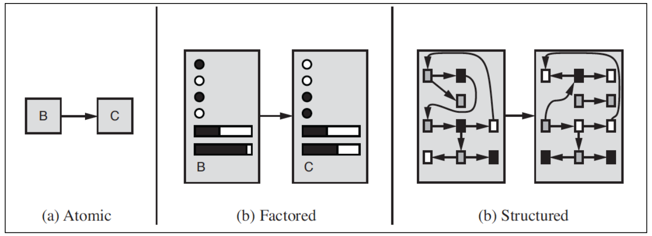
TDDC17

Fö 4
Constraint Satisfaction Problems

Representing States



So far:
Unformed search
Heuristic search

Today:
Constraints

- 10 binary variables can describe 2^10 = 10,024 worlds
- 20 binary variables can describe 2^20 = 1,048,576 worlds
- 30 binary variables can describe 2^30= 1,073,741,824
- 100 binary variables can describe 2^100 = 1,267,650,600,228,401,496,703,205,376 worlds

An Example Problem

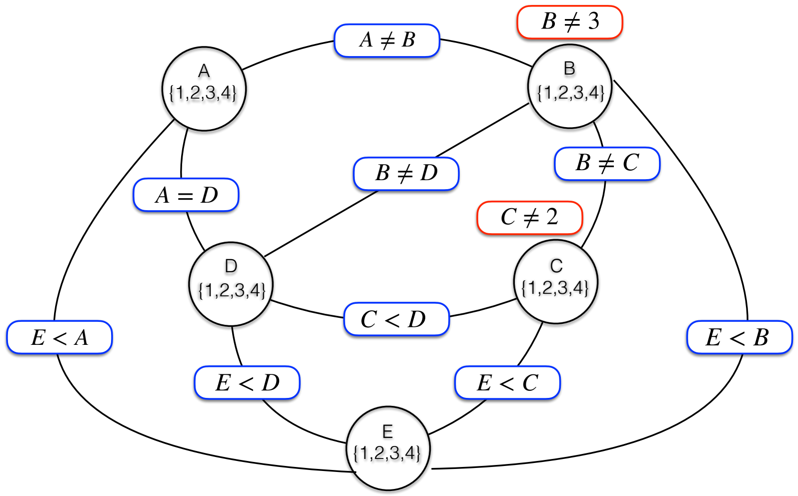
Suppose a delivery robot must carry out a number of Delivery activities, a,b,c,d,e. Suppose that each activity Can happen in the next four hours at hour 1,2,3,or 4.

Let A,B,C,D,E, be variables representing the time activities a,b,c,d,e start, respectively.

Variable domains for each activity start-time will be {1,2,3,4}
Assume the following constraints on start times:

$A \neq B$ $\quad A = D$ $\quad B \neq D$ $\quad B \neq C$ $\quad C < D$ $\quad E < A$ $\quad E < B$ $\quad E < D$ $\quad E < C$ $\quad B \neq 3$

Delivery Robot: Constraint graph



Constraint Satisfaction Problem

X is a set of variables $\{X_1, \ldots, X_n\}$
D is a set of domains $D_1, \ldots, D_n$, one for each variable
C is a set of constraints on X

The constraints restrict the values variables can simultaneously take.

Solution to a CSP
An assignment of a value from its domain to each variable, in such a way that all the constraints are satisfied

One may want to find 1 solution, all solutions, an optimal solution, or a good solution based on an objective function defined in terms of some or all variables.

Delivery Robot: CSP Formulation

```
RobD = CSP({'A':{1,2,3,4},'B':{1,2,3,4},'C':{1,2,3,4},
            'D':{1,2,3,4}, 'E':{1,2,3,4}},
           [Constraint((‘B',), ne_(3)),
            Constraint((‘C',), ne_(2)),
            Constraint((‘A','B'), ne),
            Constraint((‘B','C'), ne),
            Constraint((‘C','D'), lt),
            Constraint((‘A','D'), eq),
            Constraint((‘A','E'), gt),
            Constraint((‘B','E'), gt),
            Constraint((‘C','E'), gt),
            Constraint((‘D','E'), gt),
            Constraint((‘B','D'), ne)])
```


