Algorithms & Data Structures

Lesson 3: Asymptotic Analysis

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Efficiency

- What does it mean for an algorithm to be *efficient*?
  - We primarily care about *time* (and sometimes *space*)
- Is the following a good definition?
  - "An algorithm is efficient if, when implemented, it runs quickly on real input instances"
  - Where and how well is it implemented?
  - What constitutes "real input"?
  - How does the algorithm *scale* as input size changes?

Gauging efficiency (performance)

- Well, why not just run the program and time it?
  - Too much *variability*, not reliable or *portable*:
    - Hardware: processor(s), memory, etc.
    - OS, Java version, libraries, drivers
    - Other programs running
    - Implementation dependent
  - Choice of input
    - Testing (inexhaustive) may *miss* worst-case input
    - Timing does not *explain* relative timing among inputs (what happens when n doubles in size)
- Often want to evaluate an *algorithm*, not an implementation
  - Even *before* creating the implementation

Comparing algorithms

When is one *algorithm* (not *implementation*) better than another?

- Various possible answers (clarity, security, …)
- But a big one is *performance*: for sufficiently large inputs, runs in less time (our focus) or less space

We will focus on large inputs because probably any algorithm is “plenty good” for small inputs (if *n* is 10, probably anything is fast)

- Time difference really shows up as n grows

Answer will be *independent* of CPU speed, programming language, coding tricks, etc.

Answer is general and rigorous, complementary to “coding it up and timing it on some test cases”

- Can do analysis before coding!

We usually care about worst-case running times

- **Has proven reasonable in practice**
  - Provides some guarantees
- **Difficult to find a satisfactory alternative**
  - What about average case?
  - Difficult to express full range of input
  - Could we use randomly-generated input?
  - May learn more about generator than algorithm

Example

2	3	5	16	37	50	73	75	126

Find an integer in a *sorted* array

```
// requires array is sorted
// returns whether k is in array
boolean find(int[]arr, int k){
    ???
}
```

Linear search

2	3	5	16	37	50	73	75	126

Find an integer in a *sorted* array

```
// requires array is sorted
// returns whether k is in array
boolean find(int[]arr, int k){
   for(int i=0; i < arr.length; ++i)
      if(arr[i] == k)
        return true;
   return false;
}
```

Best case?

k is in arr[0]

c1 steps

= *O*(1)

Worst case?

k is not in arr

c2*(arr.length)

= *O*(arr.length)

Binary search

2	3	5	16	37	50	73	75	126

Find an integer in a *sorted* array

– Can also be done non-recursively but "doesn't matter" here

```
// requires array is sorted
// returns whether k is in array
boolean find(int[]arr, int k){
   return help(arr,k,0,arr.length);
}
boolean help(int[]arr, int k, int lo, int hi) {
   int mid = (hi+lo)/2; // i.e., lo+(hi-lo)/2
   if(lo==hi)       return false;
   if(arr[mid]==k)  return true;
   if(arr[mid]< k)  return help(arr,k,mid+1,hi);
   else             return help(arr,k,lo,mid);
}
```

Binary search

Best case: c1 steps = $O(1)$

Worst case: $T(n)$ = c2 steps + $T(n/2)$ where n is **`hi-lo`**

- $O($**`log`** $n)$ where n is **`array.length`**
- Solve *recurrence equation* to know that…

```
// requires array is sorted
// returns whether k is in array
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boolean help(int[]arr, int k, int lo, int hi) {
   int mid = (hi+lo)/2;
   if(lo==hi)       return false;
   if(arr[mid]==k)  return true;
   if(arr[mid]< k)  return help(arr,k,mid+1,hi);
   else             return help(arr,k,lo,mid);
}
```

Solving Recurrence Relations

1. Determine the recurrence relation. What is the base case?
   – $T(n) = c2 + T(n/2)$ $\qquad$ $T(1) = c1$
2. "Expand" the original relation to find an equivalent general expression *in terms of the number of expansions*.
   – $T(n) = c2 + c2 + T(n/4)$
     $= c2 + c2 + c2 + T(n/8)$
     $= \ldots$
     $= c2(k) + T(n/(2^k))$
3. Find a closed-form expression by setting *the number of expansions* to a value (e.g. 1) which reduces the problem to a base case
   – $n/(2^k) = 1$ means $n = 2^k$ means $k = \log_2 n$
   – So $T(n) = c2 \log_2 n + T(1)$
   – So $T(n) = c2 \log_2 n + c1$ (get to base case and do it)
   – So $T(n)$ is $O(\log n)$

Ignoring constant factors

- So binary search is $O(\texttt{log}\ n)$ and linear is $O(n)$
  - But which is faster?

