A Linear Programming Representation of the 1 Dimensional PMCLP-PCR Problem

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1 Index Sets

1. Supply Zone index $i \in \{1, \dots, p\}$
2. Demand Zone index $j \in \{1, \dots, n\}$

2 Parameters

1. Weight of target j: v_j.
2. Length of DZ j: l_j.
3. Beginning point of DZ on axis j: c_j.
4. Length of SZ i: s_i.

3 Decision Variables

1. Where SZ i begins along DZ j: $x_{i,j,f}$.
2. Where SZ j ends along DZ j: $x_{i,j,t}$.

There are therefore $2np$ many variables at this stage, linearization will induce more.

4 Objective Function

The objective is to maximize coverage. The coverage function is complicated by the fact that multiple coverage must be accounted for.

k'th order coverage is defined as area on a particular DZ covered by exactly k zones simultaneously.

k'th order coverage among k given SZ's may be detected by looking at difference between the minimum "to" value among all SZ's and the maximum "from" value among SZ's in consideration.

In math notation, let I be a set of supply zone indices, and let $a = |I|$. Then the simultaneous coverage of those a supply zones over DZ j can be determined as:

$$\min_{i\in I}[x_{i,j,t}] - \max_{i'\in I}[x_{i',j,f}]$$

To determine a'th order coverage, we must sum the area covered simultaneously by all possible a-tuples. There are $\binom{p}{a}$ such tuples.

There are therefore $\sum_{k=1}^{p}\binom{p}{a}$ many terms in the objective function, partitioned into p parts by "order" of coverage, for each DZ, giving $n\sum_{k=1}^{p}\binom{p}{a}$ terms in total.

Let us define $\mathcal{I}_k$ as the set of all sets of indices of cardinality k.

Calculating coverage is good and well, but we need to relate this coverage back to the objective function. In calculating first order coverage, we double count areas covered by two SZ's, so we must subtract double coverage. However, subtracting double coverage gets rid of areas covered by three SZ's twice, so we must add back triple covered area, which counts four-times-covered area twice, which we must subtract, and so on…

Therefore, the objective function is to maximize the following expression:

$$\text{maximize } \sum_{j=1}^{n}\sum_{k=1}^{p}\sum_{I\in\mathcal{I}_k}(-1)^{k-1}(\min_{i\in I}[x_{i,j,t}] - \max_{i'\in I}[x_{i',j,f}])$$

4.1 Reformulation as a Linear Objective

Though the objective function is presently nonlinear, we may linearize it with the aide of some constraints and slack variables.

We first need to change all max's in the objective to min's. This may be done by negating min's arguments, and then negating the entire extremum statement, i.e. $\max\{a : a \in A\}$ becomes $-\min\{-a : a \in A\}$.

At this stage, the objective function becomes:

$$\text{maximize } \sum_{j=1}^{n}\sum_{k=1}^{p}\sum_{I\in\mathcal{I}_k}(-1)^{k-1}(\min_{i\in I}[x_{i,j,t}] + \min_{i'\in I}[-x_{i',j,f}])$$

It is possible to linearize min statements under a maximization problem by adding a surplus variable which is constrained to be less than its arguments, that is:

$$\text{maximize: } \min\{a : a \in A\}$$

becomes

$$\text{maximize } x$$

$$st : -a + x \leq 0 \forall a \in A$$

.

Given some ordering r of the sets in each $\mathcal{I}_k$, we may therefore reformulate our objective as:

$$\text{maximize} \sum_{j=1}^{n} \sum_{k=1}^{p} \sum_{r=1}^{|\mathcal{I}_k|} (-1)^{k-1} (y_{k,j,r,t} + y_{k,j,r,f})$$

This is a linear function, as desired. However, as we just discussed, we need to add constraints. These will be discussed in the next section.

There were $2np$ many variables prior to linearization. As discussed above, there are $n \sum_{k=1}^{p} \binom{p}{k}$ many terms in the objective, each of which has two slack variables, adding $2n \sum_{k=1}^{p} \binom{p}{k}$ variables for $2np + 2n \sum_{k=1}^{p} \binom{p}{k}$ many variables in the linearized problem in total. This may be simplified as $2np + n2^p$ many variables.

5 Constraints

1. Don't fall off the DZ axes ($2pn$ constraints):

$$x_{i,j,f} \leq l_j \forall i$$

$$x_{i,j,t} \leq l_j \forall i$$

2. DZ axis variables must map back to rectangles (pn^2 constraints):

$$x_{i,j,t} - x_{i,j',f} \leq s_i \forall i, j, j'$$

3. No "negative extent" rectangles (np constraints):

$$x_{i,j,t} - x_{i,j,f} \geq 0 \forall i, j$$

4. Nonnegativity ($2np$ constraints):

$$x_{i,j,t} \geq 0 \forall i, j$$

$$x_{i,j,f} \geq 0 \forall i, j$$

5.1 Linearizing Constraints

Further, if we wish to linearlize the objective function as discussed above, we must add these constraints on the slack variables:

$$y_{k,j,r,t} \leq x_{i,j,t} \forall k, r, j, i \in I_{r,k}$$

$$y_{k,j,r,f} \leq -x_{i,j,f} \forall k, r, j, i \in I_{r,k}$$

This induces $n \sum_{k=1}^{p} \binom{p}{k} 2k$ constraints.

In total, we therefore have $np(3 + p + 2^p)$ many constraints.