Variable, Domain and Constraint Types

Types of variables/domains
- Discrete variables
  - Finite or infinite domains
- Boolean variables
  - Finite domain
- (Continuous variables)
  - Infinite domain

Types of constraints
- Unary constraints (1)
- Binary constraints (2)
- Higher-Order contraints (>2)
- Linear constraints
- Nonlinear constraints

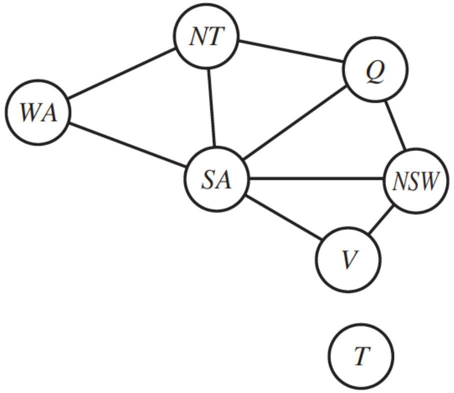
Some Special cases
- Linear programming
  - Linear inequalities forming a convex region. Continuous domains.
  - Solutions in time polynomial to the number of variables
- Integer programming
  - Linear constraints on integer variables.

Any higher-order/finite domain csp's can be translated into binary/finite domain CSPs! (In the book, R/N stick to these)

Map Coloring Problem

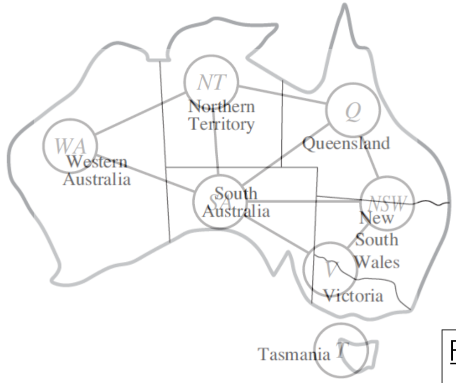
Color each of the territories/states red, green or blue with no neighboring region having the same color

Let's Abstract



Constraint Graph
Nodes are variables
Arcs are constraints

Our Representation



- Associate a variable with each region.
- Introduce a set of values the variables can be bound to.
- Define constraints on the variable/value pairs

Goal:

Find a set of legal bindings satisfying the constraints!

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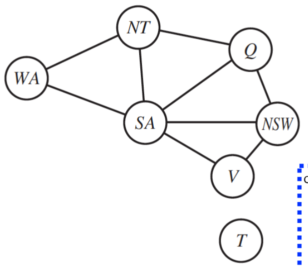
Map Coloring: Australia

Map Coloring Specification
- X = {WA, NT, SA, Q, NSW, V, T}
- D = {{red, green, blue}, ..., {red, green, blue}}
- C is a set of constraints on X

Binary Constraints

Constraints
WA ≠ NT,
WA ≠ SA,
NT ≠ Q,
NT ≠ SA,
Q ≠ SA,
Q ≠ NSW,
V ≠ SA,
V ≠ NSW




```
CSP({'WA':{'red','blue','green'},'NT':{'red','blue','green'},
     'SA':{'red','blue','green'},'Q':{'red','blue','green'},
     'NSW':{'red','blue','green'},'V':{'red','blue','green'},
     'T':{'red','blue','green'}},
        [Constraint(('WA','NT'),ne),Constraint(('WA','SA'),ne),
         Constraint(('NT','SA'),ne),Constraint(('NT','Q'),ne),
         Constraint(('SA','Q'),ne),Constraint(('SA','NSW'),ne),
         Constraint(('SA','V'),ne), Constraint(('Q','NSW'),ne) ])
```

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Sudoku

	1	2	3	4	5	6	7	8	9
A			3		2		6		
B	9			3		5			1
C			1	8		6	4		
D			8	1		2	9		
E	7								8
F			6	7		8	2		
G			2	6		9	5		
H	8			2		3			9
I			5		1		3		

	1	2	3	4	5	6	7	8	9
A	4	8	3	9	2	1	6	5	7
B	9	6	7	3	4	5	8	2	1
C	2	5	1	8	7	6	4	9	3
D	5	4	8	1	3	2	9	7	6
E	7	2	9	5	6	4	1	3	8
F	1	3	6	7	9	8	2	4	5
G	3	7	2	6	8	9	5	1	4
H	8	1	4	2	5	3	7	6	9
I	6	9	5	4	1	7	3	8	2

