Ray intersections

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1 Ray intersections in $\mathbb{R}^2$

This section deals with intersections with rays $r : \mathbb{R} \to \mathbb{R}^2$ and their intersection with different paramtrised functions $f : \mathbb{R} \to \mathbb{R}^2$.

1.1 Linesegments

The linesegment $l : \mathbb{R} \to \mathbb{R}^2$ between points $a = (a_x, a_y)$ and $b = (b_x, b_y)$ in $\mathbb{R}^2$ with $a \neq b$ is parametrized as

$$l(t) = ta + (1-t)b, t \in [0,1]$$

The ray $r : \mathbb{R} \to \mathbb{R}^2$ with origin o and direction $d \neq O$ is given by

$$r(s) = o + ds, s \geq 0$$

We now proceeds to find the intersection between the ray and the linesegment

$$l(t) = r(s) = o + ds = \begin{pmatrix} o_x \\ o_y \end{pmatrix} + \begin{pmatrix} d_x \\ d_y \end{pmatrix} s \tag{1}$$

with $d_x \neq 0 \wedge d_y \neq 0$. When split into its components that leaves us two equations with two unknown t and s. For now, let's assume that $d_x \neq 0$ and we'll solve for s using the first component which gives

$$s = \frac{l_x(t) - o_x}{d_x} \tag{2}$$

Inserting that into the second component gives us

$$\begin{aligned} l_y(t) = o_y + d_y \frac{l_x(t) - o_x}{d_x} \quad &\Rightarrow \\ d_x(ta_y + (1-t)b_y) = d_x o_y + d_y(ta_x + (1-t)b_x - o_x) \quad &\Rightarrow \\ t(d_x a_y - b_y d_x - d_y a_x + d_y b_x) = -d_x b_y + d_x o_y + d_y b_x - d_y o_x \quad &\Rightarrow \\ t = \frac{d_x(o_y - b_y) - d_y(o_x - b_x)}{d_x(a_y - b_y) - d_y(a_x - b_x)} \end{aligned}$$

Find the corresponding s using (2). The ray intersects the linesegment if $t \in [0,1]$ and $s \geq 0$. The above fraction for t is valid too if $d_x = 0$. In that case we would have supposed $d_y \neq 0$ and instead used second component to solve the first. The symmetry of the fraction makes it the same when all x becomes y and vice-versa.

1.2 Quadratic Bézier curves

The quadratic Bézier curve is the points traced by the function $b : \mathbb{R} \to \mathbb{R}^2$ given control points p_0, p_1 and p_2 with $p_0 = (p_{0x}, p_{0y})$, etc.

$$b(t) = (1-t)^2 p_0 + 2t(1-t)p_1 + t^2 p_2, t \in [0,1] \tag{3}$$

The ray $r : \mathbb{R} \to \mathbb{R}^2$ with origin o and direction $d \neq O$ is given by

$$r(s) = o + ds, s \geq 0$$

Rewriting $b(t)$ as a quadratic polynomial with regard to t gives us

$$\begin{aligned} b(t) &= (1-t)^2 p_0 + 2t(1-t)p_1 + t^2 p_2 && (4) \\ &= (t^2 + 1 - 2t)p_0 + 2p_1 t - 2p_1 t^2 + p_2 t^2 && (5) \\ &= (p_0 - 2p_1 + p_2)t^2 + (2p_1 - 2p_0)t + p_0 && (6) \end{aligned}$$

We now proceed to find the intersection between the ray and the curve

$$\begin{pmatrix} b_x(t) \\ b_y(t) \end{pmatrix} = b(t) = r(s) = \begin{pmatrix} o_x \\ o_y \end{pmatrix} + \begin{pmatrix} d_x \\ d_y \end{pmatrix} s \tag{7}$$

with $d_x \neq 0 \wedge d_y \neq 0$. When split into its components that leaves us two equations with two unknown t and s. For now, let's assume that $d_x \neq 0$ and we'll solve for s using the first component which gives

$$s = \frac{b_x(t) - o_x}{d_x} \tag{8}$$

Which we use to expand the second component, leaving us with a equation with the only unknown being t

$$o_y + d_y \frac{b_x(t) - o_x}{d_x} = b_y(t) \Rightarrow$$

$$d_y(b_x(t) - o_x) = d_x(b_y(t) - o_y)$$

Expanding this using (6) and collecting the terms leaves t as the solutions to

$$At^2 + Bt + C = 0, t \in [0, 1] \tag{9}$$

with coefficents

$$\begin{aligned}
A &= d_x(p_{0y} - 2p_{1y} + p_{2y}) - d_y(p_{0x} - 2p_{1x} + p_{2x}) \\
B &= d_x(2p_{1y} - 2p_{0y}) - d_y(2p_{1x} - 2p_{0x}) \\
C &= d_x(p_{0y} - o_y) - d_y(p_{0x} - o_x)
\end{aligned}$$

The values for s can be found by substituting the roots t into (8). Throw away the root t where that substitution yields an $s < 0$.

On the other hand, when $d_x = 0$ and thus $d_y \neq 0$ we'll use the symmetry of the above to realize that the coefficient for (9) simply are $-A$, $-B$, $-C$. The values of t found then have matching values of s given by

$$s = \frac{b_y(t) - o_y}{d_y} \tag{10}$$

2 Ray intersections in $\mathbb{R}^3$

2.1 Swept quadratic Bézier cuve

Sweeping a quadratic Bézier curve a distance $e \neq 0$ along the positive z-axis yields a quadratic Bézier patch $p : \mathbb{R}^2 \to \mathbb{R}^3$, that can be parametrized as

$$p(t, u) = \begin{pmatrix} b(t) \\ eu \end{pmatrix}, u, t \in [0, 1]$$

with the two-dimensional $b(t)$ as defined in (3).

Describing our ray $r : \mathbb{R} \to \mathbb{R}^3$ in 3 dimensional space as well using origin $o = (o_x, o_y, o_z)$ and direction $d = (d_x, d_y, d_z)$ as $r(s) = o + ds$ with $s \geq 0$, we now look for the intersections where $p(u, v) = r(s)$. Each dimension gives an equation, so we have three equations with three unknown. The third component yields

$$o_z + d_z s = eu \Rightarrow u = \frac{o_z + d_z s}{e} \tag{11}$$

We're already done now. Equation (9) gives us the values of t. After discarding values outside $[0, 1]$, we'll use (8) to get corresponding values for s. We'll discard negative values for s and use (11) above to find the corresponding values for the parameter u at the intersection points. Throw away values of s that makes u fall outside of $[0, 1]$.