Calculations Pertaining to Analytically Determining Arc Lengths of Gaussian Porcesses

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1 Lines as parametric equations

In higher dimensional spaces, lines need to be expressed as parametric equations.

In particular, the line segment connecting a and b may be expressed as such:

$$f(t) = a + t(b - a), t \in [0, 1]$$

It may also be desireable to scale t such that a one unit change in t corresponds to a movement of one unit in euclidean terms in the high D space:

$$f(t) = a + \frac{b - a}{||b - a||_2} t, t \in [0, ||b - a||_2]$$

2 Generalized Arc Length of a GP Mean Surface for General Kernel

In this section, we will semi-rigorously derive the arc length along a line for $\mathbb{R}^n$ equipped with an p-norm along a GP mean surface, a slight generalization of arc length in Euclidean space, which follows a very similar derivation.

Begin with the limit definition of arc-length from a to b, $a, b \in \mathbb{R}^n$. Let $t_i = a + \frac{b-a}{n} i$ (that is, t_i represents some point along the line from a to b), and $s(x, y)$ represent the line segment from x to y.

$$\text{ArcLen} = \lim_{n \to \infty} \sum_{i=1}^{n} ||s(\mu(t_i), \mu(t_{i-1})||_p$$

That is, simply the sum of the vanishingly small line segments along the path.

For p-norms, we can separate the norm as such:

$$\text{ArcLen} = \lim_{n \to \infty} \sum_{i=1}^{n} (||s(t_i, t_{i-1})||_p^p + (|\mu(t_i) - \mu(t_{i-1})|)^p)^{\frac{1}{p}}$$

and re-arrange the limit:

$$= \lim_{n\to\infty} \sum_{i=1}^{n} (||s(t_i, t_{i-1})||_p^p + (|\mu(t_{i-1} + \frac{b-a}{n}) - \mu(t_{i-1})|)^p)^{\frac{1}{p}}$$

$$= \lim_{n\to\infty} \sum_{i=1}^{n} ((\frac{||b-a||_p}{n})^p + (|\mu(t_{i-1} + \frac{||b-a||_p}{n} \frac{b-a}{||b-a||_p}) - \mu(t_{i-1})|)^p)^{\frac{1}{p}}$$

$$= \lim_{n\to\infty} \sum_{i=1}^{n} ((\frac{||b-a||_p}{n})^p + (|\frac{||b-a||_p}{n} \frac{\mu(t_{i-1} + \frac{||b-a||_p}{n} \frac{b-a}{||b-a||_p}) - \mu(t_{i-1})}{\frac{||b-a||_p}{n}}|)^p)^{\frac{1}{p}}$$

$$= ||b-a||_p \lim_{n\to\infty} \frac{1}{n} \sum_{i=1}^{n} (1 + (|\frac{\mu(t_{i-1} + \frac{||b-a||_p}{n} \frac{b-a}{||b-a||_p}) - \mu(t_{i-1})}{\frac{||b-a||_p}{n}}|)^p)^{\frac{1}{p}}$$

We notice that, $n \to \infty \implies \frac{||b-a||_p}{n} \to 0$, which means that:

$$\lim_{n\to\infty} \frac{\mu(t_{i-1} + \frac{||b-a||_p}{n} \frac{b-a}{||b-a||_p}) - \mu(t_{i-1})}{\frac{||b-a||_p}{n}} = \lim_{h\to 0} \frac{\mu(t_{i-1} + h\frac{b-a}{||b-a||_p}) - \mu(t_{i-1})}{h}$$

$= (\nabla_{b-a}\mu)(t_{i-1})$, the directional derivative of μ in the direction of $b-a$ evaluated at t_{i-1}.

Now for the hand-wavy part: speaking very loosely:

$$\lim_{n\to\infty} \frac{1}{n} \sum_{i=1}^{n} f(t_i) - f(t_{i-1}) = \int_a^b f(t)dt$$

so we have, in our case:

$$||b-a||_p \lim_{n\to\infty} \frac{1}{n} \sum_{i=1}^{n} (1 + (|\frac{\mu(t_{i-1} + \frac{||b-a||_p}{n} \frac{b-a}{||b-a||_p}) - \mu(t_{i-1})}{\frac{||b-a||_p}{n}}|)^p)^{\frac{1}{p}}$$

$$= \int_0^1 (1 + (|\nabla_{b-a}\mu)(a + t(b-a))|)^p)^{1/p} dt$$

, our desired result.