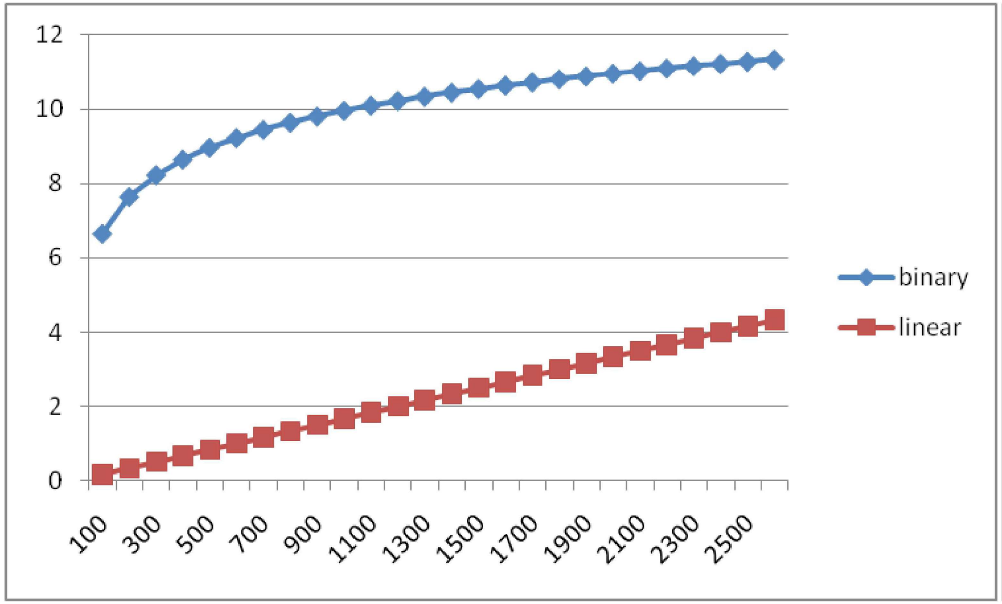
- Could depend on constant factors
  - How *many* assignments, additions, etc. for each n
    - *E.g.* $T(n) = 5{,}000{,}000n$ vs. $T(n) = 5n^2$
  - And could depend on overhead unrelated to n
    - *E.g.* $T(n) = 5{,}000{,}000 + \log n$ vs. $T(n) = 10 + n$

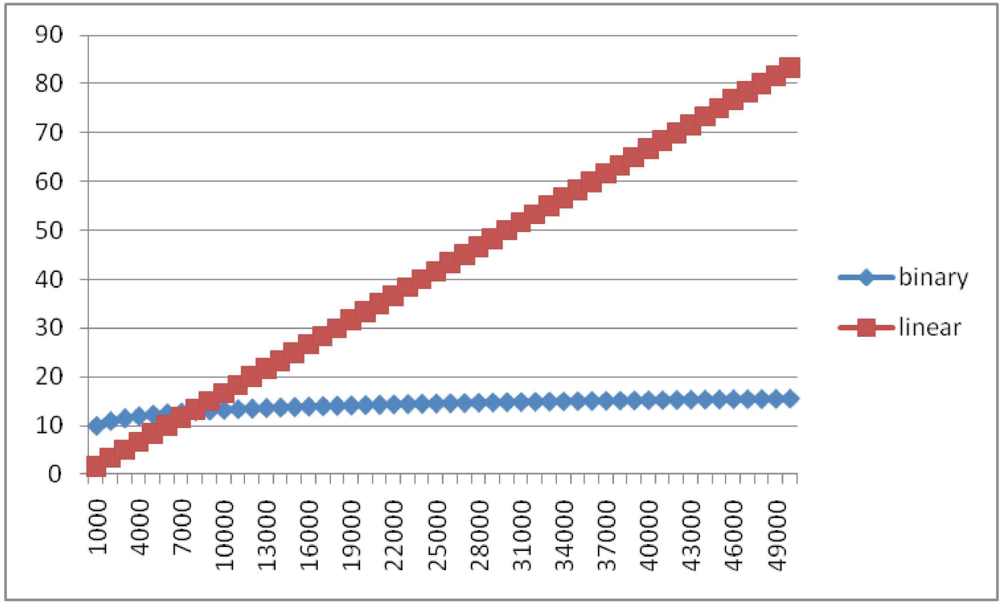
- But there exists some n_0 such that for all $n > n_0$ binary search wins