Variables
81(one for each cell)

Constraints:
Alldiff() for each row
Alldiff() for each column
Alldiff for each 9 cell area

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Another Example

- Suppose our territories are coverage areas, each with a sensor that monitors the area.
- Each sensor has N possible radio frequencies
- Sensors overlap if they are in adjacent areas
- If sensors overlap, they can not use the same frequency

Find a solution where each sensor uses a frequency that does not interfere with adjacent coverage areas

This is an N-map coloring problem!

Advantages of CSPs

- Representation is closer to the original problem.
- Representation is the same for all constraint problems.
- Algorithms used are domain independent with the same general purpose heuristics for all problems
- Algorithms are simple and often find solutions quite rapidly for large problems
  - CSPs often more efficient than regular state-space search because it can quickly eliminate large parts of the search space
  - Many problems intractable for regular state-space search can be solved efficiently with a CSP formulation.

Solving a CSP: Types of Algorithms

Search
(choose a new variable assignment)

Inference
Constraint Propagation
(reduce the # of legal values for a variable and propagate to other variables)

Constraint Propagation

Sometimes solves the problem without search!

Constraint Propagation

Pre-Processing

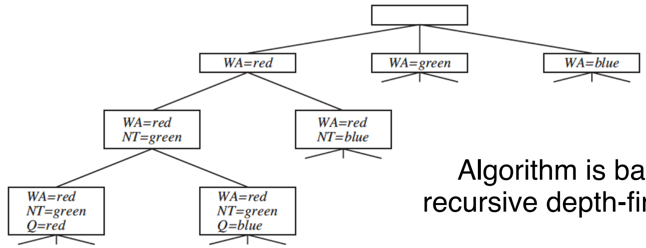
Search

Constraint Propagation

Interleave

Search

Simple Backtracking Search Algorithm for CSPs



Algorithm is based on recursive depth-first search

If a value assignment to a variable leads to failure then it is removed from the current assignment and a new value is tried (backtrack)

The algorithm will interleave inference with search

Backtracking Algorithm (Search with Inference)

```
function BACKTRACKING-SEARCH(csp) returns a solution, or failure
  return BACKTRACK({ }, csp)

function BACKTRACK(assignment, csp) returns a solution, or failure
  if assignment is complete then return assignment
  var ← SELECT-UNASSIGNED-VARIABLE(csp)
  for each value in ORDER-DOMAIN-VALUES(var, assignment, csp) do
      if value is consistent with assignment then
          add {var = value} to assignment
          inferences ← INFERENCE(csp, var, value)
          if inferences ≠ failure then
             add inferences to assignment
             result ← BACKTRACK(assignment, csp)
             if result ≠ failure then
                return result
      remove {var = value} and inferences from assignment
  return failure
```

Domain Independent Heuristics

Inference

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Backtracking Algorithm (Search with Inference)

```
def backtrack_search_constraints(cspproblem):

  def backtrack(assignment,cspproblem):
    if assignment.complete(cspproblem):
      return assignment
    var = select_unassigned_variable(cspproblem,assignment)
    for value in order_domain_values(var,assignment,cspproblem):
      if assignment.consistent_with(var,value, cspproblem):
        assignment.add(var,value)
        infer = inferences(cspproblem,var,assignment)
        if not infer == 'failure':
          assignment.add_inferences(infer)
          result = backtrack(assignment, cspproblem)
          if not result == 'failure':
            return result
        assignment.remove(var,value)
        assignment.remove_inferences(infer)
    return 'failure'

  return backtrack(assignment(),cspproblem)

def select_unassigned_variable(cspproblem,assignment):
  return assignment.unassigned_variables(cspproblem)[0]

def inferences(cspproblem,var,assignment):
  return {}

def order_domain_values(var,assignment,cspproblem):
  return list(cspproblem.domains[var])
```

b_s_f calls:

Domain Independent Heuristics

Need to instantiate!

