$:

On appelle A le point de la droite (PM) qui coupe la sphère. A est à une distance R_0 de l'origine.

$$V(\vec{r_M}) - V(\vec{r_P}) = -\int_{\vec{r_P} \to \vec{r_M}} \vec{E} \cdot \hat{t}\, dl$$

on $V(\vec{r_P}) = 0$ et $\vec{r_P} \to \vec{r_M} = \underbrace{\vec{r_P} \to \vec{r_A}}_{\text{hors de la sphère}}$ puis $\underbrace{\vec{r_A} \to \vec{r_M}}_{\text{Dans la sphère}}$

$$V(\vec{r_M}) = -\left(\int_{\vec{r_P} \to \vec{r_A}} \vec{E} \cdot \hat{t}\, dl + \int_{\vec{r_A} \to \vec{r_M}} \vec{E} \cdot \hat{t}\, dl\right)$$

$$= \underbrace{-\int_{\vec{r_P} \to \vec{r_A}} \vec{E} \cdot \hat{t}\, dl}_{V(\vec{r_A})} - \int_{\vec{r_A} \to \vec{r_M}} \vec{E} \cdot \hat{t}\, dl$$

$$V_{r_M} = \frac{Q_0}{4\pi\varepsilon_0} \frac{1}{R_0} - \int_{\vec{r_A} \to \vec{r_M}} \vec{E} \cdot \hat{t}\, dl \quad (*)$$

Calculons :

$-\int_{\vec{r_A} \to \vec{r_M}} \vec{E} \cdot \hat{t}\, dl$: Dans la sphère

$$\vec{E} = \frac{1}{4\pi\varepsilon_0} \frac{Q_0 r}{R_0} \hat{e_r} \qquad t = -\hat{e}_r \text{ et } dl = dr$$

$$t \cdot \hat{e_r} = -1$$

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$$-\int_{\vec{r_A} \to \vec{r_M}} \vec{E}\cdot\hat{r}\,dl = \int_{r_M}^{r_A} \frac{1}{4\pi\varepsilon_0} \frac{Q_0\, r}{R_0^3}\,dr$$

$$= \frac{Q_0}{4\pi\varepsilon_0 R_0^3} \int_{r_M}^{R_0} r\,dr$$

$$= \frac{Q_0}{4\pi\varepsilon_0 R_0^3} \left[\frac{r^2}{2}\right]_{r_M}^{R_0}$$

$$= \frac{Q_0}{4\pi\varepsilon_0 R_0^3} \frac{R_0^2 - r_M}{2} \quad (**)$$

(*) et (**)

$$V(\vec{r_M}) = \frac{Q_0}{4\pi\varepsilon_0} \frac{1}{R_0} + \frac{Q_0}{4\pi\varepsilon_0 R_0^3} \left(\frac{R_0^2 - r_M}{2}\right)$$

$$= \frac{Q_0}{4\pi\varepsilon_0} \left(\frac{1}{R_0} + \frac{R_0^2 - r_M^2}{2R_0^3}\right)$$

$$= \frac{Q_0}{4\pi\varepsilon_0} \left(\frac{2R_0^2 + R_0^2 - r_M^2}{2R_0^3}\right)$$

$$= \frac{Q_0}{4\pi\varepsilon_0} \left(\frac{3R_0^2 - r_M^2}{2R_0^3}\right)$$

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4.1 (Suite)

$$\boxed{r > R_0}$$

Raccourci ?

Conditions : $\vec{E} = E_r(r)\vec{e_r}$

On sait que $\vec{E}(x, y, z) = -\vec{grad}(V(x, y, z))$
$$= -\left(\frac{\partial V}{\partial x}\vec{e_x} + \frac{\partial V}{\partial y}\vec{e_y} + \frac{\partial V}{\partial z}\vec{e_z}\right)$$

$$\vec{E}(r, \varphi, \theta) = -\vec{grad}(V(r, \varphi, \theta))$$
$$= -\left(\frac{\partial V}{\partial r}\vec{e_r} + \frac{1}{r}\frac{\partial V}{\partial \theta}\vec{e_\theta} + \frac{1}{r\sin\theta}\frac{\partial V}{\partial \varphi}\vec{e_\varphi}\right)$$