- Let's play with a couple plots to get some intuition…

Example

- Let's try to "help" linear search
  - Run it on a computer 100x as fast (say 2014 model vs. 1994)
  - Use a new compiler/language that is 3x as fast
  - Be a clever programmer to eliminate half the work
  - So doing each iteration is 600x as fast as in binary search





Big-Oh relates functions

We use O on a function f(n) (for example n^2) to mean *the set of functions with asymptotic behavior* ***less than or equal to*** f(n)

So $(3n^2+17)$ **is in** $O(n^2)$

- $3n^2+17$ and n^2 have the same asymptotic behavior

Confusingly, we also say/write:

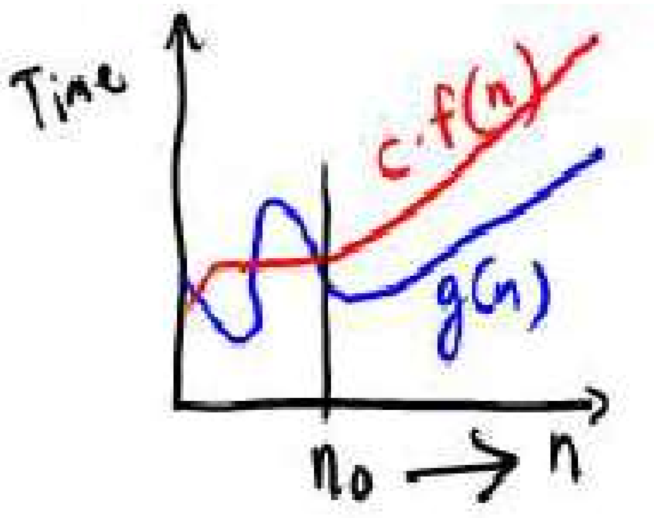
- $(3n^2+17)$ **is** $O(n^2)$
- $(3n^2+17) = O(n^2)$

But we would never say $O(n^2) = (3n^2+17)$

Big-O, formally

Definition: $g(n)$ is in $O(\ f(n)\ )$ if there exist positive constants c and n_0 such that

$$g(n) \leq c\, f(n) \qquad \text{for all } n \geq n_0$$



- To show $g(n)$ is in $O(\ f(n)\ )$, pick a c large enough to "cover the constant factors" and n_0 large enough to "cover the lower-order terms"
  - Example: Let $g(n) = 3n^2+17$ and $f(n) = n^2$

    c=5 and n_0=10 is more than good enough

    $(3*10^2)+17 \leq 5*10^2$ so $3n^2+17$ is $O(n^2)$
- This is "less than or equal to"
  - So $3n^2+17$ is also $O(n^5)$ and $O(2^n)$ etc.
    - But usually we're interested in the tightest upper bound.

Example 1, using formal definition

- Let $g(n) = 1000n$ and $f(n) = n^2$
  - To prove $g(n)$ is in $O(f(n))$, find a valid c and n_0
  - The "cross-over point" is $n=1000$
    - $g(n) = 1000*1000$ and $f(n) = 1000^2$
  - So we can choose $n_0=1000$ and $c=1$
    - Many other possible choices, e.g., larger n_0 and/or c

> Definition: $g(n)$ is in $O(\ f(n)\ )$ if there exist positive constants c and n_0 such that
>
> $g(n) \leq c\ f(n)$ for all $n \geq n_0$

Example 2, using formal definition

- Let $g(n) = n^4$ and $f(n) = 2^n$
  - To prove $g(n)$ is in $O(f(n))$, find a valid c and n_0
  - We can choose $n_0=20$ and $c=1$
    - $g(n) = 20^4$ vs. $f(n) = 1*2^{20}$

- Note: There are many correct possible choices of c and n_0

> Definition: $g(n)$ is in $O(\, f(n)\,)$ if there exist positive constants c and n_0 such that
>
> $g(n) \leq c\, f(n)$ for all $n \geq n_0$