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Potential Problems with backtracking search

- Variable choice and value assignment is arbitrary
  - Which variable should be assigned?
    - SELECT-UNASSIGNED-VARIABLE()
  - Which values should be assigned first?
    - ORDER-DOMAIN-VALUES()
  - Conflicts detected too late (empty value domain)
    - Conflicts not detected until they actually occur.
    - What are the implications of current variable assignments for the other unassigned variables?
      - INFERENCE()
  - Thrashing
    - Major reason for failure is conflicting variables, but these conflicts are continually repeated throughout the search
    - When a path fails, can the search avoid repeating the failure in subsequent paths?
      - One solution: Intelligent Backtracking

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Variable Selection Strategies

- Variable Selection Strategy
  - SELECT-UNASSIGNED-VARIABLE():
  - Minimum Remaining Values (MRV) heuristic
    - Choose the variable with the fewest remaining legal values.
    - Try first where you are most likely to fail (fail early!..hard cases 1st)
      - Fail-First heuristic
      - Will knock out large parts of the search tree.
  - Degree Heuristic
    - Select the variable that is involved in the largest number of constraints on other unassigned variables.
    - Hard cases first!
    - Tie breaker when MRV can't be applied.

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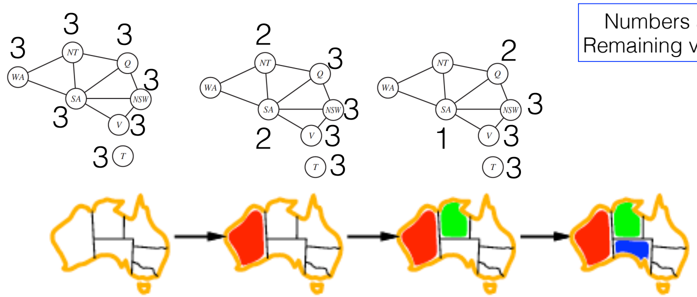
Minimal Remaining Values (MRV)

"Attempts to fail early, thus removing parts of the search tree"

Numbers are Remaining values

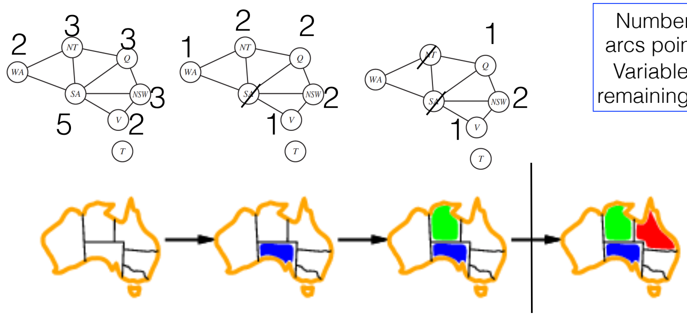


Actually, if we used degree heuristic to break ties: then SA would be chosen here instead of WA

Degree Heuristic

"Attempts to reduce the branching factor in search tree"

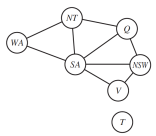
Numbers are arcs pointing to Variables with remaining values



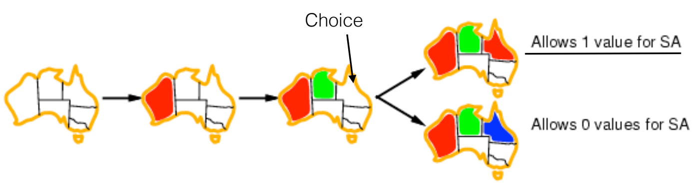
Can't use MRV: All have same number

MRV: Only one choice of color!

Value Selection Strategies



- Value Selection Strategy
  - ORDER-DOMAIN-VALUES()
  - Least-constraining-value heuristic
    - Choose the value that rules out the fewest choices of values for the neighboring variables in the constraint graph.
      - Fail Last heuristic
    - Maximize the number of options….least commitment.
      - Only useful when searching for one solution.



Inference in CSPs

Key Idea:

- Treat each variable as a node and each binary constraint as an arc in our constraint graph.
- Enforcing local consistency in each part of the graph eliminates inconsistent values throughout the graph.
- The less local we get when propagating the more expensive inference becomes.

