

# A Linear Programming Representaiton of the 1 Dimensional PMCLP-PCR Problem

Nathan Wycoff

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## 1 Index Sets

1. Supply Zone index  $i \in \{1, \dots, p\}$
2. Demand Zone index  $j \in \{1, \dots, n\}$

## 2 Parameters

1. Weight of target  $j$ :  $v_j$ .
2. Length of DZ  $j$ :  $l_j$ .
3. Beginning point of DZ on axis  $j$ :  $c_j$ .
4. Length of SZ  $i$ :  $s_i$ .

## 3 Decision Variables

1. Where SZ  $i$  begins along DZ  $j$ :  $x_{i,j,f}$ .
2. Where SZ  $j$  ends along DZ  $j$ :  $x_{i,j,t}$ .

## 4 Objective Function

The objective is to maximize coverage. The coverage function is complicated by the fact that multiple coverage must be accounted for.

$k$ 'th order coverage is defined as area on a particular DZ covered by exactly  $k$  zones simultaneously.

$k$ 'th order coverage among given SZ's may be detected by looking at difference between the minimum "to" value among all SZ's and the maximum "from" value among SZ's in consideration.

In math notation, let  $I$  be a set of supply zone indecies, and let  $a = |I|$ . Then the simultaneous covelage of those  $a$  supply zones can be determined as for DZ  $j$  is:

$$\min_{i \in I} [x_{i,j,t}] - \max_{i' \in I} [x_{i',j,f}]$$

To determine  $a$ 'th order coverage, we must sum the area covered simultaneously by all possible  $a$ -tuples. There are  $\binom{p}{a}$  such tuples.

There are therefore  $(\sum_{a=1}^p \binom{p}{a})$  many terms in the objective function, partitioned into  $p$  parts by "order" of coverage.

Let us define  $\mathcal{I}_k$  as the set of all sets of  $k$  indices.

Calculating coverage is good and well, but we need to relate this coverage back to the objective function. In calculating first order coverage, we double count areas covered by two SZ's, so we must subtract double coverage. However, subtracting double coverage gets rid of areas covered by three SZ's, so we must add back four-times-covered area, which removes five-times-covered area twice, which we must add back, and so on. . .

Therefore, the objective function is to maximize the following expression:

$$\text{maximize } \sum_{j=1}^n \sum_{k=1}^p \sum_{I \in \mathcal{I}_k} (-1)^{k-1} (\min_{i \in I} [x_{i,j,t}] - \max_{i' \in I} [x_{i',j,f}])$$

#### 4.1 Reformulation as a Linear Objective

Though the objective function is presently nonlinear, we may linearize it with the aide of some constraints and slack variables.

We first need to change all  $\max$ 's in the objective to  $\min$ 's. This may be done by negating  $\min$ 's arguments, and then negating the entire extremum statement, i.e.  $\max\{a : a \in A\}$  becomes  $-\min\{-a : a \in A\}$ .

At this stage, the objective function becomes:

$$\text{maximize } \sum_{j=1}^n \sum_{k=1}^p \sum_{I \in \mathcal{I}_k} (-1)^{k-1} (\min_{i \in I} [x_{i,j,t}] + \min_{i' \in I} [-x_{i',j,f}])$$

It is possible to linearize  $\min$  statements under a maximization problem by adding a surplus variable which is constrained to be less than its arguments, that is:

$$\text{maximize: } \min\{a : a \in A\}$$

becomes

$$\text{maximize } x$$

$$st : -a + x \leq 0 \forall a \in A$$

Given some ordering  $r$  of the sets in each  $\mathcal{I}_k$ , we may therefore reformulate our objective as:

$$\text{maximize } \sum_{j=1}^n \sum_{k=1}^p \sum_{r=1}^{|\mathcal{I}_k|} (-1)^{k-1} (y_{k,j,r,t} + y_{k,j,r,f})$$

This is a linear function, as desired. However, as we just discussed, we need to add constraints. These will be discussed in the next section.

## 5 Constraints

1. Don't fall off the DZ axes ( $2pn$  constraints):

$$x_{i,j,f} \leq l_j \forall i$$

$$x_{i,j,t} \leq l_j \forall i$$

2. DZ axis variables must map back to rectangles ( $pn^2$  constraints):

$$x_{i,j,t} - x_{i,j',f} \leq s_i \forall i, j, j'$$

3. No "negative extent" rectangles ( $np$  constraints):

$$x_{i,j,t} - x_{i,j,f} \geq 0 \forall i, j$$

4. Nonnegativity ( $2np$  constraints):

$$x_{i,j,t} \geq 0 \forall i, j$$

$$x_{i,j,f} \geq 0 \forall i, j$$

### 5.1 Linearizing Constraints

Further, if we wish to linearize the objective function as discussed above, we must add these constraints on the slack variables:

$$y_{k,j,r,t} \leq x_{i,j,t} \forall k, r, j, i \in I_{r,k}$$

$$y_{k,j,r,f} \leq x_{i,j,f} \forall k, r, j, i \in I_{r,k}$$

This induces  $n \sum_{k=1}^p \binom{p}{k} 2k$  constraints.