Comme $\vec{E} = E_r\vec{e_r}$ alors : $\frac{1}{r}\frac{\partial V}{\partial \theta} = 0$ et $\frac{1}{r\sin\theta}\frac{\partial V}{\partial \varphi} = 0$

$$\vec{E} = E_r\vec{e_r} = -\frac{\partial V}{\partial r}\vec{e_r}$$

$$V(r) = -\int E_r(r)\,dr + Cst$$

$$= -\int \frac{1}{4\pi\varepsilon_0}\frac{Q_0}{r^2}\,dr + Const$$

$$= -\frac{1}{4\pi\varepsilon_0} Q_0 \int \frac{1}{r^2}\,dr = \frac{1}{4\pi\varepsilon_0}\frac{Q_0}{r} + \underset{\substack{\| \\ V(\infty) \\ \| \\ 0}}{Cst}$$

Electromag :

Fin exercice 4.1 :

Raccourci pour $r \leq R_0$:

$$\vec{E} = E_r(r)\hat{e}_r$$

⋮ m̂ chose que cas précédent

$$V(r) = -\int E_r(r)dr + \text{Cste}$$

$$= -\int \frac{1}{4\pi\varepsilon_0} Q_0 \frac{r}{R_0^3} dr + \text{Cste}$$

$$= -\frac{Q_0}{4\pi\varepsilon_0} \times \frac{1}{R_0^3} \int r dr + \text{Cste}$$

$$= \frac{-1}{4\pi\varepsilon_0} \frac{Q_0 r^2}{2R_0^3} + \text{Cste}.$$

Or V est une fonction continue et on se place en $r = R_0$. Ainsi : $V_{int}(R_0) = V_{ext}(R_0)$.

$$\Leftrightarrow \frac{-1}{4\pi\varepsilon_0} \times \frac{Q_0 \cancel{R_0^2}}{2\cancel{R_0^3}} + \text{Cste} = \frac{1}{4\pi\varepsilon_0} \frac{Q_0}{R_0}$$

$$\Leftrightarrow \text{Cste} = \frac{1}{4\pi\varepsilon_0}\left(\frac{Q_0}{R_0} + \frac{Q_0}{2R_0}\right)$$

$$\Leftrightarrow \text{Cste} = \frac{Q_0}{4\pi\varepsilon_0 R_0}\left(1 + \frac{1}{2}\right).$$

$$\Leftrightarrow \text{Cste} = \frac{3}{2} \frac{Q_0}{4\pi\varepsilon_0 R_0}$$

Conclusion : Pour $r \leq R_0$.

$$\boxed{V(r) = \frac{-1}{4\pi\varepsilon_0} \frac{Q_0 r^2}{2R_0^3} + \frac{3}{2} \frac{Q_0}{4\pi\varepsilon_0 R_0}}$$

$$J = \int_{V_{ext}} \varepsilon_0 . E^2(\vec{r})\,dV$$

$$= \varepsilon_0 \int_{V_{ext}} \left(\frac{1}{4\pi\varepsilon_0}\frac{Q_0}{r}\right)^2 dV$$

$$= \frac{\cancel{\varepsilon_0}\, Q_0^2}{16\pi^2 \varepsilon_0^{\cancel{2}}} \int_{V_{ext}} \frac{1}{r^2}\,dV$$

$$= \frac{Q_0^2}{16\pi^2 \varepsilon_0} \int_{R_0}^{+\infty} \frac{1}{r^2}\,dr \int_0^{\pi} \sin\theta\, d\theta \int_0^{2\pi} d\varphi$$

$$= \frac{Q_0^2}{16\pi^2 \varepsilon_0} \frac{1}{R_0} 4\pi$$

$$= \frac{Q_0^2}{4\pi\varepsilon_0 R_0}.$$

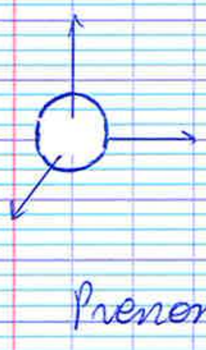
Finalement : $U_e = \frac{1}{2}\left(I + J\right)$

$$= \frac{1}{2}\left(\frac{Q_0^2}{20\pi\varepsilon_0 R_0} + \frac{Q_0^2}{4\pi\varepsilon_0 R_0}\right)$$

$$= \frac{Q_0^2}{\pi\varepsilon_0 R_0}\left(\frac{1}{40} + \frac{1}{8}\right)$$

$$\boxed{U_e = \frac{3}{20\pi\varepsilon_0 R_0} Q_0^2}$$

Exercice 4.2 :



a) Symetrie : axiale

Coordonnée : sphérique.

$$\vec{E}(\vec{r}) = \begin{cases} E(r)\,\hat{e}_r & \text{si } r > R_0 \\ -E(r)\,\hat{e}_r & r \leq R_0 \end{cases}$$