Node Consistency
A single variable is *node consistent* if all values in the variable's domain satisfy the variables *unary* constraints

WA ≠ green

WA={red, ~~green~~,blue}

Arc Consistency

Definition

Arc (V_i,V_j) is arc consistent if for every value **x** in the domain of V_i there is some value **y** in the domain of V_j such that V_i = x and V_j = y satisfies the constraints between V_i and V_j.

A constraint graph is *arc-consistent* if all its arcs are arc consistent

- The property is not symmetric.
- Arc consistent constraint graphs do not guarantee consistency of the constraint graph and thus guarantee solutions. They do help in reducing search space and in early identification of inconsistency.
- AC-3 (O(n^2d^3)), AC-4 (O($n^2 * d^2$)) are polynomial algorithms for arc consistency, but 3SAT (in NP) is a special case of CSPs, so it is clear that AC-3, AC-4 do not guarantee (full) consistency of the constraint graph.

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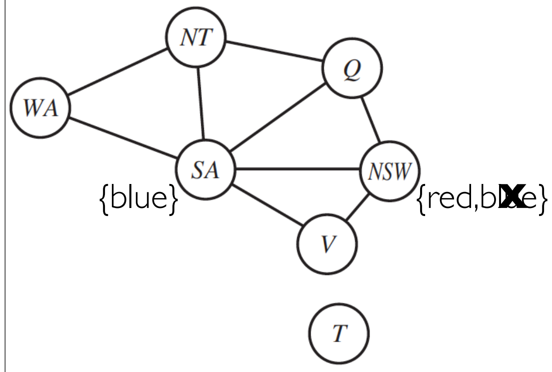
Arc Consistency is not Symmetric



SA ⟶ NSW

Is arc-consistent

NSW ⟶ SA

Is not arc-consistent

Remove blue from NSW

NSW ⟶ SA

Is now arc-consistent

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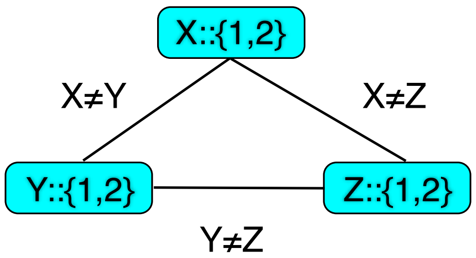
Arc Consistency does not guarantee a solution



Arc consistent constraint graph with no solutions

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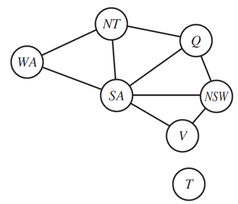
Simple Inference: Forward Checking

Whenever a variable X is assigned, look at each unassigned variable Y that is connected to X by a constraint and delete from Y's domain any value that is inconsistent with the value chosen for X. [make all Y's arc consistent with X]

Forward check each time A variable binding is added:

	WA	*NT*	*Q*	*NSW*	*V*	*SA*	*T*
Initial domains	R G B	R G B	R G B	R G B	R G B	R G B	R G B
After *WA=red*	R	G B	R G B	R G B	R G B	G B	R G B
After *Q=green*	R	B	G	R B	R G B	B	R G B
After *V=blue*	R	B	G	R	B		R G B



Note1: After *WA*=red, Q=green, *NT* and *SA* both have single values. This eliminates branching.

Note 2: After *WA*=red, *Q*=green, there is an inconsistency between *NT*, *SA*, but it is not noticed.

Note 3: After *V*=blue, an inconsistency is detected

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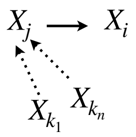
AC3 Algorithm

AC-3 propagates inferences

function AC-3(*csp*) **returns** false if an inconsistency is found and true otherwise
  inputs: *csp*, a binary CSP with components $(X,\ D,\ C)$
  local variables: *queue*, a queue of arcs, initially all the arcs in *csp*

  while *queue* is not empty **do**
    $(X_i,\ X_j) \leftarrow$ REMOVE-FIRST(*queue*)
    if REVISE(*csp*, X_i, X_j) **then**
      if size of $D_i = 0$ **then return** *false*
      for each X_k **in** X_i.NEIGHBORS - $\{X_j\}$ **do**
        add $(X_k,\ X_i)$ to *queue*
  return *true*

function REVISE(*csp*, X_i, X_j) **returns** true iff we revise the domain of X_i
  revised $\leftarrow$ *false*
  for each x **in** D_i **do**
    if no value y in D_j allows (x,y) to satisfy the constraint between X_i and X_j **then**
      delete x from D_i
      revised $\leftarrow$ *true*
  return *revised*



