Calculations Pertaining to Analytically Determining Arc Lengths of Gaussian Porcesses

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1 Lines as parametric equations

In higher dimensional spaces, lines need to be expressed as parametric equations.

In particular, the line segment connecting a and b may be expressed as such:

$$f(t) = a + t(b - a), t \in [0, 1]$$

It may also be desireable to scale t such that a one unit change in t corresponds to a movement of one unit in euclidean terms in the high D space:

$$f(t) = a + \frac{b - a}{||b - a||_2} t, t \in [0, ||b - a||_2]$$

2 Arc Length of a GP Mean Surface for General Kernel

I will seek to derive the equation for the arc length along a line segment from a to b for a GP with arbitrary kernel function.

In general, to derive the arc length of a function $f : \mathbb{R}^n \to \mathbb{R}$ over the curve S, this is the quantity of interest:

$$\int_{s \in S} \sqrt{||s||_2 + \nabla f(s)^2} ds$$

In our case, the mean surface of a GP is defined as such:

$$\mu(x) = k(x)' K^{-1} y$$

Where $k(x)$ is the kernel function evaluated with respect to each training point, stored in a vector, K is the kernel matrix evaluated for training points, and y is the observed training responses.

Also, the curve is simply a line from a to b:

$$S = \{s : \exists t \in [0, 1] \text{ s.t. } s = a + t(b - a)\}$$

So we can rewrite the integral above as:

$$\int_{t=0}^{1} \sqrt{||s(t)||_2 + (k(x)'K^{-1}y)^2} = \int_{t=0}^{1} \sqrt{||a + t(b-a)||_2 + (k(x)'K^{-1}y)^2}$$

We will denote $K^{-1}y$ as α, an n-vector.