Prenons une sphère fictive de rayon $r \leq R_0$

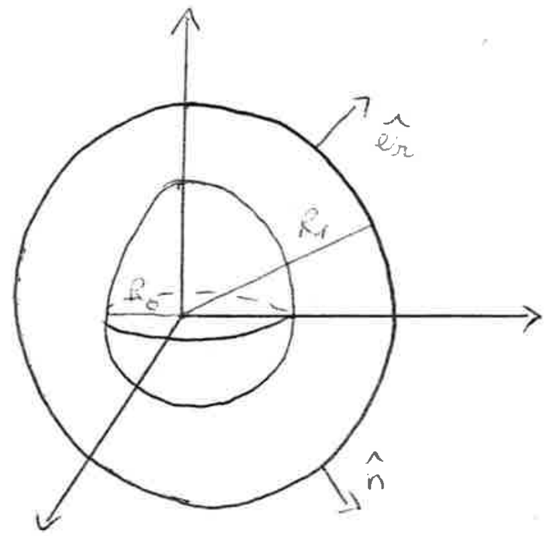
Gauss : $\int E(\vec{r}) \cdot \vec{n}\,dS = \frac{Q_{int}}{\varepsilon_0}$ or $Q_{int} = 0$

donc $\vec{E}(\vec{r}) = \vec{0}$ si $r \leq R_0$

Exercice 4.2 :

a) Méthode de Gauss :

Symétrie axiale : coord sphériques



$$\vec{E}(\vec{r}) = \begin{cases} E(\vec{r})\,\hat{e_r} & r > R_0 \\ -E(\vec{r})\,\hat{e_r} & r \leq R_0 \end{cases}$$

Sphère rayon $r \leq R_0$:

$$\oint_S \vec{E}(\vec{r}).\hat{n}\,dS = \frac{Q_{int}}{\varepsilon_0}$$

Or $Q_{int} = 0$

Donc $\vec{E}(\vec{r}) = \vec{0}$ si $r \leq R_0$

Sphère rayon $R_1 > R_0$:

$$\oint_S \vec{E}(\vec{r}).\hat{n}\,dS = \frac{Q_{int}}{\varepsilon_0} = \frac{Q_0}{\varepsilon_0} \quad (Q_{int} = Q_0)$$

$$\Rightarrow \oint_S E(\vec{r})\,\hat{e_r}.\hat{n}\,dS = \frac{Q_0}{\varepsilon_0}$$

$$\Rightarrow E(R_1)\oint_S dS = \frac{Q_0}{\varepsilon_0}$$

$$\Rightarrow E(R_1)\,S = \frac{Q_0}{\varepsilon_0}$$

$$\Rightarrow E(R_1)\,S = \frac{Q_0}{\varepsilon_0} \quad \text{avec } S = 4\pi R_1^2$$

$$\Rightarrow E(R_1) = \frac{Q_0}{4\pi\varepsilon_0 R_1^2}$$ valable pour tout $R_1 > R_0$ (on l'appelle r)

Donc $\vec{E}(\vec{r}) = \dfrac{1}{4\pi\varepsilon_0}\dfrac{Q_0}{r^2}\,\hat{e_r}$

Donc $\vec{E}(\vec{r}) = \begin{cases} \vec{0} & \text{si } r \leq R_0 \\ \dfrac{1}{4\pi\varepsilon_0}\dfrac{Q_0}{r^2}\hat{e}_r & \text{si } r > R_0 \end{cases}$

b) $V(\vec{r_B}) - V(\vec{r_A}) = -\displaystyle\int_{\vec{r_A}\to\vec{r_B}} \vec{E}(\vec{r}).\hat{t}\,dl = \dfrac{W_{A\to B}}{q}$

$V(\vec{r_A}) = 0$ car A est à l'infini

Pour $r > R_0$:

Donc $V(\vec{r_B}) = -\displaystyle\int_{\vec{r_A}\to\vec{r_B}} \vec{E}(\vec{r}).\hat{t}\,dl = -\int_{\infty}^{r} \frac{1}{4\pi\varepsilon_0}\frac{Q_0}{r^2}\underbrace{\hat{e}_r.\hat{t}}_{-1}\,dl$

$$= \int_{r}^{\infty} \frac{1}{4\pi\varepsilon_0}\frac{Q_0}{r^2}\,dl$$

$$= \frac{Q_0}{4\pi\varepsilon_0}\int_{r}^{\infty} \frac{1}{r^2}\,dl \qquad \text{Or } dl = dr$$

Donc $V(\vec{r_B}) = \dfrac{Q_0}{4\pi\varepsilon_0}\left[\dfrac{-1}{r}\right]_r^{\infty} = \dfrac{1}{4\pi\varepsilon_0}\dfrac{Q_0}{r}$ valable pour tout $r > R_0$