Returns an arc consistent binary constraint graph or false because a variable domain is empty (and thus no solution)

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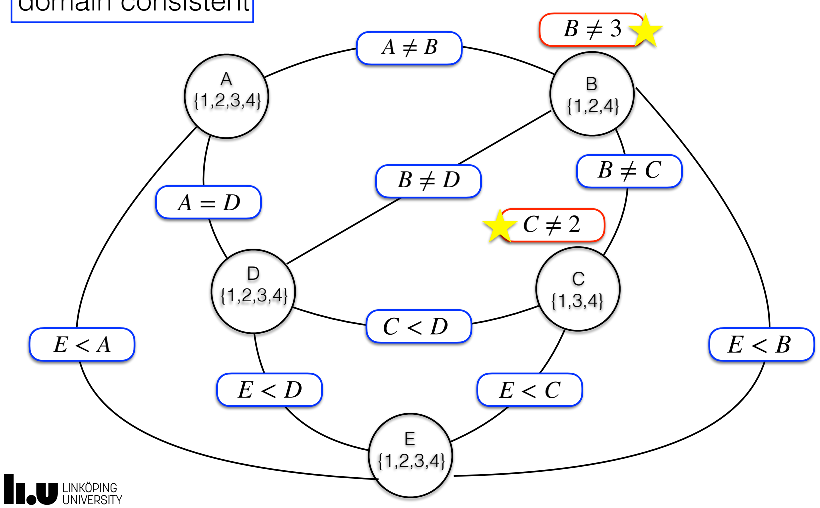
Version of AC-3 in Python

```
def ac3(csp):
    queue = fifo_queue()
    init_queue(csp,queue)
    domains = dict(csp.domains)

    def revise(xi,xj):
        revised = False
        for con in csp.constraints:
            constraint = [con for con in csp.constraints if con.scope == (xi,xj)][0]
        for value in domains[xi]:
            consistent = any(constraint.holds({xi:value,xj:val}) for val in domains[xj])
            if not consistent:
                domain = list(domains[xi])
                domain.remove(value)
                domains[xi] = set(domain)
                revised = True
        return revised

    def neighbors(xi,xj):
        neighbors = [con.scope[0] for con in csp.constraints if con.scope[1] == xi and not con.scope[0] == xj]
        return neighbors

    while queue.queue():
        arc_ij = queue.pop()
        xi = arc_ij[0]
        xj = arc_ij[1]
        if revise(xi,xj):
            if len(csp.domains[xi]) == 0:
                return [False,None]
            for xk in neighbors(xi,xj):
                queue.insert((xk,xi))

    return [True,domains]
```

Robot Delivery Example

First make domain consistent



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Apply AC-3

Initial Queue

~~{1, 2, 3, 4}~~	~~:~~	~~('A', 'B')~~	~~:~~	~~{1, 2, 4}~~
~~{1, 2, 3, 4}~~	~~:~~	~~('A', 'D')~~	~~:~~	~~{1, 2, 3, 4}~~
{ 2, 3, 4}	:	('A', 'E')	:	{1, 2, 3, 4}
~~{1, 2, 4}~~	~~:~~	~~('B', 'A')~~	~~:~~	~~{ 2, 3, 4}~~
~~{1, 2, 4}~~	~~:~~	~~('B', 'C')~~	~~:~~	~~{1, 3, 4}~~
~~{1, 2, 4}~~	~~:~~	~~('B', 'D')~~	~~:~~	~~{1, 2, 3, 4}~~
{ 2, 4}	:	('B', 'E')	:	{1, 2, 3, 4}
~~{1, 3, 4}~~	~~:~~	~~('C', 'B')~~	~~:~~	~~{ 2, 4}~~
{1, 3, }	:	('C', 'D')	:	{1, 2, 3, 4}
{ 3, 4}	:	('C', 'E')	:	{1, 2, 3, 4}
{ 2, 3, 4}	:	('D', 'A')	:	{ 2, 3, 4}
~~{ 2, 3, 4}~~	~~:~~	~~('D', 'B')~~	~~:~~	~~{ 2, 4}~~
{ 2, 3, 4}	:	('D', 'C')	:	{ 3, }
{ 2, 3, 4}	:	('D', 'E')	:	{1, 2, 3, 4}
{1, 2, 3, 4}	:	('E', 'A')	:	{ 2, 3, 4}
{1, 2, 3, 4}	:	('E', 'B')	:	{ 2, 4}
{1, 2, 3, 4}	:	('E', 'C')	:	{ 3, }
{1, 2, 3, 4}	:	('E', 'D')	:	{ 2, 3, 4}