What's with the c

- The constant multiplier c is what allows functions that differ only in their largest coefficient to have the same asymptotic complexity
- Consider:

  $g(n) = 7n+5$

  $f(n) = n$

– These have the same asymptotic behavior (linear)
  - So $g(n)$ is in $O(f(n))$ even through $g(n)$ is always larger
  - The c allows us to provide a coefficient so that $g(n) \leq c\, f(n)$

– In this example:
  - To prove $g(n)$ is in $O(f(n))$, have $c = 12$, $n_0 = 1$

    $(7*1)+5 \leq 12*1$

What you can drop

- Eliminate coefficients because we don't have units anyway
  - $3n^2$ versus $5n^2$ doesn't mean anything when we have not specified the cost of constant-time operations

- Eliminate low-order terms because they have vanishingly small impact as n grows

- Do NOT ignore constants that are not multipliers
  - n^3 is not $O(n^2)$
  - 3^n is not $O(2^n)$

(This all follows from the formal definition)

More Asymptotic Notation

- Upper bound: $O(\, f(n)\,)$ is the set of all functions asymptotically less than or equal to $f(n)$
  - $g(n)$ is in $O(\, f(n)\,)$ if there exist constants c and n_0 such that

    $g(n) \leq c\, f(n)$ for all $n \geq n_0$

- Lower bound: $\Omega(\, f(n)\,)$ is the set of all functions asymptotically greater than or equal to $f(n)$
  - $g(n)$ is in $\Omega(\, f(n)\,)$ if there exist constants c and n_0 such that

    $g(n) \geq c\, f(n)$ for all $n \geq n_0$

- Tight bound: $\theta(\, f(n)\,)$ is the set of all functions asymptotically equal to $f(n)$
  - $g(n)$ is in $\theta(\, f(n)\,)$ if **both** $g(n)$ is in $O(\, f(n)\,)$ **and**

    $g(n)$ is in $\Omega(\, f(n)\,)$

Correct terms, in theory

A common error is to say $O(\ f(n)\ )$ when you mean $\theta(\ f(n)\ )$

- Since a linear algorithm is also $O(n^5)$, it's tempting to say "this algorithm is exactly $O(n)$"
- That doesn't mean anything, say it is $\theta(n)$
- That means that it is not, for example $O(\texttt{log}\ n)$

Less common notation:

- "little-oh": intersection of "big-Oh" and *not* "big-Theta"
  - *For all c*, there exists an n_0 such that… $\leq$
  - Example: array sum is $o(n^2)$ but not $o(n)$
- "little-omega": intersection of "big-Omega" and *not* "big-Theta"
  - *For all c*, there exists an n_0 such that… $\geq$
  - Example: array sum is $\omega(\texttt{log}\ n)$ but not $\omega(n)$

What we are analyzing

- The most common thing to do is give an *O* upper bound to the worst-case running time of an algorithm

- Example: binary-search algorithm
  - Common: $O(\log n)$ running-time in the worst-case
  - Less common: $\theta(1)$ in the best-case (item is in the middle)
  - Less common (but very good to know): the find-in-sorted-array ***problem*** is $\Omega(\log n)$ in the worst-case
    - *No* algorithm can do better
    - A ***problem*** cannot be $O(f(n))$ since you can always make a slower algorithm

Other things to analyze

- **Space instead of time**
  - Remember we can often use space to gain time

- **Average case**
  - Sometimes only if you assume something about the *probability distribution* of inputs
  - Sometimes uses randomization in the algorithm
    - Will see an example with sorting
  - Sometimes an *amortized guarantee*
    - Average time over any sequence of operations
    - Will discuss in a later lecture

Big-Oh Caveats

- Asymptotic complexity focuses on behavior for large *n* and is independent of any computer / coding trick

- But you can "abuse" it to be misled about trade-offs

- Example: $n^{1/10}$ vs. **`log`** *n*
  - Asymptotically $n^{1/10}$ grows more quickly
  - But the "cross-over" point is around $5 * 10^{17}$
  - So if you have input size less than 2^{58}, prefer $n^{1/10}$

- For *small n*, an algorithm with worse asymptotic complexity might be faster. If you care about performance for small *n* then the constant factors can matter

Summary

Analysis can be about:

- The problem or the algorithm (usually algorithm)
- Time or space (usually time)
- Best-, worst-, or average-case (usually worst)
- Upper-, lower-, or tight-bound (usually upper or tight)