Pour $r \leq R_0$:

$V(\vec{r_B}) = -\displaystyle\int_{\infty}^{R_0} \vec{E}(\vec{r}).\hat{t}\,dl \underbrace{- \int_{R_0}^{r} \vec{E}(\vec{r}).\hat{t}\,dl}_{0}$

$= -\displaystyle\int_{\infty}^{R_0} \vec{E}(\vec{r}).\hat{t}\,dl = V(R_0) = \dfrac{1}{4\pi\varepsilon_0}\dfrac{Q_0}{R_0}$ valable pour tout $r \leq R_0$

Donc $V(\vec{r}) = \begin{cases} \dfrac{1}{4\pi\varepsilon_0}\dfrac{Q_0}{R_0} & \text{si } r \leq R_0 \\ \dfrac{1}{4\pi\varepsilon_0}\dfrac{Q_0}{r} & \text{si } r > R_0 \end{cases}$

c) $U_e = \frac{1}{2}\int_V \rho(\vec{r})\, V(\vec{r})\, dV \qquad (4)$

$U_e = \frac{1}{2}\int_V \varepsilon_0 E^2(\vec{r})\, dV \qquad (5)$

(4) $U_e = \frac{1}{2}\int_V \rho(\vec{r})\, V(\vec{r})\, dV$

$= \frac{1}{2}\oint_S \rho(\vec{r})\, V(\vec{r})\, dS$

$= \frac{\rho}{2}\oint_S V(\vec{r})\, dS$

$= \frac{1}{2}\rho V(R_0)\oint_S dS$

$= \frac{1}{2}\rho V(R_0) S$

$= \frac{1}{2}\frac{Q_0}{S} V(R_0) S$

$= \frac{Q_0}{2}\frac{1}{4\pi\varepsilon_0}\frac{Q_0}{R_0}$

$= \frac{1}{8\pi\varepsilon_0}\frac{Q_0^2}{R_0}$

(5) $U_e = \frac{1}{2}\int_V \varepsilon_0 E^2(\vec{r})\, dV$

$= \frac{1}{2}\varepsilon_0 \int_V \left(\frac{1}{4\pi\varepsilon_0}\frac{Q_0}{r^2}\right)^2 dV$

$= \frac{Q_0^2}{32\pi^2\varepsilon_0}\int_V \frac{1}{r^4}\, dV$

avec $dV = r^2 \sin\theta\, dr\, d\theta\, d\varphi$

Donc $U_e = \frac{Q_0^2}{32\pi^2\varepsilon_0}\int_{R_0}^{\infty}\frac{1}{r^2}\, dr \int_0^{\pi}\sin\theta\, d\theta \int_0^{2\pi} d\varphi$

$= \frac{Q_0^2}{32\pi^2\varepsilon_0}\frac{1}{R_0}\; 2 \times 2\pi$

$= \frac{1}{8\pi^2\varepsilon_0}\frac{Q_0^2}{R_0}$

d) $R_0 = 30\,cm$

$E_{max} = 3\,MV.m^{-1}$

$\vec{E}(\vec{r}) = \frac{Q_0}{4\pi\varepsilon_0}\frac{1}{r^2} \quad r > R_0$

On fait tendre r vers R_0

Donc $\vec{E}(\vec{r}) = \frac{Q_0}{4\pi\varepsilon_0}\frac{1}{R_0^2}$

Donc $Q_0 = 4\pi\varepsilon_0 R_0^2 \vec{E}(\vec{r}) = 4\pi \times 8{,}85\times10^{-12} \times (30\times10^{-2})^2 \times 3\times10^{6}$

$= 30\,\mu C$

$$V(R_0) = \frac{1}{4\pi\varepsilon_0}\frac{Q_0}{R_0} = \frac{1}{4\pi \times 8{,}85 \times 10^{-12}} \times \frac{30 \times 10^{-6}}{30 \times 10^{-2}} = 900\,kV$$

$$\frac{Q_0}{V(R_0)} = \frac{30 \times 10^{-6}}{900 \times 10^{3}} = 33{,}33\,pF$$

$$V(1m) = \frac{1}{4\pi\varepsilon_0}\frac{Q_0}{1} = \frac{30 \times 10^{-6}}{4\pi \times 8{,}85 \times 10^{-12}} = 270\,kV$$

$$V(2m) = \frac{V(1m)}{2} = 135\,kV$$

$$V(3m) = \frac{V(1m)}{3} = 90\,kV$$