Final Result:

{'A': {4}, 'B': {2}, 'C': {3}, 'D': {4}, 'E': {1}}

{2, 3, 4}	:	('A', 'B')	:	{2, 4}
{1, 2, 4}	:	('B', 'C')	:	{ , 3, }
{2, 4}	:	('B', 'D')	:	{2, 3, 4}
{3, 4}	:	('C', 'D')	:	{2, 3, 4}

Note: Domains are not part of the Arch queue.

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Path Consistency

{r,g,b} WA --> NT {r,g,b}
{r,g,b} NT --> WA {r,g,b}
{r,g,b} WA --> SA {r,g,b}
{r,g,b} SA --> WA {r,g,b}
{r,g,b} NT --> SA {r,g,b}
{r,g,b} SA --> NT {r,g,b}

Note that arc consistency does not help us out for the map coloring problem!

It only looks at pairs of variables

Definition

A two variable set $\{X_i, X_j\}$ is path consistent with respect to a 3rd variable X_m if, for every assignment $\{X_i=a, X_j=b\}$ consistent with the constraints on $\{X_i, X_j\}$, there is an assignment to X_m that satisfies the constraints on $\{X_i, X_m\}$ and $\{X_m, X_j\}$.

$X_i \longrightarrow X_m \longrightarrow X_j$

K-Consistency

A CSP is **k-consistent** if, for any set of **k-1** variables and for any consistent assignment to those variables, a consistent value can always be found for the *k*th variable.

1-consistency: node consistency
2-consistency: arc consistency
3-consistency: path consistency

A CSP is strongly k-consistent if it is k-consistent and is also k-1 consistent, k-2 consistent, ..., 1-consistent.

In this case, we can find a solution in $O(n^2d)$! but establishing n-consistency takes time exponential in n in the worst case and space exponential in n!

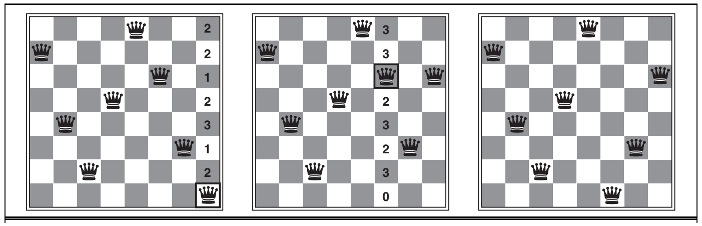
Local Search for CSPs

```
function MIN-CONFLICTS(csp, max_steps) returns a solution or failure
  inputs: csp, a constraint satisfaction problem
          max_steps, the number of steps allowed before giving up

  current ← an initial complete assignment for csp
  for i = 1 to max_steps do
     if current is a solution for csp then return current
     var ← a randomly chosen conflicted variable from csp.VARIABLES
     value ← the value v for var that minimizes CONFLICTS(var, v, current, csp)
     set var = value in current
  return failure
```

Conflicts(var, v, current, csp) - Counts the number of constraints violated by a particular value, given the rest of the current assignment.

A Local Search CSP Example



- At each step a queen is chosen for reassignment in its column
- The number of conflicts is shown in each square (# of attacking queens for a square)
- Move the queen to the min-conflicts square
- Above: A two step solution

For n-queens: runtime is independent of problem size!!!!
Can solve million queens problem in an average of 50 steps!!!

Hubble Space Telescope: Reduced time to schedule a week of observations from three weeks to 10 minutes!!